Hart, S. and Rowley, P.J.

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Zero Excess and Minimal Length in Finite Coxeter Groups

S.B. Hart and P.J. Rowley*

Abstract

Let $\mathcal{W}$ be the set of strongly real elements of W, a Coxeter group. Then for $w \in \mathcal{W}$, $e(w)$, the excess of w, is defined by $e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1\}$. When W is finite we may also define $E(w)$, the reflection excess of w. The main result established here is that if W is finite and X is a W-conjugacy class, then there exists $w \in X$ such that w has minimal length in X and $e(w) = 0 = E(w)$. (MSC2000: 20F55)

1 Introduction

Suppose that W is a Coxeter group and let $\mathcal{W}$ denote the set of strongly real elements of W. So

$$\mathcal{W} = \{w \in W \mid w = xy \text{ where } x, y \in W \text{ and } x^2 = 1 = y^2\}.$$

Let $w \in \mathcal{W}$. Then the *excess* of w, $e(w)$, is defined by

$$e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1\}.$$

In [7] it is shown that any element $w \in \mathcal{W}$ is W-conjugate to an element whose excess equals zero. Or, in other words, w is W-conjugate to an element xy where x and y are involutions or the identity element and $\ell(xy) = \ell(x) + \ell(y)$. The present paper explores this theme further in the case when W is finite. When W is finite it is well-known that $W = \mathcal{W}$. From Carter's seminal paper [3], it follows that, in fact, each $w \in W$ can be expressed in the form $w = xy$ where $x^2 = y^2 = 1$ and

$$V_{-1}(x) \cap V_{-1}(y) = \{0\}.$$

Here V is a reflection module for W (to be defined in Section 2) and $V_\lambda(x)$ denotes the λ-eigenspace of x on V. We note [3] only considers the Weyl groups – for a proof covering all finite Coxeter groups see Lemma 2.4. Writing $L(w)$ for the reflection length of w, this means that

$$L(w) = L(x) + L(y).$$

With this in mind we define the *reflection excess* of w, denoted $E(w)$, to be

$$E(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1, L(w) = L(x) + L(y)\}.$$

It is clear that $E(w) \geq e(w)$. However, it is not always the case that $E(w) = e(w)$. Consider the element $w = (145)(236)$ of $\mathrm{Sym}(6) \cong W(A_5)$. We have $L(w) = 4$, $\ell(w) = 10$ and Table

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x	y	$L(x)+L(y)$	$\ell(x)+\ell(y)$
(14)(23)	(15)(26)	4	$6+12=18$
(14)(36)	(15)(23)	4	$8+8=16$
(14)(26)	(15)(36)	4	$10+10=20$
(15)(23)	(45)(26)	4	$8+8=16$
(15)(36)	(45)(23)	4	$10+2=12$
(15)(26)	(45)(36)	4	$12+6=18$
(45)(23)	(14)(26)	4	$2+10=12$
(45)(36)	(14)(23)	4	$6+6=12$
(45)(26)	(14)(36)	4	$8+8=16$
(12)(46)(35)	(13)(24)(56)	6	$5+5=10$
(13)(24)(56)	(16)(25)(34)	6	$5+15=20$
(16)(25)(34)	(12)(35)(46)	6	$15+5=20$

Table 1: $(145)(236) = xy$

1 gives the possibilities for x and y. As can be seen, $E(w) = 2$ but $e(w) = 0$. Surprisingly, the difference between $E(w)$ and $e(w)$ can be arbitrarily large, as is shown in Proposition 3.3.

The main theorem in this paper is

Theorem 1.1 *Suppose that W is a finite Coxeter group. If X is a conjugacy class of W, then there exists $w \in X$ of minimal length in X such that $e(w) = E(w) = 0$.*

We do not know if it is true that for an arbitrary Coxeter group and a strongly real conjugacy class X, there exists $w \in X$ with w of minimal length in X and $e(w) = 0$. However we note that it does hold for an arbitrary Coxeter group when X is a strongly real conjugacy class whose elements have finite order. This may be seen by combining a theorem of Tits (Chap V, Section 4, Ex 2d of [1]) with Lemma 2.3 (which holds in general) and Theorem 1.1.

This paper is arranged as follows. Our next section gathers together relevant background material while reviewing much of the standard notation used for Coxeter groups. Section 3, apart from proving Proposition 3.3, focusses on the proof of Theorem 1.1. Part of the proof involves checking, with the aid of Magma [2], all the cuspidal classes of the exceptional finite irreducible Coxeter groups. The data resulting from these calculations is documented in the appendix. At present, when studying minimal elements in conjugacy classes, this case-by-case approach is often the best we can do –see for example Chapter 3 of Geck and Pfeiffer [6].

