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TRIGONOMETRY DEVELOPMENT IN ANCIENT AND MEDIEVAL INDIA

MATH 464WI:
HISTORY OF MATHEMATICS WITH DR. RICHARD DELAWARE

ABSTRACT:

This paper explores the history and possible mathematical methods behind the development of trigonometry in ancient and medieval India. Specifically, it describes possible methods for the construction of the Sine tables, as well as methods of interpolating sine values from these tables.

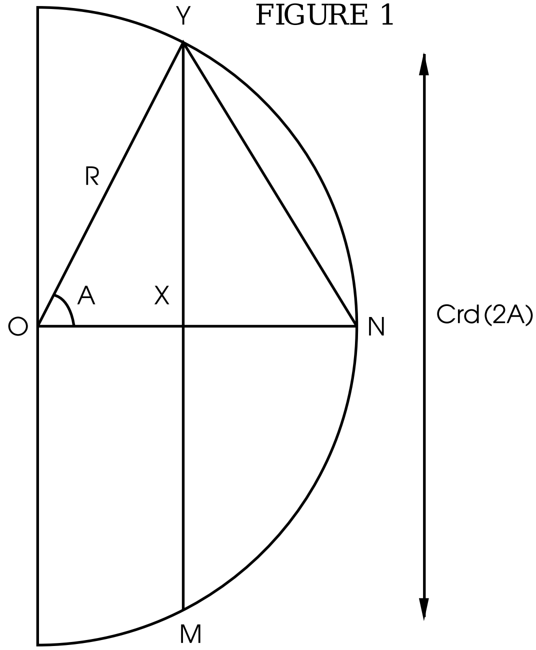
Trigonometry developed in India for the same reason that it did in the rest of the ancient world: to solve astronomy problems. The earliest reference to trigonometry in India occurs in the text "Suryasidhanta" (c. 400) (1 p.229). Indian knowledge of trigonometry most likely first derived from the ancient Greeks, as the Indian standard radius of a circle was 3438, which is the same radius R as was used by the ancient Greek astronomer Hipparchus (5 p.252). While it may be that ancient Indian astronomers simply inherited this standard radius from Hipparchus (c. 190 BC - 120 BC), another conjecture, based on several different Indian texts, is that the value of R was found by the equation $2\pi R = C$ where C is a circle's circumference and R is the circle's radius. As Indian astronomers had an astonishingly accurate value of π (3.1416) as early as 499, and C was expressed in minutes (360° = 21600 minutes, which is written 21600'), we obtain:

$$R = \frac{C}{2\pi} \approx \frac{21600'}{2(3.1416)} \approx \left(3437' + \frac{967'}{1309}\right) \approx 3438'$$

This diffusion of knowledge from ancient Greece is thought to have possibly occurred along Roman trade routes (5 p.253), but regardless of how trigonometry first came to India, its mathematicians were quick to make vast improvements. Indian mathematicians are credited with not only being the first to use sine and cosine functions (As well as the lesser-known versed-sine or "versine", see Figure 1) (5 p.252), but also produced accurate sine tables, developed multiple algorithms for approximating sine, and thus cosine, and eventually discovered approximations for sine and cosine that are equivalent to today's Maclaurin series expansions for sine and cosine.

The following figure shows the definition of the different trigonometric functions, specifically cosine, sine, chord, and versine:

FIGURE 1



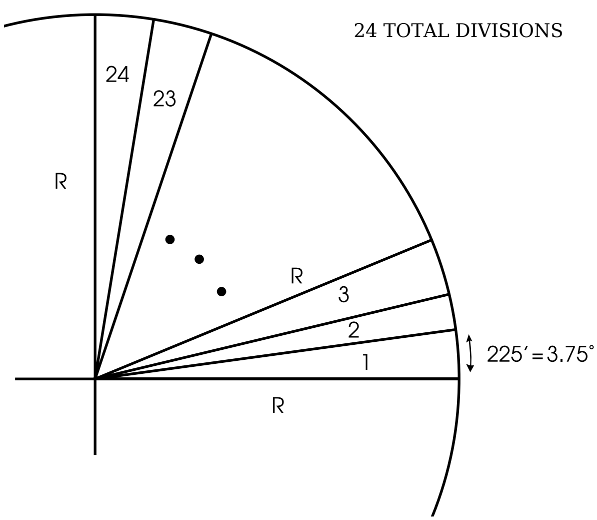
- *OY* = *ON* = *R*, where *R* is the radius of the circle, typically 3438
- *OX* = *R cos(A):* We will refer to this "Indian cosine" as the capitalized *Cos(A)*
- *XY* = *R sin(A):* Again, we will refer to this as the capitalized *Sin(A)*
- *YM* = *Crd(2A)* = *2Sin(A)*, where "crd" is the chord of the circle
- *XN* = *Rvcrsine(A)* = *R-Rcos(A)*: The "versed-sine" or "versine" of *A*, referred to as *Vers(A)*

