Constant Proportion Portfolio Insurance and Related Topics WITH EMPIRICAL STUDY

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled:

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a candidate for the degree of Doctor of Philosophy and hereby certify that, in their opinion, it is worthy of acceptance.

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ABSTRACT

The concept of Constant Proportion Portfolio Insurance (CPPI) in terms of jump-diffusion, as well as the associated mean-variance hedging problem, has been studied. Three types of risk related to: the probability of loss, the expected loss, and the loss distribution are being analyzed. Both the discrete trading time case and the continuous trading time case have been studied. Next, CPPI with stochastic dynamic floors are being discussed. The concept of exponential proportion portfolio insurance is being introduced. Finally CPPI associated with the fractional Brownian market is being studied.

Chapter 1

Introduction

Constant Proportion Portfolio Insurance (CPPI) was introduced by [61] for equity instruments, and has been further analyzed by many scholars(such as [10]). An investor invests in a portfolio and wants to protect the portfolio value from falling below a pre-assigned value. The investor shifts his asset allocation over the investment period among a risk-free asset plus a collection of risky assets. The CPPI strategy is based on the dynamic portfolio allocation of two basic assets: a riskless asset (usually a treasury bill) and a risky asset (a stock index for example). This strategy relies crucially on the concept of a *cushion* C, which is defined as the difference between the *portfolio value* V and the *floor* F. This later one corresponds to a guaranteed amount at any time t of the management period $[0, T]$. The key assumption is that the amount e invested on the risky asset, called the *exposure*, is equal to the cushion multiplied by a fixed coefficient m, called the *multiple*. The floor and the multiple can be chosen according to the investor's risk tolerance.

In chapter 2

In this chapter, we introduce the background and concept of the CPPI and EPPI modeling by a diffusion process. In section 2.1, we consider the simplest CPPI and its background. In this case the risky asset model is the classical Black-Scholes model. Both continuous and discrete are considered. We introduce the concept of EPPI (Exponential Proportion Portfolio Insurance). In section 2.3, we consider the case when the stock model satisfies a GARCH model. We also consider both the discrete and continuous trading cases. In section 2.4, we consider the EPPI in GARCH.

In chapter 3

In this chapter, we discuss the CPPI-jump-diffusion model when the trading time is continuous. The jump-diffusion model was introduced and widely studied by [58] and [65].

Let $Y_n > -1$ be the percentage of the size of n-th jump, and S_t be the process who represent the stock price at time t. Thus, $S_{T_n} = S_{T_n^-}(1 + Y_n)$. Between two jumps, we assume the risky asset model satisfies Black-Scholes. The number of jumps upper to time t is a Poisson processes N_t with intensity λ_t. Then our model becomes

$$S_t = S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1 + Y_n)\right].$$

We usually assume $\ln(1 + Y_n)$ is i.i.d. and has density function f_Q.

Our outline of this section is following.

In section 3.1, we set up the jump-diffusion model, calculate the density function and discuss the martingale measure. In section 3.2, we describe the CPPI strategy and then calculate the CPPI portfolio value, its expectation and variance. In section 3.3, we consider the CPPI portfolio as a hedging tool. [16] considers the situation in

Black-scholes model. Our discussion is a generalization of it. Both the PDE/PIDE approach and the martingale approach are studied there. However, because of the introduction of the jump term in the model, the calculation is much more complex. In section 3.2 and subsection 3.3, both short-sell and negative exposure are allowed. In section 3.4, we consider the mean-variance hedging for a given contingent claim H. In our jump-diffusion model, the market is not complete and then H is not attainable. Thus, we consider the mean-variance hedging which is a kind of quadratic hedging. [67] is a review paper about quadratic hedging, we adopt the symbol and definition from it. We consider H as the function of portfolio value V_T and measure the risk in probability $\mathbb{Q}$. Our optimal problem is following

$$\min_{(Z_0,\vartheta)\in\mathbb{R}\times\Theta} \mathbb{E}^{\mathbb{Q}}\left(\tilde{H} - Z_0 - \int_0^T \vartheta_u d\tilde{V}_u\right)^2.$$

We adopt the method in Chapter 10 in [18] and give the explicit form optimal solution of Z_0 and ϑ_t. In section 3.2, section 3.3 and section 3.4, both short-sell and negative exposure are allowed.

The main contribution of this chapter is in section 3.3 and section 3.4.

In chapter 4

In this chapter, we continue to discuss the CPPI in the jump-diffusion model.

In section 4.2, we discuss the **Gap risk** which is defined as the amount which represents how much that the portfolio value is below the floor at the terminal time. In this case, we do not allow short-sell and negative exposure. It is deduced that the gap happens only when the jump is negatively large enough such that $1 + mY_i \leq 0$. The probability of loss, expected loss and loss distribution are introduced to measure the gap. [17] has discuss the case in a more general model as

$$\frac{dS_t}{S_{t-}} = dZ_t,$$

where Z_t is a Levy process. Our jump-diffusion model could be treated as a special case. Thus, the conclusion in this section is a special case of [17]. However, we deduce more explicit expression as compare with [17], which is more appreciated in simulation. We will show that the conclusion of the probability of loss is consistant the conclusion in [17]; our conclusion for the expected loss is more explicit and the method is similar as [17]; our conclusion for the loss distribution is explicit and our method is different from [17]. In section 4.3, we consider the conditional multiple from the view of Probability of loss. Its idea is similar as the Value-at-Risk([27]). Four kinds of conditional floor are also discussed from the view of expected loss and loss distribution.

In chapter 5

In this chapter, we will study the jump-diffusion model when the trading time is discrete.

The risky asset model is similar as that in chapter 3 and 4.

In section 5.2, we calculate the CPPI portfolio value and its expectation and variance. Gap risks exist because the risky model has jumps and also the trading time is discrete.

In section 5.3, we measure the gap risk with respect to three aspects: probability of loss, expected loss and loss distribution. We give out their explicit forms.

In section 5.4, we define the conditional multiples associated with the probability of loss, conditional floors associated with expected loss and loss distribution.

In section 5.5, we prove that as the interval length of the trading times tends to zero, the CPPI strategies in discrete trading time will convergent to the CPPI strategies in continuous time.

In chapter 6

In this chapter, we investigate several types of stochastic floors and dynamic floors. In [59], they have considered the cases of diffusion models without jumps. Here we generalize it to the jump-diffusion case.

In section 6.2, we consider the case when the stochastic floor is equal to the maximum of its past value and a given percentage of the portfolio value. The idea is that when the portfolio value is large enough, the level of the floor rises. Both the continuous trading and discrete trading time cases will be analyzed. We will calculate the distribution of the time when the floor is increased.

In section 6.3, we consider the case when stochastic floor is indexed with respect to the given portfolio performance. The idea is similar as section 6.2. Both the continuous trading and discrete trading time cases will also be analyzed. We will also calculate the distribution of the first-time-change of the floor.

In section 6.4, we will deal with the Ratchet and Margin CPPI strategies with time change related to the exposition variance. We will show in discrete trading time case, the Ratchet CPPI is equivalent to the stochastic floor index on the given portfolio performance. The idea of CPPI with margin is that when the floor is close to the portfolio value, the exposure will be very small and we will reduce the floor. We will discuss the distribution of the first-change-time of the floor in the continuous trading time case.

In chapter 7

In this chapter, we consider the CPPI in a fractional Brownian Market.

Fractional Black-Scholes market was introduced by [35] where they utilize the wick product and thus redefined many market concepts such as **portfolio**, **value process**,

self-financing, **admissible**, **arbitrage** and **complete**. In Section 7.1, we adopt the fractional Brownian markets and new markets concepts as in [35]. Under this new market, we calculate the CPPI portfolio value, its expectation and variance in Section 7.2. In Section 7.3, we calculate the CPPI option. Moreover, we consider the associate hedging problem by PDE approach in Section 7.4.

In Chapter 8

In this chapter, we consider the CPPI in a fractional Brownian Markets with jumps. This chapter could be treated as an extenstion of Chapter 7. In Section 8.1, we setup the fractional Brownian markets with jumps and redifined many market concepts as in Chapter 7. We also deduce the Girsanov Formula in fractional Black-Scholes model with jumps. In Section 8.2, we calculate the CPPI portfolio value, its expectation and variance.

Chapter 2

CPPI and EPPI in Diffusion model

2.1 CPPI in the Black-Scholes model

2.1.1 The continuous trading time case

The CPPI (Constant Proportion Portfolio Insurance) strategy is based on a dynamic portfolio allocation on two basic assets: a riskless asset (usually a treasury bill) and a risky asset (a stock index for example).

This strategy depends crucially on the *cushion* C, which is defined as the difference between the portfolio value V and the *floor* F. This later one corresponds to a guaranteed amount at any time t of the management period $[0, T]$. The key assumption is that the amount e invested on the risky asset, called the *exposure*, is equal to the cushion multiplied by a fixed coefficient m, called the *multiple*. The floor and the multiple can be chosen according to the investors risk tolerance. The risk-aversion investor will choose a small multiple or/and a high floor and vice versa. The higher the multiple, the more the investor will benefit from increases in stock prices. Nevertheless, the higher the multiple, the higher the risk that the portfolio value becomes smaller than the floor if the risky asset price drops suddenly. As the cushion value is approximately equal to zero, exposure is near zero too. In the continuous-time case, if the asset dynamics has no jump, then the portfolio value does not fall below the floor. We define:

interest rate: r;

time: t;

time period: $[0, T]$;

floor: F;

floor at time t: F_t;

portfolio value: V;

portfolio value at time t: V_t;

cushion C;

cushion at time t: C_t;

multiple m;

exposure e;

exposure at time t: e_t;

riskless asset at time t: B_t.

where

$$C = V - F \qquad e = mC.$$

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ be a probability space satisfying the "usual assumption". In the simple CPPI continuous time case we assume that the risky asset satisfies the Black-Scholes model, i.e.

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = s, \tag{2.1}$$

where $V_t = mC_t + (V_t - mC_t)$.

Let the interest rate be r and floor at time t be $F_t = F_0 e^{rt} = F_T e^{-r(T-t)}$. We denote $F_T = G$.

Here are a list of their relation:

$$\begin{aligned} C_t &= V_t - F_t; \\ e_t &= mC_t; \\ B_t &= V_t - e_t. \end{aligned}$$

Proposition 2.1. *The portfolio value of CPPI under the Black-Scholes model in continuous time trading is*

$$V_t = \left(V_0 - Ge^{-rT}\right) \exp\left\{(r + m(\mu - r))t - \frac{m^2\sigma^2 t}{2} + m\sigma W_t\right\} + G \times \exp\{-r(T-t)\} \tag{2.2}$$

where $G = F_T$*.*

Proof. We have

$$\begin{aligned} V_t &= mC_t + (V_t - mC_t) \\ &= V_t\left(\frac{mC_t}{V_t} + \left(1 - \frac{mC_t}{V_t}\right)\right), \end{aligned}$$

and by the assumption of self-financing, we have

$$dV_t = V_t\left(\frac{mC_t}{V_t}\frac{dS_t}{S_t} + \left(1 - \frac{mC_t}{V_t}\right)\frac{dB_t}{B_t}\right),$$

thus

$$\begin{aligned} dC_t &= d(V_t - F_t) \\ &= V_t\left(\frac{mC_t}{V_t}\frac{dS_t}{S_t} + \left(1 - \frac{mC_t}{V_t}\right)\frac{dB_t}{B_t}\right) - F_t\frac{dB_t}{B_t} \\ &= C_t\left(\frac{mdS_t}{S_t} - (m-1)rdt\right) \\ &= C_t(m(\mu dt + \sigma dW_t) - (m-1)rdt) \end{aligned}$$

$$= \quad C_t((r + m(\mu - r))dt + m\sigma dW_t).$$

Then

$$C_t = C_0 \exp\left\{(r + m(\mu - r))t - \frac{m^2\sigma^2 t}{2} + m\sigma W_t\right\},$$

therefore, we have

$$\begin{aligned} V_t &= C_t + F_t \\ &= C_0 \exp\left\{(r + m(\mu - r))t - \frac{m^2\sigma^2 t}{2} + m\sigma W_t\right\} + G \times \exp\{-r(T-t)\} \\ &= \left(V_0 - Ge^{-r(T-t)}\right) \exp\left\{(r + m(\mu - r))t - \frac{m^2\sigma^2 t}{2} + m\sigma W_t\right\} + G \times \exp\{-rT\}. \end{aligned}$$

□

The expectation and variance of the CPPI portfolio value are obviously two important values to describe the strategies.
We know that $\exp\left(m\sigma W_t - \frac{1}{2}m^2\sigma^2 t\right)$ is an exponential martingale. Thus, we get the expectation of the CPPI portfolio value in the following proposition.

Proposition 2.2. *The expectation of CPPI portfolio value under the Black-Scholes model in continuous time trading is*

$$Ge^{-rT} + \left(V_0 - Ge^{-rT}\right) \exp\{(r + m(\mu - r))t\}$$

Proof.

$$\begin{aligned} \mathbb{E}[V_t] &= Ge^{-r(T-t)} + C_0 \exp\{(r + m(\mu - r))t\}\mathbb{E}\left[\exp\left(m\sigma W_t - \frac{1}{2}m^2\sigma^2 t\right)\right] \\ &= Ge^{-r(T-t)} + \left(V_0 - Ge^{-rT}\right) \exp\{(r + m(\mu - r))t\}. \end{aligned}$$

□

In order to calculate the variance, we will use the following lemma.

Lemma 2.3. *Let* $h_t = \exp\left(m\sigma W_t - \frac{1}{2}m^2\sigma^2 t\right)$*, then* $\mathbb{E}[h_t] = 1$ *and* $Var(h_t) = \exp(b^2 t) - 1$.

Proof. By Ito formula, $dh_t = bh_t dW_t$, then

$$h_t - h_0 = \int_0^t bh_s dW_s,$$

then h_t is a martingale and then $\mathbb{E}[h_t] = \mathbb{E}[h_0] = 1$. We have

$$\begin{aligned}
\mathrm{Var}(h_t) &= \mathbb{E}(h_t - \mathbb{E}(h_t))^2 = \mathbb{E}(h_t - h_0)^2 \\
&= \mathbb{E}\left(\int_0^t bh_s dW_s\right)^2 = \mathbb{E}\left(\int_0^t b^2 h_s ds\right) \\
&= b^2\left(\int_0^t \mathbb{E}\left(h_s^2\right) ds\right) = b^2\left(\int_0^t \mathbb{E}\left(\exp\left(2m\sigma W_t - m^2\sigma^2 t\right)\right) ds\right) \\
&= b^2\left(\int_0^t \exp\left(b^2 s\right) \mathbb{E}\left(\exp\left(2m\sigma W_t - \frac{(2b)^2}{2} m^2\sigma^2 t\right)\right) ds\right) \\
&= b^2\left(\int_0^t \exp\left(b^2 s\right) ds\right) = \exp\left(b^2 t\right) - 1.
\end{aligned}$$

□

Using the above lemma, we could calculate the variance of the CPPI portfolio value in the following proposition. (Referent [16].)

Proposition 2.4. *The variance of the CPPI portfolio value under the Black-Scholes model in continuous time trading is*

$$\left(V_0 - Ge^{-rT}\right)^2 \exp(2(r + m(\mu - r)t)\left(\exp\left(m^2\sigma^2 t\right) - 1\right).$$

Proof.

$$\mathrm{Var}[V_t] \quad = \quad \mathrm{Var}[C_t]$$

$$
\begin{aligned}
&= C_0^2 \exp(2(r+m(\mu-r)t)\mathrm{Var}\left[\exp\left(m\sigma W_t - \frac{1}{2}m^2\sigma^2 t\right)\right] \\
&= C_0^2 \exp(2(r+m(\mu-r)t)\mathrm{Var}[h_t] \\
&= \left(V_0 - Ge^{-rT}\right)^2 \exp(2(r+m(\mu-r)t)\left(\exp\left(m^2\sigma^2 t\right)-1\right).
\end{aligned}
$$

□

It is interesting at this point to wonder how the leverage regime modifies the return/risk profile of the product. As our intuition suggests, an increase in the gearing constant (multiple) which determines the leverage regime amplifies heavily the volatility.

Proposition 2.5. *The expected portfolio value and the variance of the CPPI portfolio's value, increase with the multiple m. In particular it is true that for any* $t \in [0, T]$*,*

$$
\begin{aligned}
\lim_{m\to\infty} \mathbb{E}[V_t] &= +\infty; \\
\lim_{m\to\infty} Var[V_t] &= +\infty;
\end{aligned}
$$

and

$$
\lim_{m\to\infty} \frac{\mathbb{E}[V_t]}{Var[V_t]} = 0,
$$

with an order of $o\left(\exp((r+m(\mu-r)t)\frac{1}{\exp(m^2\sigma^2 t)-1}\right)$.

Proof. The first two equations is obviously.

and for the third one

$$
\begin{aligned}
\lim_{m\to\infty} \frac{\mathbb{E}[V_t]}{\mathrm{Var}[V_t]} &\sim \frac{\exp\{(r+m(\mu-r))t}{\exp(2(r+m(\mu-r)t)(\exp(m^2\sigma^2 t)-1)} \\
&\sim \frac{1}{\exp((r+m(\mu-r)t)\,(\exp(m^2\sigma^2 t)-1)} \\
&\sim 0. \quad (m\to\infty)
\end{aligned}
$$

□

The following proposition shows there is no fallen risk for the continuous trading time CPPI defined on the continuous model.

Proposition 2.6. *Let the risky asset model S_t be $\mathbb{P}$ almost sure continuous and the trading time be continuous. If the CPPI defined on this model, then the portfolio value V_t is almost sure greater than the floor F_t.*

Proof. In the proof of Proposition 2.1, we have got

$$dC_t = C_t \left(\frac{mdS_t}{S_t} - (m-1)rdt \right).$$

Then

$$\ln(C_t) - \ln(C_0) = m(\ln(S_t) - \ln(S_0)) - (m-1)rt.$$

We have

$$C_t = C_0 \exp\left(\ln \frac{S_t}{S_0} - (m-1)rt \right)$$

and it is $\mathbb{P}$ almost sure positive. □

2.1.2 The discrete trading time case

Here we continue to assume that our risky asset satisfies the Black-Scholes model. In addition, let $\tau^N = \{t_0 = 0 < t_1 < t_2 < ... < t_n = T\}$ denote a sequence of equidistant refinements of the interval $[0, T]$, where $t_{k+1} - t_k = \frac{T}{n}$ for $k = 0, ..., n-1$. We assume now that trading is restricted to the discrete set τ^n. We have

$$C_{t_{k+1}} = C_{t_k} \left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/n} \right),$$

then

$$C_T = C_{t_n} = C_0 \prod_{k=0}^{n-1} \left(m \frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/n} \right),$$

thus

$$V_T = C_T + G.$$

Since $\frac{S_{t_{k+1}}}{S_{t_k}}$, $k = 0.1.2...n-1$ are mutually independent and also they have the identity distribution. Then we have

$$\mathbb{E}\left[\frac{S_{t_{k+1}}}{S_{t_k}}\right] = \mathbb{E}\left[\exp\left(\mu\frac{T}{n} + \sigma W_{T/n} - \frac{1}{2}\sigma^2\frac{T}{n}\right)\right] = \exp\left(\mu\frac{T}{n}\right);$$

and

$$\begin{aligned}
\mathbb{E}\left[\frac{S_{t_{k+1}}}{S_{t_k}}\right]^2 &= \mathbb{E}\left[\exp\left(2\mu\frac{T}{n} + 2\sigma W_{T/n} - \sigma^2\frac{T}{n}\right)\right] \\
&= \mathbb{E}\left[\exp\left(2\mu\frac{T}{n} + \sigma^2\frac{T}{n} + 2\sigma W_{T/n} - \frac{1}{2}(2\sigma)^2\frac{T}{n}\right)\right] \\
&= \exp\left(2\mu\frac{T}{n} + \sigma^2\frac{T}{n}\right).
\end{aligned}$$

In the discrete case, it is possible that $V_{t_i} \leq F_{t_i}$ for some t_i. We generally allow the possibility of short-sell and negative cushion. However, this also means that the CPPI-insured portfolio would incur a loss.

Proposition 2.7. *The expected terminal CPPI portfolio value under Black-Scholes model in the discrete trading is*

$$\left(V_0 - Ge^{-rT}\right)\left(m\exp\left(\mu\frac{T}{n}\right) - (m-1)e^{rT/n}\right)^n + G. \tag{2.3}$$

Proof.

$$\mathbb{E}[V_T] = \mathbb{E}[C_T] + G = \mathbb{E}\left[C_0 \prod_{k=0}^{n-1}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/n}\right)\right] + G$$

$$
\begin{aligned}
&= C_0 \prod_{k=0}^{n-1} \left(m\mathbb{E}\left[\frac{S_{t_{k+1}}}{S_{t_k}}\right] - (m-1)e^{rT/n} \right) + G \\
&= C_0 \prod_{k=0}^{n-1} \left(m \exp\left(\mu \frac{T}{n}\right) - (m-1)e^{rT/n} \right) + G \\
&= \left(V_0 - Ge^{-rT}\right) \left(m \exp\left(\mu \frac{T}{n}\right) - (m-1)e^{rT/n} \right)^n + G.
\end{aligned}
$$

□

In order to calculate the variance of the terminal CPPI portfolio value, we need the following lemma.

Lemma 2.8. *Let A_i, i=1,2,...n be independent random variables, then we have*

$$
Var\left[\prod_{k=1}^{n} A_i\right] = \prod_{k=1}^{n} \left(\mathbb{E}A_i^2\right) - \prod_{k=1}^{n} \left(\mathbb{E}A_i\right)^2.
$$

Proof. We have

$$
\begin{aligned}
\text{Var}\left[\prod_{k=1}^{n} A_i\right] &= \mathbb{E}\left(\prod_{k=1}^{n} A_i\right)^2 - \left(\mathbb{E}\prod_{k=1}^{n} A_i\right)^2 \\
&= \mathbb{E}\left(\prod_{k=1}^{n} A_i^2\right) - \left(\prod_{k=1}^{n} \mathbb{E}A_i\right)^2 \\
&= \prod_{k=1}^{n} \left(\mathbb{E}A_i^2\right) - \prod_{k=1}^{n} \left(\mathbb{E}A_i\right)^2.
\end{aligned}
$$

□

By the above lemma, we could calculate the variance of the CPPI terminal portfolio value in the following proposition.

Proposition 2.9. *The variance of the CPPI terminal portfolio value under Black-*

Scholes model in the discrete trading is

$$\begin{aligned}
& (V_0 - Ge^{-rT})^2 \Bigg(\bigg(m^2 \exp\left(2\mu\frac{T}{n} + \sigma^2\frac{T}{n}\right) + (m-1)^2 e^{2rT/n} \\
- \quad & 2m(m-1)\exp\left(\mu\frac{T}{n}\right)e^{rT/n}\bigg)^n - \left(m\exp\left(\mu\frac{T}{n}\right) - (m-1)e^{rT/n}\right)^{2n}\Bigg).
\end{aligned}$$

Proof. Since

$$\begin{aligned}
& \mathbb{E}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/n}\right)^2 \\
= \quad & \mathbb{E}\left(m^2\left(\frac{S_{t_{k+1}}}{S_{t_k}}\right)^2 + (m-1)^2e^{2rT/n} - 2m(m-1)\frac{S_{t_{k+1}}}{S_{t_k}}e^{rT/n}\right) \\
= \quad & m^2\left(\mathbb{E}\frac{S_{t_{k+1}}}{S_{t_k}}\right)^2 + (m-1)^2e^{2rT/n} - 2m(m-1)\mathbb{E}\frac{S_{t_{k+1}}}{S_{t_k}}e^{rT/n} \\
= \quad & m^2\exp\left(2\mu\frac{T}{n} + \sigma^2\frac{T}{n}\right) + (m-1)^2e^{2rT/n} - 2m(m-1)\exp\left(\mu\frac{T}{n}\right)e^{rT/n},
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{Var}[V_T] \quad = \quad & \mathrm{Var}[C_T] \\
= \quad & \mathrm{Var}\left[C_0\prod_{k=0}^{n-1}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/n}\right)\right] \\
= \quad & C_0^2\Bigg[\prod_{k=0}^{n-1}\mathbb{E}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/n}\right)^2 \\
& -\prod_{k=0}^{n-1}\left(\mathbb{E}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/n}\right)\right)^2\Bigg] \\
= \quad & (V_0 - Ge^{-rT})^2 \Bigg(\bigg(m^2 \exp\left(2\mu\frac{T}{n} + \sigma^2\frac{T}{n}\right) + (m-1)^2 e^{2rT/n} \\
- \quad & 2m(m-1)\exp\left(\mu\frac{T}{n}\right)e^{rT/n}\bigg)^n - \left(m\exp\left(\mu\frac{T}{n}\right) - (m-1)e^{rT/n}\right)^{2n}\Bigg).
\end{aligned}$$

□

Probability of Loss

In the case of discrete-time trading, it is possible that the portfolio value falls below the floor. i.e. $V_t \leq F_t$ which is equivalent to $C_t \leq 0$, happens only at time t_i. We call it the **Probability of Loss**.

There are two possible causes for gap risks. One is the existence of jumps in the risky asset model and the other is because of the trading time is not continuous. In this section, we consider the case when the gap risk happens at discontinuous trading time. In section 4.2, we will consider the presence of jumps and the trading time is continuous. In section 5.2, we will consider the co-existence of the above two situations.

Proposition 2.10. *The probability of the CPPI portfolio value under Black-Scholes model in the discrete trading going below the floor taking happen is given by*

$$\mathbb{P}[\exists\, t_i : V_{t_i} \leq F_{t_i}] = 1 - \Psi^n\left(-\frac{1}{\sigma}\left(\sqrt{\frac{n}{T}}\ln\left(\frac{m-1}{m}\right) + \left(r - \mu + \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}}\right)\right) \quad (2.4)$$

where

$$\Psi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

Proof.

$$\begin{aligned}
&\mathbb{P}[\exists\, t_i : V_{t_i} \leq F_{t_i}] = \mathbb{P}[\forall \quad t_i : C_{t_i} \leq 0] \\
= \;& 1 - \mathbb{P}[\exists \quad t_i : C_{t_i} > 0] = 1 - \mathbb{P}\left[\bigcap_{i=1}^{n}\{C_{t_i} > 0\}\right] = 1 - \prod_{i=1}^{n}\mathbb{P}[\{C_{t_i} > 0\}] \\
= \;& 1 - \prod_{i=1}^{n}\mathbb{P}\left[\left\{m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{rT/n} > 0\right\}\right] \\
= \;& 1 - \prod_{i=1}^{n}\mathbb{P}\left[\left\{m\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma W_{\frac{T}{n}}\right\} - (m-1)e^{rT/n} > 0\right\}\right]
\end{aligned}$$

$$= 1 - \left(\mathbb{P} \left[\sigma W_{\frac{T}{n}} \geq \ln \left(\frac{m-1}{m} \right) + \left(r - \mu + \frac{\sigma^2}{2} \right) \frac{T}{n} \right] \right)^n$$

$$= 1 - \Psi^n \left(-\frac{1}{\sigma} \left(\sqrt{\frac{n}{T}} \ln \left(\frac{m-1}{m} \right) + \left(r - \mu + \frac{\sigma^2}{2} \right) \sqrt{\frac{T}{n}} \right) \right)$$

where

$$\Psi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

□

Proposition 2.11. *The probability of the CPPI portfolio value under the Black-Scholes model in the discrete trading given by* (2.4) *is monotone increased function as the multiple* m*.*

Proof. We have

$$m \uparrow \Rightarrow \tfrac{m-1}{m} = 1 - \tfrac{1}{m} \uparrow \Rightarrow$$
$$-\tfrac{1}{\sigma} \left(\sqrt{\tfrac{n}{T}} \ln \left(\tfrac{m-1}{m} \right) + \left(r - \mu + \tfrac{\sigma^2}{2} \right) \sqrt{\tfrac{T}{n}} \right) \downarrow \Rightarrow (2.4) \uparrow .$$

□

More description of the gap risk and their applications will be discussed in chapter 4 and chapter 5.

2.2 EPPI in Black-Scholes model

2.2.1 The discrete trading time case

It is sometimes not practical to assume that the multiple m is a constant. We consider the case when the multiple is a function of time. Let $m_{t_k} = \eta + e^{a \ln(S_{t_k}/S_{t_{k-1}})}$ where $a > 1$. i.e. at time t_k, we employ the multiple m_{t_k}, where $\eta \geq 0$ is a constant. We may as well assume $\eta = m - 1\ e^{a\ln(S_{t_k}/S_{t_{k-1}})} = (S_{t_k}/S_{t_{k-1}})^a$.

When $S_{t_k} > S_{t_{k-1}}$, the stock price increases. Then

$$(S_{t_k}/S_{t_{k-1}}) > 1, \quad (S_{t_k}/S_{t_{k-1}})^a > 1.$$

Thus, $m_{t_k} > m$. This means that we will invest more money into the stock market.

When $S_{t_k} < S_{t_{k-1}}$, the stock price is decreases. Then

$$(S_{t_k}/S_{t_{k-1}}) < 1, \quad (S_{t_k}/S_{t_{k-1}})^a < 1.$$

Thus, $m_{t_k} < m$. This means that we will invest less money into the stock market.

When $S_{t_k} = S_{t_{k-1}}$, the stock price is not changed. Then

$$(S_{t_k}/S_{t_{k-1}}) = 1, \quad (S_{t_k}/S_{t_{k-1}})^a = 1.$$

Thus, $m_{t_k} = m$. This means that we keep the strategy as before.

We call the new strategy an Exponential Proportion Portfolio Insurance (EPPI). This is practical in real markets, the investor would like to invest more money when the stock is increasing and less money when the stock is decreasing. Here the $a > 1$ is just like a multiplier of the effect of the change of stock market. When we assume $a = 0$, then $m_{t_k} = m$ everywhere. The EPPI becomes CPPI. Thus, we can treat EPPI as an extension of CPPI.

Proposition 2.12. *The cushion of EPPI under the Black-Scholes model in the discrete trading satisfies*

$$C_{t_{k+1}} = C_{t_k}\left(m_{t_k}\frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1)e^{rT/n}\right).$$

Proof. We have

$$\begin{aligned}
V_{t_{k+1}} &= \frac{m_{t_k} C_{t_k}}{S_{t_k}} S_{t_{k+1}} + (V_{t_k} - m_{t_k} C_{t_k}) \frac{B_{t_{k+1}}}{B_{t_k}} \\
&= m_{t_k} C_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} + (V_{t_k} - m_{t_k} C_{t_k}) \frac{B_{t_{k+1}}}{B_{t_k}} \\
&= (V_{t_k} - C_{t_k}) \frac{B_{t_{k+1}}}{B_{t_k}} - (m_{t_k} - 1) C_{t_k} \frac{B_{t_{k+1}}}{B_{t_k}} + m_{t_k} C_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} \\
&= F_{t_k} \frac{B_{t_{k+1}}}{B_{t_k}} + C_{t_k} \left(m_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1) \frac{B_{t_{k+1}}}{B_{t_k}} \right) \\
&= F_{t_{k+1}} + C_{t_k} \left(m_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1) \frac{B_{t_{k+1}}}{B_{t_k}} \right).
\end{aligned}$$

Since

$$V_{t_{k+1}} = F_{t_{k+1}} + C_{t_{k+1}},$$

then we have

$$C_{t_{k+1}} = C_{t_k} \left(m_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1) e^{rT/n} \right).$$

□

Therefore, we have

$$\begin{aligned}
C_T &= C_{t_n} = C_0 \prod_{k=0}^{n-1} \left(m_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1) e^{rT/n} \right) \\
&= \left(V_0 - G e^{-rT} \right) \prod_{k=0}^{n-1} \left(m_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1) e^{rT/n} \right),
\end{aligned}$$

and since

$$V_T = C_T + G,$$

thus we get

Proposition 2.13. *The EPPI terminal portfolio value under the Black-Scholes model* *in the discrete trading is*

$$\left(V_0 - Ge^{-rT}\right)\prod_{k=0}^{n-1}\left(m_{t_k}\frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k}-1)e^{rT/n}\right) + G.$$

Monte Carlo simulation techniques We want to simulate both the CPPI strategy and EPPI strategy under the Black-scholes model. In the discrete case,

$$\ln\frac{S_{k+1}}{S_k} \sim N\left(\mu\frac{T}{n} - \frac{1}{2}\sigma^2\frac{T}{n}, \sigma^2\frac{T}{n}\right).$$

The algorithm could be

Generate $(Z_0, ..., Z_{n-1}) \sim N(0, I)$;
for $i = 0,\ 1 \ldots n-1$;
$A_i \leftarrow \exp\mu\frac{T}{n} - \frac{1}{2}\sigma^2\frac{T}{n} + \sqrt{\sigma^2\frac{T}{n}}Z_i$;
$B_i \leftarrow mA_i - (m-1)e^{rT/n}$;
for $j = 0,\ 1, \ldots n-1$
$V_j^{\text{CPPI}} \leftarrow (V_0 - Ge^{-rT})B_0 * B_1 * \ldots * B_{j-1} + Ge^{(n-j)T/n}$;
$m_0 = \eta + 1$; for $i = 1,\ 2, \ldots n-1$
$m_{t_k} = \eta + e^{a\ln(A_i)}$;
$B_i \leftarrow m_{t_k}A_i - (m_{t_k}-1)e^{rT/n}$;
for $j = 0,\ 1, \ldots n-1$ $V_j^{\text{EPPI}} \leftarrow (V_0 - Ge^{-rT})B_0 * B_1 *$
$\ldots * B_{j-1} + Ge^{(n-j)T/n}$;
plot(V_j^{CPPI}, V_j^{EPPI});

We use matlab to implement the algorithm. (Figure 2.1)

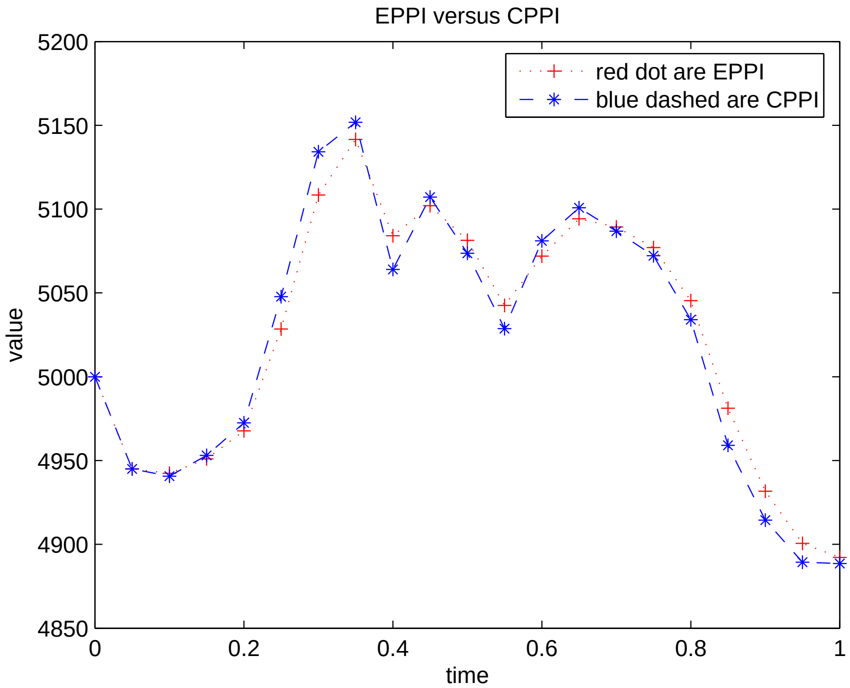

Figure 2.1: We design the function [Vn, Vn2]=EPPIBS (r, mu, sigma, T, n, s0, m, a, v0, G) with arguments in Matlab to implement the simulation.
When in EPPIBS (0.01, 0.02, 0.1, 1, 20, 10, 4, 5, 5000, 4500), this is in particular, here we assume $r = 0.01$, $\mu = 0.02$, $\sigma = 0.1$, $T = 1$, $n = 20$ $m = 4$, $a = 5$, $V(0) = 5000$ and floor $G = 4500$.