2 Preliminary Results and Notation

From now on we assume that W is a finite Coxeter group and quickly review standard notation and facts about Coxeter groups. So W has a presentation of the form

$$W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R\rangle$$

where $m_{rs} = m_{sr} \in \mathbb{N}$, $m_{rr} = 1$ and $m_{rs} \geq 2$ for $r, s \in R, r \neq s$. The length of an element w of W, denoted by $\ell(w)$, is defined to be

$$\ell(w) = \begin{cases} \min\{l \mid w = r_1 \cdots r_l, r_i \in R\} \text{ if } w \neq 1 \\ 0 \text{ if } w = 1. \end{cases}$$

Taking V to be a real vector space with basis $\Pi = \{\alpha_r \mid r \in R\}$, for $r, s \in R$

$$\langle \alpha_r, \alpha_s \rangle = -\cos\left(\frac{\pi}{m_{rs}}\right).$$

defines a symmetric bilinear form $\langle \ , \ \rangle$ on V. Letting $r, s \in R$ we define

$$r \cdot \alpha_s = \alpha_s - 2\langle \alpha_r, \alpha_s \rangle \alpha_r.$$

This then extends to an action of W on V which is both faithful and respects the bilinear form $\langle \ , \ \rangle$ (see [8]). The elements of R act as reflections upon V and V is referred to as a reflection module for W. The subset $\Phi = \{w \cdot \alpha_r \mid r \in R, w \in W\}$ of V is the root system of W, and $\Phi^+ = \{\sum_{r \in R} \lambda_r \alpha_r \in \Phi \mid \lambda_r \geq 0 \text{ for all } r\}$ and $\Phi^- = -\Phi^+$ are, respectively, the positive and negative roots of Φ. For $w \in W$, let $N(w) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\}$ – it is an important fact that $\ell(w) = |N(w)|$. In a similar vein we have

Lemma 2.1 *Let $g, h \in W$. Then*

$$N(gh) = N(h) \setminus [-h^{-1}N(g)] \cup h^{-1}[N(g) \setminus N(h^{-1})].$$

Hence $\ell(gh) = \ell(g) + \ell(h) - 2 \mid N(g) \cap N(h^{-1}) \mid$.

Proof See Lemma 2.2 in [7]. □

For J a subset of R define W_J to be the subgroup generated by J. Such a subgroup of W is referred to as a standard parabolic subgroup. Standard parabolic subgroups are Coxeter groups in their own right with root system

$$\Phi_J = \{w \cdot \alpha_r \mid r \in J, w \in W_J\}$$

(see Section 5.5 of [8] for more on this). A conjugate of a standard parabolic subgroup is called a parabolic subgroup of W. Finally, a *cuspidal* element of W is an element which is not contained in any proper parabolic subgroup of W. Equivalently, an element is cuspidal if its W-conjugacy class has empty intersection with all the proper standard parabolic subgroups of W.

Theorem 2.2 *Let $0 \neq v \in V$. Then the stabilizer of v in W is a proper parabolic subgroup of W.*

Proof Consult Ch V §3 Proposition 2 of [1]. □

Lemma 2.3 *Suppose that $J \subseteq I$ and $w \in W_J$. Then*

$$\min\{\ell(h^{-1}wh) \mid h \in W_J\} = \min\{\ell(g^{-1}wg) \mid g \in W\}.$$

Proof See Lemma 3.1.14 of [6]. $\square$

Lemma 2.4 *Suppose $w \in W$. Then there exist $x, y \in W$ with $w = xy$, $x^2 = y^2 = 1$ and $V_{-1}(x) \cap V_{-1}(y) = \{0\}$.*

Proof The proof is by induction on the rank of W, the rank 1 case being trivial. Let $x, y \in W$ be such that $w = xy$ with $x^2 = y^2 = 1$. (For W a Weyl group, this is possible by [3]. The case when W is a dihedral group is straightforward to verify while types H_3 or H_4 may be checked using [2].) If $V_{-1}(x) \cap V_{-1}(y) = \{0\}$, we are done. So suppose $0 \neq v \in V_{-1}(x) \cap V_{-1}(y)$. By Theorem 2.2, w is contained in a proper parabolic subgroup of W. Hence w is conjugate to an element u of some proper standard parabolic subgroup W_J of W. By induction $u = ab$ for some $a, b \in W_J$ where $a^2 = b^2 = 1$ and $V_{-1}(a) \cap V_{-1}(b) = \{0\}$. The appropriate conjugates of a and b will have the same properties with respect to w, so proving the lemma. $\square$

Lemma 2.5 *Let $w = r_1 r_2 \dots r_n$ and let X denote the W-conjugacy class of w. Then*

(i) the minimal length in X is n; and

(ii) the product of $r_1, \dots, r_n$ in any order is an element of X.