Many ancient astronomy problems involved solving triangles by using a table of chords, rather than sines and cosines (Recall that chords are line segments that connect two points on a circle). Often, an astronomer would have to make calculations with the half-chord of double the angle [$\frac{1}{2}Crd(2A) = Sin(A)$ for a circle of radius *R;* see figure 1](5 p.252), and one of the first Indian improvements to trigonometry was to tabulate values as not a table of chords, but a table of half-chords. These "half-chords" are known today as sines (2 p.7).

An interesting note is that, in Sanskrit, "jya-ardha" means chord-half, and this was frequently abbreviated to just "jya". Indian works were eventually translated to Arabic, and when an Arabic work was translated to Latin, the term was mistranslated to mean "bosom or breast", and thus was written as the Latin word "sinus". This is how we get our term "sine" (5 p.253).

CONSTRUCTION OF SINE TABLES

As mentioned above, one of the first improvements by Indian mathematicians to trigonometry was that of constructing tables of half-chords, or sines. The tables split a quarter-circle of radius *R* into 24 equivalent arcs (see below), which increased in increments of 225' (225', or 225 minutes, is equivalent to $3\frac{3}{4}^{\circ}$, and $24 \times 3\frac{3}{4}^{\circ} = 90^{\circ}$).



From these tabulated sine values, other important values could be derived using [Recall that we are using capital trigonometric functions to represent $R\times$ (Function), e.g. $Rcos(A) = Cos(A)$]:

$$Cos(A) = \sqrt{R^2 - Sin^2(A)}$$
$$Vers(A) = R - Cos(A)$$

The first sine table to show up in India appeared in the early 5th century in the text "Paitamahasiddhanta" (5 p.252). The first well-preserved text containing a somewhat accurate sine table is in the work of Aryabhata I's "Aryabhatiya", written in 499. Here are the first nine values of Aryabhata I's table of sines (As found in 1, p.247), and each Sine is given a number.

SINE NUMBER	ANGLE (MINUTES)	SIN(ANGLE)	SINE DIFFERENCE
1	225′	225′	225′
2	450′	449′	224′
3	675′	671′	222′
4	900′	890′	219′
5	1125′	1105′	215′
6	1350′	1315′	210′
7	1575′	1520′	205′
8	1800′	1719′	199′
9	2025′	1910′	191′

In the last column, the difference between the current Sine and the one before it is given (e.g. the Sine difference of the fifth Sine and the sixth Sine is $Sin(1350') - Sin(1125') = 1315'-1105' = 210'$). Note: Even though Sine number 1 is the "first sine", it is still known that $Sin(0')=0'$, and thus the Sine difference between Sine 1 and Sine 0 is simply

$$Sin(225') - Sin(0') = 225' - 0' = 225'$$

Aryabhata I also gave a rule for calculating the sine values. The translation of this rule is as follows: [Notes in brackets are mine.]

STANZA II, 12: "By what number the second Sine [difference] is less than the first Sine [Sine number 1 from above table], and by the quotient obtained by dividing the sum of the preceding Sine [differences] by the first Sine, by the sum of these two quantities the following Sine [difference] [is] less than the first Sine."

The “second Sine [difference]” is the Sine difference of the previously calculated Sine. For example, when calculating *Sin* (675') (see above table), the “second Sine [difference]” would be 224’, the Sine difference associated with previous Sine, *Sin* (450'). The “first Sine” is Sine number 1, or *Sin* (225') which always equals 225'. The “sum of the preceding Sine [differences]” is simply the sum of all the previously calculated Sine differences, so to continue our example of calculating $Sin(675')$, the “sum of the preceding Sine [differences]” is $(225' + 224')$.