2.2.2 The continuous trading time case

We still assume the stock price satisfies the Black-Scholes model. Let

$$0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$$

where $t_{k+1} - t_k = \frac{T}{n}$ for $k = 0, ..., n-1$. We reconsider the multiple only at time t_i which $i = 0, 1, ..., n$. Let

$$\begin{aligned} m_0 &= \eta + 1; \\ m_{t_k} &= \eta + e^{aln(S_{t_k}/S_{t_{k-1}})} \text{ when } k \geq 1; \\ m_t &= m_{t_k} \text{ when } t \in [t_k, t_{k+1}). \end{aligned}$$

In this case, in every interval $[t_k, t_{k+1})$, the strategy is standard CPPI. If we let $a = 0$, then it is same as standard CPPI. Therefore, we can treat this strategy also as an extension of standard CPPI. We deduced

$$C_t = C_{t_k} \exp\left\{ \left(r + m_{t_k}(\mu - r) - \frac{1}{2} m_{t_k}^2 \right) (t - t_k) + \sigma m_{t_k} (W_t - W_{t_k}) \right\}$$

when $t \in [t_k, t_{k+1})$, and thus

$$C_{t_{k+1}} = C_{t_k} \left(m_{t_k} \frac{S_{t_{k+1}}}{S_{t_k}} - (m_{t_k} - 1) e^{rT/n} \right).$$

Therefore, we have

$$\begin{aligned} C_T &= C_0 \exp\left\{ rT + \frac{m_0 + ... + m_{t_{k-1}}(\mu - r)}{n} T - \frac{\sigma^2}{2} \frac{m_0^2 + ... + m_{t_{k-1}}^2}{n} T \right. \\ &\quad \left. + \sigma \left(\Sigma_{i=0}^{n-1} m_{t_i} (W_{t_{i+1}} - W_{t_i}) \right) \right\} \\ &= \left(V_0 - G e^{-rT} \right) \exp\left\{ rT + \frac{m_0 + ... + m_{t_{k-1}}}{n} (\mu - r) T \right. \\ &\quad \left. - \frac{\sigma^2}{2} \frac{m_0^2 + ... + m_{t_{k-1}}^2}{n} T + \sigma \left(\Sigma_{i=0}^{n-1} m_{t_i} (W_{t_{i+1}} - W_{t_i}) \right) \right\}. \end{aligned}$$

Since

$$V_T = C_T + G,$$

thus we get

Proposition 2.14. *The terminal EPPI portfolio value under the Black-Scholes model in the discrete trading is*

$$\left(V_0 - G e^{-rT} \right) \exp\left\{ rT + \frac{m_0 + ... + m_{t_{k-1}}}{n} (\mu - r) T - \frac{\sigma^2}{2} \frac{m_0^2 + ... + m_{t_{k-1}}^2}{n} T \right.$$
$$\left. + \sigma \left(\Sigma_{i=0}^{n-1} m_{t_i} (W_{t_{i+1}} - W_{t_i}) \right) \right\} + G.$$

Monte Carlo simulation techniques In this case, the algorithm could be

```
Generate $(Z_1, ..., Z_n) \sim N(0, I)$;
for $i = 0, 1 \ldots n-1$;
$A_i \leftarrow \exp \mu \frac{T}{n} - \frac{1}{2}\sigma^2 \frac{T}{n} + \sqrt{\sigma^2 \frac{T}{n}} Z_i$;
$m_0 = \eta + 1$; $C_{t_0} = V_{t_0} - Ge^{-rT}$;
for $i = 1, 2, \ldots n-1$
$m_{t_k} = \eta + e^{a \ln(A_i)}$;
$C_{t_{k+1}} = C_{t_k} \left( m_{t_k} A_i - (m_{t_k} - 1) e^{rT/n} \right)$
$V_{t_{k+1}} = C_{t_{k+1}} + Ge^{(n-k-1)rT/n}$;
plot(V);
```

2.3 CPPI in GARCH model

2.3.1 The Continuous trading time cases

Here instead of treating the volatility as a constant, we consider the following model. The ARCH/GARCH model considers the volatility which depend on the past history. In particular consider the **GARCH(p,q)** model:

$$\ln \frac{S_t}{S_{t-1}} = \mu(\sigma_t) - \frac{\sigma_t^2}{2} + \sigma_t \epsilon_t,$$

where μ is a given function, ϵ_1, ϵ_2,...is a sequence of i.i.d. standard normal random variables, and σ_t satisfies:

$$\sigma_t^2 = \omega + \Sigma_{i=1}^{p} \alpha_i (\sigma_{t-i} \epsilon_{t-i})^2 + \Sigma_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

$\omega, \alpha_1, ...\alpha_p, \beta_1, ..., \beta_q$ being fixed constants.

By an embedding methodology, we can recover the continuous-time GARCH$(1,1)$ model. (Refer to [44])

$$
\begin{aligned}
F(X)_t = &\ \left\{ \begin{array}{l} \sigma_0 \quad for \quad 0 \leq t < 1 \\ \left(\omega + \alpha \left(X_{[t]-X_{[t]-1}}\right)^2 + \beta F(x)^2_{[t]-1}\right)^{1/2} \quad \text{for } t \geq 1 \end{array} \right. ; \\
S_t = &\ S_0 \exp\left\{ \int_0^t (\mu(\sigma_{s-}) - \frac{\sigma^2_{s-}}{2}) ds + X_t \right\}; \\
X_t = &\ \int_0^t \sigma_{s-} dB_s; \\
\sigma := &\ F(x).
\end{aligned}
$$

We have

$$dS_t = S_t \left(\mu(\sigma_{t-})dt + \sigma_{t-} dB_t\right).$$

Next we have,

Proposition 2.15. *The CPPI cushion under GARCH$(1,1)$ model in the continuous trading time case above satisfies*

$$C_t = C_0 \exp\left(\int_0^t \left(\mu(\sigma_{s-}) - \frac{m^2 \sigma^2_{s-}}{2} \right) ds + m \int_0^t \sigma_{s-} dB_s - (m-1)rt \right). \qquad (2.5)$$

Proof. Since the strategy is self-financing, thus, we have

$$V_t = V_t \left(\frac{mC_t}{V_t} + \left(1 - \frac{mC_t}{V_t} \right) \right)$$

and

$$dV_t = V_t \left(\frac{mC_t}{V_t} \frac{dS_t}{S_t} + \left(1 - \frac{mC_t}{V_t} \right) \frac{dB_t}{B_t} \right).$$

Then

$$
\begin{aligned}
dC_t &= d(V_t - F_t) \\
&= V_t\left(\frac{mC_t}{V_t}\frac{dS_t}{S_t} + \left(1 - \frac{mC_t}{V_t}\right)\frac{dB_t}{B_t}\right) - F_t\frac{dB_t}{B_t} \\
&= C_t\left(\frac{mdS_t}{S_t} - (m-1)rdt\right) \\
&= C_t((\mu(\sigma_{t-})dt + \sigma_{t-}dB_t) - (m-1)rdt).
\end{aligned}
$$

Hence, we get

$$
C_t = C_0 \exp\left(\int_0^t \left(\mu(\sigma_{s-}) - \frac{m^2\sigma_{s-}^2}{2}\right) ds + m\int_0^t \sigma_{s-}dB_s - (m-1)rt\right).
$$

□

We then got the *portfolio value* is

$$
V_t = C_t + F_t = C_t + G\exp\{-r(T-t)\}.
$$

The following proposition is the property of GARCH$(1,1)$ model.

Proposition 2.16. *Let $n \in \mathbb{N}$, we have*

$$
\mathbb{E}[\sigma_n^2] = (\alpha+\beta)^{n-1}\left(\sigma_0^2 + \frac{\omega}{\alpha+\beta-1}\right) - \frac{\omega}{\alpha+\beta-1}. \tag{2.6}
$$

Proof. By definition

$$
\begin{aligned}
F(X)_t &= \begin{cases} \sigma_0 \text{ for } 0 \le t < 1 \\ \left(\omega + \alpha\left(X_{[t]-X_{[t]-1}}\right)^2 + \beta F(x)_{[t]-1}^2\right)^{1/2} \text{ for } t \ge 1 \end{cases}; \\
S_t &= S_0 \exp\left\{\int_0^t (\mu(\sigma_{s-}) - \frac{\sigma_{s-}^2}{2})ds + X_t\right\}; \\
X_t &= \int_0^t \sigma_{s-}dB_s;
\end{aligned}
$$

$$\sigma := F(x).$$

then

$$\begin{cases} \sigma_t = \sigma_0 \text{ for } 0 \leq t < 1 \\ \sigma_t^2 = \sigma_{[t]}^2 \omega + \alpha \left(\int_{[t]-1}^{[t]} \sigma_{s-} dB_s \right)^2 + \beta \sigma_{[t]-1}^2 \text{ for } t \geq 1 \end{cases},$$

thus,

$$\begin{aligned} \sigma_n^2 &= \omega + \alpha \left(\int_{n-1}^{n} \sigma_{s-} dB_s \right)^2 + \beta \sigma_{n-1}^2 \\ &= \omega + \alpha \left(\sigma_{n-1}(B_n - B_{n-1}) \right)^2 + \beta \sigma_{n-1}^2 \\ &= \omega + \sigma_{n-1}^2 (\alpha (B_n - B_{n-1})^2 + \beta), \end{aligned}$$

then

$$\mathbb{E}\sigma_n^2 = \omega + E\sigma_{n-1}^2(\alpha + \beta),$$

and hence

$$\mathbb{E}[\sigma_n^2] + \frac{\omega}{\alpha + \beta - 1} = (\alpha + \beta) \left(\mathbb{E}[\sigma_{n-1}^2] + \frac{\omega}{\alpha + \beta - 1} \right).$$

Thus, we get

$$\mathbb{E}[\sigma_n^2] = (\alpha + \beta)^{n-1} \left(\sigma_0^2 + \frac{\omega}{\alpha + \beta - 1} \right) - \frac{\omega}{\alpha + \beta - 1}.$$

□

In order to calculate the expectation of V_t explicitly, in the following, we assume $\mu(z) = \mu$ be constant function.

Lemma 2.17. *Let* $h_t = \exp\left(\int_0^t \left(m\sigma_{s-} dB_s - \int_0^t \frac{m^2 \sigma_{s-}^2}{2} \right) ds \right)$, *then* $\mathbb{E}[h_t] = 1$.

Proof. By Ito formula

$$dh_t = m\sigma_{t-}h_t dB_t,$$

thus

$$h_t - h_0 = \int_0^t m\sigma_{s-}h_s dB_s.$$

Hence h_t is a martingale and thus $\mathbb{E}[h_t] = \mathbb{E}[h_0] = 1$. □

Proposition 2.18. *Let $\mu(z) = \mu$ be constant function. Then the expectation of the CPPI portfolio value V_t under the GARCH$(1,1)$ model in the continuous trading time case is*

$$\mathbb{E}(V_t) = Ge^{-r(T-t)} + (V_0 - Ge^{-rT})e^{m\mu t-(m-1)rt}.$$

Proof.

$$\begin{aligned}\mathbb{E}(V_t) &= Ge^{-r(T-t)} + C_0 e^{m\int_0^t \mu(\sigma_{s-})ds-(m-1)rt)} \\ &\times \mathbb{E}\left[\exp\left(\int_0^t \left(m\sigma_{s-}dB_s - \int_0^t \frac{m^2\sigma_{s-}^2}{2}\right) ds\right)\right] \\ &= Ge^{-r(T-t)} + (V_0 - Ge^{-rT})e^{m\mu t-(m-1)rt}.\end{aligned}$$

□

2.3.2 The Discrete trading time case GARCH(1,1) model

In this case, the model is

$$\ln\frac{S_t}{S_{t-1}} = \mu(\sigma_t) - \frac{\sigma_t^2}{2} + \sigma_t\epsilon_t;$$

$$\sigma_t^2 = \omega + \alpha(\sigma_{t-1}\epsilon_{t-1})^2 + \beta\sigma_{t-1}^2,$$

where ω, α, β are fixed constants. We only consider the time on the integer-value, i.e. integer times unit time.

Proposition 2.19. *The CPPI cushion under GARCH(1,1) model in the discrete trading time case model satisfies*

$$C_{t+1} = C_t \left(m \frac{S_{t+1}}{S_t} - (m-1)e^r \right). \tag{2.7}$$

Proof. Since the strategy is self-financing, we have

$$\begin{aligned}
V_{t+1} &= (V_t - mC_t)\frac{B_{t+1}}{B_t} + mC_t(S_{t+1}/S_t) \\
&= (V_t - C_t)\frac{B_{t+1}}{B_t} - (m-1)C_t\frac{B_{t+1}}{B_t} + mC_t(S_{t+1}/S_t) \\
&= F_t\frac{B_{t+1}}{B_t} + C_t\left(m\frac{S_{t+1}}{S_t} - (m-1)\frac{B_{t+1}}{B_t}\right) \\
&= F_{t+1} + C_t\left(m\frac{S_{t+1}}{S_t} - (m-1)\frac{B_{t+1}}{B_t}\right),
\end{aligned}$$

and since

$$V_{t+1} = F_{t+1} + C_{t+1},$$

then

$$C_{t+1} = C_t \left(m \frac{S_{t+1}}{S_t} - (m-1)e^r \right).$$

□

We then have

$$C_n = C_0 \prod_{k=0}^{n-1} \left(m \frac{S_{k+1}}{S_k} - (m-1)e^r \right),$$

and

$$V_n = C_n + F_n = C_n + G.$$

Monte Carlo simulation techniques Our algorithm could be

Generate $(\epsilon_1,\, \epsilon_2,\, \dots \epsilon_t) \sim N(0, I)$

for $i = 1,\, \dots t$

$\sigma_t \leftarrow \sqrt{\omega + \alpha(\sigma_{t-1}\epsilon_{t-1})^2 + \beta\sigma_{t-1}^2}$;

$A_i \leftarrow \exp\left(\mu(\sigma_t) - \frac{\sigma_t^2}{2} + \sigma_t\epsilon_t\right)$;

$C_0 = Ge^{-rt}$;

for $i = 0,\, \dots t - 1$

$C_{i+1} \leftarrow C_t((mA_i - (m-1)e^r))$;

$V_{i+1} = C_{i+1} + Ge^{-(t-i)}$;

plot(V);

We use Matlab to implement the strategy according to the above algorithm.(Figure 2.2)

2.4 EPPI in GARCH$(1, 1)$ model

We consider the EPPI in GARCH$(1, 1)$ model. We assume the stock price satisfy:

$$\ln \frac{S_t}{S_{t-1}} = \mu(\sigma_t) - \frac{\sigma_t^2}{s} + \sigma_t\epsilon_t;$$

$$\sigma_t^2 = \omega + \alpha(\sigma_{t-1}\epsilon_{t-1})^2 + \beta\sigma_{t-1}^2,$$

where ω, α, β are fixed constants and the multiple is

$$m_t = \eta + \exp\{a \ln(S_t/S_{t-1})\}.$$

Proposition 2.20. *The EPPI cushion in the GARCH*$(1, 1)$ *model satisfies*

$$C_{t+1} = C_t\left(m_t\frac{S_{t+1}}{S_t} - (m_t - 1)e^r\right)$$

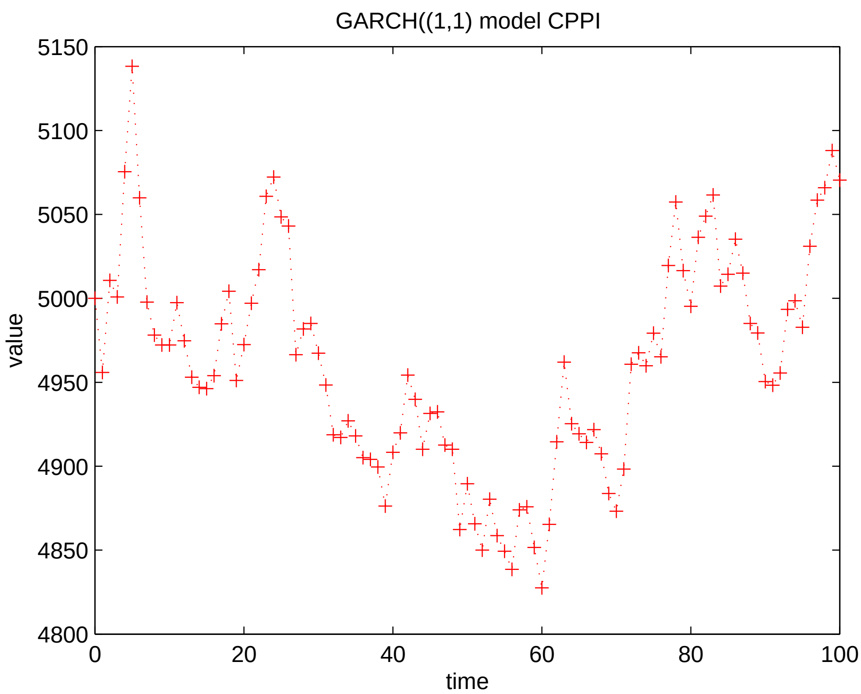

Figure 2.2: We design the function GARCHCPPI(r, mu, sigma0, alpha1, beta1,omega, n, v0, G, m) with arguments to implement the simulation.
When in GARCHCPPI(0.0001, 0.00015, 0.0003, 0.05, 0.05,0.0002, 100, 5000, 4500, 4), this is in particular, here we set $\mu(\sigma_{s-}$ is constant, $r = 0.0001$, $\mu = 0.00015$, $\sigma_0 = 0.0003$, $\alpha = 0.05$, $\beta = 0.05$, $\omega = 0.0002$, $n = 100$, $m = 4$, $V(0) = 5000$ and floor $G = 4500$.

Proof. Since the strategy is self-financing, we have

$$\begin{aligned}
V_{t+1} &= (V_t - m_t C_t)\frac{B_{t+1}}{B_t} + m_t C_t (S_{t+1}/S_t) \\
&= (V_t - C_t)\frac{B_{t+1}}{B_t} - (m_t - 1)C_t\frac{B_{t+1}}{B_t} + m_t C_t (S_{t+1}/S_t) \\
&= F_t\frac{B_{t+1}}{B_t} + C_t\left(m_t\frac{S_{t+1}}{S_t} - (m_t - 1)\frac{B_{t+1}}{B_t}\right) \\
&= F_{t+1} + C_t\left(m_t\frac{S_{t+1}}{S_t} - (m_t - 1)\frac{B_{t+1}}{B_t}\right),
\end{aligned}$$

and since

$$V_{t+1} = F_{t+1} + C_{t+1},$$

then

$$C_{t+1} = C_t \left(m_t \frac{S_{t+1}}{S_t} - (m_t - 1)e^r \right).$$

$\square$

Therefore we have

$$C_n = C_0 \prod_{k=0}^{n-1} \left(m_k \frac{S_{k+1}}{S_k} - (m_k - 1)e^r \right),$$

and

$$V_n = C_n + F_n = C_n + G.$$

Monte Carlo simulation techniques Our algorithm could be

Generate $(\epsilon_1, \epsilon_2, \ldots \epsilon_t) \sim N(0, I)$
for $i = 1, \ldots t$
$\sigma_t \leftarrow \sqrt{\omega + \alpha(\sigma_{t-1}\epsilon_{t-1})^2 + \beta\sigma_{t-1}^2}$;
$A_i \leftarrow \exp\left(\mu(\sigma_t) - \frac{\sigma_t^2}{2} + \sigma_t\epsilon_t\right)$;
$C_0 = Ge^{-rt}$;
$m_0 = m$;
for $i = 0, \ldots t-1$
$m_{i+1} = m - 1 + \exp\left(a \ln(A_i)\right)$;
$C_{i+1} \leftarrow C_t((m_i A_i - (m_i - 1)e^r))$;
$V_{i+1} = C_{i+1} + Ge^{-(t-i)}$;
plot(V);

We use Matlab to implement the strategy according the above algorithm. (Figure 2.3)

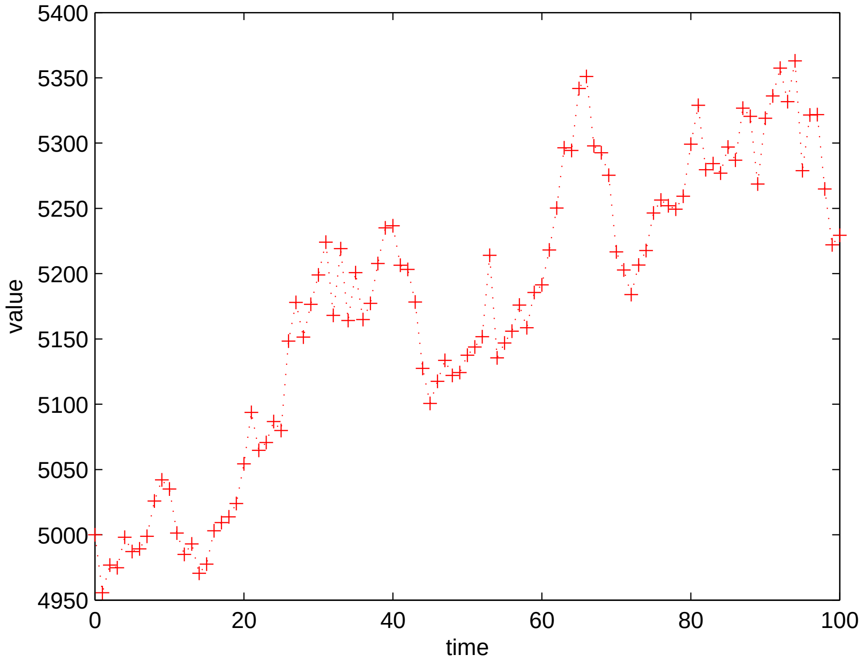

Figure 2.3: We design the function GARCHEPPI(r, mu, sigma0, alpha1, beta1, omega, n, v0, G, m,a) with arguments to implement the simulation.
When in GARCHEPPI(0.0001, 0.00015, 0.0003, 0.05, 0.05,0.0002, 100, 5000, 4500, 4,2), this is particular, here we set $\mu(\sigma_{s-})$ is constant, $r = 0.0001$, $\mu = 0.00015$, $\sigma_0 = 0.0003$, $\alpha = 0.05$, $\beta = 0.05$, $\omega = 0.0002$, $n = 100$, $m = 4$, $a = 2$, $V(0) = 5000$ and floor $G = 4500$.

The next figure draws the EPPI versus CPPI in GARCH. (Figure 2.4)

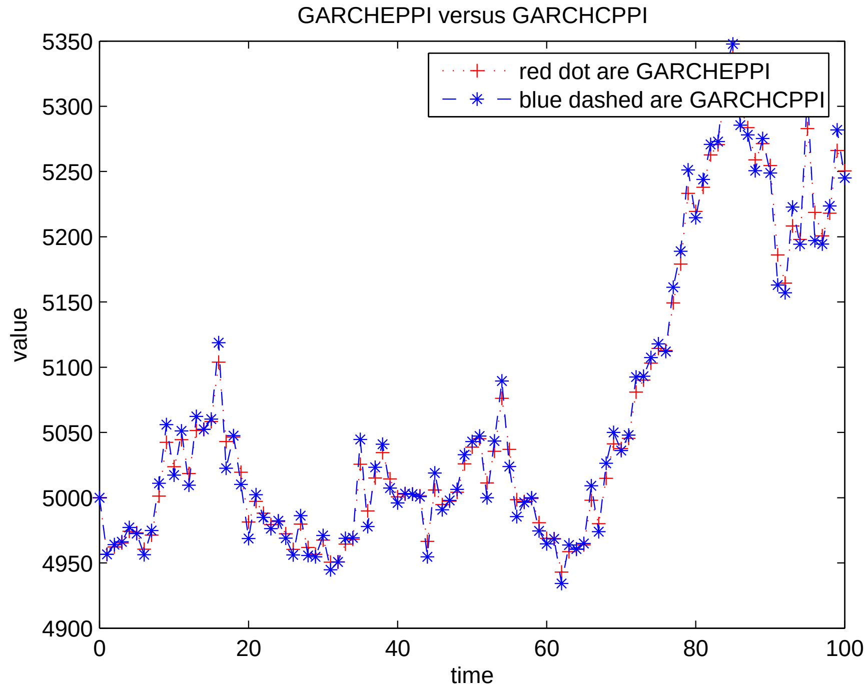

Figure 2.4: We design the function GARCHEPPIvsCPPI(r, mu, sigma0, alpha1, beta1, omega, n, v0, G, m,a)to implement the simulation.
When in [y1,y2]=GARCHEPPIvsCPPI(0.0001, 0.00015, 0.0003, 0.05, 0.05,0.0002, 100, 5000, 4500, 4,2), this is particular, here we set $\mu(\sigma_{s-})$ is constant, $r = 0.0001$, $\mu = 0.00015$, $\sigma_0 = 0.0003$, $\alpha = 0.05$, $\beta = 0.05$, $\omega = 0.0002$, $n = 100$, $m = 4$, $a = 2$, $V(0) = 5000$ and floor $G = 4500$.

Chapter 3

CPPI in the Jump-diffusion model when the trading time is continuous

3.1 Jump-diffusion model

3.1.1 Set up the model:

In this section, we consider the jump-diffusion model. It has been studied by many researchers since the Merton's Paper [58]. The model in our paper is described in section 3.1.1 of [65]. [53] is another survey paper about jump-diffusion model, which gives four reasons for choosing the jump-diffusion models. [53] also gives the shortcoming of the jump-diffusion model. We also want to mention [18], [71], [33], [57], [50] among others, for further information.

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ be a probability space satisfying the "usual assumption". Let the price S_t of a risky asset (usually stocks or their benchmark) be a right continuous with left limits stochastic process on this probability space which jumps at the random times T_1, T_2,... and suppose that the relative/proportional change in its value at a jump time is given by Y_1, Y_2,... respectively. We assume $\ln(1+Y_n)$s be i.i.d., and denote the density of $\ln(1+Y_n)$s by f_Q. We assume that, between any two consecutive jump times, the price S_t follows the Black-Scholes model. These T_ns are the jump times of a Poisson process N_t with intensity λ_t and the Y_ns are a sequence of random variables with values in $(-1, +\infty)$. We have

$$N_t = \sum_{n \geq 1} \chi_{t \geq T_n}$$

and

$$\mathbb{P}[N_t = n] = \frac{e^{-\int_0^t \lambda_s ds} \left(\int_0^t \lambda_s ds \right)^n}{n!}.$$

Then on the intervals $[T_n, T_{n+1})$, the description of the model can be formalized by letting,

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t),$$

and in exponential form:

$$S_t = S_{T_n} \exp \left[\int_{T_n}^t \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds + \int_0^t \sigma_s dW_s \right].$$

At $t = T_n$, the jump size is given by $\Delta S_n = S_{T_n} - S_{T_n^-} = S_{T_n -} Y_n$, i.e.

$$S_{T_n} = S_{T_n^-}(1 + Y_n)$$

which, by the assumption that $Y_n > -1$, leads to always positive values of the prices. At the generic time t, S_t can be expressed by the following equivalent representations

$$
\begin{aligned}
S_t &= S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s\right] \left[\prod_{n=1}^{N_t} (1 + Y_n)\right] && (3.1) \\
&= S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1 + Y_n)\right] && (3.2) \\
&= S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s + \int_0^t \ln(1 + Y_s) dN_s\right] && (3.3)
\end{aligned}
$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation, i.e.

$$Y_t = Y_n \text{ if } T_n < t \leq T_{n+1},$$

here we let $T_0 = 0$. The term $\sum_{n=1}^{N_t} \ln(1 + Y_n)]$ in (3.2) is a compound Poisson process. It has independent and stationary increments. Also because of (3.2), our jump-diffusion model is an exponential levy model. Moreover, by the generalized Ito formula, the processes S_t is the solution of

$$dS_t = S_{t-}[\mu_t dt + \sigma_t dW_t + Y_t dN_t], \tag{3.4}$$

with initial value $S_0 = s$.

3.1.2 Two special Jump-diffusion models

Two important special jump-diffusion models will be considered and we introduce them here.

The Merton's Model When we assume $\ln(1+Y_n) \sim N(\alpha, \delta^2)$. This is the **Merton's** model ([58]). The following Proposition considers the density of $\ln(\frac{S_t}{S_0})$.

Proposition 3.1. *Let $\phi(x, m, \upsilon^2)$ be a density function for a normally distributed random variable with mean m and variance υ^2, i.e. $\phi(x, m, \upsilon^2) = \frac{1}{\sqrt{2\pi\upsilon^2}} e^{-\frac{(x-m)^2}{2\upsilon^2}}$. Then, the density function of*

$$\ln\left(\frac{S_t}{S_0}\right) = \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s + \Sigma_{n=1}^{N_t} \ln(1+Y_n)$$

is:

$$p(x) = \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds} (\int_0^t \lambda_s ds)^j}{j!} \phi\left(x; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + j\alpha, \int_0^t \sigma_s^2 ds + j\delta^2\right). \quad (3.5)$$

Proof. Let $L = \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}) ds + \int_0^t \sigma_s dW_s\right)$ and $M = \Sigma_{n=1}^{N_t} \ln(1+Y_n)$.
Then we have,

$$L \sim N\left(\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds, \int_0^t \sigma_s^2 ds\right).$$

When $N_t = j$, by the properties of normal distribution, we have

$$L + M \sim N\left(\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + j\alpha, \int_0^t \sigma_s^2 ds + j\delta^2\right).$$

In general,

$$\begin{aligned} & \forall x \in \mathbb{R}, \\ & \mathbb{P}(L+M \leq x) = \mathbb{P}\left(\bigcup_{j=0}^{\infty} (L+M \leq x, N_t = j)\right) \\ = \quad & \sum_{j=0}^{\infty} \mathbb{P}(L+M \leq x, N_t = j) = \sum_{j=0}^{\infty} \mathbb{P}(L+M \leq x | N_t = j) \mathbb{P}(N_t = j) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \mathbb{P}(L+\Sigma_{n=1}^{j}\ln(1+Y_n)\le x|N_t=j)\mathbb{P}(N_t=j) \\
&= \sum_{j=0}^{\infty} \frac{\mathbb{P}((L+\Sigma_{n=1}^{j}\ln(1+Y_n)\le x, N_t=j)}{\mathbb{P}(N_t=j)})P(N_t=j) \\
&= \sum_{j=0}^{\infty} \frac{\mathbb{P}((L+\Sigma_{n=1}^{j}\ln(1+Y_n)\le x)P(N_t=j)}{\mathbb{P}(N_t=j)})P(N_t=j) \\
&= \sum_{j=0}^{\infty} \mathbb{P}(L+\Sigma_{n=1}^{j}\ln(1+Y_n)\le x)\mathbb{P}(N_t=j) \\
&= \sum_{j=0}^{\infty} \int_{-\infty}^{x} \phi\left(y; \int_0^t \left(\mu_s-\frac{\sigma_s^2}{2}\right)ds+j\alpha, \int_0^t \sigma_s^2 ds+j\delta^2\right)dy\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^j}{j!}.
\end{aligned}
$$

When $j=0$,we take $\Sigma_{n=1}^{j}\ln(1+Y_n)=0$. Each item in the above equations is positive, thus the series is absolute convergence. Thus, the density function is

$$
\begin{aligned}
& p(x) \\
&= \frac{d\left(\sum_{j=0}^{\infty}\int_{-\infty}^{x}\phi\left(y;\int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds+j\alpha,\int_0^t\sigma_s^2ds+j\delta^2\right)dy\frac{e^{-\int_0^t\lambda_s ds}(\int_0^t\lambda_s ds)^j}{j!}\right)}{dx} \\
&= \sum_{j=0}^{\infty}\frac{e^{-\int_0^t\lambda_s ds}(\int_0^t\lambda_s ds)^j}{j!}\phi\left(x;\int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds+j\alpha,\int_0^t\sigma_s^2ds+j\delta^2\right).
\end{aligned}
$$

□

The Kou's Model When we assume $Q=\ln(1+Y_n)$ has an asymmetric double exponential distribution with the density

$$
f_Q(y)=p\cdot\eta_1 e^{-\eta_1 y}\chi_{y\ge 0}+q\cdot\eta_2 e^{-\eta_2 y}\chi_{y<0}
$$

where $\eta_1>1$, $\eta_2>0$, p, $q\ge 0$ and $p+q=1$.