Proof See Proposition 3.1.6 of [6]. $\square$
Note that the minimal length elements of X in Lemma 2.5 are known as the *Coxeter elements* of W.The famous classification of irreducible finite Coxeter groups obtained by Coxeter [4] (see also [8]) states

Theorem 2.6 *An irreducible finite Coxeter group is either of type $A_n (n \geq 1)$, $B_n (n \geq 2)$, $D_n (n \geq 4)$, $\mathrm{Dih}(2m)$, E_6, E_7, E_8, F_4, H_3 or H_4.*

We next discuss concrete descriptions of the Coxeter groups of types A_n, B_n and D_n which will feature in a number of our proofs. First, $W(A_n)$ may be viewed as being $\mathrm{Sym}(n+1)$ with the set of fundamental reflections given by $\{(12), (23), \dots, (n\ n+1)\}$. The elements of $W(B_n)$ can be thought of as signed permutations of $\mathrm{Sym}(n)$. We say a cycle in an element of $W(B_n)$ is of negative sign type if it has an odd number of minus signs, and positive sign type otherwise. The set of fundamental reflections in $W(B_n)$ can be taken to be $\{(\overset{+}{1}\overset{+}{2}), (\overset{+}{2}\overset{+}{3}), \dots, (\overset{+}{n-1}\ \overset{+}{n}), (\overset{-}{n})\}$. An element w expressed as a product $g_1 g_2 \cdots g_k$ of disjoint signed cycles is *positive* if the product of all the sign types of the cycles is positive, and negative otherwise. The group $W(D_n)$ consists of all positive elements of $W(B_n)$. The fundamental reflections of $W(D_n)$ can be taken to be the set $\{(\overset{+}{1}\overset{+}{2}), (\overset{+}{2}\overset{+}{3}), \dots, (\overset{+}{n-1}\ \overset{+}{n}), (\overset{-}{n-1}\ \overset{-}{n})\}$. The positive roots of B_n are of the form $e_i \pm e_j$ for $1 \leq i < j \leq n$ and e_i for $1 \leq i \leq n$. The positive roots of D_n are of the form $e_i \pm e_j$ for $1 \leq i < j \leq n$. Therefore the root system of D_n consists of the long roots of the root system for B_n. Even if w is positive, it may contain negative cycles, which we wish on occasion to consider separately, so when considering elements of $W(D_n)$ we often work in the environment of $W(B_n)$ to avoid ending up with non-group elements. We draw the reader's attention to the fact that when regarding W as a group of permutations or signed permutations we act on the right, as is customary for permutations. We further note that, in the case of a type B_n Coxeter group described as a group of signed permutations, the roots do not all have the same length. This doesn't accord with our earlier description of Φ where all the roots have length 1. However this causes no problems here.

The following result is obtained from Propositions 3.4.6, 3.4.7, 3.4.11, 3.4.12 of [6]. The length of minimal length elements in conjugacy classes is not given explicitly, but expressions for elements of minimal length are given in Section 3.4.2 and from these the length can be easily calculated.

Proposition 2.7 *Let W be of type B_n or D_n. Then the following hold.*

(i) Conjugacy classes in W are parameterized by signed cycle type, with one class for each signed cycle type except in the case where all cycles are even length and positive, and W is of type D_n. In this case there are two conjugacy classes, which can be interchanged by the length-preserving graph automorphism.

(ii) Cuspidal conjugacy classes in W are those whose signed cycle type consists only of negative cycles.

(iii) Each cuspidal conjugacy class X corresponds to a non-increasing partition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

of n, where each λ_i is the length of a negative cycle of $w \in X$. Let $\mu_i = \sum_{j=i+1}^{k} \lambda_i$. The minimal length of an element of X is

$$\begin{cases} \sum_{i=1}^{k} (\lambda_i + 2\mu_i) & \text{if } W = W(B_n) \\ \sum_{i=1}^{k} (\lambda_i - 1 + 2\mu_i) & \text{if } W = W(D_n) \end{cases}.$$

We close this section with a few pieces of notation. For a subset X of W we define $\ell_{\min}(X) = \min\{\ell(x) \mid x \in X\}$. The longest element of W will be denoted by w_0. When $W = W(A_{n-1}) \cong \mathrm{Sym}(n)$, W acts in the usual way on the set $\Omega = \{1, 2, \ldots, n\}$. In this case, for $w \in W$, the *support* of w, $\mathrm{supp}(w)$, is given by $\mathrm{supp}(w) = \{\delta \in \Omega \mid \delta^w \neq \delta\}$. Finally, the dihedral group of order $2m$ will be denoted by $\mathrm{Dih}(2m)$.