For $Sin(675')$ subtracting the “second Sine [difference]”, $224'$, from the “first Sine”, 225’, is 225’-224’=1”. The “sum of the preceding Sine [differences]”, $225' + 224'$, divided by the “first Sine”, 225’, is $\frac{225'+224'}{225'}$. Summing these two quantities yields $1' + \frac{225'+224'}{225'} \approx 3'$. This gives us what “the following Sine [difference] [is] less than the first Sine” meaning that the Sine difference of $Sin(675')$ is 3’ less than the first Sine, which is $225' - 3' = 222'$. Since we already know the preceding Sine, $Sin(450') = 449'$:

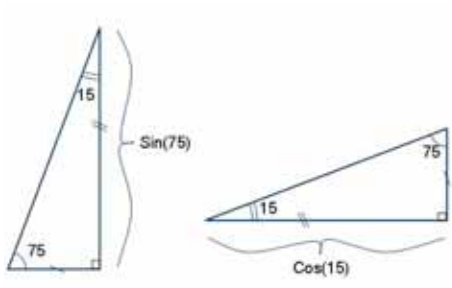
$$Sin(675') = Sin(450') + 222' = 449' + 222' = 671'.$$

Aryabhata I’s rule uses cumulative Sine differences to calculate Sine values. Starting at $Sin(225') = 225'$, it is possible to determine all 24 values of the Sine table, as each new Sine value depends only on the preceding Sine values and their differences. However, as is noted by (5, p.253) and (1, p.254), this rule leads to several discrepancies from Aryabhata I’s actual sine table as the *Sine* of larger values is taken. In fact, differences occur as early as calculating the 8th Sine, $Sin(1800')$. This value, when approximated with Aryabhata I’s rule, is 1717’, yet Aryabhata I reports $Sin(1800') = 1719'$. As the angle of the arc gets larger and larger, the Sine values obtained from this approximation method get more and more inaccurate.

This leads many scholars to believe that this was not the actual way that mathematicians such as Aryabhata I calculated their sine tables. Instead, it is thought that the sine tables were calculated by manipulating the already-known values of $Sin(30^\circ)$, $Sin(45^\circ)$, $Sin(60^\circ)$ and $Sin(90^\circ)$ with the following identities:

- Pythagorean identity: $Sin^2(A) + Cos^2(A) = R^2$, in modern notation this is $sin^2(A) + cos^2(A) = 1$, as $R = 1$].
- Indian half-angle identity: $Sin(\frac{A}{2}) = \frac{1}{2}\sqrt{Sin^2(A) + Vers^2(A)}$ [Note this is equivalent to our half-angle identity $Rsin(\frac{A}{2}) = R\sqrt{\frac{1-cosA}{2}}$.]

For example, to calculate the 10th tabulated Sine, $Sin(2250') = Sin(37.5°)$ [we will use degrees in this example for convenience], we would first use the Indian half-angle identity to calculate $Sin(\frac{30°}{2}) = Sin(15°)$, since $Sin(30°)$ is already known. Next we can use the Pythagorean identity to calculate $Cos(15°)$. By similar triangles, $Cos(15°)$ is equivalent to the Sine of its complement, $Sin(75°)$ [See below]:



Applying the Indian half-angle identity once more to $Sin(75°)$ gives us $Sin(\frac{75°}{2}) = Sin(37.5°)$. All of the 24 tabulated values of Sine and Cosine can be found in this fashion, and the values that result match that of Aryabhata I's.

Approximations for non-tabulated values: Second-Order Interpolations

Although many astronomical problems required values not found in the 24 sections of the Sine table, the first table which had Sines for arcs closer together than $3\frac{3}{4}°$ was not printed until the time of Bhaskara II, around 1150 (5 p.254). As a result, Indian astronomers relied on methods of interpolating, or estimating, Sine values in between the tabulated values. For example, in order to find *Sin(*301'*)*, a value not in the table of 24 Sine values, Indian mathematicians could find a line that passes through the known values of *Sin(*225'*)* and *Sin(*450'*)* and then use that line to approximate *Sin(*301'*)*. This linear approximation is a first-order interpolation, and while it is a good start, it is not very accurate for Sine values.