This is called the **Kou's** model([51]). We have:

Proposition 3.2. *The density function of*

$$\ln\left(\frac{S_t}{S_0}\right) = \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s + \Sigma_{n=1}^{N_t} \ln(1+Y_n)$$

is:

$$p(x) = \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^j}{j!} \int_{-\infty}^{\infty} \phi\left(x-y; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds, \int_0^t \sigma_s^2 ds\right) f_Q^{(j)}(y)dy,$$

where $f_Q^{(j)}(y)$ is the density function of $\sum_{n=1}^{j} \ln(1+Y_n)$.

Proof. Let $L = \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s$ and $M = \Sigma_{n=1}^{N_t} \ln(1+Y_n)$. Then,

$$L \sim N\left(\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds, \int_0^t \sigma_s^2 ds\right).$$

When $N_t = j$, we have the distribution of the sum of two random variables is

$$\mathbb{P}(L+M \leq x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} \phi\left(y-y_2; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds, \int_0^t \sigma_s^2 ds\right) f_Q^{(j)}(y_2)dy_2 dy.$$

We calculate the distribution of $L+M$ in general.

$$\begin{aligned}
& \forall x \in \mathbb{R} \\
& \mathbb{P}(L+M \leq x) = \mathbb{P}\left(\bigcup_{j=0}^{\infty}(L+M \leq x, N_t = j)\right) \\
= & \sum_{j=0}^{\infty} \mathbb{P}((L+M \leq x, N_t = j)) = \sum_{j=0}^{\infty} \mathbb{P}((L+M \leq x, N_t = j)) \\
= & \sum_{j=0}^{\infty} \mathbb{P}(L+M \leq x | N_t = j)\, P(N_t = j) \\
= & \sum_{j=0}^{\infty} \mathbb{P}\left(L + \Sigma_{n=1}^{j} \ln(1+Y_n) \leq x | N_t = j\right) \mathbb{P}(N_t = j) \\
= & \sum_{j=0}^{\infty} \frac{\mathbb{P}\left(L + \Sigma_{n=1}^{j} \ln(1+Y_n) \leq x, N_t = j\right)}{\mathbb{P}(N_t = j)} \mathbb{P}(N_t = j)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{\mathbb{P}(L+\Sigma_{n=1}^{j}\ln(1+Y_n)\le x)\mathbb{P}(N_t=j)}{\mathbb{P}(N_t=j)}\mathbb{P}(N_t=j) \\
&= \sum_{j=0}^{\infty} \mathbb{P}((L+\Sigma_{n=1}^{j}\ln(1+Y_n)\le x))\mathbb{P}(N_t=j) \\
&= \sum_{j=0}^{\infty} \int_{-\infty}^{x}\int_{-\infty}^{\infty} \phi\left(y-y_2;\int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds,\int_0^t\sigma_s^2 ds\right) f_Q^{(j)}(y_2)dy_2dy \\
&\quad \times \frac{e^{-\int_0^t\lambda_s ds}(\int_0^t\lambda_s ds)^j}{j!}.
\end{aligned}
$$

Each item in the above equations is positive, thus the series is absolute convergence. Hence, the density function is

$$
\begin{aligned}
& p(x) \\
&= \frac{d\left(\sum_{j=0}^{\infty}\int_{-\infty}^{x}\int_{-\infty}^{\infty}\phi\left(y-y_2;\int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds,\int_0^t\sigma_s^2ds\right)f_Q^{(j)}(y_2)dy_2dy\frac{e^{-\int_0^t\lambda_sds}(\int_0^t\lambda_sds)^j}{j!}\right)}{dx} \\
&= \sum_{j=0}^{\infty}\frac{e^{-\int_0^t\lambda_sds}(\int_0^t\lambda_sds)^j}{j!}\int_{-\infty}^{\infty}\phi\left(x-y;\int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds,\int_0^t\sigma_s^2ds\right)f_Q^{(j)}(y)dy.
\end{aligned}
$$

□

$f_Q^{(j)}(y)$ can be calculated by the convolution of k $f_Q(y)$s. i.e.

$$
f_Q^{(j)}(y) = \underbrace{f_Q(y) * f_Q(y) * \ldots f_Q(y)}_{j\text{ terms}}. \tag{3.6}
$$

Thus, the density function could be calculated explicitly. In generally, when we assume $Q_n = \ln(1+Y_n)$ have i.i.d. with density f_Q, then the density of $\sum_{n=1}^{j}\ln(1+Y_n)$ is $f_Q^{(j)}$. We have the following proposition:

Proposition 3.3. *Let $Q_n = \ln(1+Y_n)$ be i.i.d. random variables with density function f_Q. The density function of*

$$
\ln\left(\frac{S_t}{S_0}\right) = \int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds + \int_0^t\sigma_s dW_s + \Sigma_{n=1}^{N_t}\ln(1+Y_n)
$$

is:

$$p(x) = \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^j}{j!} \int_{-\infty}^{\infty} \phi\left(x - y; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds, \int_0^t \sigma_s^2 ds\right) f_Q^{(j)}(y)dy.$$

3.1.3 Martingale Measure

For our jump-diffusion model defined by (3.2), consider a predictable $\mathfrak{F}_t$-process ψ_t, such that $\int_0^t \psi_t \lambda_s ds < \infty$. Choose θ_t and ψ_t such that

$$\mu_t + \sigma_t \theta_t + Y_t \psi_t \lambda_t = r_t \tag{3.7}$$

and

$$\psi_t \geq 0.$$

From here we see that

$$\theta_t = \sigma_t^{-1}(r_t - \mu_t - Y_t \psi_t \lambda_t) \tag{3.8}$$

where ψ_t is arbitrary. Define

$$L_t = \exp\left\{\int_0^t \left[(1 - \psi_s)\lambda_s - \frac{1}{2}\theta_s^2\right] ds + \int_0^t \theta_s dW_s + \int_0^t \ln \psi_s dN_s\right\} \tag{3.9}$$

for $t \in [0, T]$ and the Radon-Nykodymn derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_T. \tag{3.10}$$

Then the $\mathbb{Q}$ is a risk neutral measure or martingale measure, i.e. a measure under which $\tilde{S}_t = \exp\{-\int_0^t r_s ds\} S_t$ is a martingale (see [65]).

Define

$$dW_t^{\mathbb{Q}} = dW_t - \theta_t dt; \tag{3.11}$$

$$dM_t^{\mathbb{Q}} = dN_t - \psi_t \lambda_t dt. \tag{3.12}$$

Then $W_t^{\mathbb{Q}}$ and $M_t^{\mathbb{Q}}$ are $\mathbb{Q}$-martingales. Also under the measure $\mathbb{Q}$, S_t satisfies

$$dS_t = S_{t-}\left[(\mu_t + \sigma_t\theta_t + Y_t\psi_t\lambda_t)dt + \sigma_t dW_t^{\mathbb{Q}} + Y_t dM_t^{\mathbb{Q}}\right]. \tag{3.13}$$

Under the measure $\mathbb{Q}$, N_t is a Poisson Processes with intensity $\lambda_t\psi_t$.

There are many risk-neutral measures $\mathbb{Q} \sim \mathbb{P}$. A special case of a risk-neutral measure, reflecting the case of a risk-neutral world, it should satisfy

$$\mathbb{E}(S(t)) = S_0 e^{rt}.$$

(See page 312 on [33], page 248-250 on [38], page 19 on [57].)

For Merton Model, since its density function has explicit expression, we will deduce it. We have deduce the density function of Merton's model in (3.5).

$$p(x) = \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^j}{j!} \phi\left(x; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + j\alpha, \int_0^t \sigma_s^2 ds + j\delta^2\right).$$

Then

$$\begin{aligned}
&\mathbb{E}(S(t)) = S_0\mathbb{E}\left(e^{\ln S_t/S_0}\right) = S_0 \int_{\mathbb{R}} e^x p(x) dx \\
= \;& S_0 \int_{\mathbb{R}} e^x \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^j}{j!} \phi\left(x; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + j\alpha, \int_0^t \sigma_s^2 ds + j\delta^2\right) dx \\
= \;& S_0 \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^j}{j!} \int_{\mathbb{R}} e^x \phi\left(x; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + j\alpha, \int_0^t \sigma_s^2 ds + j\delta^2\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= S_0 \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^j}{j!} \exp\left\{\int_0^t \mu_s ds + j\alpha + j\frac{\delta^2}{2}\right\} \\
&= S_0 \exp \int_0^t \left(\mu_s - \lambda_s + e^{\alpha+\frac{\delta^2}{2}}\lambda_s\right) ds.
\end{aligned}
$$

In case of

$$\mathbb{E}(S(t)) = S_0 e^{rt},$$

then we have

$$\mu_s - \lambda_s + e^{\alpha+\frac{\delta^2}{2}}\lambda_s = r.$$

Thus under our new risk-neutral measure $\mathbb{P}^{rn}$, we can use $r+\lambda_s-e^{\alpha+\frac{\delta^2}{2}}\lambda_s$ to substitute μ_s. The model then becomes

$$S_t = S_0 \exp\left[\int_0^t \left(r + \lambda_s - e^{\alpha+\frac{\delta^2}{2}}\lambda_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s^{rn} + \sum_{n=1}^{N_t^{(rn)}} \ln(1+Y_n)\right].$$

$W_s^{(rn)}$ is a Brownian motion and $N_t^{(rn)}$ is a Poisson process whose intensity is λ_s under the probability measure $\mathbb{P}^{rn}$. For convenient, we still denote them as W_s and N_t. Then, under the probability measure $\mathbb{P}^{rn}$, the model is

$$S_t = S_0 \exp\left[\int_0^t \left(r + \lambda_s - e^{\alpha+\frac{\delta^2}{2}}\lambda_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1+Y_n)\right].$$

3.2 The CPPI strategies

3.2.1 The constant multiple case

Recall that in the jump-diffusion model, the *exposure* e_t is equal to the cushion C_t multiplied by m. The *cushion* C_t is the difference between the *portfolio* value V_t and the *floor* F_t and $F_t = G \times \exp\{-r(T-t)\}$. It is possible to have the portfolio value less than the floor, which means that the cushion will be negative and so will be the exposure. Thus short-sell should be allowed. The following proposition describes the portfolio value under this strategy. CPPI would fail if the value of the portfolio falls below the floor. We will measure the failure.

In our strategy the portfolio value V_t consists of a riskless asset $V_t - mC_t$ and risky asset mC_t. i.e. $V_t = mC_t + (V_t - mC_t)$

Proposition 3.4. *The CPPI portfolio value under the jump-diffusion model defined by* (3.2) *is*

$$V_t = C_0 \exp\left\{\int_0^t \left(r + m(\mu_s - r) - \frac{m\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s\right\} \left[\prod_{n=1}^{N_t}(1 + mY_n)\right] + F_t,$$

where

$$C_0 = \left(V_0 - Ge^{-rT}\right);$$
$$F_t = G \times \exp\{-r(T-t)\}.$$

Proof. We have

$$\begin{aligned} V_t &= mC_t + (V_t - mC_t) \\ &= V_t\left(\frac{mC_t}{V_t} + \left(1 - \frac{mC_t}{V_t}\right)\right) \end{aligned}$$

and

$$dV_t = V_t \left(\frac{mC_t}{V_{t-}} \frac{dS_t}{S_{t-}} + \left(1 - \frac{mC_t}{V_{t-}} \right) \frac{dB_t}{B_t} \right).$$

Since B_s is continuous, then $B_{s-} = B_s$, we have

$$\begin{aligned} dC_t &= d(V_t - F_t) \\ &= V_t \left(\frac{mC_{t-}}{V_t} \frac{dS_t}{S_{t-}} + \left(1 - \frac{mC_{t-}}{V_t} \right) \frac{dB_t}{B_t} \right) - F_t \frac{dB_t}{B_t} \\ &= C_{t-} \left(\frac{mdS_t}{S_{t-}} - (m-1)rdt \right) \qquad (3.14) \\ &= C_{t-}(m(\mu_t dt + \sigma_t dW_t + Y_t dN_t) - (m-1)rdt) \\ &= C_{t-}((r + m(\mu_t - r))dt + m\sigma_t dW_t + mY_t dN_t). \end{aligned}$$

Then

$$C_t = C_0 \exp\left\{ \int_0^t \left(r + m(\mu_s - r) - \frac{m^2 \sigma_s^2}{2} \right) ds + \int_0^t m\sigma_s dW_s \right\} \left[\prod_{n=1}^{N_t} (1 + mY_n) \right].$$

Hence

$$\begin{aligned} V_t &= C_t + F_t \\ &= C_0 \exp\left\{ \int_0^t \left(r + m(\mu_s - r) - \frac{m\sigma_s^2}{2} \right) ds + \int_0^t m\sigma_s dW_s \right\} \left[\prod_{n=1}^{N_t} (1 + mY_n) \right] + F_t. \end{aligned}$$

□

If we substitute μ_s by $r + \lambda_s - e^{\alpha + \frac{\delta^2}{2}} \lambda_s$, under the probability measure $\mathbb{P}^{rn}$, we get the following corollary.

Corollary 3.5. *In the Merton's model, under the probability measure* $\mathbb{P}^{rn}$, *the CPPI* *portfolio value V_t under our jump-diffusion model is*

$$C_0 \exp\left\{ \int_0^t \left(r + m(\lambda_s - e^{\alpha+\frac{\delta^2}{2}}\lambda_s) - \frac{m\sigma_s^2}{2} \right) ds + \int_0^t m\sigma_s dW_s \right\} \left[\prod_{n=1}^{N_t} (1 + mY_n) \right] + F_t,$$

where

$$C_0 = (V_0 - Ge^{-rT});$$

$$F_t = G \times \exp\{-r(T-t)\}.$$

The expectation and variance of the CPPI portfolio value are deduced in the following two propositions.

Proposition 3.6. *The expected CPPI portfolio value at time t under the jump-diffusion model is*

$$\mathbb{E}[V_t] = C_0 \exp\left\{ \int_0^t (r + m(\mu_s - r))ds \right\} \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds} (\int_0^t \lambda_s ds)^k}{k!} \mathbb{E}\left[\prod_{n=1}^{k} (1 + mY_n) \right] + F_t.$$

Proof. Because

$$\begin{aligned}
\mathbb{P}\left[\prod_{n=1}^{N_t} (1 + mY_n) \leq x \right] &= \mathbb{P}\left[\bigcup_{k=1}^{\infty} \left[\prod_{n=1}^{N_t} (1 + mY_n) \leq x, N_t = k \right] \right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left[\prod_{n=1}^{N_t} (1 + mY_n) \leq x | N_t = k \right] \mathbb{P}[N_t = k] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left[\prod_{n=1}^{k} (1 + mY_n) \leq x | N_t = k \right] \mathbb{P}[N_t = k] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\frac{\left[\prod_{n=1}^{k} (1 + mY_n) \leq x, N_t = k \right]}{\mathbb{P}[N_t = k]} \mathbb{P}[N_t = k] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\frac{\left[\prod_{n=1}^{k} (1 + mY_n) \leq x \right] \mathbb{P}[N_t = k]}{\mathbb{P}[N_t = k]} P[N_t = k] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left[\prod_{n=1}^{k} (1 + mY_n) \leq x \right] \mathbb{P}[N_t = k]
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mathbb{P}\left[\prod_{n=1}^{k}(1+mY_n) \leq x\right],$$

we get

$$\mathbb{E}\left[\prod_{n=1}^{N_t}(1+mY_n)\right] = \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]$$

and then

$$\begin{aligned}
& \mathbb{E}[V_t] \\
= \;& C_0\mathbb{E}\left[\exp\left\{\int_0^t \left(r + m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s\right\}\right] \\
& \times\mathbb{E}\left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t \\
= \;& C_0 \exp\left\{\int_0^t (r + m(\mu_s - r))ds\right\} \mathbb{E}\left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t \\
= \;& C_0 \exp\left\{\int_0^t (r + m(\mu_s - r))ds\right\} \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right] + F_t.
\end{aligned}$$

□

Proposition 3.7. *The variance of the CPPI portfolio value at time t under jump-diffusion model is*

$$\begin{aligned}
& C_0^2 \exp\left\{\int_0^t 2(r + m(\mu_s - r) + m^2\sigma_s^2)ds\right\} \sum_{k=1}^{\infty} \mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]^2 \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \\
& - \left[\exp\left\{\int_0^t (r + m(\mu_s - r))\right\} ds \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2.
\end{aligned}$$

Proof. Similar to the proof of Prop. 3.6, we have

$$\mathbb{E}\left[\left[\prod_{n=1}^{N_t}(1+mY_n)\right]^2\right] = \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}\left[\left[\prod_{n=1}^{k}(1+mY_n)\right]^2\right].$$

Thus,

$$\begin{aligned}
& \mathrm{Var}[V_t] = \mathrm{Var}[C_t] \\
= \; & C_0^2 \mathrm{Var}\left[\exp\left\{\int_0^t \left(r+m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right)ds + \int_0^t m\sigma_s dW_s\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right] \\
= \; & C_0^2 \mathbb{E}\left[\exp\left\{\int_0^t \left(r+m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right)ds + \int_0^t m\sigma_s dW_s\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right]^2 \\
& -C_0^2\left(\mathbb{E}\left[\exp\left\{\int_0^t \left(r+m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right)ds + \int_0^t m\sigma_s dW_s\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right]\right)^2 \\
= \; & C_0^2 \mathbb{E}\left[\exp\left\{\int_0^t \left(r+m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right)ds + \int_0^t m\sigma_s dW_s\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right]^2 \\
& -C_0^2\left[\exp\{\int_0^t (r+m(\mu_s-r))ds \sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2 \\
= \; & C_0^2 \mathbb{E}\left[\exp\left\{\int_0^t 2(r+m(\mu_s-r)-m^2\sigma_s^2)ds + 2\int_0^t m\sigma_s dW_s\right\}\right]\mathbb{E}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]^2 \\
& -C_0^2\left[\exp\left\{\int_0^t (r+m(\mu_s-r))ds\right\}\sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2 \\
= \; & C_0^2 \exp\left\{\int_0^t 2(r+m(\mu_s-r)+m^2\sigma_s^2)ds\right\}\sum_{k=1}^{\infty}\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]^2 \\
& \times\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \\
& -\left[\exp\left\{\int_0^t (r+m(\mu_s-r))\right\}ds\sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2.
\end{aligned}$$

□

Remarks. (1) Another method to calculate the expectation of the portfolio value is through calculating the characteristic function of

$$\int_0^t \left(r + m(\mu_s - r) - \frac{m\sigma_s^2}{2} \right) ds + \int_0^t m\sigma_s dW_s + \prod_{n=1}^{N_t} (1 + mY_n)$$

In subsection 3.5.2, we will use this method to calculate a similar expectation.
(2) For the Merton's and Kou's model, $\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]$ and $\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]^2$ can be calculated and thus expected portfolio can be calculated explicitly. In general, if we assume $Q_n = \ln(1+Y_n)$ have i.i.d. with density f_Q, $\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]$ and $\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right]^2$ still can be calculated in terms of the function f_Q.

The following lemma gives the density function of $1 + mY_i$.

Lemma 3.8. *Let the density function of* $\ln(1+Y_n)$ *be* $f_Q(y)$*, then the density funtion* f_Q' *of the random variable* $1 + mY_i$ *is*

$$f_Q'(z) = f_Q\left(\ln\left(1 + \frac{z-1}{m}\right)\right) \frac{1}{m+z-1}.$$

Proof. Since

$$\mathbb{P}(1 + mY_i \leq z) = \mathbb{P}\left(\ln(1+Y_i) \leq \ln\left(1 + \frac{z-1}{m}\right)\right)$$
$$= \int_{-\infty}^{\ln(1+\frac{z-1}{m})} f_Q(y)dy,$$

the density f_Q' of the random variable $1 + mY_i$ is

$$f_Q'(z) = \frac{d\left(\mathbb{P}(1+mY_i \leq z)\right)}{dz} = f_Q\left(\ln\left(1 + \frac{z-1}{m}\right)\right) \frac{1}{m+z-1}.$$

□

Now we can calculate

$$\begin{aligned}
\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)\right] &= \mathbb{E}\left[\exp\left\{\sum_{n=1}^{k}\ln(1+mY_n)\right\}\right] \\
&= \int_{\mathbb{R}} \exp\left\{\underbrace{f'_Q * f'_Q * \ldots * f'_Q(x)}_{k \text{ items}}\right\} dx
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\prod_{n=1}^{k}(1+mY_n)^2\right] &= \mathbb{E}\left[\exp\left\{\sum_{n=1}^{k}2\ln(1+mY_n)\right\}\right] \\
&= \int_{\mathbb{R}} \exp\left\{2\underbrace{f'_Q * f'_Q * \ldots * f'_Q(x)}_{k \text{ items}}\right\} dx.
\end{aligned}$$

3.2.2 The case when the multiple is a function of time

Let m_t be the multiple at time t. We have similar results:

Proposition 3.9. *When the multiple is a function of time the CPPI portfolio value under the jump-diffusion model is*

$$V_t = C_0 \exp\left\{\int_0^t \left(r + m_s(\mu_s - r) - \frac{m_s^2\sigma_s^2}{2}\right) ds + \int_0^t m_s\sigma_s dW_s\right\}\left[\prod_{n=1}^{N_t}(1+m_nY_n)\right] + F_t,$$

where m_n is obtained from m_t by the formula

$$m_n = m_{T_n},$$

where $T_0 = 0$.

Proposition 3.10. *When the multiple is a function of time the expected CPPI port-*

folio value under jump-diffusion model is

$$C_0 \exp\left\{ \int_0^t (r + m_s(\mu_s - r))ds \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mathbb{E}\left[\prod_{n=1}^{k}(1 + m_n Y_n)\right] \right\} + F_t.$$

Proposition 3.11. *When the multiple is a function of time the variance of the CPPI portfolio value under jump-diffusion model is*

$$\begin{aligned}
&C_0^2 \exp\left\{\int_0^t 2\left(r + m_s(\mu_s - r) + m_s^2\sigma_s^2\right) ds\right\} \\
&\times \sum_{k=1}^{\infty} \mathbb{E}\left[\prod_{n=1}^{k}(1 + m_n Y_n)\right]^2 \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \\
&- \left[\exp\left\{\int_0^t (r + m_s(\mu_s - r))ds\right\} \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mathbb{E}\left[\prod_{n=1}^{k}(1 + m_n Y_n)\right]\right]^2.
\end{aligned}$$

Here we consider a special form of m_t. Let $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T$ where $t_{k+1} - t_k = \frac{T}{n}$ for $k = 0, ..., n-1$. We reconsider the multiple only at time t_i which $i = 0, 1, ..., n$. Let

$$\begin{aligned}
m_0 &= \eta + 1 \\
m_{t_k} &= \eta + e^{aln(S_{t_k}/S_{t_{k-1}})} \text{ When } k \geq 1 \\
m_t &= m_{t_k} \text{ When } t \in [t_k, t_{k+1})
\end{aligned}$$

Remarks. The above is called an EPPI strategy, a special case of which would be when the multiple is a function of time. However, since CPPI is a common term in financial mathematics, we still refer the above EPPI as a special case of CPPI.

3.3 The CPPI portfolio as a hedging tool

We have proved that the portfolio value is

$$V_t = C_0 \exp\left\{\int_0^t \left(r + m_s(\mu_s - r) - \frac{m_s^2\sigma_s^2}{2}\right) ds + \int_0^t m_s\sigma_s dW_s\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t.$$

The following lemma is deduced from the Ito formula and will be used to prove some later theorems.

Lemma 3.12. *Let $v(x,t) \in C^{1,2}([0,\, T] \times \mathbb{R})$ and bounded at infinity. Then the conditional expectation of the composition process $v(t, x(t))$ satisfies*

$$\begin{aligned}\mathbb{E}[v(t,S_t)|S(0)=S_0] &= v(0,S_0) + \mathbb{E}\Bigg[\int_0^t \left(\frac{\partial v}{\partial t} + \mu_t S_{u-}\frac{\partial v}{\partial x} + \frac{1}{2}(\sigma_u S_{u-})^2\frac{\partial^2 v}{\partial x^2}\right)(u,S_u) \\ &\quad + \lambda_u(v(u,S_{u-}+S_{u-}Y_u) - v(u,S_u))dN_u | S(0)=S_0\Bigg].\end{aligned}$$

Proof. Our risky asset S_t is given by

$$dS_t = S_{t-}(\mu_t dt + \sigma_t dW_t + Y_t dN_t).$$

By the Ito chain rule,

$$\begin{aligned}dv(t,S_t) &= \left(\frac{\partial v}{\partial t} + \mu_t S_{t-}\frac{\partial v}{\partial x} + \frac{1}{2}(\sigma_t S_{t-})^2\frac{\partial^2 v}{\partial x^2}\right)(t,S_t)dt \\ &\quad + S_{t-}\sigma_t\frac{\partial v}{\partial x}(t,S_t)dW_t + (v(t,S_{t-}+S_{t-}Y_t) - v(t,S_t))dN_t.\end{aligned}$$

When expressed in integral form, we have,

$$\begin{aligned}v(t,S_t) &= v(0,S_0) + \int_0^t \left(\frac{\partial v}{\partial t} + \mu_t S_{u-}\frac{\partial v}{\partial x} + \frac{1}{2}(\sigma_u S_{u-})^2\frac{\partial^2 v}{\partial x^2}\right)(u,S_u)du \\ &\quad + \int_0^t S_{u-}\sigma_u\frac{\partial v}{\partial x}dW_u + (v(u,S_{u-}+S_{u-}Y_u) - v(u,S_{u-}))dN_u.\end{aligned}$$

By taking conditional expectation on both sides, we have

$$\mathbb{E}[v(t,S_t)|S(0)=S_0] = v(0,S_0)+\mathbb{E}\Bigg[\int_0^t \left(\frac{\partial v}{\partial t}+\mu_t S_{u-}\frac{\partial v}{\partial x}+\frac{1}{2}(\sigma_u S_{u-})^2\frac{\partial^2 v}{\partial x^2}\right)(u,S_u)$$
$$+\lambda_u(v(u,S_{u-}+S_{u-}Y_u)-v(u,S_u))dN_u|S(0)=S_0\Bigg].$$

□

Remarks. The term $(v(t,S_{t-}+S_{t-}Y_t)-v(t,S_t))dN_t$ describes the difference of the portfolio value as a functional of S_t when a jump occurs.

In section 4 of [16], the CPPI portfolio is utilized as a hedging tool under the Black-scholes model. See also [26]. In this section, we generalize the above result to our jump-diffusion case.

3.3.1 PIDE Approach

Suppose that $\eta = g(S_T)$ is a contingent claim that the portfolio's manager is aiming to have at maturity. Can the CPPI portfolio be converted into a synthetic derivative with pay-off specified by $\eta = g(S_T)$?

Theorem 3.13. *If $g:\mathbb{R}\to\mathbb{R}$ is sufficiently smooth, there exists a unique self-financed $g(S_T)$ hedging CPPI portfolio V, defined by*

$$V_t = v(t,S_t) \quad t\in[0,T] \tag{3.15}$$

where $v\in C^{1,2}([0,T]\times\mathbb{R})$ is the unique solution of the following partial integro-differential equations (PIDE).

$$\frac{\partial u}{\partial t}(t,s)+(\mu_t s)\frac{\partial u}{\partial x}(t,s)+\frac{1}{2}(s\sigma_t)^2\frac{\partial^2 u}{\partial x^2}(t,s)-ru(t,s)=0 \tag{3.16}$$
$$sz\frac{\partial u}{\partial x}(t,s)=u(t,s+sz)-u(t,s) \tag{3.17}$$

$$u(T,s) = g(s), \quad (t,s) \in [0,T] \times \mathbb{R}), \quad u \in C^{1,2}([0,T] \times \mathbb{R}) \tag{3.18}$$

In particular the CPPI portfolio's gearing factor is given by:

$$m_t = \frac{\frac{\partial u}{\partial x}(t, S_t) S_{t-}}{V_{t-} - F_t}, \quad t \in [0,T]. \tag{3.19}$$

Proof. For V to be a a self-financed $g(S_T)$-hedging portfolio, it is enough to ensure that at maturity time we have

$$V_T = g(S_T), \quad a.s..$$

Choose a map $v \in C^{1,2}([0,T] \times \mathbb{R})$ and set $V_t = v(t, S_t) \quad (t \in [0,T])$. Then $v(T, S_T) = g(S_T)$ $\mathbb{P}$-a.s., therefore

$$v(T,s) = g(s), \qquad \forall s \in \mathbb{R}.$$

Second by Ito's chain rule,

$$\begin{aligned} dv(t,S_t) &= \left(\frac{\partial v}{\partial t} + \mu_t S_{t-} \frac{\partial v}{\partial x} + \frac{1}{2} (\sigma_t S_{t-})^2 \frac{\partial^2 v}{\partial x^2} \right)(t, S_t) dt \\ &\quad + S_{t-} \sigma_t \frac{\partial v}{\partial x}(t, S_t) dW_t + (v(t, S_{t-} + S_{t-} Y_t) - v(t, S_{t-})) dN_t. \end{aligned}$$

Now V_t satisfies

$$\begin{aligned} dV_t &= dC_t + dF_t \\ &= (V_{t-} - F_t)(r + m_t(\mu_t - r))dt + rF_t dt + (V_{t-} - F_t) m_t \sigma_t dW_t + (V_{t-} - F_t) m_t Y_t dN_t \\ &= (rV_{t-} + (V_{t-} - F_t) m_t (\mu_t - r))dt + (V_{t-} - F_t) m_t \sigma_t dW_t + (V_{t-} - F_t) m_t Y_t dN_t. \end{aligned}$$

A comparison of the above two equations implies that

$$m_t = \frac{\frac{\partial u}{\partial x}(t, S_t) S_{t-}}{V_{t-} - F_t}, \quad t \in [0,T]$$

and

$$\frac{\partial u}{\partial t}(t,s) + (\mu_t s)\frac{\partial u}{\partial x}(t,s) + \frac{1}{2}(s\sigma_t)^2\frac{\partial^2 u}{\partial x^2}(t,s) - ru(t,s) = 0;$$
$$sz\frac{\partial u}{\partial x}(t,s) = u(t, s+sz) - u(t,s).$$

□

In a financial turmoil, the portfolio's manager acting on the leverage regime may convert the CPPI portfolio in a suitable synthetic derivative whose price is specified by (3.15)-(3.18). Moreover the required dynamic gearing factor (multiple) can be easily determined, using (3.19). This is the PIDE/PDE approach hedging.
Another observation that reveals to be central in the analysis of possible portfolio's hedges is that at any time of the financial horizon the CPPI portfolio value may be regarded as a standard risky asset and therefore as an underlying for any convenient contingent claim:

Theorem 3.14. *Under the risk neutral measure $\mathbb{Q}$, the discounted CPPI portfolio's value $\{V_t\}_{t\in[0,T]}$*

$$\tilde{V}_t = e^{-rt}V_t, \quad t \in [0,T] \tag{3.20}$$

is a martingale.

Proof. In the proof of Theorem 3.13, we have deduced

$$dV_t = (rV_{t-} + (V_{t-} - F_t)m_t(\mu_t - r))dt + (V_{t-} - F_t)m_t\sigma_t dW_t + (V_{t-} - F_t)m_t Y_t dN_t,$$

thus we have

$$dV_t$$

$$
\begin{aligned}
&= (rV_{t-} + (V_{t-} - F_t)m_t(\mu_t - r))dt + (V_{t-} - F_t)m_t\sigma_t(dW_t^{\mathbb{Q}} + \theta_t dt) \\
&\quad +(V_{t-} - F_t)m_tY_tdN_t \\
&= (rV_{t-} + (V_{t-} - F_t)m_t(\mu_t - r + \theta_t))dt + (V_{t-} - F_t)m_t\sigma_t dW_t^{\mathbb{Q}} \\
&\quad +(V_{t-} - F_t)m_tY_tdN_t \\
&= (rV_{t-} + (V_{t-} - F_t)m_t\sigma_t)dW_t^{\mathbb{Q}} + (V_{t-} - F_t)m_t(-Y_t\psi_t\lambda_t)dt + (V_{t-} - F_t)m_tY_tdN_t \\
&= (rV_{t-} + (V_{t-} - F_t)m_t\sigma_t)dW_t^{\mathbb{Q}} + (V_{t-} - F_t)m_tY_tdM_t^{\mathbb{Q}}.
\end{aligned}
$$

Integration by parts implies that

$$
\begin{aligned}
d\tilde{V}_t &= de^{-rt}V_t = -re^{-rt}V_tdt + e^{-rt}dV_t \\
&= e^{-rt}((V_{t-} - F_t)m_t\sigma_t dW_t^{\mathbb{Q}} + (V_{t-} - F_t)m_tY_tdM_t^{\mathbb{Q}}).
\end{aligned}
$$

Thus, $\tilde{V}_t$ is a $\mathbb{Q}$-martingale. □

If we substitute μ_s by $r + \lambda_s - e^{\alpha+\frac{\delta^2}{2}}\lambda_s$, under the probability measure $\mathbb{P}^{rn}$, we get the following corollary.