3 Minimal Length and Zero Excess and Reflection Excess

We begin with an elementary lemma.

Lemma 3.1 *For $w \in W$, the following hold.*

(i) Both $e(w)$ and $E(w)$ are non-negative and even.

(ii) If w is an involution or the identity element, then $e(w) = E(w) = 0$.

(iii) $\ell(w)$ is the sum of the lengths, $e(w)$ is the sum of the excesses, and $E(w)$ is the sum of the reflection excesses of the irreducible direct factors of W.

Proof For (iii), suppose $x^2 = y^2 = 1$ and $xy = w$. Then, using Lemma 2.1, $\ell(w) = \ell(x) + \ell(y) - 2|N(x) \cap N(y)|$ and hence $\ell(x) + \ell(y) - \ell(w)$ is even and (iii) follows. The remaining parts of the lemma are straightforward. ☐

In the next lemma we encounter the following two subsets of W:-

$$\mathcal{I}_w = \{x \in W \mid x^2 = 1, w^x = w^{-1}\} \text{ and}$$

$$\mathcal{J}_w = \{x \in W \mid x^2 = 1, w^x = w^{-1}, V_1(w) \subseteq V_1(x)\}.$$

Lemma 3.2 *Suppose that $w \in W$*

(i) $\mathcal{J}_w$ is the set of x such that $w = xy$ where $x^2 = y^2 = 1$ and $L(w) = L(x) + L(y)$.

(ii) If w is cuspidal, then $\mathcal{I}_w = \mathcal{J}_w$ and, in particular, $e(w) = E(w)$.

Proof Suppose $x \in \mathcal{J}_w$. Set $y = xw$. Clearly $y^2 = 1$. Let $v \in V_{-1}(x) \cap V_{-1}(y)$. Then $y \cdot v = x \cdot v$ and hence $w \cdot v = v$. That is, $v \in V_1(w)$. But $V_1(w) \subseteq V_1(x)$ as $x \in \mathcal{J}_w$. Thus $v \in V_1(x) \cap V_{-1}(x)$ and so $v = 0$. Therefore $V_{-1}(x) \cap V_{-1}(y) = 0$, whence $L(w) = L(x) + L(y)$. On the other hand, suppose $x^2 = y^2 = 1$, $w = xy$ and $L(w) = L(x) + L(y)$. Then $V_{-1}(x) \cap V_{-1}(y) = 0$. Clearly $w^x = w^{-1}$. Suppose $v \in V_1(w)$. Then $v = w \cdot v = xy \cdot v$. Hence $x \cdot v = y \cdot v$. Therefore $x \cdot v - v = y \cdot v - v$. But $x \cdot v - v \in V_{-1}(x)$ and $y \cdot v - v \in V_{-1}(y)$, which forces $x \cdot v - v = y \cdot v - v = 0$. That is, $v \in V_1(x)$. Therefore $V_1(w) \subseteq V_1(x)$ and $x \in \mathcal{J}_w$, which establishes (i).

For (ii), w being cuspidal implies, by Theorem 2.2, that $V_1(w) = 0$. So $\mathcal{I}_w = \mathcal{J}_w$, and clearly $e(w) = E(w)$.

□

We now give examples of elements in the symmetric group which have arbitrarily large reflection excess, while having zero excess, showing that these quantities can differ by arbitrarily large amounts.

Proposition 3.3 *Let $W = W(A_{n-1}) = \mathrm{Sym}(n)$ and assume $0 < 4k \leq n$. Define*

$$\begin{aligned} w_1 &= (1\ 4\ 6\ 8 \cdots 4k-2 \quad 4k-1), \\ w_2 &= (2 \ \ 4k \quad 4k-3 \cdots 7\ 5\ 3) \end{aligned}$$

and $w = w_1 w_2$. Then $e(w) = 0$ but $E(w) \geq 4(k-1)^2$.

Proof We define two involutions x, y as follows.

$$\begin{aligned} x &= (13)(2\ 4k-1)\,(45)(67)\cdots(4k-4\ 4k-3)\,(4k-2\ 4k) \\ y &= (12)(34)\cdots(4k-1\ 4k) \end{aligned}$$

A simple check shows $w = xy$. Now $N(y) = \{e_{2i-1} - e_{2i} : 1 \leq i \leq 2k\}$. To find $N(x) \cap N(y)$, note that

$$\begin{aligned} x \cdot (e_1 - e_2) &= e_3 - e_{4k-1} \in \Phi^+ \\ x \cdot (e_3 - e_4) &= e_1 - e_5 \in \Phi^+ \\ x \cdot (e_{4k-3} - e_{4k-2}) &= e_{4k-4} - e_{4k} \in \Phi^+ \\ x \cdot (e_{4k-1} - e_{4k}) &= e_2 - e_{4k-2} \in \Phi^+ \\ x \cdot (e_{2i-1} - e_{2i}) &= e_{2i-2} - e_{2i+1} \in \Phi^+ (3 \leq i \leq 2k-2). \end{aligned}$$