By 628, Indian mathematician Brahmagupta (598 - 668) developed a method for approximating second-order interpolations of general equations. From this general method, one can find a second-order interpolation for Sine, effectively allowing Indian mathematicians to more accurately find *Sin (A+x)*, where *Sin(A)* is a known, tabulated value, and x

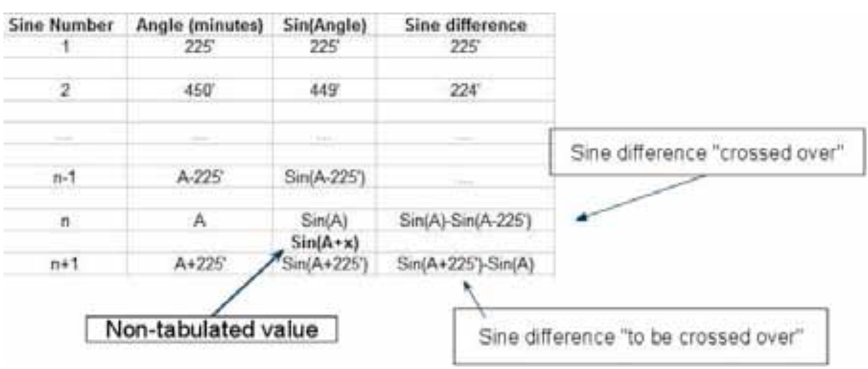
is some value such that 0' < x < 225' (3 p.87).
Brahmagupta's general rule first appeared in Sanskrit:

गत भोग्य खण्डकान्तर दल विकल वधात् शतैर्नवभिराप्तैः ।
तद्युति दलं युतोनं भोग्यादूनाधिकं भोग्यम् ॥

While this rule applies to general functions and not just Sines, we will treat it as it applies to the Sine function, specifically *Sin(A + x)* . The rule translates as follows (3 p.88): [Notes in brackets are mine.]

> "Multiply half the difference of the tabular differences crossed over [*Sin(A)* – *Sin(A* – 225')] and to be crossed over [*Sin(A* + 225') – *Sin(A)*] by the residual arc [*x*] and divide by (the common [tabulated] interval [225']). By the result (so obtained) increase or decrease half the sum of the same (two) differences, according as this [average] is less or greater than the difference to be crossed over. We get the true functional differences to be crossed over [the difference between *Sin(A)* and *Sin(A + x)]*"

Consider the following Sine table:



Sine Number	Angle (minutes)	Sin(Angle)	Sine difference
1	225'	225'	225'
2	450'	449'	224'
...	...	...	...
n-1	A-225'	Sin(A-225')	...
n	A	Sin(A)	Sin(A)-Sin(A-225')
		Sin(A+x)	
n+1	A+225'	Sin(A+225')	Sin(A+225')-Sin(A)

The "tabular difference crossed over" is *Sin(A)* – *Sin*(*A* – 225'), where *Sin(A* – 225'*)* is the tabulated value preceding *Sin(A)*, meaning we have already passed, or "crossed over" this Sine difference on our table of Sines. The tabular difference "to be crossed over" refers to *Sin(A* + 225'*)* – *Sin(A)*, where *Sin(A* + 225') is the next tabulated Sine value after *Sin(A)*, so we have not "crossed over" this Sine difference yet. The "residual arc" is x, which is 0' < x < 225, and the "common [tabulated] interval" is the constant value by which our Sine table increases, 225'.

Now, “multiply half the difference of the tabular differences crossed over [*Sin(A)* – *Sin(A* – 225')] and to be crossed over [*Sin(A* + 225') – *Sin(A)*] by the residual arc [*x*] and divide by (the common [tabulated] interval [225'])”. Since in quadrant I, *Sin(A)* – *Sin(A* – 225'*)* > *Sin(A* + 225') – *Sin(A)*, in order to stay with positive numbers the order in which we subtract matters:

$$\frac{x}{225'}\left[\frac{Sin(A) - Sin(A-225') - [Sin(A+225') - Sin(A)]}{2}\right]$$

$$= \frac{x}{2(225')}\,[2Sin(A) - Sin(A+225') - Sin(A-225')] \qquad (2.1)$$

Next, “by the result (so obtained), [the above (expression 2.1)], increase or decrease half the sum of the same (two) [“crossed over ” and “to be crossed over”] differences”. Thus we increase or decrease (expression 2.1) by

$$\frac{1}{2}[\mathrm{Sin}(A + 225') - \mathrm{Sin}(A) + \mathrm{Sin}(A) - \mathrm{Sin}(A - 225')] = \frac{1}{2}[\mathrm{Sin}(A + 225') - \mathrm{Sin}(A - 225')]:$$

$$\frac{x}{2(225')}\,[2Sin(A) - Sin(A+225') - Sin(A-225')] \pm \frac{1}{2}[\mathrm{Sin}(A + 225') - \mathrm{Sin}(A - 225')]$$