Corollary 3.15. *In Merton's model, under probability measure $\mathbb{P}^{rn}$, the discounted CPPI portfolio's value $\{V_t\}_{t\in[0,T]}$*

$$\tilde{V}_t = e^{-rt}V_t, \quad t \in [0, T]$$

is a martingale.

Proof. We have

$$
\begin{aligned}
&dV_t \\
&= (rV_{t-} + (V_{t-} - F_t)m_t(\mu_t - r))dt + (V_{t-} - F_t)m_t\sigma_t dW_t + (V_{t-} - F_t)m_tY_tdN_t \\
&= \left(rV_{t-} + (V_{t-} - F_t)m_t\left(\lambda_t - e^{\alpha+\frac{\delta^2}{2}}\lambda_t\right)\right)dt + (V_{t-} - F_t)m_t\sigma_t dW_t
\end{aligned}
$$

$$+(V_{t-} - F_t)m_t Y_t dN_t.$$

Thus

$$\begin{aligned}
d\tilde{V_t} &= de^{-rt}V_t = -re^{-rt}V_t dt + e^{-rt}dV_t \\
&= -re^{-rt}V_t dt + e^{-rt}\left(rV_{t-} + (V_{t-} - F_t)m_t\left(\lambda_t - e^{\alpha+\frac{\delta^2}{2}}\lambda_t\right)\right)dt \\
&\quad +(V_{t-} - F_t)m_t\sigma_t dW_t + (V_{t-} - F_t)m_t Y_t dN_t) \\
&= e^{-rt}(V_{t-} - F_t)m_t\left(\lambda_t - e^{\alpha+\frac{\delta^2}{2}}\lambda_t\right)dt + (V_{t-} - F_t)m_t\sigma_t dW_t \\
&\quad +(V_{t-} - F_t)m_t Y_t dN_t) \\
&= e^{-rt}(V_{t-} - F_t)m_t\left(\lambda_t - e^{\alpha+\frac{\delta^2}{2}}\lambda_t + \lambda_t Y_t\right)dt \\
&\quad +(V_{t-} - F_t)m_t\sigma_t dW_t + (V_{t-} - F_t)m_t Y_t(dN_t - \lambda_t dt)).
\end{aligned}$$

Since $dN_t - \lambda_t dt$ is a martingale and

$$\mathbb{E}[Y_t] = \mathbb{E}\left(e^{\ln(1+Y_n)} - 1\right) = e^{\alpha+\frac{\delta^2}{2}} - 1,$$

we get $\mathbb{E}[(\lambda_t - e^{\alpha+\frac{\delta^2}{2}}\lambda_t)] = 0$, so we prove $\tilde{V_t}$ is a $\mathbb{P}^{nr}$-martingale. □

Given any claim $\eta = g(V_T)$, which is a function of the terminal portfolio's price, there exists a unique self-financed $\eta = g(V_T)$-hedging strategy:

Theorem 3.16. *Let $g : \mathbb{R} \to \mathbb{R}$ sufficiently smooth. Then there exists a unique $\eta = g(V_T)$-hedging self-financed trading strategy (U, β) defined as*

$$U_t = u(t, V_t), \quad \beta_{t-} = \frac{\partial u}{\partial x}(t, V_t), \quad t \in [0, T],$$

where $u \in C^{1,2}([0, T] \times \mathbb{R})$ is the unique solution of the PIDE.

$$\frac{\partial u}{\partial t}(t, v) + rv\frac{\partial u}{\partial x}(t, v) + \frac{1}{2}m^2\sigma_t^2(v - f)^2\frac{\partial^2 u}{\partial x^2}(t, v) - ru(t, v) = 0 \qquad (3.21)$$

$$mz(v-f)\frac{\partial u}{\partial x}(t,v) = u(t,v+m(v-f)z) - u(t,v) \tag{3.22}$$

with the final condition $u(T,v) = g(v)$.

Proof. Consider an asset $\{V_t\}_{t\in[0,T]}$, and pick a self-financed $g(V_T)$ hedging strategy space $(U_t, \beta_t)_{t\in[0,T]}$ by setting:

$$dU_t = \beta_{t-}dV_t + (U_{t-} - \beta_{t-}V_{t-})rdt$$

and

$$U_T = g(V_T) \quad a.s.$$

Since

$$\begin{aligned} dV_t &= (rV_{t-} + (V_{t-} - F_t)m_t(\mu_t - r))dt + (V_{t-} - F_t)m_t\sigma_t dW_t \\ &\quad +(V_{t-} - F_t)m_t Y_t dN_t, \end{aligned}$$

the hedging portfolio's equation may be rewritten as:

$$\begin{aligned} dU_t &= \beta_{t-}(rV_{t-} + (V_{t-} - F_t)m_t(\mu_t - r)dt + (V_{t-} - F_t)m_t\sigma_t dW_t \\ &\quad +(V_{t-} - F_t)m_t Y_t dN_t) + (U_{t-} - \beta_{t-}V_{t-})rdt \\ &= (rU_{t-} + \beta_{t-}(V_{t-} - F_t)m(\mu_t - r))dt + \beta_{t-}(V_{t-} - F_t)m_t\sigma_t dW_t \\ &\quad +\beta_{t-}(V_{t-} - F_t)m_t Y_t dN_t). \end{aligned}$$

Pick $u \in C^{1,2}([0,T]\times\mathbb{R})$ and set $U_t = u(t,V_t)$, for $t \in [0,T]$.
For any $t \in [0,T]$,the Ito's formula implies that:

$$du(t,V_t) = \frac{\partial u}{\partial t}(t,V_t) + (rV_{t-} + m(\mu_t - r)(V_{t-} - F_t))\frac{\partial u}{\partial x}(t,V_t)$$

$$+\frac{1}{2}(m\sigma_t)^2(V_{t-}-F_t)^2\frac{\partial^2 u}{\partial x^2}(t,V_t)dt+m\sigma_t(V_{t-}-F_t)\frac{\partial u}{\partial x}(t,V_t)dW_t$$
$$+(u(t,V_{t-}+m(V_{t-}-F_t)Y_t)-u(t,V_{t-}))dN_t.$$

A comparison between the above two equations implies in particular

$$\beta_{t-}=\frac{\partial u}{\partial x}(t,V_t)$$

and

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,v)+(rv+m(\mu_t-r)(v-f))\frac{\partial u}{\partial x}(t,v)+\frac{1}{2}m^2\sigma_t^2(v-f)^2\frac{\partial^2 u}{\partial x^2}(t,v)\\ =\;& ru(t,v)+m(\mu_t-r)(v-f)\frac{\partial u}{\partial x}(t,v).\end{aligned}$$

Thus

$$\frac{\partial u}{\partial t}(t,v)+rv\frac{\partial u}{\partial x}(t,v)+\frac{1}{2}m^2\sigma_t^2(v-f)^2\frac{\partial^2 u}{\partial x^2}(t,v)-ru(t,v)=0$$

and

$$mz(v-f)\frac{\partial u}{\partial x}(t,v)=u(t,v+m(v-f)z)-u(t,v)$$

with the final condition $u(T,v)=g(v)$. □

The rationale in constructing self-financed trading strategies that hedge the CPPI portfolio's terminal price, is that there are contingent claims particularly useful to control both the closing-out-effect and the gap risk. As an example consider the case of a Vanilla option based on the CPPI portfolio's value. For instance being long in an at-the-money put option on the portfolio with a strike at least equal to the protection required is a natural way to hedge gap risk. Similarly being long in an at-the-money call option on the portfolio is a natural way to invest in a CPPI's portfolio preserving the capability to not purse forward the investment in the case of closed out.

3.3.2 Fourier Transformation Approach

[14] and [54] do research on how to use Fourier transform to value option when we know the characteristic function. We refer to their results to value our CPPI option. Under the martingale measure $\mathbb{Q}$, the discounted stock price $\tilde{S}_t = e^{-rt}S_t$ is a martingale. Consider the European option with the pay-off as the function of $\tilde{S}_T$, i.e. $G(\tilde{S_T})$, and denote by h its log-payoff function $G(e^x) \equiv g(x)$ and by Φ the characteristic function of $\ln(\tilde{S}_t)$. Proposition 10 in [74] states the following result.

Proposition 3.17. *Suppose that there exists $R \neq 0$ such that*
$h(x)e^{Rx}$ has finite variance on $\mathbb{R}$, $h(x)e^{-Rx} \in L^1(\mathbb{R})$, $\mathbb{E}^{\mathbb{Q}}\left[e^{RX_{T-t}}\right] < \infty$ and
$\int_{\mathbb{R}} \frac{|\Phi_{T-t}(u-iR)|}{1+|u|}du < \infty$.
Then the price at time t of the European option with pay-off function G satisfies

$$\begin{aligned} P(t, \tilde{S}_t) &:= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G(\tilde{S}_T)|\mathfrak{F}_t\right] \\ &= \frac{e^{-r(T-t)}}{2\pi}\int_{\mathbb{R}} \hat{h}(u+iR)\Phi_{T-t}(-u-iR)\hat{\tilde{S}}_t^{R-iu}du, \end{aligned} \tag{3.23}$$

where $\hat{h}(u) := \int_{\mathbb{R}} e^{iux}h(x)ds$.

We are interested in considering the European option whose pay-off a function depends on the discounted CPPI portfolio $\tilde{V}_T$, i.e. $G(\tilde{V}_T)$. Since $V_t = C_t + F_t$ and C_t are in exponential forms, it is more convenient to treat it as a function of the cushion $\tilde{C}_T$. Let $G_2(e^x) = h_2(x)$ and

$$\varepsilon_t = C_0 \exp\left\{\int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right)ds + \int_0^t m\sigma_s dW_s\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]. \tag{3.24}$$

In subsection 4.2.2, we will show that the characteristic function $\phi_t(u)$ of $\ln(\frac{\varepsilon_t}{C_0})$ is given by

$$\begin{aligned}\phi_t(u) &= \exp\left\{i\left(\int_0^t\left(m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right)ds\right)u-\frac{1}{2}\left(\int_0^t m\sigma_s^2 ds\right)u^2\right\}\\ &\times \exp\left\{t\lambda\int_{\mathbb{R}}\left(e^{iux}-1\right)f_Q\left(\ln\left(1+\frac{e^x-1}{m}\right)\right)\frac{e^x}{m+e^x-1}dx\right\}.\end{aligned}$$

Thus the characteristic function of $\ln(C_T)$ is $C_0\phi_t(u)$. We then have

Proposition 3.18. *Suppose that there exists $R \neq 0$ such that*
$h_2(x)e^{Rx}$ has finite variance on $\mathbb{R}$, $h_2(x)e^{-Rx} \in L^1(\mathbb{R})$, $\mathbb{E}^{\mathbb{Q}}\left[e^{RX_{T-t}}\right] < \infty$
and $\int_{\mathbb{R}} \frac{|C_0\phi_{T-t}(u-iR)|}{1+|u|}du < \infty$.
Then the price at time t of the European option with pay-off function G_2 satisfies

$$\begin{aligned}P(t,\tilde{V}_t) &:= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G_2(\tilde{C}_T)|\mathfrak{F}_t\right]\\ &= \frac{e^{-r(T-t)}}{2\pi}\int_{\mathbb{R}}\hat{h_2}(u+iR)C_0\phi_{T-t}(-u-iR)\hat{\tilde{C}}_t^{R-iu}du,\end{aligned} \tag{3.25}$$

where $\hat{h}(u) := \int_{\mathbb{R}} e^{iux}h(x)ds$.

Remarks. The European call option has pay-off $G_2(C_T) = (C_T+F_T-K)^+$, therefore, we have for all $R > 1$

$$\hat{h_2}(u+iR) = \frac{(K-F_T)^{iu+1-R}}{(R-iu)(R-1-iu)}.$$

3.3.3 Martingale Approach

It is possible to obtain a Black-Sholes type formula for pricing Vanilla options based on the CPPI portfolio:
We first consider the general case. We assume that the $\ln(1+Y_i)$ are i.i.d. with common density function f_Q.

Proposition 3.19. *Let the density of* $\ln(1+Y_i)$ *be* $f_Q(x)$ *and the density function of*

$$\ln(L_t) = \int_0^t \left[(1-\psi_s)\lambda_s - \frac{1}{2}\theta_s^2\right] ds + \int_0^t \theta_s dW_s + \int_0^t \ln \psi_s dN_s$$

be f^{L_t}*, where* L_t *is defined by* (3.9)*. Then the vanilla call/put option on the whole CPPI portfolio's value at maturity is completely determined by:*

$$\begin{aligned} & Call(0, v, T, K) \\ = & \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \psi_s \lambda_s ds} (\int_0^t \psi_s \lambda_s ds)^k}{k!} \right) \times \int_{\varsigma}^{\infty} \left(C_0 e^x + F_0 - e^{-rT} K\right) p^{(k)} dx \end{aligned}$$

and

$$\begin{aligned} & Put(0, v, T, K) \\ = & \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \psi_s \lambda_s ds} (\int_0^t \psi_s \lambda_s ds)^k}{k!} \right) \times \int_{\varsigma}^{\infty} (-C_0 e^x - F_0 + e^{-rT} K) p^{(k)} dx, \end{aligned}$$

where $K > F_T$ *and*

$$p^{(k)} = f_1 * \underbrace{f_{Q'}^{\mathbb{Q}} * ... * f_{Q'}^{\mathbb{Q}}}_{k \text{ terms}},$$

where $f_{Q'}$ *and* $f_{Q'}^{\mathbb{Q}}$ *have the following relation:*

$$\int_{\mathbb{R}} \exp\left\{iu f_{Q'}^{\mathbb{Q}}(z)\right\} dz = \int_{\mathbb{R}} \exp\left\{\left[f_{Q'}(\frac{z}{iu})\frac{z}{iu}\right] * f^{L_T}(z)\right\} dz$$

and f_1 *is the density function of the normal distribution*

$$\mathcal{N}\left(\cdot, \int_0^T \left((m(-Y\psi_s\lambda_s) - \frac{m^2\sigma_s^2}{2}\right) ds, \int_0^T m\sigma_s dW_s^{\mathbb{Q}}\right)$$

and $\varsigma = \ln\left(\frac{e^{-rT}K - F_0}{C_0}\right)$.

Proof. Consider the process:

$$\begin{aligned} V_t &= C_0 \exp\left\{\int_0^t \left(r + m(\mu_s - r) - \frac{m_s^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s\right\} \left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t \\ &= C_0 \exp\left\{\int_0^t \left(r + m(\mu_s - r) + m\sigma_s\theta_s - \frac{m^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s^{\mathbb{Q}}\right\} \\ &\quad \times \left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t \\ &= C_0 \exp\left\{\int_0^t \left(r - mY_s\psi_s\lambda_s - \frac{m^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s^{\mathbb{Q}}\right\} \left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t. \end{aligned}$$

In the case of $N_T = k$, we denote

$$\begin{aligned} L^{(k)} &= e^{-rT}\frac{V_T^k - F_T}{C_0} = \exp\left\{\int_0^T \left(m(-Y\psi_s\lambda_s) - \frac{m^2\sigma_s^2}{2}\right) ds \right. \\ &\quad \left. + \int_0^T m\sigma_s dW_s^Q + \left[\sum_{n=1}^{k}\ln(1+mY_n)\right]\right\}. \end{aligned}$$

(see the remark (3) below the proof.) Because

$$\mathbb{P}(\ln(1+mY_i) \le z) = \mathbb{P}\left(\ln(1+Y_i) \le \ln\left(1 + \frac{e^z - 1}{m}\right)\right) = \int_{-\infty}^{\ln\left(1+\frac{e^z-1}{m}\right)} f_Q(y)dy,$$

the density function $f_{Q'}$ of the random variable $\ln(1+mY_i)$ under the probability measure $\mathbb{P}$ is

$$f_{Q'}(z) = \frac{d(\mathbb{P}(\ln(1+mY_i) \le z))}{dz} = f_Q\left(\ln\left(1+\frac{e^z-1}{m}\right)\right)\frac{e^z}{m+e^z-1}.$$

Suppose the density function of $\ln(1+mY_i)$ under the measure $\mathbb{Q}$ is $f_{Q'}^{\mathbb{Q}}$. By the properties of the Radon-Nikodym derivative and the characteristic function, we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\exp\{iu\ln(1+mY_i)\}] &= \mathbb{E}[\exp\{iu\ln(1+mY_i)\}L_T] \\ &= \mathbb{E}[\exp\{iu\ln(1+mY_i) + \ln L_T\}] \end{aligned}$$

Under $\mathbb{P}$, the density function of $iu\ln(1+mY_i)$ is $f_{Q'}\left(\frac{z}{iu}\right)\frac{z}{iu}$, thus the density function of $iu\ln(1+mY_i)+\ln L_T$ under $\mathbb{P}$ is

$$\left[f_{Q'}\left(\frac{z}{iu}\right)\frac{z}{iu}\right]*f^{L_T}(z)$$

and thus $f_{Q'}$ and $f_{Q'}^{\mathbb{Q}}$ have the following relation:

$$\int_{\mathbb{R}}\exp\left\{iuf_{Q'}^{\mathbb{Q}}(z)\right\}dz=\int_{\mathbb{R}}\exp\left\{\left[f_{Q'}\left(\frac{z}{iu}\right)\frac{z}{iu}\right]*f^{L_T}(z)\right\}dz.$$

Since

$$\begin{aligned}&\int_0^T\left(m(-Y\psi_s\lambda_s)-\frac{m^2\sigma_s^2}{2}\right)ds+\int_0^T m\sigma_s dW_s^{\mathbb{Q}}\\ \sim\ &\mathcal{N}\left(\cdot,\int_0^T\left(m(-Y\psi_s\lambda_s)-\frac{m^2\sigma_s^2}{2}\right)ds,\int_0^T m\sigma_s dW_s^{\mathbb{Q}}\right),\end{aligned}$$

we denote its density function by

$$f_1(x)=\phi\left(x,\int_0^T\left(m(-Y\psi_s\lambda_s)-\frac{m^2\sigma_s^2}{2}\right)ds,\int_0^T m\sigma_s dW_s^{\mathbb{Q}}\right)$$

under the probability measure $\mathbb{Q}$ where $\phi(x,m,\upsilon)=\frac{1}{\sqrt{2\pi\upsilon^2}}e^{-\frac{(x-m)^2}{2\upsilon^2}}$. Then the density function $p^{(k)}(x)$ of $L^{(k)}$ is

$$p^{(k)}=f_1*\underbrace{f_{Q'}^{\mathbb{Q}}*...*f_{Q'}^{\mathbb{Q}}}_{k\text{ terms}}.$$

We have

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(V_T^{(k)}-K)^+\right)=\int_{\varsigma}^{\infty}\left(C_0e^x+F_0-e^{-rT}K\right)p^{(k)}dx,$$

where

$$\varsigma = \ln\left(\frac{e^{-rT}K - F_0}{C_0}\right),$$

thus

$$\begin{aligned}
& \text{Call}(0, v, T, K) \\
= & \ \mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(V_T - K)^+\right) = \sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(V_T^{(k)} - K)^+\right)\left(\frac{e^{-\int_0^t \psi_s\lambda_s ds}(\int_0^t \psi_s\lambda_s ds)^k}{k!}\right) \\
= & \ \sum_{k=0}^{\infty}\left(\frac{e^{-\int_0^t \psi_s\lambda_s ds}(\int_0^t \psi_s\lambda_s ds)^k}{k!}\right) \times \int_{\varsigma}^{\infty}(C_0e^x + F_0 - e^{-rT}K)p^{(k)}dx.
\end{aligned}$$

Similarly,

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(K - V_T^{(k)})^+\right) = \int_{\varsigma}^{\infty}(-C_0e^x - F_0 + e^{-rT}K)p^{(k)}dx$$

and

$$\begin{aligned}
& \text{Put}(0, v, T, K) \\
= & \ \mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(K - V_T)^+\right) = \sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(K - V_T^{(k)})^+\right)\left(\frac{e^{-\int_0^t \psi_s\lambda_s ds}(\int_0^t \psi_s\lambda_s ds)^k}{k!}\right) \\
= & \ \sum_{k=0}^{\infty}\left(\frac{e^{-\int_0^t \psi_s\lambda_s ds}(\int_0^t \psi_s\lambda_s ds)^k}{k!}\right) \times \int_{\varsigma}^{\infty}(-C_0e^x - F_0 + e^{-rT}K)p^{(k)}dx.
\end{aligned}$$

□

Remarks. (1) The expression is not very explicit since they contain measure transformations and convolutions.

(2)When $\mathbb{Q}$ is the risk neutral measure, the price of a Vanilla call option is given by

$$\text{Call}(t, v, T, K) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(V_T^{t,v} - K)^+\right] = E^{\mathbb{Q}}\left[e^{-r(T-t)}(V_T - K)^+ | V_t = v\right],$$

for any $t \in [0, T]$. The CPPI portfolio's value $\{V_t\}$ is a Markov process so that

$$\text{Call}(t, v, T, K) = \text{Call}(0, v, T - t, K), \text{for } t \in [0, T]$$

and it is sufficient to cover the case of the Vanilla call option's price at zero.

(3) The value of $1 + mY_n$ might be negative, in this case $\ln(1 + mY_n)$ is an imaginary number.

Corollary 3.20. *In Merton's Model and under the probability measure $\mathbb{P}^{rn}$, let the density of $\ln(1 + Y_i)$ be $\phi(x, \alpha, \delta^2)$. Then the Vanilla call/put option on the whole CPPI portfolio's value at maturity is completely determined by*

$$Call(0, v, T, K) = \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \lambda_s ds} (\int_0^t \lambda_s ds)^k}{k!} \right) \times \int_{\varsigma}^{\infty} \left(C_0 e^x + F_0 - e^{-rT} K \right) p^{(k)} dx$$

and

$$Put(0, v, T, K) = \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \lambda_s ds} (\int_0^t \lambda_s ds)^k}{k!} \right) \times \int_{\varsigma}^{\infty} \left(-C_0 e^x - F_0 + e^{-rT} K \right) p^{(k)} dx,$$

where $K > F_T$ and

$$p^{(k)} = f_1 * \underbrace{f_{Q'} * ... * f_{Q'}}_{k \text{ terms}},$$

$$f_{Q'}(z) = \phi \left(\ln \left(1 + \frac{e^z - 1}{m} \right), \alpha, \delta^2 \right) \frac{e^z}{m + e^z - 1},$$

and f_1 is the density function of the normal distribution

$$\mathcal{N} \left(\cdot, \int_0^T \left(m \left(\lambda_s - e^{\alpha + \frac{\delta^2}{2} \lambda_s} \right) - \frac{m\sigma_s^2}{2} \right) ds, \int_0^T m\sigma_s dW_s \right)$$

and $\varsigma = \ln\left(\frac{e^{-rT}K - F_0}{C_0}\right)$.

In the following proposition we consider the special case that $Y_n = Y$ is a constant. In this case, the expression is more explicit.

Proposition 3.21. *In the case that $Y_n = Y$ is a constant, the vanilla call/put option on the whole CPPI portfolio's value at maturity has the explicit expression:*

$$\begin{aligned} Call(0, v, T, K) &= \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \psi_s \lambda_s ds} \left(\int_0^t \psi_s \lambda_s ds\right)^k}{k!} \right) \\ &\times \left(C_0 e^{M^{(k)} + \frac{1}{2}\sigma_{(k)}^2} \Psi\left(\frac{M^{(k)} + \sigma_{(k)}^2 - \varsigma}{\sigma_{(k)}}\right) - (F_0 - e^{-rT}K)\Psi\left(\frac{M^{(k)} - \varsigma}{\sigma_{(k)}}\right)\right) \end{aligned}$$

and

$$\begin{aligned} Put(0, v, T, K) &= \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \psi_s \lambda_s ds} \left(\int_0^t \psi_s \lambda_s ds\right)^k}{k!} \right) \\ &\times \left(-C_0 e^{M^{(k)} + \frac{1}{2}\sigma_{(k)}^2} \Psi\left(-\frac{M^{(k)} - \sigma_{(k)}^2 + \varsigma}{\sigma_{(k)}}\right) + \left(-F_0 + e^{-rT}K\right)\Psi\left(\frac{-M^{(k)} + \varsigma}{\sigma_{(k)}}\right)\right), \end{aligned}$$

where $K > F_T$ and

$$\begin{aligned} M^{(k)} &= \int_0^T \left((m - Y\psi_s\lambda_s) - \frac{m\sigma_s^2}{2}\right) ds + k\ln(1 + mY), \\ \sigma_{(k)}^2 &= \int_0^T m\sigma_s dW_s^{\mathbb{Q}}, \end{aligned}$$

$$\varsigma = \ln\left(\frac{e^{-rT}K - F_0}{C_0}\right)$$

and

$$\Psi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

.

Proof. We have

$$\begin{aligned} V_t &= C_0 \exp\left\{ \int_0^t \left(r + m(\mu_s - r) - \frac{m\sigma_s^2}{2} \right) ds \right. \\ &\quad \left. + \int_0^t m\sigma_s dW_s + \left[\sum_{n=1}^{N_t} \ln(1 + mY_n) \right] \right\} + F_t \\ &= C_0 \exp\left\{ \int_0^t \left(r + m(-Y\psi_s\lambda_s) - \frac{m\sigma_s^2}{2} \right) ds \right. \\ &\quad \left. + \int_0^t m\sigma_s dW_s^Q + \left[\sum_{n=1}^{N_t} \ln(1 + mY_n) \right] \right\} + F_t. \end{aligned}$$

In case that $N_T = k$, we have

$$e^{-rT} \frac{V_T^k - F_T}{C_0} = \exp\left\{ \int_0^T \left(m(-Y\psi_s\lambda_s) - \frac{m\sigma_s^2}{2} \right) ds \right.$$
$$\left. + \int_0^T m\sigma_s dW_s^Q + \left[\sum_{n=1}^{N_T} \ln(1 + mY_n) \right] \right\}.$$

Then we have

$$\ln\left(e^{-rT} \frac{V_T^k - F_T}{C_0} \right) \sim \mathcal{N}\left(.; M^{(k)}, \sigma_{(k)}^2\right),$$

where

$$\begin{aligned} M^{(k)} &= \int_0^T \left(m(-Y\psi_s\lambda_s) - \frac{m\sigma_s^2}{2} \right) ds + k\ln(1 + mY) \\ \sigma_{(k)}^2 &= \int_0^T m\sigma_s dW_s^{\mathbb{Q}}. \end{aligned}$$

Thus

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(V_T^{(k)} - K)^+ \right) = \int_{\varsigma}^{\infty} \left(C_0 e^x + F_0 - e^{-rT} K \right) d\left(\mathcal{N}\left(x; M^{(k)}, \sigma_{(k)}^2\right) \right)$$

$$= \quad C_0 e^{M^{(k)}+\frac{1}{2}\sigma^2_{(k)}} \Psi\left(\frac{M^{(k)}+\sigma^2_{(k)}-\varsigma}{\sigma_{(k)}}\right) - \left(F_0 - e^{-rT}K\right)\Psi\left(\frac{M^{(k)}-\varsigma}{\sigma_{(k)}}\right),$$

where

$$\varsigma = \ln\left(\frac{e^{-rT}K - F_0}{C_0}\right).$$

and

$$\Psi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

Then

$$\begin{aligned}
& Call(0, v, T, K) \\
= \quad & \mathbb{E}^{\mathbb{Q}}(e^{-rT}(V_T - K)^+) = \sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}}(^{-rT}(V_T^{(k)} - K)^+)(\frac{e^{-\int_0^t \psi_s \lambda_s ds}(\int_0^t \psi_s \lambda_s ds)^k}{k!}) \\
= \quad & \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \psi_s \lambda_s ds}\left(\int_0^t \psi_s \lambda_s ds\right)^k}{k!}\right) \\
& \times \left(C_0 e^{M^{(k)}+\frac{1}{2}\sigma^2_{(k)}} \Psi\left(\frac{M^{(k)}+\sigma^2_{(k)}-\varsigma}{\sigma_{(k)}}\right) - (F_0 - e^{-rT}K)\Psi\left(\frac{M^{(k)}-\varsigma}{\sigma_{(k)}}\right)\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left(e^{-rT}(K - V_T^{(k)})^+\right) = \int_{-\infty}^{\varsigma} \left(-C_0 e^x - F_0 + e^{-rT}K\right) d\left(\mathcal{N}\left(x; M^{(k)}, \sigma^2_{(k)}\right)\right) \\
= \quad & -C_0 e^{M^{(k)}+\frac{1}{2}\sigma^2_{(k)}} \Psi\left(-\frac{M^{(k)}-\sigma^2_{(k)}+\varsigma}{\sigma_{(k)}}\right) + \left(-F_0 + e^{-rT}K\right)\Psi\left(\frac{-M^{(k)}+\varsigma}{\sigma_{(k)}}\right)
\end{aligned}$$

and

$$Put(0, v, T, K)$$

$$
\begin{aligned}
&= \mathbb{E}^{\mathbb{Q}}(e^{-rT}(K - V_T)^+) = \sum_{k=0}^{\infty} \mathbb{E}^{\mathbb{Q}}(^{-rT}(K - V_T^{(k)})^+)(\frac{e^{-\int_0^t \psi_s \lambda_s ds}(\int_0^t \psi_s \lambda_s ds)^k}{k!}) \\
&= \sum_{k=0}^{\infty} \left(\frac{e^{-\int_0^t \psi_s \lambda_s ds} \left(\int_0^t \psi_s \lambda_s ds \right)^k}{k!} \right) \\
&\quad \times \left(-C_0 e^{M^{(k)} + \frac{1}{2}\sigma_{(k)}^2} \Psi \left(-\frac{M^{(k)} - \sigma_{(k)}^2 + \varsigma}{\sigma_{(k)}} \right) + \left(-F_0 + e^{-rT} K \right) \Psi \left(\frac{-M^{(k)} + \varsigma}{\sigma_{(k)}} \right) \right).
\end{aligned}
$$

□

Remarks. The assumption of the jump Y_n be constant is not reasonable, however, it is looks like the option formula in Black-Scholes model with constant coefficient.

3.4 Mean-variance Hedging

3.4.1 Introduction

Given a contingent claim H and suppose there is no arbitrage opportunities, then in a complete market H is attainable, i.e. there exists a self-financing strategy with final portfolio value $Z_T = H$, $\mathbb{P}$-a.s. However, when in our jump-diffusion model, the market is not complete and so H is not attainable. In this case we consider quadratic hedging. There are two approaches. One approach is risk-minimization; the other approach is mean-variance hedging. See [67]. We employ the notations from that paper.

We consider the mean-variance hedging. For any contingent claim, let the payoff at T be H. Our jump-diffusion model of the risky asset price S is a semimartingale under $\mathbb{P}$. The following definition is taken from section 4 in [67].

Definition 3.22. We denote by Θ_2 the set of all $\vartheta \in L(S)$ such that the stochastic integral process $G(\vartheta) := \int \vartheta dS$ satisfies $G_T \in L^2(\mathbb{P})$. For a fixed linear subspace Θ of Θ_2, a Θ-strategy is a pair $(Z_0, \vartheta) \in \mathbb{R} \times \Theta$ and its value process is $Z_0 + G(\vartheta)$.

A Θ-strategy $(\tilde{Z}_0, \tilde{\vartheta})$ is called Θ-*mean-variance optimal* for a given contingent claim $H \in L^2$ if it minimizes $||H - Z_0 - G_T(\vartheta)||_{L^2}$ over all Θ-strategies (Z_0, ϑ) and $\tilde{Z}_0$ is then called the Θ-*approximation price* for H.

The linear subspace

$$\mathcal{G} := G_T(\Theta) = \left\{ \int_0^T \vartheta_u dS_u | \vartheta \in \Theta \right\}$$

of L^2 describes all outcomes of self-financing Θ-strategies with initial wealth $Z_0 = 0$ and

$$\mathcal{A} = \mathbb{R} + \mathcal{G} = \left\{ Z_0 + \int_0^T \vartheta_u dS_u | (Z_0, \vartheta) \in (\mathbb{R} \times \Theta) \right\}$$

is the space of contingent claims replicable by self-financing Θ-strategies. Our goal in mean-variance hedging is to find the projection in L^2 of H on $\mathcal{A}$ and this can be studied for a general linear subspace $\mathcal{G}$ of L^2 space. In analogy to the above definition, we introduce a $\mathcal{G}$-*mean-variance optimal* pair $(\tilde{Z}_0, \tilde{g}) \in \mathbb{R} \times \mathcal{G}$ for $H \in L^2$ and call $\tilde{Z}_0$ the $\mathcal{G}$-*approximation price* for H. Our goal is to find

$$\min_{(Z_0, \vartheta) \in \mathbb{R} \times \Theta} ||H - Z_0 - G_T(\vartheta)||_{L^2}.$$

Since

$$dS_t = S_{t-}[\mu_t dt + \sigma_t dW_t + Y_t dN_t],$$

we have

$$
\begin{aligned}
&\min_{(Z_0,\vartheta)\in\mathbb{R}\times\Theta}\left|\left|H - Z_0 - \int_0^T \vartheta_u dS_u\right|\right|_{L^2} \\
&\min_{(Z_0,\vartheta)\in\mathbb{R}\times\Theta}\left|\left|H - Z_0 - \int_0^T \vartheta_u S_{u-}\left[\mu_u du + \sigma_u dW_u + Y_t dN_u\right]\right|\right|_{L^2} \\
&= \min_{(Z_0,\vartheta)\in\mathbb{R}\times\Theta}\left(\mathbb{E}\left\{H - Z_0 - \int_0^T \vartheta_u S_{u-}[\mu_u du + \sigma_u dW_u + Y_t dN_u]\right\}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

[67] has pointed out that finding the optimal $\tilde{\vartheta}$ is in general an open problem. On the other hand, in the case of real contingent claim pricing, we should always using the risk-neutral measure. [20] gives the $\mathcal{G}$-mean-variance optimal pair $(\tilde{Z}_0, \tilde{g})$ when the stocks' model is an exponential levy form martingale. For similar consideration also see Chapter 10 in [18].