Hence $N(x) \cap N(y) = \emptyset$, which means $e(w) = 0$ by Lemma 2.1.
Next we consider $E(w)$. Since $w_1 = (1\ 4\ 6\ 8 \cdots 4k-2 \quad 4k-1)$ and $1 < 4 < \cdots < 4k-1$, we can compare w_1 seen as an element of $\mathrm{Sym}(\mathrm{supp}(w_1))$ with the element $(1\ 2 \cdots 2k)$ as an element of $\mathrm{Sym}(2k)$. This allows us to use [Proposition 2.7(iv), [7]] and Lemma 3.2(ii) to deduce that $E(w_1) \geq E((12 \cdots 2k)) = 2(k-1)^2$. This is because whenever $w_1 = \sigma_1 \tau_1$ where σ_1, τ_1 are involutions in $\mathrm{Sym}(\mathrm{supp}(w_1))$, we have that

$$(N(\sigma_1) \cap N(\tau_1))\,|_{\mathrm{Sym}(\mathrm{supp}(w_1))} \subseteq N(\sigma_1) \cap N(\tau_1).$$

Similarly, by comparing $w_2 = (2\ \ 4k\ \ \ 4k{-}3 \cdots 7\ 5\ 3)$ to $(1\ \ 2k\ \ 2k{-}1 \cdots 3\ 2) = (1\ 2 \cdots 2k)^{-1}$ we deduce that $E(w_2) \geq 2(k-1)^2$.

Suppose $w = \sigma\tau$ with $L(\sigma) + L(\tau) = L(w)$. Then $\sigma = \sigma_1\sigma_2$, $\tau = \tau_1\tau_2$ where $w_1 = \sigma_1\tau_1$, $w_2 = \sigma_2\tau_2$, $\mathrm{supp}(\sigma_1) \cup \mathrm{supp}(\tau_1) \subseteq \mathrm{supp}(w_1)$ and $\mathrm{supp}(\sigma_2) \cup \mathrm{supp}(\tau_2) \subseteq \mathrm{supp}(w_2)$. Any $\{i,j\} \subseteq \mathrm{supp}(w_1)$ (and hence $w_1(i), w_1(j)$) will thus be fixed by σ_2 and τ_2. So if $e_i - e_j \in N(\sigma_1) \cap N(\tau_1)$, then $e_i - e_j \in N(\sigma) \cap N(\tau)$. Hence

$$(N(\sigma_1) \cap N(\tau_1))\,|_{\mathrm{Sym}(\mathrm{supp}(w_1))} \subseteq N(\sigma) \cap N(\tau).$$

Similarly

$$(N(\sigma_2) \cap N(\tau_2))\,|_{\mathrm{Sym}(\mathrm{supp}(w_2))} \subseteq N(\sigma) \cap N(\tau).$$

This implies that $E(w) \geq E(w_1) + E(w_2) \geq 4(k-1)^2$. $\square$

Lemma 3.4 *Let W be a finite Coxeter group, and X be the conjugacy class of Coxeter elements of W. Then there exists $w \in X$ of minimal length in X such that $e(w) = E(w) = 0$.*

Proof Since the Coxeter graph of every finite Coxeter group is a forest (that is, a disjoint union of trees), the Coxeter graph of W is 2-colourable. Therefore we may divide the fundamental reflections into two sets $R = R_1 \dot\cup R_2$, such that for $r, s \in R_i$, $rs = sr$. Now let $x = \prod_{r \in R_1} r$ and $y = \prod_{r \in R_2} r$. Then $w = xy$ is an element of X of minimal length, and $e(w) = E(w) = 0$. $\square$

We remark that if W is irreducible, its Coxeter graph can only be 2-coloured in one way and consequently there are exactly two Coxeter elements for which $e(w) = E(w) = 0$.