Dealing with the ±: “increase or decrease...according as this (average, ½[*Sin(A* + 225') – *Sin(A* – 225')]) is less or greater than the difference to be crossed over *Sin(A* + 225') – *Sin(A)*”. Recall, in quadrant I, *Sin(A)* – *Sin(A* – 225') > *Sin(A* + 225') – *Sin(A)*. Thus the average of the left side and the right side will always be greater than the right side, namely the difference to be crossed over (If $x > y$, then always $x > \frac{x+y}{2} > y$), so we will use a “_” rather than a “±”:

$$\frac{x}{2(225')}\,[2Sin(A) - Sin(A + 225') - Sin(A - 225')] - \frac{1}{2}[\mathrm{Sin}(A + 225') - \mathrm{Sin}(A - 225')]$$

This gives us “the true functional differences to be crossed over”. Multiplying this by $\frac{x}{225'}$ and adding it to *Sin(A)* gives us our second order interpolation for *Sin(A - x)*:

$$Sin(A) + \frac{x}{225'}\left(\frac{1}{2}[Sin(A+225') - Sin(A-225')] + \frac{x}{225'}\left[\frac{Sin(A+225') - 2Sin(A) + Sin(A-225')}{2}\right]\right)$$

$$= Sin(A) + x\,\frac{Sin(A+225') - 2Sin(A-225')}{2(225')} + \left[\frac{x^2}{2}\right]\,\frac{Sin(A+225') - 2Sin(A) + Sin(A-225')}{(225')^2}$$

Thus we get our end result:

$$Sin(A+x) \approx Sin(A) + x\frac{Sin(A+225') - Sin(A-225')}{2(225')} + \left[\frac{x^2}{2}\right]\frac{Sin(A+225') - 2Sin(A) + Sin(A-225')}{(225')^2}$$

It is interesting to note that Brahmagupta's 2nd-order interpolation method relies on the differences of tabulated values, as did the Aryabhata I method for calculating Sines based on Sine differences.

225' is the common difference in our tabulated values. But, if we replace that with some variable α, measured in radians, and take the limit of Brahmagupta's approximation as $\alpha \to 0$, then we get the second-order Taylor polynomial for *Sin(A + x)* [The proof is omitted here]:

$$Sin(A+x) \approx Sin(A) + (x)Cos(A) - \left[\frac{x^2}{2}\right] Sin(A).$$

Approximations for non-tabulated values: Third-Order Interpolations

This second-order interpolation for *Sin(A + x)* was a definite stepping stone towards Indian mathematicians finding the Taylor series expansion for sine and cosine. The most probable next step was discovered in Paramsevara's "Siddhanta-dipika". Paramsevara was a student of Madhava's, and this work, which was a commentary of a commentary of the early seventh-century text "Mahabhaskariya", gives a more accurate approximation for *Sin(A + x)* than Brahamagupta's. Paramsevara's approximation formula is nearly equivalent to the function's third-order Taylor series approximation; there is a divisor of *4* in the fourth term of his approximation, rather than a 6 (4 p.289).

Again, we are calculating *Sin(A + x)*, where *Sin(A)* is a known tabulated value, and the residual arc x is such that $0' < x < 225'$. The explication of Paramsevara's rule is as follows (4 p.288): [Notes in brackets are mine.]

"The semi-diameter [the radius R] divided by the residual arc [x] becomes the divisor [$\frac{R}{x}$]. Put down [write down for later use] the Sine and again the Cosine at the end of the arc traversed".

[The "arc traversed" is A, so "put down" $Sin(A)$ and $Cos(A)$ for later use.]

"From the Cosine, subtract half the quotient obtained from the divisor-divided Sine [which is] increased by half the quotient obtained from the Cosine by the divisor [$\frac{R}{x}$]".