3.4.2 Our Problem

Now we consider H as a function of V_T and denote $H = g(V_T)$. For any martingale measure $\mathbb{Q}$ defined in (3.10), we have proved that $\tilde{V}_t = e^{-rt}V_t$ is a $\mathbb{Q}$-martingale. Denote $\tilde{H} = e^{-rT}H$. We want to consider the following optimization problem.

$$
\min_{(Z_0,\vartheta)\in\mathbb{R}\times\Theta} \mathbb{E}^{\mathbb{Q}}\left(\tilde{H} - Z_0 - \int_0^T \vartheta_u d\tilde{V}_u\right)^2. \tag{3.26}
$$

Proposition 3.23. *The solution of the optimization problem* (3.26) *is*

$$
Z_0 = \mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right];
$$

$$
\vartheta_t = \frac{\sigma_t(\mathcal{C}_x(t,V_t)) + (\mathcal{C}(t, V_t + (V_{t-} - F_t)m_t Y_t) - \mathcal{C}(t,V_t))Y_t\lambda_t\psi_t}{\sigma_t + (V_{t-} - F_t)m_t Y_t^2\lambda_t\psi_t}.
$$

Proof. We have

$$\mathbb{E}^{\mathbb{Q}}\left(\tilde{H}-Z_0-\int_0^T \vartheta_u d\tilde{V}_u\right)^2 = \mathbb{E}^{\mathbb{Q}}\left(\mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right]-Z_0+\tilde{H}-\mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right]-\int_0^T \vartheta_u d\tilde{V}_u\right)^2$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right]-Z_0\right)^2\right]+\mathbb{E}^{\mathbb{Q}}\left(\tilde{H}-\mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right]-\int_0^T \vartheta_u d\tilde{V}_u\right)^2.$$

We see that the optimal value for the initial capital is $Z_0 = \mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right]$.
Define $\mathcal{C}(t,x) = e^{rt}\mathbb{E}^{\mathbb{Q}}\left[\tilde{H}|V_t = x\right]$ and $\tilde{\mathcal{C}}(t,x) = e^{-rt}\mathcal{C}(t,x)$. By construction, $\tilde{\mathcal{C}}(t,x)$ is a $\mathbb{Q}$-martingale. We have deduced that

$$\begin{aligned} dV_t &= (rV_{t-}+(V_{t-}-F_t)m_t(\mu_t-r))dt+(V_{t-}-F_t)m_t\sigma_t dW_t \\ &\quad +(V_{t-}-F_t)m_tY_tdN_t, \end{aligned}$$

and

$$d\tilde{V}_t = e^{-rt}\left((V_{t-}-F_t)m_t\sigma_t dW_t^{\mathbb{Q}}+(V_{t-}-F_t)m_tY_tdM_t^{\mathbb{Q}}\right).$$

Then by Ito's formula we have

$$\begin{aligned} & d\tilde{\mathcal{C}}(t,V_t) \\ = & \Big(-re^{-rt}\mathcal{C}(t,V_t)+e^{-rt}\mathcal{C}_t(t,V_t)+(rV_{t-}+(V_{t-}-F_t)m_t(\mu_t-r))e^{-rt}\mathcal{C}_x(t,V_t) \\ & +\frac{1}{2}(V_{t-}-F_t)^2m_t^2\sigma_t^2e^{-rt}\mathcal{C}_{xx}(t,V_t)\Big)dt+(V_{t-}-F_t)m_t\sigma_te^{-rt}\mathcal{C}_x(t,V_t)dW_t \\ & +\left(e^{-rt}\mathcal{C}(t,V_t+(V_{t-}-F_t)m_tY_t)-e^{-rt}\mathcal{C}(t,V_t)\right)dN_t \\ = & (V_{t-}-F_t)m_t\sigma_te^{-rt}\mathcal{C}_x(t,V_t)dW_t^{\mathbb{Q}} \\ & +\left(e^{-rt}\mathcal{C}(t,V_t+(V_{t-}-F_t)m_tY_t)-e^{-rt}\mathcal{C}(t,V_t)\right)dM_t^{\mathbb{Q}}. \end{aligned}$$

Thus we have

$$
\begin{aligned}
& \tilde{H} - \mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right] - \int_0^T \vartheta_u d\tilde{V}_u \\
= \ & \tilde{\mathcal{C}}(T, V_T) - \tilde{\mathcal{C}}(0, V_0) - \int_0^T \vartheta_t e^{-rt}\left((V_{t-} - F_t)m_t\sigma_t dW_t^{\mathbb{Q}} + (V_{t-} - F_t)m_t Y_t dM_t^{\mathbb{Q}}\right) \\
= \ & e^{-rt}\Bigg(\int_0^T (V_{t-} - F_t)m_t\sigma_t(\mathcal{C}_x(t, V_t) - \vartheta_t)dW_t^{\mathbb{Q}} \\
& + \int_0^T ((\mathcal{C}(t, V_t + (V_{t-} - F_t)m_t Y_t) - \mathcal{C}(t, V_t)) - \vartheta_t(V_{t-} - F_t)m_t Y_t)dM_t^{\mathbb{Q}}\Bigg).
\end{aligned}
$$

By the Isometry formula, we have

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left(\tilde{H} - \mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right] - \int_0^T \vartheta_u d\tilde{V}_u\right)^2 \\
= \ & e^{-2rt}\Bigg(\mathbb{E}^{\mathbb{Q}}\left[\int_0^T ((V_{t-} - F_t)m_t\sigma_t(\mathcal{C}_x(t, V_t) - \vartheta_t))^2 dt\right] \\
& + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T (((\mathcal{C}(t, V_t + (V_{t-} - F_t)m_t Y_t) - \mathcal{C}(t, V_t)) - \vartheta_t(V_{t-} - F_t)m_t Y_t))^2 \lambda_t\psi_t dt\right]\Bigg).
\end{aligned}
$$

This is the minimizing problem with respect to ϑ_t. Differentiating the above expression with respect to ϑ_t and letting the first order derivative equal to 0, we have

$$
\begin{aligned}
& (V_{t-} - F_t)m_t\sigma_t(\mathcal{C}_x(t, V_t) - \vartheta_t) + (((\mathcal{C}(t, V_t + (V_{t-} - F_t)m_t Y_t) \\
& -\mathcal{C}(t, V_t)) - \vartheta_t(V_{t-} - F_t)m_t Y_t))(V_{t-} - F_t)m_t Y_t\lambda_t\psi_t = 0,
\end{aligned}
$$

thus

$$
\vartheta_t = \frac{\sigma_t(\mathcal{C}_x(t, V_t)) + (\mathcal{C}(t, V_t + (V_{t-} - F_t)m_t Y_t) - \mathcal{C}(t, V_t))Y_t\lambda_t\psi_t}{\sigma_t + (V_{t-} - F_t)m_t Y_t^2\lambda_t\psi_t}
$$

□

Remarks. When the contingent claim is the call option with the strike price K, i.e.

$H = (V_T - K)^+$, then

$$Z_0 = \mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right] = \text{Call}(0, V_0, T, K)$$

and

$$\mathcal{C}(t, x) = e^{rt}\mathbb{E}^{\mathbb{Q}}\left[\tilde{H}|V_t = x\right] = \text{Call}(t, x, T, K);$$

when the contingent claim is the put option with the strike price K, i.e. $H = (K - V_T)^+$, then

$$Z_0 = \mathbb{E}^{\mathbb{Q}}\left[\tilde{H}\right] = \text{Put}(0, V_0, T, K)$$

and

$$\mathcal{C}(t, x) = e^{rt}\mathbb{E}^{\mathbb{Q}}\left[\tilde{H}|V_t = x\right] = \text{Put}(t, x, T, K).$$

This is consitent with the calculation of call and put options.

Chapter 4

Gap risks

4.1 Introduction

Let

$$\frac{dS_t}{S_{t-}} = dZ_t. \tag{4.1}$$

where Z_t is a Levy process, a special case of which would be our jump diffusion. We will show the probability of loss we obtain is consistant with [17] and our result on the expected loss is more explicit and the method is similar to [17]; the result we obtain for the loss distribution is explicit and our method is different from [17].

Two kinds of conditional floors will be introduced in section 4.3. Its idea is similar to the Value-at-Risk considered in [27]. Meanwhile, four kinds of conditional floor are discussed associated with expected loss and loss distribution.

4.2 Gap risk Measure for CPPI strategies in Jump-diffusion model

4.2.1 Probability of Loss

In practice, a CPPI-insured portfolio incurs a loss (breaks through the floor) if, for some $t \in [0, T]$, $V_t \leq F_t$. The event $V_t \leq F_t$ is equivalent to $C_t \leq 0$. It happens at time T_i, associated with the i-th jump of the risky asset, $1 + mY_i \leq 0$. We have

Proposition 4.1. *Let the density of* $\ln(1+Y_n)$ *be* $f_Q(y)$*. The probability of the CPPI portfolio value going below the floor taking happen during time* $[0, T]$ *is given by*

$$\mathbb{P}\left[\exists t \in [0,T] : V_t \leq F_t\right] = 1 - \exp\left\{\int_0^T \lambda_s ds \left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y)dy - 1\right)\right\} \tag{4.2}$$

Proof. Since

$$\mathbb{P}(1 + mY_i > 0) = \mathbb{P}\left(\ln(1+Y_i) > \ln\left(1 - \frac{1}{m}\right)\right) = \int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y)dy,$$

then

$$\begin{aligned}
&\mathbb{P}[\exists\, t \in [0,T] :\ V_t \leq F_t] = \mathbb{P}(\exists\, t \in [0,T] :\ C_t \leq 0) \\
= \ & \mathbb{P}(\exists\, T_i,\ 1 + mY_i \leq 0) = 1 - \mathbb{P}(\forall\, T_i,\ 1 + mY_i > 0) \\
= \ & 1 - \mathbb{P}\left(\bigcup_{j=0}^{\infty} [\forall\, T_i\ 1 + mY_i > 0, N_T = j]\right) \\
= \ & 1 - \sum_{j=0}^{\infty} \mathbb{P}\left([\forall\, T_i\ 1 + mY_i > 0, N_T = j]\right) \\
= \ & 1 - \sum_{j=0}^{\infty} \mathbb{P}(\forall\, T_i\ 1 + mY_i > 0 | N_T = j)\mathbb{P}(N_T = j) \\
= \ & 1 - \sum_{j=0}^{\infty} \mathbb{P}(\forall\, T_1,\ T_2 \ldots T_j,\ 1 + mY_i > 0)\mathbb{P}(N_T = j)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \sum_{j=0}^{\infty} \mathbb{P}\left(\bigcap_{i=1}^{j} (1 + mY_i > 0) \right) \mathbb{P}(N_T = j) \\
&= 1 - \sum_{j=0}^{\infty} \prod_{i=1}^{j} \mathbb{P}(1 + mY_i > 0) \mathbb{P}(N_T = j) \\
&= 1 - \sum_{j=0}^{\infty} \frac{e^{-\int_0^T \lambda_s ds} (\int_0^T \lambda_s ds)^j}{j!} \left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y) dy \right)^j \\
&= 1 - \exp \left\{ \int_0^T \lambda_s ds \left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y) dy - 1 \right) \right\}.
\end{aligned}
$$

□

Remarks. (1) When $\lambda_s = \lambda$, the probability of loss is

$$
1 - \exp \left\{ T\lambda \left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y) dy - 1 \right) \right\} \tag{4.3}
$$

Our conclusion is a special case of Corollary 3.1 in [17], where the probability of loss is given by

$$
1 - \exp \left(-T \int_{-\infty}^{\ln(1-1/m)} \nu(dx) \right).
$$

In our case the levy measure ν is $\nu(dx) = \lambda f_Q(x) dx$ (See Page 75, [18] or Page 14, [57]), then

$$
\begin{aligned}
& 1 - \exp \left(-T \int_{-\infty}^{\ln(1-1/m)} \nu(dx) \right) \\
&= 1 - \exp \left(-T \int_{-\infty}^{\ln(1-1/m)} \lambda f_Q(x) dx \right) \\
&= 1 - \exp \left\{ T\lambda \left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y) dy - 1 \right) \right\}.
\end{aligned}
$$

(2) This proposition describes the loss takes place before the mature time T. Thus, it is naturally to generalize it to the time $t \in [0, T]$. We have the following corollary,

Corollary 4.2. *Assume $\lambda = \lambda_s$ and let $\tau \leq T$ if the loss take happen i.e. $C_\tau \leq 0$ and $\tau = \infty$ otherwise. The distribution of τ is, for $t \in [0, T]$*

$$\mathbb{P}(\tau \leq t) = 1 - \exp\left\{ t\lambda \left(\int_{\ln(1-\frac{1}{m})}^{\infty} f_Q(y)dy - 1 \right) \right\}$$

and the density function f_τ of τ is

$$f_\tau = -\exp\left\{ t\lambda \left(\int_{\ln(1-\frac{1}{m})}^{\infty} f_Q(y)dy - 1 \right) \right\} \left(\lambda \left(\int_{\ln(1-\frac{1}{m})}^{\infty} f_Q(y)dy - 1 \right) \right). \quad (4.4)$$

Proof. The first one is obvious and the second one is the derivative with respect to t. □

4.2.2 Expected Loss

Let τ be the first time when $C_\tau \leq 0$ and we let $\tau = \infty$ if the loss never happens. Let

$$\varepsilon_t = C_0 \exp\left\{ \int_0^t \left(m(\mu_s - r) - \frac{m^2 \sigma_s^2}{2} \right) ds + \int_0^t m\sigma_s dW_s \right\} \left[\prod_{n=1}^{N_t} (1 + mY_n) \right] \quad (4.5)$$

If a loss takes place, then at time τ, the cushion $C_\tau \leq 0$. If we do not allow short-sell, then, at time $\tau+$, we let the exposure be 0. Then, we have the discounted cushion:

$$C_T^* = \varepsilon_T \chi_{\tau > T} + \varepsilon_\tau (1 + mY_\tau) \chi_{\tau \leq T} \quad (4.6)$$

Remarks. In subsection 3.2 we allowed negative exposure to happen and we have the expression for the cushion C_t and the portfolio value V_t. When the CPPI portfolio is considered as an hedging tool in subsection 3.3, short-selling is allowed.
In this subsection, we take the exposure to be 0 at the time when there is a loss and we measure the gap.

When $t < \tau$, $1 + mY_i > 0$ for $T_i \leq t$. We first calculate the characteristic function of $\ln\left(\frac{\varepsilon_t}{C_0}\right)$ for $t < \tau$.

Since $1 + mY_i > 0$ when $t < \tau$, we have

$$\ln\left(\frac{\varepsilon_t}{C_0}\right) = \int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1 + mY_n).$$

Proposition 4.3. *Let the density function of* $\ln(1 + Y_i)$ *be* $f_Q(x)$*. When* $t < \tau$*, the characteristic function* $\phi_t(u)$ *of* $\ln\left(\frac{\varepsilon_t}{C_0}\right)$ *is*

$$\begin{aligned}\phi_t(u) &= \exp\left\{i\left(\int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds\right) u - \frac{1}{2}\left(\int_0^t m\sigma_s^2 ds\right) u^2\right\} \\ &\times \exp\left\{t\lambda \int_{\mathbb{R}} (e^{iux} - 1) f_Q\left(\ln\left(1 + \frac{e^x - 1}{m}\right)\right) \frac{e^x}{m + e^x - 1} dx\right\}.\end{aligned}$$

Proof. Since $\int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s$ and $\sum_{n=1}^{N_t} \ln(1 + mY_n)$ are independent, thus the characteristic function of the sum of two random variables is the production of characteristic function of each random variables.

$\int_0^t((m(\mu_s - r)) - \frac{m^2\sigma_s^2}{2})ds + \int_0^t m\sigma_s dW_s$ is normal distribution with mean $\int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds$ and variance $\int_0^t m^2\sigma_s^2 ds$ and hence its characteristic function is

$$\phi_{1,t}(u) = \exp\left\{i\left(\int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds\right) u - \frac{1}{2}\left(\int_0^t m^2\sigma_s^2 ds\right) u^2\right\}. \quad (4.7)$$

In section 3.3, we have deduced the density function f'_Q of the random variable $\ln(1+ mY_i)$ is

$$f'_Q(z) = f_Q\left(\ln\left(1 + \frac{e^z - 1}{m}\right)\right) \frac{e^z}{m + e^z - 1}.$$

We denote the characteristic function of f'_Q by $\hat{f}'_Q$. Then, the characteristic function

$\phi_{2,t}(u)$ of $\sum_{n=1}^{N_t} \ln(1+mY_n)$ is

$$\begin{aligned}
\phi_{2,t}(u) &= \mathbb{E}\left[\exp iu\left(\sum_{n=1}^{N_t} \ln(1+mY_n)\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp iu.\left(\sum_{n=1}^{N_t} \ln(1+mY_n)\right)\right]|N_t\right] = \mathbb{E}\left[\left(\hat{f'_Q}(u)\right)^{N_t}\right] \\
&= \sum_{j=0}^{\infty} \frac{e^{-\lambda t}(-\lambda t)^j \left(\hat{f'_Q}(u)\right)^j}{j!} = \exp\left\{\lambda t\left(\hat{f'_Q}(u)-1\right)\right\} \\
&= \exp\left\{t\lambda \int_{\mathbb{R}} \left(e^{iux}-1\right) f'_Q(dx)\right\} \\
&= \exp\left\{t\lambda \int_{\mathbb{R}} \left(e^{iux}-1\right) f_Q\left(\ln\left(1+\frac{e^x-1}{m}\right)\right) \frac{e^x}{m+e^x-1} dx\right\}.
\end{aligned}$$

Then, the characteristic function $\phi_t(u)$ of $\ln\left(\frac{\varepsilon_t}{C_0}\right)$ is

$$\begin{aligned}
\phi_t(u) &= \phi_{1,t}(u)\phi_{2,t}(u) \\
&= \exp\left\{i\left(\int_0^t \left(m(\mu_s-r)-\frac{m^2\sigma_s^2}{2}\right) ds\right) u - \frac{1}{2}\left(\int_0^t m^2\sigma_s^2 ds\right) u^2\right\} \\
&\times \exp\left\{t\lambda \int_{\mathbb{R}} (e^{iux}-1) f_Q\left(\ln\left(1+\frac{e^x-1}{m}\right)\right) \frac{e^x}{m+e^x-1} dx\right\}.
\end{aligned}$$

□

Definition 4.4. The conditional expectation of the discounted cushion is called the **conditional expected loss** and we assume that $\mathbb{E}[C_T^*|\tau \leq T]$; while the **unconditional expected loss** is represented by $\mathbb{E}[C_T^* \chi_{\tau \leq T}]$.

We have:

Proposition 4.5. *The expectation of loss conditioned on the fact that a loss has occured is*

$$\mathbb{E}[C_T^*|\tau \leq T] = \frac{\int_{-\infty}^{\ln(1-1/m)} f_Q(y)dy \int_0^T C_0\phi_t(-i) f_\tau dt}{1-\exp\left\{T\lambda\left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y)dy - 1\right)\right\}} \tag{4.8}$$

and the unconditional expected loss satisfies

$$\mathbb{E}[C_T^* \chi_{\tau \leq T}] = \int_{-\infty}^{\ln(1-1/m)} f_Q(y)dy \int_0^T C_0 \phi_t(-i) f_\tau dt \tag{4.9}$$

where f_τ is the density function of τ and defined by (4.4) *and ϕ_t is the characteristic function of* $\ln\left(\frac{\varepsilon_t}{C_0}\right)$.

Proof. First the discounted cushion is

$$C_T^* \;=\; \varepsilon_T \chi_{\tau > T} + \varepsilon_\tau (1 + mY_\tau) \chi_{\tau \leq T}.$$

Then

$$\mathbb{E}[C_T^* \chi_{\tau \leq T}] = \mathbb{E}[(1 + mY_\tau)] \mathbb{E}[\varepsilon_\tau].$$

$(1 + mY_\tau)$ is the size of the first jump which size is $Y_i < -1/m$. Thus,

$$\mathbb{E}\left[(1 + mY_\tau)\left(\chi_{\tau \leq T}\right)\right] = \int_{-\infty}^{\ln(1-1/m)} f_Q(y)dy.$$

By the property of the characteristic function, we have

$$\mathbb{E}[\varepsilon_t] = C_0 \phi_t(-i).$$

Therefore

$$\begin{aligned} &\mathbb{E}[C_T^* \chi_{\tau \leq T}] = \mathbb{E}\left[(1 + mY_\tau)\left(\chi_{\tau \leq T}\right)\right] \mathbb{E}[\varepsilon_\tau \chi_{\tau \leq T}] \\ = &\int_{-\infty}^{\ln(1-1/m)} f_Q(y)dy \int_0^T C_0 \phi_t(-i) f_\tau dt, \end{aligned}$$

where f_τ is the density function of τ and defined by (4.4). Furthermore, by the property of conditional expectation, we have

$$\mathbb{E}[C_T^*|\tau \le T] = \frac{\mathbb{E}[C_T^*\chi_{\tau\le T}]}{\mathbb{P}[\tau \le T]} = \frac{\int_{-\infty}^{\ln(1-1/m)} f_Q(y)dy \int_0^T C_0\phi_t(-i)f_\tau dt}{1-\exp\left\{T\lambda\left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y)dy-1\right)\right\}}.$$

□

4.2.3 Loss Distribution

To compute risk measures, we consider, for $x < 0$, the quantity

$$\mathbb{P}[C_T^* < x|\tau \le T]. \tag{4.10}$$

This is called the **Loss Distribution**. We next have:

Proposition 4.6. *Let the density of* $\ln(1+Y_n)$ *be* $f_Q(y)$ *and* C_T^* *be the discounted cushion. For* $x < 0$*, the unconditional loss distribution is*

$$\begin{aligned}
&\mathbb{P}\left[C_T^*\chi_{\tau\le T} < x\right] \\
&= \int_0^{\ln\left(-\frac{x}{C_0}\right)} \int_0^T \phi\left(z, \int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right)ds, \int_0^t m\sigma_s^2 ds\right) f_\tau dt\bigg|_{z=l} \\
&\times \left(-f_Q\left(\ln\left(1+\frac{-e^z-1}{m}\right)\right)\frac{e^z}{-m+e^z+1}\right)\bigg|_{z=\ln(-\frac{x}{C_0})-l} dl.
\end{aligned} \tag{4.11}$$

and the loss distribution is

$$\mathbb{P}[C_T^* < x|\tau \le T] = \frac{\mathbb{P}[C_T^*\chi_{\tau\le T} < x]}{1-\exp\left\{T\lambda\left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y)dy-1\right)\right\}}, \tag{4.12}$$

where f_τ *is the density function of* τ *and defined by (31) and* $\phi(x, m, v^2)$ *is the density function of normal distribution which is same as defined in subsection 3.1.1.*

Proof. For $x < 0$, the unconditional loss distribution is

$$\begin{aligned}
&\mathbb{P}[C_T^* \chi_{\tau\le T} < x] = \mathbb{P}[\varepsilon_\tau(1+mY_\tau)\chi_{\tau\le T} < x] \\
= \ & \mathbb{P}\left[\frac{\varepsilon_\tau}{C_0}(-(1+mY_\tau))\chi_{\tau\le T} > -\frac{x}{C_0}\right] \\
= \ & \mathbb{P}\left[\ln\left(\frac{\varepsilon_\tau}{C_0}\chi_{\tau\le T}\right) + \ln\left((-(1+mY_\tau))\chi_{\tau\le T}\right) > \ln\left(-\frac{x}{C_0}\right)\right] \\
= \ & \int_0^{\ln\left(-\frac{x}{C_0}\right)} \frac{d}{dz}\left(\mathbb{P}\left(\ln\left(\frac{\varepsilon_\tau}{C_0}\chi_{\tau\le T}\right) < z\right)\right)\Bigg|_{z=l} \\
& \times \frac{d}{dz}\left(\mathbb{P}\left(\ln\left((-(1+mY_\tau))\chi_{\tau\le T}\right) < z\right)\right)\Bigg|_{z=\ln\left(-\frac{x}{C_0}\right)-l} dl.
\end{aligned}$$

The last step is by the property of the distribution of the sum of two random variables. Since

$$\begin{aligned}
&\frac{d}{dz}\left(\mathbb{P}\left(\ln\left(\frac{\varepsilon_\tau}{C_0}\chi_{\tau\le T}\right) < z\right)\right) = \frac{d}{dz}\left(\int_0^T \mathbb{P}\left(\ln\left(\frac{\varepsilon_\tau}{C_0}\right) < z\right) f_\tau dt\right) \\
= \ & \int_0^T \phi\left(z, \int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds, \int_0^t m\sigma_s^2 ds\right) f_\tau dt
\end{aligned}$$

where f_τ is the density function of τ and defined by (31) and $\phi(x, m, v^2)$ is the density function of normal distribution which is same as defined in subsection 3.1.1 and

$$\begin{aligned}
&\frac{d}{dz}(\mathbb{P}(\ln((-(1+mY_\tau))\chi_{\tau\le T}) < z)) \\
= \ & \frac{d}{dz}\left(\mathbb{P}\left((1+mY_\tau)\chi_{\tau\le T} > -e^z\right)\right) \\
= \ & \frac{d}{dz}\left(\mathbb{P}\left(\ln\left((1+Y_\tau)\chi_{\tau\le T}\right) > \ln\left(1 + \frac{-e^z-1}{m}\right)\right)\right) \\
= \ & \frac{d}{dz}\left(\int_{\ln\left(1+\frac{-e^z-1}{m}\right)}^{\ln\left(1-\frac{1}{m}\right)} f_Q(y)dy\right) \\
= \ & -f_Q\left(\ln\left(1+\frac{-e^z-1}{m}\right)\right)\frac{e^z}{-m+e^z+1},
\end{aligned}$$

substitute the above two expressions, we get

$$
\begin{aligned}
&\mathbb{P}\left[C_T^*\chi_{\tau\le T} < x\right] \\
= \quad & \int_0^{\ln\left(-\frac{x}{C_0}\right)} \frac{d}{dz}\left(\mathbb{P}\left(\ln\left(\frac{\varepsilon_\tau}{C_0}\chi_{\tau\le T}\right) < z\right)\right)\Big|_{z=l} \\
& \times \frac{d}{dz}\left(\mathbb{P}(\ln((-(1+mY_\tau))\chi_{\tau\le T}) < z)\right)\Big|_{z=\ln(-\frac{x}{C_0})-l} dl. \\
= \quad & \int_0^{\ln\left(-\frac{x}{C_0}\right)} \int_0^T \phi\left(z, \int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds, \int_0^t m\sigma_s^2 ds\right) f_\tau dt\Big|_{z=l} \\
\times \quad & \left(-f_Q\left(\ln\left(1+\frac{-e^z-1}{m}\right)\right)\frac{e^z}{-m+e^z+1}\right)\Big|_{z=\ln(-\frac{x}{C_0})-l} dl.
\end{aligned}
$$

Moreover, the loss distribution is

$$
\mathbb{P}[C_T^* < x|\tau \le T] = \frac{\mathbb{P}[C_T^*\chi_{\tau\le T} < x]}{\mathbb{P}[\tau \le T]} = \frac{\mathbb{P}[C_T^*\chi_{\tau\le T} < x]}{1 - \exp\left\{T\lambda\left(\int_{\ln(1-\frac{1}{m})}^{\infty} f_Q(y)dy - 1\right)\right\}}.
$$

□

4.3 Conditional Floor and Conditional Multiple of CPPI in the Jump-diffusion Model

4.3.1 Introduction

We want to control the level of the gap by suitably adjusting the floor or/and multiple. For example, if we take $m = 1$, then the portfolio value is always greater than the floor, and thus there is no gap risk in the case. Another case is if we make the floor equal to initial portfolio measure, then there is also no gap risk. Risk occurs when we choose large enough multiples or low floors which result in more exposures. [1] and [2] describe how the conditional multiple and conditional floor control gap risks in the continuous case. Risks occur because the trading time is discrete.

4.3.2 Probability of Loss

Let the density of $\ln(1+Y_n)$ be $f_Q(y)$. Recall that the probability that the CPPI portfolio value falls below the floor during the time interval $[0,T]$ is given by (4.2). We see that the probability of loss is irrelevant to the floor. It is also irrelevant to the continuous part of the risky asset model. It is only related to the jump part of risky asset model and the multiple m. Moreover, we have the following proposition.

Proposition 4.7. *The probability of loss given in* (4.2) *is monotone increased function as the multiple* m*.*

Proof. For $m > 1$ in general,

$$\begin{gathered} m \text{ increased} \Longrightarrow \\ \ln\left(1-\tfrac{1}{m}\right) \text{ is increasing} \Longrightarrow \\ \textstyle\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y)dy - 1 \text{ is decreasing} \Longrightarrow \\ 1-\exp\left\{\textstyle\int_0^T \lambda_s ds \left(\int_{\ln\left(1-\frac{1}{m}\right)}^{\infty} f_Q(y)dy - 1\right)\right\} \text{ is increasing.} \end{gathered}$$

□

Like for the Value-at-Risk (VaR) (See [27]), we define:

Definition 4.8. For $\epsilon > 0$, the multiple $m = m_0$ makes

$$\mathbb{P}[\exists t \in [0,T] : V_t \leq F_t] = \epsilon$$

is called the **ϵ-conditional multiple**.

m_0 can be treated as a quantile. Since the probability of loss is monotone increased as the function of the multiple m. Then for $m < m_0$,

$$\mathbb{P}\left[\exists t \in [0,T] : V_t \leq F_t\right] < \epsilon.$$

Let the distribution function of $\ln(1+Y_n)$ be F_Q, then the quantile point m_0 is given by:

$$
\begin{aligned}
& \mathbb{P}\left[\exists t \in [0,T] : V_t \leq F_t\right] = \epsilon \\
\iff \quad & 1 - \exp\left\{ \int_0^T \lambda_s ds \left(\int_{\ln(1-\frac{1}{m_0})}^{\infty} f_Q(y)dy - 1 \right) \right\} = \epsilon \\
\iff \quad & -\frac{\ln(1-\epsilon)}{\int_0^T \lambda_s ds} = \int_{-\infty}^{\ln\left(1-\frac{1}{m_0}\right)} f_Q(y)dy \\
\iff \quad & \ln\left(1 - \frac{1}{m_0}\right) = F_Q^{-1}\left(-\frac{\ln(1-\epsilon)}{\int_0^T \lambda_s ds} \right) \\
\iff \quad & m_0 = \frac{1}{1 - \exp\left\{ F_Q^{-1}\left(-\frac{\ln(1-\epsilon)}{\int_0^T \lambda_s ds} \right) \right\}}
\end{aligned}
$$

When we know the distribution function of the jump-part, it is easy to determine the ϵ-conditional multiple and hence the strategies accordingly.

4.3.3 Expected Loss

Through the notion of Probability of Loss, we determine the conditional multiple and hence control the risk of the gap occurrence. From (4.8) and (4.9), we have the following proposition:

Proposition 4.9. *Given a fixed multiple, both the conditional expected loss given by* (4.8) *and the unconditional expected loss given by* (4.9) *are monotone decreased functions of the initial floor* F_0.

Proof. From (4.8) and (4.9), we see that they are increasing functions of the initial cushion C_0, and $C_0 = V_0 - F_0$. □

Similar to the concept of ϵ-conditional multiple, we define the following:

Definition 4.10. For $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c1}$ which causes

$$\mathbb{E}[C_T^* | \tau \leq T] = \varrho$$

is called the **first type ϱ-m_0-conditional floor**.

and

Definition 4.11. For $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c2}$ which causes

$$\mathbb{E}[C_T^* \chi_{\tau \leq T}] = \varrho$$

is called the **Second type ϱ-m_0-conditional floor**.

From (4.8) and (4.9), we can solve the two conditional floors easily.

4.3.4 Loss Distribution

Similar to the case of expected loss, we define the conditional floor in terms of loss distribution. Equation (4.11) gives the unconditional loss distribution and equation (4.12) gives the conditional loss distribution. The following propositions are immediate:

Proposition 4.12. *Given a fixed multiple and $x < 0$, the expressions* (4.11) *and* (4.12) *are monotone increasing functions of the initial floor F_0*

Proof. From equations (4.11) and (4.12), we see that they are increasing functions of C_0. Since $C_0 = V_0 - F_0$,

$$F_0 \text{ increased} \Longrightarrow$$
$$C_0 \text{ decreased} \Longrightarrow$$
$$\ln\left(-\frac{x}{C_0}\right) \text{ increased} \Longrightarrow$$
$$\text{Both the expression (4.11) and (4.12) are increased.}$$

□

As in the case of ϵ-conditional multiple, we define the following two concepts:

Definition 4.13. For $\epsilon > 0$, $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c3}$ associated with the condition

$$\mathbb{E}[C_T^* \chi_{\tau \leq T} < \varrho] = \epsilon$$

is called the **third type ϵ-ϱ-m_0-conditional floor**.

and

Definition 4.14. For $\epsilon > 0$, $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c4}$ which gives

$$\mathbb{E}[C_T^* < \varrho | \tau \leq T] = \epsilon$$

is called the **fourth type ϵ-ϱ-m_0-conditional floor**.