Proposition 3.5 *Let W be irreducible of type A_{n-1}, B_n or D_n and X be a conjugacy class of W. Then there exists $w \in X$ such that $e(w) = E(w) = 0$ and w has minimal length in X.*

Proof It suffices to consider the case where X is cuspidal.
Suppose $W = W(A_{n-1})$. The only cuspidal class is the class of Coxeter elements, and by Lemma 3.4 Proposition 3.5 holds. It remains to deal with the case where W is type B_n or type D_n and X is cuspidal. Here, by Proposition 2.7, the elements of X contain only negative cycles of lengths $\lambda_1 \geq \ldots \geq \lambda_k$, where $\sum \lambda_i = n$. We will construct an element $w \in X$ of minimal length. Let $\nu_1 = 0$ and for $2 \leq i \leq k$ let $\nu_i = \sum_{j=1}^{i-1} \lambda_i$. As in Proposition 2.7, for $1 \leq i \leq k$, let $\mu_i = \sum_{j=i+1}^{k} \lambda_i$. Note that $n = \mu_i + \lambda_i + \nu_i$. By Proposition 2.7, the minimal length of elements of X, $l_{\min}(X)$, is given by

$$l_{\min}(X) = \begin{cases} \sum_{i=1}^{k} (\lambda_i + 2\mu_i) & \text{if } W = W(B_n); \\ \sum_{i=1}^{k} (\lambda_i - 1 + 2\mu_i) & \text{if } W = W(D_n). \end{cases}$$

Now for $1 \leq i \leq k$ define two involutions, τ_i and σ_i as follows.

$$\tau_i = \begin{cases} (\nu_i \overset{+}{+} 1\ \ \nu_i \overset{+}{+} 2)(\nu_i \overset{+}{+} 3\ \ \nu_i \overset{+}{+} 4) \cdots (\nu_i + \overset{+}{\lambda_i} - 2\ \ \nu_i + \overset{+}{\lambda_i} - 1)(\nu_i \overset{-}{+} \lambda_i) & \text{if } \lambda_i \text{ is odd} \\ (\nu_i \overset{+}{+} 2\ \ \nu_i \overset{+}{+} 3)(\nu_i \overset{+}{+} 4\ \ \nu_i \overset{+}{+} 5) \cdots (\nu_i + \overset{+}{\lambda_i} - 2\ \ \nu_i + \overset{+}{\lambda_i} - 1)(\nu_i \overset{-}{+} \lambda_i) & \text{if } \lambda_i \text{ is even} \end{cases}$$

$$\sigma_i = \begin{cases} (\nu_i \overset{+}{+} 2\ \ \nu_i \overset{+}{+} 3)(\nu_i \overset{+}{+} 4\ \ \nu_i \overset{+}{+} 5) \cdots (\nu_i + \overset{+}{\lambda_i} - 1\ \ \nu_i \overset{+}{+} \lambda_i) & \text{if } \lambda_i \text{ is odd} \\ (\nu_i \overset{+}{+} 1\ \ \nu_i \overset{+}{+} 2)(\nu_i \overset{+}{+} 3\ \ \nu_i \overset{+}{+} 4) \cdots (\nu_i + \overset{+}{\lambda_i} - 1\ \ \nu_i \overset{+}{+} \lambda_i) & \text{if } \lambda_i \text{ is even} \end{cases}$$

Define $w_i = \tau_i\sigma_i$, $\tau = \tau_1\cdots\tau_k$, $\sigma = \sigma_1\cdots\sigma_k$ and $w = \tau\sigma = w_1\cdots w_k$. Note that if $w \in W(D_n)$, then k is even and hence τ and σ are both elements of $W(D_n)$. If λ_i is odd, then

$$w_i = (\overset{+}{\nu_i+1}\ \ \overset{+}{\nu_i+2}\ \ \overset{+}{\nu_i+4}\ \ \cdots \overset{+}{\nu_i+\lambda_i-3}\ \ \overset{-}{\nu_i+\lambda_i-1}\ \ \overset{+}{\nu_i+\lambda_i}\ \ \overset{+}{\nu_i+\lambda_i-2}\cdots \overset{+}{\nu_i+5}\ \ \overset{+}{\nu_i+3})$$

whereas if λ_i is even, then

$$w_i = (\overset{+}{\nu_i+1}\ \ \overset{+}{\nu_i+3}\ \ \overset{+}{\nu_i+5}\cdots \overset{+}{\nu_i+\lambda_i-3}\ \ \overset{-}{\nu_i+\lambda_i-1}\ \ \overset{+}{\nu_i+\lambda_i}\ \ \overset{+}{\nu_i+\lambda_i-2}\cdots \overset{+}{\nu_i+4}\ \ \overset{+}{\nu_i+2}).$$