[First, find "half the quotient obtained from the Cosine by the divisor" $= \frac{1}{2}\frac{Cos(A)}{\frac{R}{x}}$, then add this to $Sin(A)$ giving $Sin(A) + \frac{Cos(A)}{2\frac{R}{x}}$ which is the "Sine [which is] increased...". Next we divide this by "the divisor" $\frac{R}{x}$ to obtain the "divisor-divided Sine", and subtract half of that quotient from $Cos(A)$]:

$$Cos(A) - \frac{1}{2}\left[\frac{Sin(A) + \frac{Cos(A)}{2\frac{R}{x}}}{\frac{R}{x}}\right]$$

"Again, [the quotient] obtained from that [above difference] by dividing by the divisor [$\frac{R}{x}$] becomes the true Sine-difference".

$$\frac{1}{\frac{R}{x}}\left(Cos(A) - \frac{1}{2}\left[\frac{Sin(A) + \frac{Cos(A)}{2\frac{R}{x}}}{\frac{R}{x}}\right]\right)$$

"The Sine at the end of the arc traversed [$Sin(A)$] increased by that [true Sine-difference] becomes the desired Sine for a [given] arc [$Sin(A+x)$]."

$$Sin(A+x) \approx Sin(A) + \frac{1}{\frac{R}{x}}\left(Cos(A) - \frac{1}{2}\left[\frac{Sin(A) + \frac{Cos(A)}{2\frac{R}{x}}}{\frac{R}{x}}\right]\right)$$

$$= Sin(A) + \frac{Cos(A)}{\frac{R}{x}} - \frac{Sin(A)}{2(\frac{R}{x})^2} - \frac{Cos(A)}{4(\frac{R}{x})^3}$$

Recalling $Sin(A) = Rsin(A)$, and $Cos(A) = Rcos(A)$:

$$Rsin(A + x) \approx Rsin(A) + \frac{Rcos(A)}{\frac{R}{x}} - \frac{Rcos(A)}{2(\frac{R}{x})^2} - \frac{Rcos(A)}{4(\frac{R}{x})^3}$$

$$= Rsin(A) + xcos(A) - x^2\frac{sin(A)}{2R} - x^3\frac{cos(A)}{4(R^2)}$$

If we now let $R = 1$, we see that

$$sin(A + x) \approx sin(A) + xcos(A) - x^2\frac{sin(A)}{2} - x^3\frac{cos(A)}{4}$$

This is nearly equivalent to the third-order Taylor series approximation for $sin(A + x)$, which is

$$sin(A + x) \approx sin(A) + xcos(A) - x^2\frac{sin(A)}{2} - x^3\frac{cos(A)}{3!}$$

The only difference between the two equations is in the denominator of the fourth term, which is 4 as opposed to $3! = 6$.

When working with radians, this third-order approximation for Sine is accurate up to four decimal places. Yet, as nautical navigation required more and more accurate Sine values, better approximations were derived. Indian mathematicians, specifically Madhava, eventually found approximations for Sine and Cosine which are equivalent to today's Taylor series expansions. Although none of Madhava's own works on the subject remain intact, we know of this through the work of his students. From their commentaries and works, we know these Sine and Cosine expansions allowed Madhava to find remarkably accurate values of Sine, Cosine, and π (5 p.256).

TIMELINE

- *c. 400*: The earliest reference to trigonometry in India occurs in the text "Suryasidhanta" (1 p.229)
- *c. 400*: The first sine table, appears in the text "Paitamahasiddhanta" (5 p.252)
- *499*: Arybahata writes "Aryabhatiya", which introduces sines and versed sines, as well as a Sine table (1 p.15)
- *505*: Varahamihira gives the Pythagorean identity and Indian half-angle identity, as well as values for $Sin(30^o)$, $Sin(45^o)$, $Sin(60^o)$, and $Sin(90^o)$ (1 p.255)
- *665*: Brahmagupta discovers a formula for second-order interpolations, which allows $Sin(x + \epsilon)$ to be computed with an equation that is equivalent to the second-order Taylor series approximation of $Sin(x + \epsilon)$ (3 p.87)
- *1150*: The first sine table which had Sines for arcs closer together than 3¾° is printed (5 p.254)
- *c. 1400*: Madhava develops an approximation for $Sin\theta$ which is equivalent to its Taylor series expansion (2 p.9)
- *c. 1400*: Paramsevara, a student of Madhava's, writes "Siddhanta-dipika" in which he discusses an approximation for $Sin(x + \epsilon)$ which is nearly equal to its third-order Taylor series approximation (4 p.287)
- *c. 1550*: Jyesthadeva writes "Yuktibhasa" including an approximation for $Sin\theta$ that is equivalent to its Maclaurin series, which is credited to Madhava (2 p.8)

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