4.3.5 Conclusion

The conditional multiple and four conditional floors defined in our section can be used to the investment. The investor can determine them according to their risk-aversion level.

Chapter 5

CPPI in the jump-diffusion model when the trading time is discrete

5.1 Introduction

In this chapter we discuss the case of discrete trading time. The risky asset model is the same as in chapter 3 and 4.

In section 5.2, as in section 3.2, we calculate the CPPI portfolio value, its expectation and variance.

The gap risks are occurred because the risky model has jumps and also the trading time is discrete. As in section 4.2, we measure the gap risk from three aspects in section 5.3: the probability of loss,the expected loss and the loss distribution.

In section 5.4, similar to the ideas given in section 4.3, we define the conditional multiples associated with the probability of loss as well as the conditional floors from the views of expected loss and loss distribution. It could be treated as an application of 5.2.

In section 5.5, we prove that as the interval of the trading times tends to zero, the CPPI strategies in discrete trading time is agrees with the CPPI strategies in continuous time.

5.2 The strategy

Let $\tau^N = \{t_0 = 0 < t_1 < t_2 < ... < t_N = T\}$ be a sequence of equidistant refinements of the interval $[0, T]$, where $t_{k+1} - t_k = \frac{T}{N}$ for $k = 0, ..., N-1$. Suppose that the trading times are restricted to the discrete set τ^N. Futhermore we suppose

$$\mathbb{P}[T_i = t_j] = 0 \quad \forall i = 0, 1, 2, 3, ... \text{ and } j = 0, 1, 2, ..., N.$$

Hence we may assume $T_i \neq t_j$ for $\forall i = 0, 1, 2, 3, ...$ and $j = 0, 1, 2, ..., N$. We have

$$C_{t_{k+1}} = C_{t_k} \left(m \frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N} \right), \tag{5.1}$$

then

$$C_T = C_{t_N} = C_0 \prod_{k=0}^{N-1} \left(m \frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N} \right),$$

thus

$$V_T = C_T + G.$$

Since $\frac{S_{t_{k+1}}}{S_{t_k}}$, $k = 0, 1, 2, ..., n-1$ are manually independent and also they have the identity distribution, i.e.

$$\frac{S_{t_{k+1}}}{S_{t_k}} = \exp\left[\int_{t_k}^{t_{k+1}} \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds + \int_{t_k}^{t_{k+1}} \sigma_s dW_s + \sum_{n_k = N_{t_k}}^{N_{t_{k+1}}} \ln(1 + Y_{n_k}) \right],$$

then we have

$$\mathbb{E}\left[\frac{S_{t_{k+1}}}{S_{t_k}} \right] = \mathbb{E}\left[\exp(\mu \frac{T}{N} + \sigma W_{T/N} - \frac{1}{2}\sigma^2 \frac{T}{N}) \right] \mathbb{E} \prod_{n_k = N_{t_k}}^{N_{t_{k+1}}} (1 + Y_{n_k})$$

$$= \exp\left(\mu\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k});$$

and

$$\begin{aligned}\mathbb{E}\left[\frac{S_{t_{k+1}}}{S_{t_k}}\right]^2 &= \mathbb{E}\left[\exp\left(2\mu\frac{T}{N}+2\sigma W_{T/N}-\sigma^2\frac{T}{N}\right)\right]\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})^2\\ &= \mathbb{E}\left[\exp\left(2\mu\frac{T}{N}+\sigma^2\frac{T}{N}+2\sigma W_{T/N}-\frac{1}{2}(2\sigma)^2\frac{T}{N}\right)\right]\mathbb{E}\prod_{n=N_{t_k}}^{N_{t_{k+1}}}(1+Y_n)^2\\ &= \exp\left(2\mu\frac{T}{N}+\sigma^2\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})^2.\end{aligned}$$

Lemma 5.1. *Let the density function of* $\ln(1+Y_n)$ *be* f_Q*, then we have*

$$\begin{aligned}\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k}) =& \sum_{j=1}^{\infty}\frac{e^{-\int_{t_k}^{t_{k+1}}\lambda_s ds}(\int_{t_k}^{t_{k+1}}\lambda_s ds)^j}{j!}\\ &\times\int_{\mathbb{R}}\exp\left\{\underbrace{f_Q * f_Q * ... * f_Q(x)}_{j\ items}\right\}dx\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})^2 =& \sum_{j=1}^{\infty}\frac{e^{-\int_{t_k}^{t_{k+1}}\lambda_s ds}(\int_{t_k}^{t_{k+1}}\lambda_s ds)^j}{j!}\\ &\times\int_{\mathbb{R}}\exp\left\{2\underbrace{f_Q * f_Q * ... * f_Q(x)}_{j\ terms}\right\}dx.\end{aligned} \tag{5.3}$$

Proof. As the proof of proposition 3.6, we have

$$\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \frac{e^{-\int_{t_k}^{t_{k+1}} \lambda_s ds} (\int_{t_k}^{t_{k+1}} \lambda_s ds)^j}{j!} \mathbb{E}\left[\prod_{n_k=1}^{j} (1+mY_{n_k})\right] \\
&= \sum_{j=1}^{\infty} \frac{e^{-\int_{t_k}^{t_{k+1}} \lambda_s ds} (\int_{t_k}^{t_{k+1}} \lambda_s ds)^j}{j!} \mathbb{E}\left[\exp\left\{\sum_{n_k=1}^{j} (1+mY_{n_k})\right\}\right] \\
&= \sum_{j=1}^{\infty} \frac{e^{-\int_{t_k}^{t_{k+1}} \lambda_s ds} (\int_{t_k}^{t_{k+1}} \lambda_s ds)^j}{j!} \int_{\mathbb{R}} \exp\left\{\underbrace{f_Q * f_Q * ... * f_Q(x)}_{j \text{ items}}\right\} dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E} \prod_{n_k=N_{t_k}}^{N_{t_{k+1}}} (1+Y_{n_k}) \\
&= \sum_{j=1}^{\infty} \frac{e^{-\int_{t_k}^{t_{k+1}} \lambda_s ds} (\int_{t_k}^{t_{k+1}} \lambda_s ds)^j}{j!} \mathbb{E}\left[\prod_{n_k=1}^{j} (1+mY_{n_k})^2\right] \\
&= \sum_{j=1}^{\infty} \frac{e^{-\int_{t_k}^{t_{k+1}} \lambda_s ds} (\int_{t_k}^{t_{k+1}} \lambda_s ds)^j}{j!} \mathbb{E}\left[\exp\left\{2\sum_{n_k=1}^{j} (1+mY_{n_k})\right\}\right] \\
&= \sum_{j=1}^{\infty} \frac{e^{-\int_{t_k}^{t_{k+1}} \lambda_s ds} (\int_{t_k}^{t_{k+1}} \lambda_s ds)^j}{j!} \int_{\mathbb{R}} \exp\left\{2\underbrace{f_Q * f_Q * ... * f_Q(x)}_{j \text{ items}}\right\} dx.
\end{aligned}
$$

□

Next we calculate the expectation and variance of the terminal CPPI portfolio value:

Proposition 5.2. *The expected terminal CPPI portfolio value in discrete trading time case under the jump-diffusion model is*

$$
\mathbb{E}[V_T] = C_0 \prod_{k=0}^{N-1} \left(m \left[\exp\left(\mu \frac{T}{N}\right) \mathbb{E} \prod_{n_k=N_{t_k}}^{N_{t_{k+1}}} (1+Y_{n_k}) \right] - (m-1)e^{rT/N} \right) + G,
$$

where $\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})$ *is given by (5.2).*

Proof.

$$\begin{aligned}
&\mathbb{E}[V_T] = \mathbb{E}[C_T] + G \\
= \quad & C_0\mathbb{E}\left[\prod_{k=0}^{N-1}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right)\right] + G \\
= \quad & C_0\prod_{k=0}^{N-1}\left(m\mathbb{E}\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right) + G \\
= \quad & C_0\prod_{k=0}^{N-1}\left(m\left[\exp\left(\mu\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})\right] - (m-1)e^{rT/N}\right) + G,
\end{aligned}$$

where $\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})$ is given by (5.2). □

Proposition 5.3. *The variance of terminal CPPI portfolio value in discrete time case under the jump-diffusion model is*

$$\begin{aligned}
Var[V_T] = C_0^2\Bigg[\prod_{k=0}^{N-1}\Bigg(\Bigg[m^2\left(\exp\left(2\mu\frac{T}{N} + \sigma^2\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})^2\right) \\
+(m-1)^2e^{2rT/N} - 2m(m-1)e^{rT/N}\left(\exp\left(\mu\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})\right)\Bigg]\Bigg) \\
-\prod_{k=0}^{N-1}\left(\left[m\left(\exp\left(\mu\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})\right) - (m-1)e^{rT/N}\right]\right)^2\Bigg],
\end{aligned}$$

where $\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})$ *is given by* (5.2) *and* $\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})^2$ *is given by* (5.3).

Proof. By Lemma 2.7, we have

$$\begin{aligned}
&\mathrm{Var}[V_T] = \mathrm{Var}[C_T] = \mathrm{Var}\left[C_0\prod_{k=0}^{N-1}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right)\right] \\
= \quad & C_0^2\left[\prod_{k=0}^{N-1}\left(\mathbb{E}\left[m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right]^2\right) - \prod_{k=0}^{N-1}\left(\mathbb{E}\left[m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right]\right)^2\right] \\
= \quad & C_0^2\left[\prod_{k=0}^{N-1}\left(\left[m^2\mathbb{E}\left(\frac{S_{t_{k+1}}}{S_{t_k}}\right)^2 + (m-1)^2e^{2rT/N} - 2m(m-1)e^{rT/N}\mathbb{E}\frac{S_{t_{k+1}}}{S_{t_k}}\right]\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& -\prod_{k=0}^{N-1}\left(\left[m\mathbb{E}\frac{S_{t_{k+1}}}{S_{t_k}}-(m-1)e^{rT/N}\right]\right)^2\Bigg] \\
= \;& C_0^2\Bigg[\prod_{k=0}^{N-1}\Bigg(\Bigg[m^2\left(\exp\left(2\mu\frac{T}{N}+\sigma^2\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})^2\right) \\
& +(m-1)^2e^{2rT/N}-2m(m-1)e^{rT/N}\left(\exp\left(\mu\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})\right)\Bigg]\Bigg) \\
& -\prod_{k=0}^{N-1}\left(\left[m\left(\exp\left(\mu\frac{T}{N}\right)\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})\right)-(m-1)e^{rT/N}\right]\right)^2\Bigg],
\end{aligned}
$$

where $\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})$ is given by (5.2) and $\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})^2$ is given by (5.3). □

5.3 Measure the Gap risk for CPPI strategies in the jump-diffusion model-the discrete time case

5.3.1 Probability of Loss

In practice,suppose that a CPPI-insured portfolio incurs a loss. That is, for some $t_i \in \tau^N$, $V_{t_i} \leq F_{t_i}$, which is equivalent to $C_{t_i} \leq 0$. We consider the following probabilities:

Definition 5.4. The probability

$$\mathbb{P}^{PLL}_{t_i,t_{i+1}} := \mathbb{P}(V_{t_{i+1}} \leq F_{t_{i+1}} | V_{t_i} > F_{t_i}) \tag{5.4}$$

is called the **probability of local loss**.

and

Definition 5.5. The probability

$$\mathbb{P}^{PL} := \mathbb{P}(\text{if for some } t_i \in \tau^N : V_{t_i} \leq F_{t_i}) \tag{5.5}$$

is called the **probability of loss**.

Remarks. We refer the definition of *probability of local loss* to page 209 in [5].

The following proposition gives a relation between the **probability of local loss** and the **probability of loss**.

Proposition 5.6. *The probability of loss defined by* (5.5) *and probability of local loss defined by* (5.4) *have the following relation:*

$$\mathbb{P}^{PL} = 1 - \prod_{i=1}^{N} \left(1 - \mathbb{P}^{PLL}_{t_{i-1}, t_i}\right). \tag{5.6}$$

Proof. We have

$$\begin{aligned} \mathbb{P}^{PL} &= \mathbb{P}(\text{if for some }, t_i \in \tau^N : V_{t_i} \leq F_{t_i}) \\ &= 1 - \mathbb{P}(\forall\, t_i \in \tau^N : V_{t_i} > F_{t_i}) = 1 - \mathbb{P}\left(\bigcap_{i=1}^{N} \{V_{t_i} > F_{t_i} | V_{t_{i-1}} > F_{t_{i-1}}\}\right) \\ &= 1 - \prod_{i=1}^{N} \mathbb{P}(\{V_{t_i} > F_{t_i} | V_{t_{i-1}} > F_{t_{i-1}}\}) \\ &= 1 - \prod_{i=1}^{N} \left(1 - \mathbb{P}(\{V_{t_i} \leq F_{t_i} | V_{t_{i-1}} > F_{t_{i-1}}\})\right) = 1 - \prod_{i=1}^{N} \left(1 - \mathbb{P}^{PLL}_{t_{i-1}, t_i}\right). \end{aligned}$$

□

Proposition 5.7. *The probability of local loss defined by* (5.4) *is given by*

$$\mathbb{P}^{PLL}_{t_i, t_{i+1}} = \int_{-\infty}^{\ln(\frac{m-1}{m}) + \frac{rT}{N}} p^{(i)}(x) dx, \tag{5.7}$$

where

$$\begin{aligned} p^{(i)}(x) &= \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds} (\int_{t_i}^{t_{i+1}} \lambda_s ds)^j}{j!} \\ &\times \int_{-\infty}^{\infty} \phi\left(x - y; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds, \int_0^t \sigma_s^2 ds\right) f_Q^{(j)}(y) dy, \end{aligned} \tag{5.8}$$

where $f_Q^{(j)}(y) = \underbrace{f_Q(y) * f_Q(y) * ...f_Q(y)}_{Convolved\ j\ times}.$

Proof. We have

$$\begin{aligned}
&\mathbb{P}^{PLL}_{t_i,t_{i+1}} = \mathbb{P}(V_{t_{i+1}} \leq F_{t_{i+1}} | V_{t_i} > F_{t_i}) = \mathbb{P}(C_{i+1} \leq 0 | C_i > 0) \\
= \ & \mathbb{P}\left(m\frac{S_{t_{i+1}}}{S_{t_i}} - (m-1)e^{rT/N}) \leq 0\right) \\
= \ & \mathbb{P}\Bigg(m \exp\left[\int_{t_i}^{t_{i+1}} \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_{t_i}^{t_{i+1}} \sigma_s dW_s + \sum_{n=N_{t_i}}^{N_{t_{i+1}}} \ln(1+Y_n)\right] \\
& -(m-1)e^{rT/N} \leq 0\Bigg) \\
= \ & \mathbb{P}\Bigg(\int_{t_i}^{t_{i+1}} \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_{t_i}^{t_{i+1}} \sigma_s dW_s + \sum_{n=N_{t_i}}^{N_{t_{i+1}}} \ln(1+Y_n) \leq \ln\left(\frac{m-1}{m}\right) + \frac{rT}{N}\Bigg).
\end{aligned}$$

The proof of Proposition 3.2 shows the density function $p^{(i)}(x)$ of

$$\int_{t_i}^{t_{i+1}} \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_{t_i}^{t_{i+1}} \sigma_s dW_s + \sum_{n=N_{t_i}}^{N_{t_{i+1}}} \ln(1+Y_n)$$

is

$$\begin{aligned}
p^{(i)}(x) = & \sum_{j=0}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}\left(\int_{t_i}^{t_{i+1}} \lambda_s ds\right)^j}{j!} \\
& \times \int_{-\infty}^{\infty} \phi\left(x-y; \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds, \int_0^t \sigma_s^2 ds\right) f_Q^{(j)}(y) dy,
\end{aligned}$$

where $f_Q^{(j)}(y) = \underbrace{f_Q(y) * f_Q(y) * ...f_Q(y)}_{j \text{ terms}}.$

Thus,

$$\begin{aligned}
&\mathbb{P}^{PLL}_{t_i,t_{i+1}} = \mathbb{P}(V_{t_{i+1}} \leq F_{t_{i+1}} | V_{t_i} > F_{t_i}) \\
= \ & \mathbb{P}\Bigg(\int_{t_i}^{t_{i+1}} \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_{t_i}^{t_{i+1}} \sigma_s dW_s + \sum_{n=N_{t_i}}^{N_{t_{i+1}}} \ln(1+Y_n) \leq \ln\left(\frac{m-1}{m}\right) + \frac{rT}{N}\Bigg)
\end{aligned}$$

$$= \int_{-\infty}^{\ln\left(\frac{m-1}{m}\right)+\frac{rT}{N}} p^{(i)}(x)dx$$

□

By (5.6), the probability of loss $\mathbb{P}^{PL}$ can be obtained.

5.3.2 Expected Loss

Suppose that the first loss takes place at τ. I.e. $C_\tau \leq 0$. We let $\tau = \infty$ if a loss never happens. i.e.

$$\tau = t_i \text{ if } V_{t_i} \leq F_{t_i} \text{ and } V_{t_j} > F_{t_j} \text{ for } j = 0\,,1\,,2\,,...,i-1;$$
$$\tau = +\infty \text{ if } V_{t_j} > F_{t_j} \text{ for } j = 0\,,1\,,2\,,...,N.$$

Since $V_0 > F_0$, then

$$\tau = +\infty \text{ if } V_{t_j} > F_{t_j} \text{ for } j = 0\,,1\,,2\,,...,N$$

which is equivalent to

$$\tau = +\infty \text{ if } V_{t_j} > F_{t_j} \text{ for } j = 1\,,2\,,...,N.$$

By the definition, τ is a stopping time.

We consider the following situation. If a loss happens at time τ, the cushion $C_\tau \leq 0$. If we do not allow the short-sell, then at this trading time τ, we take the exposure to be 0. Let

$$\varepsilon_{t_i} = C_0 \prod_{k=0}^{i-1} \left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N} \right), \tag{5.9}$$

where

$$\frac{S_{t_{k+1}}}{S_{t_k}} = \exp\left[\int_{t_k}^{t_{k+1}} \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds + \int_{t_k}^{t_{k+1}} \sigma_s dW_s + \sum_{n_k=N_{t_k}}^{N_{t_{k+1}}} \ln(1+Y_{n_k}) \right].$$

Then the discounted cushion is

$$C_T^* = \exp(-rT)\varepsilon_{t_N}\chi_{\tau>T} + \exp(-rT)\sum_{j=0}^{N-1}\varepsilon_{t_j}\chi_{\tau=t_j}. \tag{5.10}$$

Definition 5.8. If a loss happens, the conditional expectation of the discounted cushion is called the **conditional expected loss** and we denote this by $\mathbb{E}[C_T^*|\tau \leq T]$. The expectation of the discounted cushion is called the **unconditional expected loss** and we use $\mathbb{E}[C_T^*\chi_{\tau\leq T}]$ to represent it.

The following proposition gives the distribution of the break time τ.

Proposition 5.9. *The distribution of τ defined above is*

$$\mathbb{P}(\tau = t_i) = \mathbb{P}^{PLL}_{t_i-1,\,t_i} \times \prod_{j=1}^{i-1}\left(1 - \mathbb{P}^{PLL}_{t_{j-1},\,t_j}\right) \tag{5.11}$$

Remarks. In the above, if $j-1<0$, let $\mathbb{P}^{PLL}_{t_{j-1},\,t_j} = 0$. In this case, $\mathbb{P}(\tau = t_0) = 0$ as expected.

Proof.

$$\begin{aligned}
&\mathbb{P}(\tau = t_i) = \mathbb{P}(V_{t_i} \leq F_{t_i},\ \text{and}\ V_{t_j} > F_{t_j}\ \text{for}\ j = 1\,,2\,,...,i-1) \\
=\ & \mathbb{P}\left(\{V_{t_i} \leq F_{t_i}|V_{t_{i-1}} > F_{t_{i-1}}\}\bigcap\bigcap_{j=1}^{i-1}\{V_{t_j} > F_{t_j}|V_{t_{j-1}} > F_{t_{j-1}}\}\right) \\
=\ & \mathbb{P}\left(\{V_{t_i} \leq F_{t_i}|V_{t_{i-1}} > F_{t_{i-1}}\}) \times \prod_{j=1}^{i-1}\mathbb{P}(\{V_{t_j} > F_{t_j}|V_{t_{j-1}} > F_{t_{j-1}}\}\right) \\
=\ & \mathbb{P}\left(\{V_{t_i} \leq F_{t_i}|V_{t_{i-1}} > F_{t_{i-1}}\}) \times \prod_{j=1}^{i-1}(1 - \mathbb{P}(\{V_{t_j} \leq F_{t_j}|V_{t_{j-1}} > F_{t_{j-1}}\})\right) \\
=\ & \mathbb{P}^{PLL}_{t_i-1,\,t_i} \times \prod_{j=1}^{i-1}\left(1 - \mathbb{P}^{PLL}_{t_{j-1},\,t_j}\right).
\end{aligned}$$

□

Lemma 5.10.

$$\mathbb{E}[\varepsilon_{t_i}] = C_0 \prod_{k=0}^{i-1} \left(m \left[\exp\left(\mu \frac{T}{N}\right) \mathbb{E} \prod_{n_k=N_{t_k}}^{N_{t_{k+1}}} (1+Y_{n_k}) \right] - (m-1)e^{rT/N} \right) + G. \quad (5.12)$$

where $\mathbb{E}\prod_{n_k=N_{t_k}}^{N_{t_{k+1}}}(1+Y_{n_k})$ *is given by* (5.2).

Proof. This is an corollary of Proposition 4.2 when substitute i to N. □

Proposition 5.11. *The expectation of loss conditional on the fact that a loss occur is*

$$\mathbb{E}[C_T^*|\tau \le T] = \frac{\exp(-rT)\sum_{j=0}^{N-1}\mathbb{E}[\varepsilon_{t_j}]\mathbb{P}[\tau = t_j]}{\sum_{i=1}^{N}\mathbb{P}[\tau = t_i]} \quad (5.13)$$

and the unconditional expected loss satisfies

$$\mathbb{E}[C_T^*\chi_{\tau\le T}] = \exp(-rT)\sum_{j=0}^{N-1}\mathbb{E}[\varepsilon_{t_j}]\mathbb{P}[\tau = t_j], \quad (5.14)$$

where $\mathbb{E}[\varepsilon_{t_j}]$ *is given by* (5.12) *and* $\mathbb{P}[\tau = t_j]$ *is given by* (5.11).

Proof.

$$\begin{aligned} \mathbb{E}[C_T^*\chi_{\tau\le T}] &= \mathbb{E}\left[\exp(-rT)\sum_{j=0}^{N-1}\varepsilon_{t_j}\chi_{\tau=t_j}\right] \\ &= \exp(-rT)\sum_{j=0}^{N-1}\mathbb{E}[\varepsilon_{t_j}\chi_{\tau=t_j}] = \exp(-rT)\sum_{j=0}^{N-1}\mathbb{E}[\varepsilon_{t_j}]\mathbb{P}[\tau = t_j]. \end{aligned}$$

$\mathbb{E}[\varepsilon_{t_j}]$ is given by (5.12) and $\mathbb{P}[\tau = t_j]$ is given by (5.11). Thus we prove (5.14). Moreover, by the property of conditional expectation, we have

$$\mathbb{E}[C_T^*|\tau \le T] = \frac{\mathbb{E}[C_T^*\chi_{\tau\le T}]}{\mathbb{P}[\tau \le T]} = \frac{\exp(-rT)\sum_{j=0}^{N-1}\mathbb{E}[\varepsilon_{t_j}]\mathbb{P}[\tau = t_j]}{\sum_{i=1}^{N}\mathbb{P}[\tau = t_i]}.$$

This is (5.13). □

5.3.3 Loss Distribution

In order to compute risk measures, we utilize the distribution function of the loss. We compute, for $x < 0$, the quantity

$$\mathbb{P}[C_T^* < x | \tau \le T]. \tag{5.15}$$

We call it the **Loss Distribution**. For $x < 0$, the quantity

$$\mathbb{P}[C_T^* \chi_{\tau \le T} < x] \tag{5.16}$$

is called **unconditional loss distribution**.

Proposition 5.12. *Let the density of* $\ln(1 + Y_n)$ *be* $f_Q(y)$ *and* C_T^* *be the discounted cushion. For* $x < 0$*, the unconditional loss distribution is*

$$\begin{aligned} \mathbb{P}[C_T^* \chi_{\tau \le T} < x] = \sum_{j=0}^{N-1} \Bigg[\int_{y_{i-1}}^{+\infty} \int_{y_{i-2}}^{+\infty} \cdots \int_{y_0}^{+\infty} \sum_{k=0}^{i-2} \ln\left(m x_k - (m-1) e^{rT/N}\right) \\ + \ln\left(-\left(m x_{i-1} - (m-1)e^{rT/N}\right)\right) p^{(0)}(x) dx_0 p^{(1)}(x_1) dx_1 \\ \ldots p^{(i-2)}(x_{i-2}) dx_{i-2} p^{(i-1)}(x_{i-1}) dx_{i-1} \mathbb{P}[\tau = t_j] \Bigg] \end{aligned} \tag{5.17}$$

and the loss distribution is

$$\mathbb{P}[C_T^* < x | \tau \le T] = \frac{\mathbb{P}[C_T^* \chi_{\tau \le T} < x]}{\sum_{i=1}^{N} \mathbb{P}[\tau = t_i]}, \tag{5.18}$$

where $p^{(i)}(x)$ *is given by* (5.8) *and* $\mathbb{P}[\tau = t_j]$ *is given by* (5.11) *and*

$$(y_0,\, y_1,\, y_2,\, \ldots y_{i-1}) \;\in\; \Big\{ (y_0,\, y_1,\, y_2,\, \ldots y_{i-1}) \in \mathbb{R}^i$$
$$y_k > \frac{m-1}{m} e^{rT/N} \;\; \textit{for } k = 0,\, 1,\, 2,\, \ldots i-2$$

$$\sum_{k=0}^{i-2} \ln(my_k - (m-1)e^{rT/N})$$
$$+ \ln\left(-\left(my_{i-1} - (m-1)e^{rT/N}\right)\right) > \ln \frac{-x}{C_0} + rT \Big\}.$$

Proof. We have

$$\mathbb{P}[C_T^* \chi_{\tau \leq T} < x] = \mathbb{P}\left[\exp(-rT) \sum_{j=0}^{N-1} \varepsilon_{t_j} < x\right]$$
$$= \sum_{j=0}^{N-1} \mathbb{P}[\exp(-rT)\varepsilon_{t_j} < x|\tau = t_j]\mathbb{P}[\tau = t_j]].$$

We now calculate $\mathbb{P}[\exp(-rT)\varepsilon_{t_j} < x|\tau = t_j]$.

$$\mathbb{P}[\exp(-rT)\varepsilon_{t_j} < x|\tau = t_j]$$
$$= \mathbb{P}\left[\exp(-rT) \prod_{k=0}^{i-1} C_0 \left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right) < x|\tau = t_j\right]$$
$$= \mathbb{P}\Bigg[\exp(-rT) \left(\prod_{k=0}^{i-2} C_0 \left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right)\right)$$
$$\left(m\frac{S_{t_{i+1}}}{S_{t_i}} - (m-1)e^{rT/N}\right) < x|\tau = t_j\Bigg]$$
$$= \mathbb{P}\Bigg[\exp(-rT) \left(\prod_{k=0}^{i-2} C_0 \left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right)\right)$$
$$\left(-\left(m\frac{S_{t_{i+1}}}{S_{t_i}} - (m-1)e^{rT/N}\right)\right) > -x|\tau = t_j\Bigg]$$
$$= \mathbb{P}\Bigg[\sum_{k=0}^{i-2} \ln\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right)$$
$$+ \ln\left(-\left(m\frac{S_{t_{i+1}}}{S_{t_i}} - (m-1)e^{rT/N}\right)\right) > \ln\frac{-x}{C_0} + rT|\tau = \tau_j\Bigg]$$
$$= \int_{y_{i-1}}^{+\infty} \int_{y_{i-2}}^{+\infty} ... \int_{y_0}^{+\infty} \sum_{k=0}^{i-2} \ln(mx_k - (m-1)e^{rT/N})$$
$$+ \ln\left(-\left(mx_{i-1} - (m-1)e^{rT/N}\right)\right)$$
$$p^{(0)}(x_0)dx_0 p^{(1)}(x_1)dx_1 ... p^{(i-2)}(x_{i-2})dx_{i-2} p^{(i-1)}(x_{i-1})dx_{i-1},$$

where

$$
\begin{aligned}
(y_0,\, y_1,\, y_2,\, \dots y_{i-1}) \quad \in \quad & \Big\{ (y_0,\, y_1,\, y_2,\, \dots y_{i-1}) \in \mathbb{R}^i \\
& y_k > \frac{m-1}{m} e^{rT/N} \;\text{ for }\; k = 0,\, 1,\, 2,\, \dots i-2 \\
& \sum_{k=0}^{i-2} \ln\left(m y_k - (m-1) e^{rT/N}\right) \\
& + \ln\left(-\left(m y_{i-1} - (m-1) e^{rT/N}\right)\right) > \ln \frac{-x}{C_0} + rT \Big\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \mathbb{P}[C_T^* \chi_{\tau \le T} < x] = \mathbb{P}\left[\exp(-rT) \sum_{j=0}^{N-1} \varepsilon_{t_j} < x\right] \\
= \; & \sum_{j=0}^{N-1} \Bigg[\int_{y_{i-1}}^{+\infty} \int_{y_{i-2}}^{+\infty} \dots \int_{y_0}^{+\infty} \sum_{k=0}^{i-2} \ln(m x_k - (m-1) e^{rT/N}) \\
& + \ln(-(m x_{i-1} - (m-1) e^{rT/N})) p^{(0)}(x_0) dx_0 p^{(1)}(x_1) dx_1 \\
& \dots p^{(i-2)}(x_{i-2}) dx_{i-2} p^{(i-1)}(x_{i-1}) dx_{i-1} \mathbb{P}[\tau = t_j] \Bigg].
\end{aligned}
$$

Thus, we obtain (5.17). Through the property of conditional probability, we obtain (5.18). □

5.3.4 Conclusion

The definition of probability of loss, expected loss and loss distribution in the jump-diffusion model with discrete trading time is corresponding to the continuous trading time case.

5.4 Conditional Floor and Conditional Multiple of CPPI under Jump-diffusion Model in Discrete Trading Time

5.4.1 Introduction

In this section we study the conditional floor and conditional multiple from three aspects: the probability of loss, expected loss and loss distribution.

5.4.2 Probability of Loss

Similar to proposition 1 in [2], we have the following proposition.

Proposition 5.13. *The condition $C_{t_k} > 0$ is satisfied at any time t_k of the management period with probability 1 if and only if:*

$$1 - e^{-rT/N} \min_{k=0,\,1,\,\dots\,N-1} \frac{S_{t_{k+1}}}{S_{t_k}} < \frac{1}{m}. \tag{5.19}$$

Proof. C_{t_i} has the relation in (5.1).

$$C_{t_{k+1}} = C_{t_k} \left(m \frac{S_{t_{k+1}}}{S_{t_k}} - (m-1) e^{rT/N} \right).$$

The condition $C_{t_k} > 0$ is true for any time t_k, if and only if

$$m \frac{S_{t_{k+1}}}{S_{t_k}} - (m-1) e^{rT/N} > 0$$

for all $k = 0,\, 1,\, \dots\, N-1$. This is equivalent to

$$1 - e^{-rT/N} \frac{S_{t_{k+1}}}{S_{t_k}} < \frac{1}{m}$$

for all $k = 0, 1, \ldots N-1$ or equivalent to

$$1 - e^{-rT/N} \min_{k=0,1,\ldots N-1} \frac{S_{t_{k+1}}}{S_{t_k}} < \frac{1}{m}.$$

□

Proposition 5.14. *The probability of loss defined by* (5.5) *and probability of local loss defined by* (5.4) *are monotone increasing functions of the multiple m. Moreover both of them are irrelevant with the floor F_t.*

Proof. We have proved the probability of local loss defined by (5.4) is given by

$$\mathbb{P}^{PLL}_{t_i, t_{i+1}} = \int_{-\infty}^{\ln(\frac{m-1}{m}) + \frac{rT}{N}} p^{(i)}(x)dx.$$

$$m \text{ increased} \Longrightarrow \ln(\tfrac{m-1}{m}) \text{ increased}$$
$$\Longrightarrow \int_{-\infty}^{\ln(\frac{m-1}{m})+\frac{rT}{N}} p^{(i)}(x)dx \text{ increased for each } i = 0, 1, \ldots N-1$$
$$\Longrightarrow \mathbb{P}^{PLL}_{t_i, t_{i+1}} \text{ increased for each } i = 0, 1, \ldots N-1$$
$$\text{and by (5.6) implies } \mathbb{P}^{PL} \text{ increased.}$$

From the expressions in (5.4), (5.5) and (5.6), we see both of them are irrelevant with the floor F_t. □

Similar to the Value-at-Risk (VaR) concept (See [27]), and as in the continuous trading time case, we define:

Definition 5.15. For $\epsilon > 0$, the multiple $m = m_0$ which satisfies

$$\mathbb{P}[\exists t_i \in \tau^N : V_{t_i} \le F_{t_i}] = \epsilon$$

is called the **ϵ-conditional multiple**.

m_0 can be treated as a quantile. Since the probability of loss is monotone increasing as a function of the multiple m, then for $m < m_0$,

$$\mathbb{P}\left[\exists t_i \in \tau^N : V_{t_i} \leq F_{t_i}\right] < \epsilon.$$

For

$$\mathbb{P}^{PL} = 1 - \prod_{i=1}^{N} \left(1 - \mathbb{P}^{PLL}_{t_{i-1},t_i}\right) = \epsilon,$$

if we assume all the probability of local losses are the same, we obtain

$$\mathbb{P}^{PLL}_{t_{i-1},t_i} = 1 - (1-\epsilon)^{\frac{1}{N}}.$$

From (5.7), we obtain the expression for m_0.