In either case, w_i is a negative λ_i-cycle. Hence $w \in X$. We now consider $\ell(\tau)$ and $\ell(\sigma)$. Firstly, note that σ is a product of distinct mutually commuting fundamental reflections. Hence $\ell(\sigma) = \sum_{i=1}^{k}\lceil\frac{\lambda_i-1}{2}\rceil$. Now τ can be split into a product $\tau = \tau'\tau''$ where τ' consists of the 2-cycles of τ and $\tau'' = (\bar{\nu_1})(\bar{\nu_2})\cdots(\bar{\nu_k})$. Now $\ell(\tau') = \sum_{i=1}^{k}\lfloor\frac{\lambda_i-1}{2}\rfloor$ and $N(\tau)$ consists of roots of the form $e_a - e_b$ where for some i, $\{a,b\} \subseteq \{\nu_i+1,\ldots,\nu_i+\lambda_i\}$. Also it is not hard to see that, thinking of τ'' as an element of $W(B_n)$,

$$N(\tau'') = \cup_{i=1}^{k}\left[\{e_{\nu_i+\lambda_i}\} \cup \{e_{\nu_i+\lambda_i} \pm e_b : \nu_i+\lambda_i < b \leq n\}\right].$$

Therefore $N(\tau') \cap N(\tau'') = \emptyset$. Hence $\ell(\tau) = \ell(\tau') + \ell(\tau'')$. If $W = W(B_n)$ we get $\ell(\tau'') = k + 2\sum_{i=1}^{k}\mu_i$ and hence $\ell(\tau) = \sum_{i=1}^{k}\lfloor\frac{\lambda_i-1}{2}\rfloor + k + 2\sum_{i=1}^{k}\mu_i$. If $W = W(D_n)$ then $\ell(\tau'') = 2\sum_{i=1}^{k}\mu_i$ and hence $\ell(\tau) = \sum_{i=1}^{k}\lfloor\frac{\lambda_i-1}{2}\rfloor + 2\sum_{i=1}^{k}\mu_i$.
Now $w = \tau\sigma$, so $\ell(w) \leq \ell(\tau) + \ell(\sigma)$. If $W = W(B_n)$, then

$$\begin{aligned}
\ell(w) &\leq \ell(\tau) + \ell(\sigma) \\
&= \sum_{i=1}^{k}\lfloor\frac{\lambda_i-1}{2}\rfloor + k + 2\sum_{i=1}^{k}\mu_i + \sum_{i=1}^{k}\lceil\frac{\lambda_i-1}{2}\rceil \\
&= \sum_{i=1}^{k}\lambda_i + 2\mu_i \\
&= l_{\min}(X) \leq \ell(w).
\end{aligned}$$

A similar calculation when $W = W(D_n)$ shows again that $\ell(w) \leq \ell(\tau) + \ell(\sigma) = l_{\min}(X) \leq \ell(w)$. Therefore in each case, $\ell(w) = \ell(\tau) + \ell(\sigma) = l_{\min}(X)$. Hence $e(w) = 0$ and w has minimal length in X. Since w is cuspidal, $E(w) = e(w) = 0$ and we have therefore proved the proposition. □

Proof of Theorem 1.1 By Lemma 3.1(v) we only need establish the theorem for W an irreducible Coxeter group. This has been done for W of type A_{n-1}, B_n and D_n in Proposition 3.5. The case when W is isomorphic to Dih$(2m)$ is easy to handle. For the remaining exceptional irreducible Coxeter groups we had to resort to using MAGMA [2]. The sizes of $X^0_{\min}$, where $X^0_{\min}$ denotes the set of elements in X which have minimal length in X and excess zero, are itemized in the appendix as well as giving a representative element of this set. In all cases $X^0_{\min}$ is non-empty, whence Theorem 1.1 is proven.

□

Appendix

Tables 3–6 give cuspidal classes in exceptional Weyl groups. Each class is labeled using the system in Carter [3]. Detailed information about these classes is given in [6], Tables B.1 – B.6. For a conjugacy class X let $X_{\min}$ be the set of elements of minimal length in X and $X^0_{\min}$ be the set of $w \in X_{\min}$ for which $e(x) = E(x) = 0$. For each class X we give $|X_{\min}|$, $|X^0_{\min}|$ and a representative $w \in X^0_{\min}$. Tables 7 and 8 give cuspidal classes in H_3 and H_4. For these tables the label of the class is its address in the CHEVIE list [5]. See also [6]. And finally, to improve readability, we write the element, say, $r_1r_3r_2r_4$ as 1324.