5.4.3 Expected Loss

First we have, from (5.13) and (5.14), the following proposition:

Proposition 5.16. *Given a fixed multiple, both the conditional expected loss given by* (5.13) *and the unconditional expected loss given by* (5.14) *are monotone decreasing functions of the initial floor* F_0.

Proof. They are direct consequences of (5.13) and (5.14). □

Similar to the ϵ-conditional multiple, we define following two concepts.

Definition 5.17. For $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c1}$ which satisfies

$$\mathbb{E}[C_T^*|\tau \leq T] = \varrho$$

is called the **first type ϱ-m_0-conditional floor**.

and

Definition 5.18. For $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c2}$ which satisfies

$$\mathbb{E}[C_T^* \chi_{\tau \leq T}] = \varrho$$

is called the **Second type ϱ-m_0-conditional floor**.

Similar to (5.13) and (5.14), we can solve for the two conditional floors immediately.

5.4.4 Loss Distribution

Proposition 5.19. *Given a fixed multiple and $x < 0$, the expressions* (5.17) *and* (5.18) *are monotone increasing function of the initial floor F_0*

Proof. Since $C_0 = V_0 - F_0$.

$$F_0 \text{ increased} \Longrightarrow$$
$$C_0 \text{ decreased} \Longrightarrow$$
$$\ln(-\tfrac{x}{C_0}) \text{ increased} \Longrightarrow$$

That is, both the expressions (5.17) and (5.18) are increasing.

□

Next we define the following two concepts.

Definition 5.20. For $\epsilon > 0$, $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c3}$ which satisfies

$$\mathbb{E}[C_T^* \chi_{\tau \leq T} < \varrho] = \epsilon$$

is called the **third type ϵ-ϱ-m_0-conditional floor**.

and

Definition 5.21. For $\epsilon > 0$, $\varrho < 0$ and $m = m_0$, the floor $F_0 = F^{c4}$ which satisfies

$$\mathbb{E}[C_T^* < \varrho | \tau \leq T] = \epsilon$$

is called the **fourth type** ϵ**-**ϱ**-**m_0**-conditional floor**.

The above two conditional floors are useful in numerical computations.

5.5 Convergence

In this section, we consider the relation between the case when the trading time is continuous and the case when the trading time is disccrete.

Recall (5.1).

$$C_{t_{k+1}} = C_{t_k}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{rT/N}\right),$$

When $N \to \infty$, $\Delta t = T/N \to 0$

$$\exp\left(r\frac{T}{N}\right) \sim 1 + r\frac{T}{N}.$$

Thus, we got

$$\frac{C_{t_{k+1}} - C_{t_k}}{C_{t_k}} + 1 = \left(m\left[\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} + 1\right] - (m-1)e^{rT/N}\right).$$

Let $N \to \infty$, we have

$$\frac{dC_t}{C_{t-}} + 1 = \left(m\left[\frac{dS_t}{S_{t-}} + 1\right] - (m-1)(1 + rdt)\right),$$

and this is equivalent to

$$\frac{dC_t}{C_{t-}} = \left(m\frac{dS_t}{S_{t-}} - (m-1)(rdt)\right).$$

This is consistant with the continuous case (3.14). We have the following proposition:

Proposition 5.22. *For $N \to \infty$, the portfolio value in discrete trading time converges a.s. to the portfolio value in continuous trading time.*

Chapter 6

Stochastic and dynamic floors

6.1 Introduction

In section 6.2, we will consider the case of stochastic floor which is equal to the maximum of its past value and a given percentage of the portfolio value. The idea is that when the portfolio value is large enough, we will increase the level of the floor. Both the continuous and discrete trading time cases will be analyzed. We will also calculate the distribution of the time.

In section 6.3, we will consider the case of stochastic floor which is indexed by the given portfolio performance. The idea is similar to that as in section 6.2. We will also calculate the distribution of the first-time-change of the floor.

In section 6.4, we will deal with Ratchet and Margin CPPI strategies with the time-change of strategy defined on the exposition variance. We will show that in the discrete trading time case, the Ratchet CPPI is equivalent to the stochastic floor which is indexed by the given portfolio performance. In the cases of CPPI with margin when the floor is close to the portfolio value, the exposure will be very small and we will reduce the floor. We will discuss the distribution of the first-change-time of the floor when the trading time is continuous.

6.2 When the floor equals to the maximum of its past value and a given percentage of the portfolio value

In this section, the current floor value is the maximum of the past floor value and a given percentage of the current portfolio value.

6.2.1 Discrete-time case with fixed multiple

Let

$$\tau^n = \{t_0 = 0 < t_1 < t_2 < ... < t_n = T\}$$

denote a sequence of equidistant refinements of the interval $[0, T]$, where $t_{k+1} - t_k = \frac{T}{n} =: \Delta$ for $k = 0, ..., n-1$.
Let

$$F_{t_k} = \max\{F_{t_{k-1}} \exp(r\Delta), \quad xV_{t_k}\}$$

and the initial floor $F_0 = Ge^{-rT}$ be the same as before and suppose x is an arbitrary but fixed percentage of the portfolio value. This definition means that the floor is equal to the maximum of its past value and a given percentage of the portfolio value. As the portfolio value increases and if we keep the floor unchanged, the cushion will be very big. Our idea is that as the portfolio value increase to a specific level, we will also increase the level of the floor. In general, we assume $xV_0 \leq F_0$.
Let $T_1 = \min\{t > 0 : F_t = xV_t\}$. Denote respectively by θ_0^B and θ_0^S the shares invested

on the riskless and risky assets. We have:

$$\begin{aligned}\theta_0^B &= (V_0 - \theta_0^S S_0)/B_0,\\ \theta_0^S &= m(V_0 - F_0)/S_0.\end{aligned}$$

The following proposition calculates the probability of the first-time-change of the floor taking place at t_1.

Proposition 6.1. *For the jump-diffusion model, if we assume $x_{i+1} = \ln(\frac{S_{i+1}}{S_i})$, $i = 0, 1, 2, ...$ be i.i.d. and their density function be $p(x)$. Then the probability of the first-time-change of the floor which takes happen at t_1 is*

$$\mathbb{P}[T_1 = t_1] = \int_{\ln(e^{r\Delta}\frac{F_0/x - \theta_0^B B_0}{m(V_0 - F_0)})}^{\infty} p(x)dx.$$

Proof.

$$\begin{aligned}
& \mathbb{P}[T_1 = t_1] = \mathbb{P}[F_{t_1} \le xV_{t_1}] = \mathbb{P}[F_0 e^{r\Delta} \le x(F_{t_1} + e_{t_1})] \\
= \;& \mathbb{P}\Bigg[F_0 e^{r\Delta} \le x\Bigg(\theta_0^B B_0 e^{r\Delta} + m(V_0 - F_0) \\
& \times \exp\left[\int_0^{t_1}\left(\mu_s - \frac{\sigma_s^2}{2}\right)ds + \int_0^{t_1}\sigma_s dW_s + \sum_{n=1}^{N_{t_1}}\ln(1+Y_n)\right]\Bigg)\Bigg] \\
= \;& \mathbb{P}\left[\exp\left[\int_0^{t_1}\left(\mu_s - \frac{\sigma_s^2}{2}\right)ds + \int_0^{t_1}\sigma_s dW_s + \sum_{n=1}^{N_{t_1}}\ln(1+Y_n)\right] \ge e^{r\Delta}\frac{F_0/x - \theta_0^B B_0}{m(V_0 - F_0)}\right] \\
= \;& \int_{\ln\left(e^{r\Delta}\frac{F_0/x - \theta_0^B B_0}{m(V_0 - F_0)}\right)}^{\infty} p(x)dx.
\end{aligned}$$

□

Remarks. In the simple CPPI case, $Y_n = 0$, μ_s and $\sigma_s = \sigma$, then

$$\mathbb{P}[F_{t_1} = xV_{t_1}] = \mathbb{P}\left[\exp\left(\mu - \frac{1}{2}\sigma^2\right)t_1 + \sigma W_{t_1} \ge e^{r\Delta}\frac{F_0/x - \theta_0^B B_0}{m(V_0 - F_0)}\right]$$

$$= 1 - N\left(\frac{1}{\sigma\sqrt{\Delta}}\left(\ln\left[e^{r\Delta}\frac{F_0/x - \theta_0^B B_0}{m(V_0 - F_0)}\right] - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta\right)\right).$$

This is the first part of the proposition 1 on [59].

In the following, we consider the probability that $T_1 = t_N$.

Proposition 6.2. *For the jump-diffusion model, if assume the density function of x_i is $p(x)$, the the probability of the first-time-change of floor at t_N is*

$$\mathbb{P}[T_1 = t_N] = \int ... \int_{D_N} p(u_i)...p(u_N)du_1du_2...du_N,$$

where

$$\begin{aligned}
&(u_1, ..., u_N) \in D_n \textit{ iff} \\
&\forall i \leq N-1,\ F_0e^{ri\Delta} > x\left[F_0e^{ri\Delta} + C_0\prod_{j=1}^{i} g(u_j)\right]; \\
&\textit{for } i = N,\ F_0e^{rN\Delta} \leq x\left[F_0e^{rN\Delta} + C_0\prod_{j=1}^{i} g(u_N)\right].
\end{aligned}$$

Proof. We have

$$\mathbb{P}[T_1 = t_N] = P[F_{t_1} > xV_{t_1}, .., F_{t_{N-1}} > xV_{t_{N-1}}, F_{t_N} \leq xV_{t_N}]$$

and

$$\begin{aligned}
V_{t_i} &= \theta^B_{t_{i-1}} B_{t_i} + \theta^S_{t_{i-1}} S_{t_i} = F_{t_i} + C_{t_i} \\
&= F_0e^{rt_i} + C_0\prod_{t=1}^{t_i}\left[1 + (1-m)\frac{B_t - B_{t-1}}{B_{t-1}} + m\frac{S_t - S_{t-1}}{S_{t-1}}\right].
\end{aligned}$$

Let $g(x) = 1 + (1-m)(e^{r\Delta} - 1) + m(e^x - 1)$, then

$$V_{t_i} = F_0e^{rt_i} + C_0\prod_{t=1}^{t_i} g(x_t).$$

Thus,

$$\mathbb{P}[T_1 = t_N] = \mathbb{P}[F_{t_1} > xV_{t_1}, .., F_{t_{N-1}} > xV_{t_{N-1}}, F_{t_N} \leq xV_{t_N}]$$

$$= \mathbb{P}\Bigg[F_0 e^{r\Delta} > x[F_0 e^{r\Delta} + C_0 g(x_1)], ..., F_0 e^{r(N-1)\Delta} >$$

$$x\left[F_0 e^{r(N-1)\Delta} + C_0 \prod_{t=1}^{t_{N-1}} g(x_t)\right], F_0 e^{rN\Delta} \leq x\left[F_0 e^{rN\Delta} + C_0 \prod_{t=1}^{t_N} g(x_t)\right]\Bigg].$$

Let

$$\begin{aligned}
&(u_1, ..., u_N) \in D_n \text{ iff} \\
&\forall i \leq N-1,\ F_0 e^{ri\Delta} > x\left[F_0 e^{ri\Delta} + C_0 \prod_{j=1}^{i} g(u_j)\right]; \\
&\text{for } i = N,\ F_0 e^{rN\Delta} \leq x\left[F_0 e^{rN\Delta} + C_0 \prod_{j=1}^{i} g(u_N)\right].
\end{aligned}$$

For the jump-diffusion model, when the density function of x_i is $p(x)$, we have

$$\mathbb{P}[T_1 = t_N] = \int ... \int_{D_N} p(u_i)...p(u_N)du_1 du_2 ... du_N.$$

□

Remarks. The second part of proposition 1 on [59] is a special case. Also, when the density function $p(x)$ of x_i is given, the associated probability can be calculated explicitly.

Next we have the following proposition (see also [59]):

Proposition 6.3. *For any t_i, the stochastic floor F is equal to the stochastic floor Q defined by:*

$$Q_{t_i} = \max\left[\tilde{F}_{t_i}, \sup_{j \leq i} e^{r(t_i - t_j)} V_{t_j}\right].$$

Proof. (1) Firstly, the stochastic floor F is above the deterministic floor $\tilde{F}$

$$F_{t_i} \geq \tilde{F}_{t_i} = P_0 e^{rt_i},$$

and secondly, we have:

$$F_{t_i} \geq x \sup_{j \leq i} e^{r(t_i - t_j)} V_{t_j}.$$

Indeed, by recursion we have:

$$\begin{aligned} F_{t_i} &\geq e^{r\delta} F_{t_{i-1}} \; and \; F_{t_i} \geq x V_{t_i}, \\ F_{t_{i-1}} &\geq e^{r\delta} F_{t_{i-2}} \; and \; F_{t_{i-1}} \geq x V_{t_{i-1}}. \end{aligned}$$

Thus,

$$F_{t_i} \geq \max(e^{r\delta} V_{t_{i-1}}; V_{t_i}),$$

which, by iteration, leads to the inequality $F_{t_i} \geq Q_{t_i}$.
(2) Conversely, if $F_{t_i} = x V_{t_i}$, then

$$V_{t_i} = \sup_{j \leq i} e^{r(t_i - t_j)} V_{t_j}.$$

Therefore, since we have $Q_{t_i} \geq e^{r(t_i - t_j)} V_{t_j}$ for all $j \leq i$, we deduce that $F_{t_i} \leq Q_{t_i}$. □

The proposition shows that the previous CPPI strategy with floor F is the discrete-time version of Time Invariant Portfolio Protection strategy(TIPP).

6.2.2 Continuous-time case with a fixed multiple

As in the previous section, when the current floor value is the maximum of the past floor value and a given percentage of the current portfolio value, the strategy is equivalent to the TIPP strategy. Standard convergence results lead to the following model, in continuous-time:

$$\begin{aligned} F_t &= \max\left[\tilde{F}_t, x \sup_{s\le t} e^{r(t-s)} V_s\right], \\ e_t &= mC_t = m(V_t - F_t). \end{aligned}$$

Define

$$T_1^c = \inf\left[t \le T:\, F_t = x\sup_{s\le t} e^{r(t-s)} V_s\right].$$

This is **the first-time-change** of floor. We will consider the probability distribution of T_1^c.

Before T_1^c, we have

$$\begin{aligned} V_t &= C_0 \exp\left\{\int_0^t \left(r + m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds \right. \\ &\quad \left. + \int_0^t m\sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1+mY_n)\right\} + F_0 e^{rt}. \end{aligned}$$

Denote

$$X_t = \int_0^t \left(m(\mu_s - r) - \frac{m^2\sigma_s^2}{2}\right) ds + \int_0^t m\sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1+mY_n).$$

Thus, we have

$$\begin{aligned} T_1^c &= \inf\left[t \le T:\, F_t = x\sup_{s\le t} e^{r(t-s)} V_s\right] \\ &= \inf\left[t \le T:\, x\sup_{s\le t} e^{r(t-s)} V_s = x\left(C_0 e^{rt} \exp\left\{\sup_{s\le t} X_s\right\} + e^{rt} F_0\right) = F_0 e^{rt}\right] \end{aligned}$$

$$= \inf\left[t \le T : \sup_{s\le t} X_s \ge \ln\left[\frac{F_0}{C_0}\left(\frac{1}{x}-1\right)\right]\right].$$

When $\mu_s = \mu$ and $\sigma_s = \sigma$ is constant and then

$$X_t = \left(m(\mu - r) - \frac{1}{2}m^2\sigma^2\right)t + m\sigma W_t + \sum_{n=1}^{N_t}\ln(1+mY_n).$$

Let

$$A = \frac{(\mu - r) - \frac{1}{2}m\sigma^2}{\sigma}$$

and

$$W_s^{(A)} = \left(As + W_s + \frac{1}{m\sigma}\sum_{n=1}^{N_t}\ln(1+mY_n)\right).$$

Then, we can calculate the distribution of $\sup_{s\le t} W_s^{(A)}$ is

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\le t} W_s^{(A)} \le y\right)\\
&= \mathbb{P}\left[\bigcup_{k=1}^{\infty}\left(\sup_{s\le t}(As + W_s) + \frac{1}{m\sigma}\sum_{n=1}^{N_t}\ln(1+mY_n) \le y, N_t = k\right)\right]\\
&= \sum_{k=1}^{\infty}\mathbb{P}\left(\sup_{s\le t}(As + W_s) + \frac{1}{m\sigma}\sum_{n=1}^{N_t}\ln(1+mY_n) \le y | N_t = k\right)\mathbb{P}(N_t = k)\\
&= \sum_{k=1}^{\infty}\frac{\mathbb{P}\left(\sup_{s\le t}(As + W_s) + \frac{1}{m\sigma}\sum_{n=1}^{N_t}\ln(1+mY_n) \le y, N_t = k\right)}{\mathbb{P}(N_t = k)}\mathbb{P}(N_t = k)\\
&= \sum_{k=1}^{\infty}\frac{\mathbb{P}\left(\sup_{s\le t}(As + W_s) + \frac{1}{m\sigma}\sum_{n=1}^{N_t}\ln(1+mY_n) \le y\right)\mathbb{P}(N_t = k)}{\mathbb{P}(N_t = k)}\mathbb{P}(N_t = k)\\
&= \sum_{k=1}^{\infty}\mathbb{P}\left(\sup_{s\le t}(As + W_s) + \frac{1}{m\sigma}\sum_{n=1}^{N_t}\ln(1+mY_n) \le y\right)\mathbb{P}(N_t = k)\\
&= \sum_{k=1}^{\infty}\mathbb{P}\left(\sup_{s\le t}(As + W_s) + \frac{1}{m\sigma}\sum_{n=1}^{k}\ln(1+mY_n) \le y\right)\frac{e^{-\int_0^t \lambda_s ds}\left(\int_0^t \lambda_s ds\right)^k}{k!}.
\end{aligned}$$

Recall that a property possessed by the maximum value of the Brownian motion with drift gives:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\leq t}(As+W_s)+\frac{1}{m\sigma}\sum_{n=1}^{k}\ln(1+mY_n)\leq y\right)\\
=&\int_{-\infty}^{\infty}\left(1-\frac{1}{2}\text{Erfc}\left(\frac{y-y_2}{\sqrt{2t}}-A\frac{\sqrt{t}}{\sqrt{2}}\right)\right.\\
&\left.-\frac{1}{2}e^{2A(y-y_2)}\text{Erfc}\left(\frac{y-y_2}{\sqrt{2t}}+A\frac{\sqrt{t}}{\sqrt{2}}\right)\right)dF_k(y_2),
\end{aligned}$$

where the function *Erfc* is given by:

$$\text{Erfc}(x)=\frac{2}{\sqrt{\pi}}\int_{x}^{\infty}e^{-u^2}du$$

and the $F_k(y_2)$ is the distribution function of $\frac{1}{m\sigma}\sum_{n=1}^{k}\ln(1+mY_n)$.

Then we have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\leq t}W_s^{(A)}\leq y\right)\\
=&\sum_{k=1}^{\infty}\frac{e^{-\int_0^t\lambda_s ds}\left(\int_0^t\lambda_s ds\right)^k}{k!}\int_{-\infty}^{\infty}(1-\frac{1}{2}\text{Erfc}\left(\frac{y-y_2}{\sqrt{2t}}-A\frac{\sqrt{t}}{\sqrt{2}}\right)\\
&-\frac{1}{2}e^{2A(y-y_2)}\text{Erfc}\left(\frac{y-y_2}{\sqrt{2t}}+A\frac{\sqrt{t}}{\sqrt{2}})\right)dF_k(y_2).
\end{aligned}$$

Therefore, we have deduced the following proposition:

Proposition 6.4. *When assume $\mu_s=\mu$ and $\sigma_s=\sigma$ be constant, the cdf of the first time T_1^c before maturity T at which $F_t=\sup_{s\leq t}e^{r(t-s)}V_s$ is given by:*

$$\mathbb{P}[T_1^c\leq t]=\sum_{k=1}^{\infty}\frac{e^{-\int_0^t\lambda_s ds}\left(\int_0^t\lambda_s ds\right)^k}{k!}\int_{-\infty}^{\infty}\left(\frac{1}{2}Erfc\left(\frac{y-y_2}{\sqrt{2t}}-A\frac{\sqrt{t}}{\sqrt{2}}\right)\right.$$

$$+\frac{1}{2}e^{2A(y-y_2)}Erfc\left(\frac{y-y_2}{\sqrt{2t}}+A\frac{\sqrt{t}}{\sqrt{2}}\right)\Big)dF_k(y_2).$$

Proposition 5 of [59] is a special case for the diffusion model without jump.

6.2.3 Capped CPPI

Assume that the portfolio manager does not want selling short on the money market account (condition $\theta_t^B \geq 0$).

Therefore the exposure e is bounded by a fixed proportion ϖ of the portfolio value V. We call it the **Capped CPPI**. This leads to the following conditions on the CPPI strategy in continuous time:

(a)

$$F_t = \max\left[\tilde{F}_t, x\sup_{s\leq t} e^{r(t-s)}V_s\right]$$

The floor equals to the maximum of its past value and a given percentage of the portfolio value.

(b)

$$e_t = \inf(\varpi V_t, mC_t)$$

We always assume that

$$C_t \quad = \quad V_t - F_t.$$

There are four cases have to be analyzed:

Case 1 (C1): $F_t = \tilde{F}_t$ and $e_t = mC_t$ (Standard CPPI);

Case 2 (C2): $F_t = \tilde{F}_t$ and $e_t = \varpi V_t$ (Standard capped CPPI);

Case 3 (C3): $F_t = x\sup_{s\leq t} e^{r(t-s)V_s}$ and $e_t = mC_t$;

Case 4 (C4): $F_t = x \sup_{s \le t} e^{r(t-s)V_s}$ and $e_t = \varpi V_t$.

For (C1):

We have $F_t = \tilde{F}_t$, Thus, $\tilde{F}_t \geq xV_t$ or equivalently $-\tilde{F}_t \leq -xV_t$.

Since $e_t = mC_t$, we have $\varpi V_t \geq mC_t$. Then $e_t = mC_t = m(V_t - \tilde{F}_t)$ with $-\tilde{F}_t \leq -xV_t$.

Therefore, we get:

$$e_t \leq m(1-x)V_t.$$

Additionally, since $C_t \leq (1-x)V_t$ and $\varpi V_t \geq mC_t$, we deduce:

$$C_t \leq \min\left[(1-x), \frac{\varpi}{m}\right] V_t.$$

For (C2):

We have $F_t = \tilde{F}_t$ and $e_t = \varpi V_t$. Thus, $\varpi V_t \leq mC_t$. Then:

$$\varpi V_t \leq m(V_t - \tilde{F}_t) \leq m(1-x)V_t$$

Consequently, we have:

$$\varpi \leq m(1-x).$$

Equivalently, if $F_t = \tilde{F}_t$ and $\varpi > m(1-x)$, then $e_t = mC_t$, which means that the TIPP strategy does not need to be capped, in that case.

For (C3) and (C4), whenever there exists a ratchet effect, the portfolio value V_t satisfies $V_t = \sup_{s \le t} e^{r(t-s)} V_s$. we have discuss the (C3) on section 6.2 and for the (C4) we will do it on section 6.4.

6.3 CPPI with a floor indexed on a given portfolio performance

In this section, the floor value is indexed accordingly on a given portfolio performance.

6.3.1 Discrete-time with a fixed multiple

For $\tau \in \{t_0 = 0, ..., t_N = T\}$, the CPPI strategy is defined as follows. The floor is now assumed to be standard (deterministic) until the portfolio return $\frac{V_{t_i}}{V_0}$ becomes higher than a deterministic value αe^{rt} where the coefficient α is higher than 1. As soon as $\frac{V_{t_i}}{V_0} > \alpha e^{rt_i}$, the floor is equal to a fixed proportion β of the portfolio value with $0 < \beta < 1$. Therefore, the floor F is determined as follows. Denote by $T_1^{d,\alpha}$ the first time at which the portfolio return $\frac{V_{t_i}}{V_0}$ is higher than αe^{rt_i}.

Proposition 6.5. *Under above assumption, the time $T_1^{d,\alpha}$ is characterized by the relation:*

$$T_1^{d,\alpha} = \inf\{t_i \leq T : V_{t_i} \geq \alpha V_0 e^{rt_i}\}.$$

Thus the floor is given by:

$$\begin{aligned} F_{t_j} &= \tilde{F_{t_j}} = F_0 e^{rt_j} \quad for \quad t_j \leq T_1^{d,\alpha}; \\ F_{t_j} &= \beta V_{T_1^{d,\alpha}} e^{r(t_j - T_1^{d,\alpha})} \quad for \quad t_j > T_1^{d,\alpha}. \end{aligned}$$

In the following, we calculate the the probability of $T_1^{d,\alpha} = t_1$.

Proposition 6.6. *For the jump-diffusion model, if we assume $x_{i+1} = \ln(\frac{S_{i+1}}{S_i})$, $i = 0, 1, 2, ...$ is i.i.d. and their density function is p(x), then the probability of the first-time-change of floor which takes happen at t_1 is*

$$\mathbb{P}[V_{t_1} \geq \alpha V_0 e^{r\Delta}] = \int_{\ln\left(e^{r\Delta}\frac{\alpha V_0 - \theta_0^B B_0}{m(V_0 - F_0)}\right)}^{\infty} p(x)dx.$$

Proof. We have

$$
\begin{aligned}
\mathbb{P}[T_1^{d,\alpha} = t_1] &= \mathbb{P}\left[V_{t_1} \geq \alpha V_0 e^{r\Delta}\right] \\
&= \mathbb{P}\Bigg[\theta_0^B B_0 e^{r\Delta} + m(V_0 - F_0) \times \exp\left[\int_0^{t_1}\left(\mu_s - \frac{\sigma_s^2}{2}\right)ds + \int_0^{t_1}\sigma_s dW_s \right. \\
&\quad \left. + \sum_{n=1}^{N_{t_1}} \ln(1+Y_n)\right] \geq \alpha V_0 e^{r\Delta}\Bigg] \\
&= \mathbb{P}\Bigg[\exp\left[\int_0^{t_1}\left(\mu_s - \frac{\sigma_s^2}{2}\right)ds + \int_0^{t_1}\sigma_s dW_s + \sum_{n=1}^{N_{t_1}} \ln(1+Y_n)\right] \\
&\quad \geq e^{r\Delta}\frac{\alpha V_0 - \theta_0^B B_0}{m(V_0 - F_0)}\Bigg]. \\
&= \int_{\ln\left(e^{r\Delta}\frac{\alpha V_0 - \theta_0^B B_0}{m(V_0-F_0)}\right)}^{\infty} p(x)dx.
\end{aligned}
$$

□

Remarks. For the simple CPPI case, $Y_n = 0$, μ_s and $\sigma_s = \sigma$, then

$$
\begin{aligned}
\mathbb{P}\left[V_{t_1} \geq \alpha V_0 e^{r\Delta}\right] &= \mathbb{P}\left[\exp\left(\mu - \frac{1}{2}\sigma^2\right)t_1 + \sigma W_{t_1} \geq e^{r\Delta}\frac{F_0/x - \theta_0^B B_0}{m(V_0 - F_0)}\right] \\
&= 1 - N\left(\frac{1}{\sigma\sqrt{\Delta}}\left(\ln\left[e^{r\Delta}\frac{\alpha V_0 - \theta_0^B B_0}{m(V_0 - F_0)}\right] - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta\right)\right).
\end{aligned}
$$

In the following, we consider the probability that $T_1 = t_N$.

Proposition 6.7. *For the jump-diffusion model, if we assume* $x_{i+1} = \ln\left(\frac{S_{i+1}}{S_i}\right)$, $i = 0, 1, 2, ...$ *is i.i.d. and their density function is* $p(x)$*, then the probability of the first-time-change of floor which takes happen at* t_N *is*

$$
\mathbb{P}\left[T_1^{d,\alpha} = t_N\right] = \int ... \int_{D_N} p(u_i)...p(u_N)du_1du_2...du_N,
$$

where

$$
\begin{aligned}
&(u_1, ..., u_N) \in D_n \ \textit{iff} \\
&\forall i \leq N-1,\ F_0 e^{ri\Delta} + C_0 \prod_{t=1}^{t_i} g(x_t) < \alpha V_0 e^{ri\Delta}; \\
&\textit{for } i = N,\ F_0 e^{rN\Delta} + C_0 \prod_{t=1}^{t_N} g(x_t) \geq \alpha V_0 e^{rN\Delta}.
\end{aligned}
$$

Proof. We have

$$
\mathbb{P}[T_1 = t_N] = \mathbb{P}\left[F_{t_1} > xV_{t_1}, .., F_{t_{N-1}} > xV_{t_{N-1}}, F_{t_N} \leq xV_{t_N}\right]
$$

and

$$
\begin{aligned}
V_{t_i} &= \theta^B_{t_{i-1}} B_{t_i} + \theta^S_{t_{i-1}} S_{t_i} = F_{t_i} + C_{t_i} \\
&= F_0 e^{rt_i} + C_0 \prod_{t=1}^{t_i} \left[1 + (1-m)\frac{B_t - B_{t-1}}{B_{t-1}} + m\frac{S_t - S_{t-1}}{S_{t-1}}\right].
\end{aligned}
$$

Let $g(x) = 1 + (1-m)(e^{r\Delta} - 1) + m(e^x - 1)$, then

$$
V_{t_i} = F_0 e^{rt_i} + C_0 \prod_{t=1}^{t_i} g(x_t)
$$

and

$$
\begin{aligned}
&\mathbb{P}[T_1^{d,\alpha} = t_N] \\
= \ &\mathbb{P}\left[V_{t_1} < \alpha V_0 e^{r\Delta}, .., V_{t_{N-1}} < \alpha V_0 e^{r(N-1)\Delta}, V_{t_N} \geq \alpha V_0 e^{rN\Delta}\right] \\
= \ &\mathbb{P}\Bigg[F_0 e^{r\Delta} + C_0 g(x_1) < \alpha V_0 e^{r\Delta}, ..., F_0 e^{r(N-1)\Delta} + C_0 \prod_{t=1}^{t_{N-1}} g(x_t) < \alpha V_0 e^{r(N-1)\Delta}, \\
&F_0 e^{rN\Delta} + C_0 \prod_{t=1}^{t_N} g(x_t) \geq e^{rN\Delta}\Bigg].
\end{aligned}
$$

Assume

$$\begin{aligned}
&(u_1, ..., u_N) \in D_n \text{ iff} \\
&\forall i \leq N-1,\ F_0 e^{ri\Delta} + C_0 \prod_{t=1}^{t_i} g(x_t) < \alpha V_0 e^{ri\Delta}; \\
&\text{For } i = N,\ F_0 e^{rN\Delta} + C_0 \prod_{t=1}^{t_N} g(x_t) \geq \alpha V_0 e^{rN\Delta},
\end{aligned}$$

then, we have

$$\mathbb{P}\left[T_1^{d,\alpha} = t_N\right] = \int ... \int_{D_N} p(u_i)...p(u_N) du_1 du_2 ... du_N.$$

□

6.3.2 Continuous-time case

The floor is now assumed to be standard (deterministic) until the portfolio return V_t/V_0 is higher than a deterministic value of the form αe^{rt} where the coefficient α is higher than 1. As soon as $V_t/V_0 > \alpha e^{rt}$, the floor is equal to a fixed proportion β of the portfolio value with $0 < \beta < 1$. Therefore, the floor F is determined as follows. Denote by $T_1^{c,\alpha}$ the first time at which the portfolio return V_t/V_0 is higher than αe^{rt}.

Proposition 6.8. *Under above assumption, the time $T_1^{c,\alpha}$ is characterized by the relation:*

$$T_1^{c,\alpha} = \inf\{t \leq T : V_t \geq \alpha V_0 e^{rt}\}.$$

Thus, the floor is given by:

$$\begin{aligned}
F_t &= \tilde{F}_t = F_0 e^{rt} \textit{ for } t \leq T_1^{c,\alpha}; \\
F_t &= \beta V_{T_1^{c,\alpha}} e^{r(t-T_1^{c,\alpha})} \textit{ for } t > T_1^{c,\alpha}.
\end{aligned}$$

The stochastic floor is also defined by:

$$F_t = \tilde{F}_t \chi_{t \le T_1^{c,\alpha}} + \beta V_{T_1^{c,\alpha}} e^{r(t - T_1^{c,\alpha})} \chi_{t > T_1^{c,\alpha}}.$$

We assume that the exposure satisfies: $e_t = mC_t$. Therefore at time $T_1^{c,\alpha}$, the portfolio value is such that $V_{T_1^{c,\alpha}} \ge \alpha V_0 e^{rT_1^{c,\alpha}}$. Thus, at time $T_1^{c,\alpha}$, the floor is equal to $\beta V_{T_1^{c,\alpha}}$ and the cushion is equal to $(1-\beta)V_{T_1^{c,\alpha}}$.