Class X	$\lvert X_{\min}\rvert$	$\lvert X^0_{\min}\rvert$	$w \in X^0_{\min}$
F_4	8	2	1324
B_4	14	2	124323
$F_4(a_1)$	16	16	12132343
D_4	8	6	1232343234
$C_3 + A_1$	8	6	1213213234
$D_4(a_1)$	12	12	121321343234
$A_3 + \tilde{A}_1$	16	12	12132132343234
$A_2 + \tilde{A}_2$	16	16	1213213432132343
$4A_1$	1	1	w_0

Table 2: Cuspidal Classes in F_4

Class X	$\lvert X_{\min}\rvert$	$\lvert X^0_{\min}\rvert$	$w \in X^0_{\min}$
E_6	32	2	142365
$E_6(a_1)$	80	4	12542346
$E_6(a_2)$	144	36	231423154654
$A_5 + A_1$	48	10	23423454231465
$3A_2$	80	80	123142314542314565423456

Table 3: Cuspidal Classes in E_6

Class X	$\lvert X_{\min}\rvert$	$\lvert X^0_{\min}\rvert$	$w \in X^0_{\min}$
E_7	64	2	1423657
$E_7(a_1)$	160	4	125423476
$E_7(a_2)$	280	18	12465423457
$E_7(a_3)$	366	40	1231542365476
$D_6 + A_1$	96	10	123423465423457
A_7	316	58	12365423476542345
$E_7(a_4)$	800	422	234354231435465765431
$D_6(a_2) + A_1$	708	190	13542345654231435765423
$A_5 + A_2$	420	194	1234231436542314576542345
$D_4 + 3A_1$	32	20	1234234542345654234567654234567
$2A_3 + A_1$	360	326	423454365423143542654317654234567
$7A_1$	1	1	w_0

Table 4: Cuspidal Classes in E_7

Class X	$\lvert X_{\min}\rvert$	$\lvert X^0_{\min}\rvert$	$w \in X^0_{\min}$
E_8	128	2	14682357
$E_8(a_1)$	320	4	1425423768
$E_8(a_2)$	624	10	435423145768
$E_8(a_4)$	732	40	12316542376548
$E_8(a_5)$	1516	66	1231654237687654
$E_7 + A_1$	192	10	2423454231437658
D_8	852	18	242345423165438765
$E_8(a_3)$	2696	238	23423546542314768765
$D_8(a_1)$	2040	178	1245423654765423876543
$E_8(a_7)$	2360	316	3542314654231765423148
$E_8(a_6)$	3370	422	235423143546765423145876
$E_7(a_2) + A_1$	1758	100	134234542314354265423876
$E_6 + A_2$	840	194	12345423143546542314354876
$D_8(a_2)$	4996	362	31423547654234587654231456
A_8	2816	592	1234231436542314765876542345
$D_8(a_3)$	7748	910	231423145423657654231435487654
$D_6 + 2A_1$	256	48	34354231435426542345687654234567
$A_7 + A_1$	2080	422	2314254236542345676542314354265478
$E_8(a_8)$	4480	4480	1231431542345654231456765423143546787654
$E_7(a_4) + A_1$	11592	5158	123435423456542317654231435426543765876543
$2D_4$	4070	1262	12425423456542765423456787654231435426543768
$E_6(a_2) + A_2$	16374	7438	34231454234654231435465765428765423143542657
$A_5 + A_2 + A_1$	3752	1910	3142345423654234567654234567876542314354265478
$D_5(a_1) + A_3$	15134	4900	4254231435465423143542678765423143542654317687
$2A_4$	7952	4058	134542314354265423143546765423143567876542345678
$2D_4(a_1)$	15120	15120	142354265423176542314354265487654231435426543176· ·542348765423
$D_4 + 4A_1$	56	42	124231542314365423143542765423143542654318765423· ·1435426543176543
$2A_3 + 2A_1$	1260	1090	123142315431654765423143542654387654231435426543· ·176542345876542345
$4A_2$	4480	4480	123142314542314354265437654231435426543176876542· ·31435426543176542345678765423456
$8A_1$	1	1	w_0

Table 5: Cuspidal Classes in E_8

Class X	$\lvert X_{\min}\rvert$	$\lvert X^0_{\min}\rvert$	$w \in X^0_{\min}$
6	4	2	312
8	6	4	21231
9	6	6	121232123
10	1	1	w_0

Table 6: Cuspidal Classes in H_3

Class X	$\lvert X_{\min}\rvert$	$\lvert X^0_{\min}\rvert$	$w \in X^0_{\min}$
11	8	2	1324
14	12	4	212431
15	18	8	32434121
16	22	10	2321234121
18	24	24	121213212343
19	34	26	32121321234121
21	12	10	2132121321234121
22	24	20	2321234321234121
23	38	30	213212134321234121
24	40	40	1212132121321432 1234
25	36	28	21321213212343212 34121
26	24	24	121213212134321213212343
27	56	52	12123212132123432121321234
28	40	38	121213212343212132123432 1234
29	60	60	12121321213212432121321432 1234
30	24	24	121213212132432121321234321213212343
31	36	34	12121321213212343212132123432121321234
32	40	40	12121321213212432121321243212132143 21234
33	24	24	121213212132143212132123432121321234 3212· ·13212343
34	1		w_0

Table 7: Cuspidal Classes in H_4

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