As before we have

Proposition 6.9. *The portfolio value after $T_1^{c,\alpha}$ ($T_1^{c,\alpha} < t \le T$) is*

$$(1-\beta)V_{T_1^{c,\alpha}} \exp\left\{ \int_{T_1^{c,\alpha}}^t \left((r + m(\mu_s - r)) - \frac{m\sigma_s^2}{2} \right) ds + \int_{T_1^{c,\alpha}}^t m\sigma_s dW_s \right\} \left[\prod_{n=N_{T_1^{c,\alpha}}}^{N_t} (1 + mY_n) \right] + \beta V_{T_1^{c,\alpha}}.$$

and

Proposition 6.10. *When assume $\mu_s = \mu$ and $\sigma_s = \sigma$ be constant, the cumulative distribution function of $T_1^{c,\alpha}$ is given by*

$$\mathbb{P}(T_1^{c,\alpha} \le t) = 1 - \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds} \left(\int_0^t \lambda_s ds \right)^k}{k!} \int_{-\infty}^{\infty} \left(1 - \frac{1}{2} Erfc\left(\frac{y - y_2}{\sqrt{2t}} - A\frac{\sqrt{t}}{\sqrt{2}} \right) - \frac{1}{2} e^{2A(y-y_2)} Erfc\left(\frac{y - y_2}{\sqrt{2t}} + A\frac{\sqrt{t}}{\sqrt{2}} \right) \right) dF_k(y_2),$$

where the $F_k(y_2)$ is the distribution function of $\frac{1}{m\sigma}\sum_{n=1}^k \ln(1 + mY_n)$ and

$$y = \frac{1}{m\sigma} \ln \frac{(\alpha - 1)F_0 + \alpha C_0}{C_0}.$$

Proof. We have

$$\mathbb{P}(T_1^{c,\alpha} \leq t) = 1 - \mathbb{P}(T_1^{c,\alpha} > t)$$

Before $T_1^{c,\alpha}$, we have

$$\begin{aligned} V_t &= C_0 \exp\left\{ \int_0^t \left((r + m(\mu_s - r)) - \frac{m^2 \sigma_s^2}{2} \right) ds \right. \\ &\quad \left. + \int_0^t m\sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1 + mY_n) \right\} + F_0 e^{rt}. \end{aligned}$$

Denote

$$X_t = \int_0^t \left(m(\mu_s - r) - \frac{m^2 \sigma_s^2}{2} \right) ds + \int_0^t m\sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1 + mY_n).$$

When $\mu_s = \mu$ and $\sigma_s = \sigma$ is constant, let

$$A = \frac{(\mu - r) - \frac{1}{2} m\sigma^2}{\sigma}$$

and

$$W_s^{(A)} = \left(As + W_s + \frac{1}{m\sigma} \sum_{n=1}^{N_t} \ln(1 + mY_n) \right),$$

then we have

$$\begin{aligned} &\mathbb{P}(T_1^{c,\alpha} > t) = \mathbb{P}\left(\sup_{0 \leq s \leq t} \frac{V_t}{V_s} < \alpha e^{rt} \right) = \mathbb{P}\left(\sup_{0 \leq s \leq t} \left(C_0 e^{X_s} + F_0 \right) < \alpha V_0 \right) \\ = \; &\mathbb{P}\left(\sup_{0 \leq s \leq t} X_s < \ln \frac{(\alpha V_0 - F_0)}{C_0} \right) = \mathbb{P}\left(\sup_{0 \leq s \leq t} W_s^{(A)} < \frac{1}{m\sigma} \ln \frac{(\alpha - 1)F_0 + \alpha C_0}{C_0} \right). \end{aligned}$$

By subsection 6.2.2 we deduce that

$$\begin{aligned}\mathbb{P}\left(\sup_{s\leq t} W_s^{(A)} \leq y\right) &= \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}\left(\int_0^t \lambda_s ds\right)^k}{k!} \int_{-\infty}^{\infty}\left(1-\frac{1}{2}\text{Erfc}\left(\frac{y-y_2}{\sqrt{2t}}-A\frac{\sqrt{t}}{\sqrt{2}}\right)\right.\\ &\left. -\frac{1}{2}e^{2A(y-y_2)}\text{Erfc}\left(\frac{y-y_2}{\sqrt{2t}}+A\frac{\sqrt{t}}{\sqrt{2}}\right)\right) dF_k(y_2)\end{aligned}$$

where $F_k(y_2)$ is the distribution function of $\frac{1}{m\sigma}\sum_{n=1}^{k}\ln(1+mY_n)$.

Let $y = \frac{1}{m\sigma}\ln\frac{(\alpha-1)F_0+\alpha C_0}{C_0}$, then we have the conclusion. □

6.4 CPPI with a floor indexed on the exposition variance

6.4.1 The "Ratchet" CPPI

Discrete-time case with fixed multiple

For $\tau \in \{t_0 = 0, ..., t_N = T\}$, the CPPI strategy is defined as follows. The floor is based on the difference between the two potential values of the exposure.

As usual, the exposure is defined as the minimum between the standard cushion multiplied by the multiple and a given percentage of the portfolio value:

$$e_{t_k} = \inf[m(V_{t_k} - \tilde{F}_{t_k}), \varpi V_{t_k}]. \tag{6.1}$$

We have

$$e_{t_k} = \inf[m(V_{t_k} - \tilde{F}_{t_k}), \varpi V_{t_k}] \Longleftrightarrow F_{t_k} = \max\left[\tilde{F}_{t_k}, \frac{m-\varpi}{m}V_{t_k}\right]$$

This is equivalent to the situation in subsection 4.2.1 with the percentage

$$x = \frac{m - \varpi}{m}.$$

Denote by $T_1^{c,r}$ the first time at which $m(V_{t_k} - \tilde{F}_{t_k})$ greater than ϖV_{t_k}. Its properties is a special case as in subsection 4.2.1 with the percentage

$$x = \frac{m - \varpi}{m}.$$

Continuous-time case with fixed multiple

The floor is based on the difference between the two potential values of the exposure. As usual, the exposure is defined as the minimum between the standard cushion multiplied by the multiple and a given percentage of the portfolio value:

$$e_t = \inf[mC_t, \varpi V_t]. \tag{6.2}$$

At time 0, the exposure e_0 is assumed to be equal to mC_0. Consider the first time $T_1^{d,r}$ at which mC_t becomes higher than ϖV_t. That is:

Proposition 6.11. *Under above assumption, the time $T_1^{d,r}$ is characterized by the relation:*

$$T_1^{d,r} = \inf\{t \leq T : mC_t \geq \varpi V_t\}.$$

Then, the floor is defined as follows:

$$\begin{aligned} F_t^r &= \tilde{F}_t \text{ if } t < T_1^r; \\ F_t^r &= \left(\frac{m - \varpi}{m}\right) V_{T_1^r} e^{r(t - T_1^r)} \text{ if } t \geq T_1^r. \end{aligned}$$

We have

$$
\begin{aligned}
e_t = \inf[mC_t, \varpi V_t] &\Longleftrightarrow m(V_t - F_t) = \inf[m(V_t - \tilde{F}_t), \varpi V_t] \\
\Longleftrightarrow \quad F_t &= \max\left[\tilde{F}_t, \frac{m-\varpi}{m} V_t\right]
\end{aligned}
$$

On the other hand, the standard convergence result in the discrete-time case and proposition 4.3 lead to the equivalent situation given in subsection 4.2.2 with the percentage

$$x = \frac{m - \varpi}{m}.$$

Thus, we can calculate the probability distribution of $T_1^{d,r}$ using the result in subsection 4.2.2.

6.4.2 CPPI with margin

This kind of strategy can be applied in the situation when the initial exposition is too high.

The initial floor is chosen to be higher than the reference floor. The difference, called *the margin*, can be used later if the exposure gets too small.

Denote by F_0 the initial reference level of the floor. The initial value of the stochastic floor F_0 is equal to the reference level plus an initial margin equal to M_0. Thus we have:

$$F_0 = \tilde{F}_0 + M_0.$$

The exposition e is equal to mC with $C = V - F$. Assume that $F_t = F_0 e^{rt}$ until the time T_1^{marg} at which the exposure e becomes less than or equal to 0. The floor F is then:

$$F_{T_1^{marg}} = (\tilde{F}_0 + \gamma M_0) e^{rT_1^{marg}} \text{ with } 0 < \gamma < 1. \tag{6.3}$$

That is, at time T_1^{marg}, the reduction of the floor equals to

$$(1-\gamma)M_0 e^{rT_1^{marg}}.$$

Usually, the parameter γ is set to 1/2.

The probability distribution of the time T_1^{marg} is determined as follows. We consider a "small" $\varepsilon > 0$ and examine the time $T_1^{marg}(\varepsilon)$ at which V_t is equal or less than $(F_0+\varepsilon)e^{rt}$. We have: for any $t \leq T_1^{marg}(\varepsilon)$

$$\begin{aligned} V_t &= C_0 \exp\left\{ \int_0^t \left(r + m(\mu_s - r) - \frac{m^2\sigma_s^2}{2} \right) ds \right. \\ &\quad \left. + \int_0^t m\sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1+mY_n) \right\} + F_0 e^{rt}. \end{aligned}$$

Denote

$$X_t = \int_0^t \left((m(\mu_s - r)) - \frac{m^2\sigma_s^2}{2} \right) ds + \int_0^t m\sigma_s dW_s + \sum_{n=1}^{N_t} \ln(1+mY_n)$$

and

$$T_1^{\mathrm{marg}}(\varepsilon) = \inf\left\{ t \leq T | \inf_{0\leq s\leq t}(X_s) \leq \ln\left[\frac{\varepsilon}{C_0}\right] \right\}.$$

When the $\mu_s = \mu$ and $\sigma_s = \sigma$ are constants we have

$$X_t = \left(m(\mu - r) - \frac{1}{2}m^2\sigma^2 \right) t + m\sigma W_t + \sum_{n=1}^{N_t} \ln(1+mY_n).$$

Let

$$A = \frac{(\mu - r) - \frac{1}{2}m\sigma^2}{\sigma}$$

and

$$W_s^{(A)} = \left(As + W_s + \frac{1}{m\sigma} \sum_{n=1}^{N_t} \ln(1 + mY_n) \right),$$

so that

$$T_1^{\text{marg}}(\varepsilon) = \inf \left\{ t \le T \middle| \inf_{0 \le s \le t} \left(W_s^A \right) \le \frac{\ln\left[\frac{\varepsilon}{C_0}\right]}{m\sigma} \right\}.$$

Denote

$$y = \frac{\ln\left[\frac{\varepsilon}{C_0}\right]}{m\sigma},$$

then

$$\mathbb{P}\left(\inf_{0 \le s \le t} \left(W_s^A \right) \le y \right) = \mathbb{P}\left(- \inf_{0 \le s \le t} \left(W_s^A \right) \le -y \right) = \mathbb{P}\left(\sup_{0 \le s \le t} \left(-W_s^A \right) \le -y \right).$$

Similar to our discussions in subsection 4.2.2, we get

$$\begin{aligned}
& \mathbb{P}\left(\inf_{0 \le s \le t} \left(W_s^A \right) \le y \right) = \mathbb{P}\left(\sup_{0 \le s \le t} \left(-W_s^A \right) \le -y \right) \\
= \; & \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds} \left(\int_0^t \lambda_s ds \right)^k}{k!} \int_{-\infty}^{\infty} \left(\frac{1}{2} \text{Erfc}\left(\frac{-y - y_2}{\sqrt{2t}} + A \frac{\sqrt{t}}{\sqrt{2}} \right) \right. \\
& \left. + \frac{1}{2} e^{-2A(-y-y_2)} \text{Erfc}\left(\frac{-y - y_2}{\sqrt{2t}} - A \frac{\sqrt{t}}{\sqrt{2}} \right) \right) dF_k'(y_2),
\end{aligned}$$

where $F_k'(y_2)$ is the distribution function of

$$-\frac{1}{m\sigma} \sum_{n=1}^{k} \ln(1 + mY_n).$$

Therefore, we have:

Proposition 6.12. *Suppose that $\mu_s = \mu$ and $\sigma_s = \sigma$ are constants. Then the cdf of the time $T_1^{marg}(\varepsilon)$ is given by*

$$\begin{aligned}
&\mathbb{P}\left(T_1^{marg}(\varepsilon) \leq t\right) \\
= \; &\sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \int_{-\infty}^{\infty} \left(1 - \frac{1}{2} Erfc\left(\frac{-y - y_2}{\sqrt{2}t} + A\frac{\sqrt{t}}{\sqrt{2}}\right)\right. \\
&\left. - \frac{1}{2} e^{-2A(-y-y_2)} Erfc\left(\frac{-y - y_2}{\sqrt{2}t} - A\frac{\sqrt{t}}{\sqrt{2}}\right)\right) dF_k'(y_2)
\end{aligned}$$

Proof.

$$\mathbb{P}\left(T_1^{\text{marg}}(\varepsilon) \leq t\right) = 1 - \mathbb{P}\left(T_1^{\text{marg}}(\varepsilon) > t\right) = 1 - \mathbb{P}\left(\inf_{0 \leq s \leq t} \left(W_s^A\right) \leq y\right).$$

Substitute last term and we get the conclusion. □

Chapter 7

CPPI in the Fractional Brownian Markets

7.1 Fractional Brownian Markets

Define

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}, \quad s,t \in \mathbb{R}. \tag{7.1}$$

and suppose that $B_H(t)$ is a fractional Brownian motion with Hurst parameter H in $(1/2, 1)$ defined on the probability space $(\Omega, \mathfrak{F}, \mu_\phi)$. Let $\mathfrak{F}_t^{(H)}$ be the filtration generated by $B_H(t)$.

Reference [22] discusses the fractional Ito Integrals in terms of the Wick product associated with the fractional Brownian motion having Hurst parameter in $(1/2, 1)$. i.e.

$$\int_a^b f(t,\omega)dB_H(t) = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} f(t_k,\omega) \diamond (B_H(t_{k+1}) - B_H(t_k)). \tag{7.2}$$

See also [35] for some finance applications .

For the detail of the wick product and construction of fractional Brownian motion with Hurst parameter H, see references [22] and [35].

Definition 7.1. The **fractional Black-Scholes market** has two investment components:

(1)A bank account or a bond, where the price $A(t)$ satisfies:

$$dA(t) = rA(t)dt, \quad A(0) = 1; \quad 0 \leq t \leq T. \tag{7.3}$$

(2) A stock, where the price $S(t)$ satisfies:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t); \quad S(0) = x > 0, \tag{7.4}$$

and its solution is

$$S(t) = x \exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right) \quad t \geq 0. \tag{7.5}$$

Definition 7.2. A **portfolio** or **trading strategy** $\theta(t) = \theta(t, \omega) = (u(t), v(t))$ is an $\mathfrak{F}_t^{(H)}$-adapted two-dimensional process giving the number of units $u(t)$, $v(t)$ held at time t of the bond and the stock, respectively.

We assume the corresponding **value process** $Z(t) = Z^\theta(t, \omega)$ is given by

$$Z^\theta(t, \omega) = u(t)A(t) + v(t) \diamond S(t). \tag{7.6}$$

Definition 7.3. The portfolio is called **self-financing** if

$$\begin{aligned} dZ^\theta(t, \omega) &= u(t)dA(t) + v(t) \diamond dS(t) \\ &:= u(t)dA(t) + \mu v(t) \diamond S(t)dt + \sigma v(t) \diamond S(t)dB_H(t); \quad t \in [0, T]. \end{aligned} \tag{7.7}$$

The *Girsanov theorem* for the fractional Brownian motion(Theorem 3.18 in [35]) shows that

$$\hat{B}_H(t) := \frac{\mu - r}{\sigma} t + B_H(t) \tag{7.8}$$

is a fractional Brownian motion with respect to the measure $\hat{\mu}_\phi$ defined on $\mathfrak{F}_T^H$ by

$$d\hat{\mu}_\phi(\omega) = \exp\left(-\int_0^T K(s) dB_H(s) - \frac{1}{2}|K|_\phi^2 \right) d\mu_\phi(\omega), \tag{7.9}$$

where $K(s) = K(T, s)$ is defined by the following properties: supp $K \subset [0, T]$ and

$$\int_0^T K(T, s)\phi(t, s) ds = \frac{\mu - r}{\sigma}, \quad \text{for } 0 \le t \le T. \tag{7.10}$$

For the self-financing portfolio, from (7.6) and (7.7), we have

$$\begin{aligned} dZ^\theta(t) &= rZ^\theta(t)dt + \sigma v(t) \diamond S(t) \left[\frac{\mu - r}{\sigma} dt + dB_H(t) \right] && (7.11) \\ &= rZ^\theta(t)dt + \sigma v(t) \diamond S(t) d\hat{B}_H(t) && (7.12) \end{aligned}$$

Let $\hat{L}_\phi^{1,2}(\mathbb{R})$ denote the completion of the set of all $\mathfrak{F}_t^{(H)}$-adapted processes $f(t) = f(t, \omega)$ such that

$$||f||_{\hat{L}_\phi^{1,2}(\mathbb{R})} := \mathbb{E}_{\hat{\mu}_\phi} \left[\int_\mathbb{R} \int_\mathbb{R} f(s) f(t) \phi(s, t) ds dt \right] + \mathbb{E}_{\hat{\mu}_\phi} \left[\left(\int_\mathbb{R} D_s^\phi f(s) ds \right)^2 \right] < \infty.$$

Definition 7.4. A portfolio is called **admissible** if it is self-financing and $v \diamond S \in \hat{L}_\phi^{1,2}(\mathbb{R})$.

Definition 7.5. An admissible portfolio θ is called an **arbitrage** for the market in $t \in [0, T]$ if

$$Z^\theta(0) \le 0, Z^\theta(T) \ge 0 \quad \text{a.s. and}$$

$$\mu_\phi\big(\omega : Z^\theta(T, \omega) > 0\big) > 0.$$

Definition 7.6. The market $(A(t), S(t)); t \in [0, T]$ is called **complete** if for every $\mathfrak{F}_T^{(H)}$-measurable bounded random variable $F(\omega)$ there exists $z \in \mathbb{R}$ and portfolio $\theta = (u, v)$ such that

$$F(\omega) = Z^{\theta,z}(T, \omega). \tag{7.13}$$

This is the same as the condition that

$$e^{-rT}F(\omega) = z + \int_0^T e^{-rt}\sigma v(t) \diamond S(t) d\hat{B}_H(t). \tag{7.14}$$

Reference [35] showes that the fractional Black-Scholes market (7.3) and (7.4) has no arbitrage opportunities and it is complete.

7.2 CPPI in the Fractional Black-Scholes market

Recall that V_t is the portfolio value, $F_t = rF_t dt$, $F_T = G$ is the floor, $C_t = V_t - F_t$ is the cushion, m is the multiplier and $e_t = mC_t$ is the exposure.

Proposition 7.7. *The portfolio value of CPPI under the fractional Black-Scholes model in continuous trading time is*

$$V_t = (V_0 - F_0) \exp\left[(r + m(\mu - r))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right] + F_t. \tag{7.15}$$

Proof. With the trading strategies denoted by $\theta(t) = (u(t), v(t))$, we have the portfolio value V_t

$$V_t = u_t A_t + v_t \diamond S_t, \tag{7.16}$$

$$dV_t = u_t dA_t + v_t \diamond dS_t, \tag{7.17}$$

and

$$v_t \diamond S_t = m(V_t - F_t). \tag{7.18}$$

By (7.11), we have

$$dV_t = rV_t dt + \sigma v_t \diamond S_t \left[\frac{\mu - r}{\sigma} dt + dB_H(t) \right]. \tag{7.19}$$

Substitute (7.18) into (7.19), we obtain,

$$dV_t = rV_t dt + \sigma m(V_t - F_t) \left[\frac{\mu - r}{\sigma} dt + dB_H(t) \right]. \tag{7.20}$$

Since $C_t = V_t - F_t$ and $dF_t = rF_t dt$, we have

$$d(V_t - F_t) = r(V_t - F_t)dt + \sigma m(V_t - F_t) \left[\frac{\mu - r}{\sigma} dt + dB_H(t) \right], \tag{7.21}$$

thus,

$$dC_t = rC_t dt + \sigma m C_t \left[\frac{\mu - r}{\sigma} dt + dB_H(t) \right] \tag{7.22}$$

$$= C_t(r + m(\mu - r)dt + m\sigma dB_H(t)). \tag{7.23}$$

Then

$$C_t = C_0 \exp \left[(r + m(\mu - r))t - \frac{1}{2} m^2 \sigma^2 t^{2H} + m\sigma B_H(t) \right]. \tag{7.24}$$

Therefore, we have (7.15). □

By (3.50) in [35], we have

$$\mathbb{E}_{\mu_\phi}[C_t] = C_0 \exp \left[(r + m(\mu - r))t \right].$$

Thus we have

Proposition 7.8. *The expectation of CPPI portfolio value under the fractional Black-Scholes model in continuous time trading is*

$$(V_0 - F_0)\exp\left[(r + m(\mu - r))t\right] + F_t. \tag{7.25}$$

Proposition 7.9. *The variance of the CPPI portfolio value under the fractional Black-Scholes model in continuous time trading is*

$$Var[V_t] = (V_0 - F_0)^2 \exp\left[2(r + m(\mu - r))t\right]\left[\exp\left[m^2\sigma^2 t^{2H}\right] - 1\right]. \tag{7.26}$$

Proof.

$$\begin{aligned}\mathrm{Var}[V_t] &= \mathrm{Var}[C_t] \\ &= C_0^2 \exp\left[2(r + m(\mu - r))t\right] \mathrm{Var}\left[\exp\left[-\frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right]\right].\end{aligned}$$

For $\mathrm{Var}\left[\exp\left[-\frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right]\right]$, we have

$$\begin{aligned}&\mathrm{Var}\left[\exp\left[-\frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right]\right] \\ =\ &\mathbb{E}_{\mu_\phi}\left[\exp\left[-m^2\sigma^2 t^{2H} + 2m\sigma B_H(t)\right]\right] \\ &- \left(\mathbb{E}_{\mu_\phi}\left[\exp\left[-\frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right]\right]\right)^2 \\ =\ &\exp\left[m^2\sigma^2 t^{2H}\right] - 1.\end{aligned}$$

For the last step, we have used (3.50) in [35]. Therefore, we get

$$\mathrm{Var}[V_t] = C_0^2 \exp\left[2(r + m(\mu - r))t\right]\left[\exp\left[m^2\sigma^2 t^{2H}\right] - 1\right].$$

□

7.3 CPPI Option

We consider the Vanilla options underlying the CPPI portfolio.

Proposition 7.10. *The pricing of CPPI portfolio call option under the fractional Black-Scholes model is*

$$\begin{aligned} e^{-rT}\mathbb{E}_{\hat{\mu}_\phi}[(V_T-K)^+] &= (V_0-F_0)\Phi\left(\eta+\frac{1}{2}m\sigma T^H\right) \\ &\quad -(G-K)e^{-rT}\Phi\left(\eta-\frac{1}{2}m\sigma T^H\right), \end{aligned} \tag{7.27}$$

where

$$\eta=(m\sigma)^{-1}T^{-H}\left(\ln\frac{V_0-F_0}{G-K}\right)+rT$$

and $\Phi(t)$ *is the normal distribution function.*

Proof. Since

$$e^{-rT}\mathbb{E}_{\hat{\mu}_\phi}[(V_T-K)^+]=e^{-rT}\mathbb{E}_{\hat{\mu}_\phi}[(C_T+G-K)^+]$$

and C_t has the expression (7.23), when compared with (5.2) in [35], we see that the result is the same as the one given in corollary 5.5 in [35] where we use $G-K$, $r+m(\mu-r)$, V_0-F_0, r and $m\sigma$ to substitute for c, μ, x, ρ and σ in (5.23) of [35] respectively. Therefore,

$$\begin{aligned} e^{-rT}\mathbb{E}_{\hat{\mu}_\phi}[(V_T-K)^+] &= (V_0-F_0)\Phi\left(\eta+\frac{1}{2}m\sigma T^H\right) \\ &\quad -(G-K)e^{-rT}\Phi\left(\eta-\frac{1}{2}m\sigma T^H\right), \end{aligned}$$

where

$$\eta=(m\sigma)^{-1}T^{-H}\left(\ln\frac{V_0-F_0}{G-K}\right)+rT$$

and $\Phi(t)$ is the normal distribution function. □

7.4 PDE Approach

Theorem 7.11. *For any contingent claim of the form $g(S_t)$, there exists a unique self-financed $g(S_T)$-hedging CPPI portfolio V; defined as*

$$V_t = v(t, S_t) \quad t \in [0, T] \tag{7.28}$$

for $v \in C^{1,2}([0,T] \times \mathbb{R})$ being the unique solution of the partial differential equation (PDE).

$$\frac{\partial u}{\partial t}(t,s) + rs\frac{\partial u}{\partial x}(t,s) + \sigma^2 s^2 H \frac{\partial^2 u}{\partial x^2}(t,s)t^{2H-1} - ru(t,s) = 0; \tag{7.29}$$

$$u(T,s) = g(s), \quad (t,s) \in [0,T] \times \mathbb{R}), \quad u \in C^{1,2}([0,T] \times \mathbb{R}); \tag{7.30}$$

In particular the CPPI portfolio's gearing factor is given by:

$$m = \frac{\frac{\partial u}{\partial x}(t, S_t)S_t}{V_t - F_t}, \quad t \in [0, T]. \tag{7.31}$$

Proof. In order to have V is a self-financed $g(S_T)$-hedging portfolio, it is enough to ensure that at maturity:

$$V_T = g(S_T), \quad a.s..$$

Choose $v \in C^{1,2}([0,T] \times \mathbb{R})$ and set $V_t = v(t, S_t)$ $(t \in [0,T])$.

Now, $v(T, S_T) = g(S_T)$ $\mathbb{P}$-a.s., so that:

$$v(T,s) = g(s), \quad s \in \mathbb{R}.$$

Then, by the FBM version of Ito's formula (see [25]),

$$dv(t, S_t) \quad = \quad \left[\frac{\partial v}{\partial t} + \mu S_t \frac{\partial v}{\partial x} + \sigma^2 S_t^2 H \frac{\partial^2 v}{\partial x^2} t^{2H-1}\right](t, S_t)dt + \sigma S_t \frac{\partial v}{\partial x} \diamond dB_H(s).$$

On the other hand, by (7.20), V_t satisfies

$$dV_t = rV_t dt + \sigma m(V_t - F_t)\left[\frac{\mu - r}{\sigma}dt + dB_H(t)\right],$$

A comparison between the above two equations gives

$$m = \frac{\frac{\partial v}{\partial x}(t, S_t)S_t}{V_t - F_t}$$

and

$$\frac{\partial v}{\partial t}(t, s) + \mu s \frac{\partial v}{\partial x}(t, s) + \sigma^2 s^2 H \frac{\partial^2 v}{\partial x^2}(t, s)t^{2H-1} = rv(t, s) + (\mu - r)s\frac{\partial v}{\partial x}(t, s).$$

That is

$$\frac{\partial v}{\partial t}(t, s) + rs\frac{\partial v}{\partial x}(t, s) + \sigma^2 s^2 H \frac{\partial^2 v}{\partial x^2}(t, s)t^{2H-1} - rv(t, s) = 0.$$

□

Hence given any contingent claim $\eta = g(V_T)$, there exists a unique self-financed $\eta = g(V_T)$-hedging strategy:

Theorem 7.12. *For any map $g : \mathbb{R} \to \mathbb{R}$ sufficiently smooth, there exists a unique $\eta = g(V_T)$-hedging self-financed trading strategy (U, β) defined as*

$$U_t = u(t, V_t), \quad \beta_t = \frac{\partial u}{\partial x}(t, V_t), \quad t \in [0, T],$$

where $u \in C^{1,2}([0,T] \times \mathbb{R})$ *is the unique solution of the PDE:*

$$\frac{\partial u}{\partial t}(t,v) + rv\frac{\partial u}{\partial x}(t,v) + Ht^{2H-1}(m\sigma)^2(v-f)^2\frac{\partial^2 u}{\partial x^2}(t,v) - ru(t,v) = 0 \qquad (7.32)$$

with the final condition $u(T,v) = g(v)$.

Proof. Consider $\{V_t\}_{t\in[0,T]}$ as an asset, and pick a self-financed $g(V_T)$ hedging strategy $(U_t, \beta_t)_{t\in[0,T]}$ by setting:

$$dU_t = \beta_t dV_t + (U_t - \beta_t V_t)rdt$$

and

$$U_T = g(V_T) \quad a.s.$$

Since

$$dV_t = rV_t dt + \sigma m(V_t - F_t)\left[\frac{\mu - r}{\sigma}dt + dB_H(t)\right],$$

the hedging portfolio's equation may be rewritten as:

$$\begin{aligned} dU_t &= \beta_t\left(rV_t dt + \sigma m(V_t - F_t)\left[\frac{\mu - r}{\sigma}dt + dB_H(t)\right]\right) + (U_t - \beta_t V_t)rdt \\ &= [rU_t + \beta_t(V_t - F_t)m(\mu - r)]dt + \sigma m\beta_t(V_t - F_t)dB_H(t). \end{aligned}$$

Pick $u \in C^{1,2}([0,T] \times \mathbb{R})$ and set $U_t = u(t, V_t), \quad t \in [0,T]$.

For any $t \in [0,T]$, the FBM Ito's formula implies that:

$$\begin{aligned} du(t,V_t) &= \left[\frac{\partial u}{\partial t}(t,V_t) + (rV_t + m(\mu - r)(V_t - F_t))\frac{\partial u}{\partial x}(t,V_t)\right. \\ &\quad \left.+Ht^{2H-1}(m\sigma)^2(V_t - F_t)^2\frac{\partial^2 u}{\partial x^2}(t,V_t)\right]dt \\ &\quad +m\sigma(V_t - F_t)\frac{\partial u}{\partial x}(t,V_t)dB_H(t). \end{aligned}$$

A comparison between the above two equations implies in particular

$$\beta_t = \frac{\partial u}{\partial x}(t, V_t)$$

and

$$\begin{aligned} & \frac{\partial u}{\partial t}(t,v) + (rv + m(\mu - r)(v - f))\frac{\partial u}{\partial x}(t,v) + Ht^{2H-1}(m\sigma)^2(v-f)^2\frac{\partial^2 u}{\partial x^2}(t,v) \\ = \quad & ru(t,v) + m(v-f)(\mu - r)\frac{\partial u}{\partial x}(t,v). \end{aligned}$$

Thus

$$\frac{\partial u}{\partial t}(t,v) + rv\frac{\partial u}{\partial x}(t,v) + Ht^{2H-1}(m\sigma)^2(v-f)^2\frac{\partial^2 u}{\partial x^2}(t,v) - ru(t,v) = 0$$

with the final condition $u(T, v) = g(v)$. □

Chapter 8

CPPI in Fractional Brownian Markets with Jumps

8.1 Fractional Brownian Markets with Jumps

As before consider:

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}, \quad s,t \in \mathbb{R}. \tag{8.1}$$

Let $B_H(t)$ be a fractional Brownian motion with Hurst parameter H in the interval $(1/2, 1)$, living under the probability space $(\Omega, \mathfrak{F}, \mu_\phi)$. Moreover, $\mathfrak{F}_t^{(H)}$ denotes the filtration generated by $B_H(t)$.

[22] introduces the fractional Ito Integrals in terms of the Wick product. That is,

$$\int_a^b f(t,\omega)dB_H(t) = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} f(t_k,\omega) \diamond (B_H(t_{k+1}) - B_H(t_k)). \tag{8.2}$$

Let the price S_t of a risky asset (usually stocks or their benchmark) be a right continuous with left limits stochastic process on this probability space which jumps at the random times T_1, T_2,... and suppose that the relative/proportional change in its value at a jump time is given by Y_1, Y_2,... respectively. We usually assume the $\ln(1+Y_n)$s are i.i.d. and in our paper, we denote the density function of $\ln(1+Y_n)$s by f_Q. We assume that, between any two consecutive jump times, the price S_t follows the fractional Black-Scholes model. The T_n's are the jump times of a Poisson process N_t with intensity λ_t and the Y_n's are a sequence of random variables with values in $(-1, +\infty)$. The description of the model can be formalized by letting, on the intervals $t \in [T_n, T_{n+1})$,

$$dS_t = S_t(\mu dt + \sigma dB_H(t)). \tag{8.3}$$

Where, at $t = T_n$, the jump size is given by $\Delta S_n = S_{T_n} - S_{T_n^-} = S_{T_n-}Y_n$, so that

$$S_{T_n} = S_{T_n^-}(1 + Y_n)$$

and by assumption, $Y_n > -1$, leads to positive values of the prices.
At the generic time t, S_t satisfies

$$dS(t) = S(t)(\mu dt + \sigma dB_H(t)) + S(t-)Y_t dN_t \tag{8.4}$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation, i.e.

$$Y_t = Y_n \quad \text{if} \quad T_n < t \le T_{n+1},$$

here we let $T_0 = 0$.
We have

$$S_t \;=\; S_0 \exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right)\left[\prod_{n=1}^{N_t}(1+Y_n)\right] \tag{8.5}$$

$$= S_0 \exp\left[\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H} + \sum_{n=1}^{N_t} \ln(1+Y_n)\right] \quad (8.6)$$

$$= S_0 \exp\left[\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H} + \int_0^t \ln(1+Y_s)dN_s\right]. \quad (8.7)$$

The following definition is redefined by [35] and we adopt them.

Definition 8.1. The **fractional Black-Scholes market with jumps** has two possible types of investment:

(1)A bank account or a bond, where the price $A(t)$ satisfies:

$$dA(t) = rA(t)dt, \quad A(0) = 1; \quad 0 \le t \le T. \quad (8.8)$$

(2) A stock, where the price $S(t)$ satisfies (8.4).

Definition 8.2. A **portfolio** or **trading strategy** $\theta(t) = \theta(t,\omega) = (u(t), v(t))$ is an $\mathfrak{F}_t^{(H)}$-adapted two-dimensional process giving the number of units $u(t)$, $v(t)$ held at time t of the bond and the stock, respectively.

We assume that the corresponding **value process** $Z(t) = Z^\theta(t,\omega)$ is given by

$$Z^\theta(t,\omega) = u(t)A(t) + v(t) \diamond S(t). \quad (8.9)$$

Definition 8.3. The portfolio is called **self-financing** if

$$\begin{aligned} dZ^\theta(t,\omega) &= u(t)dA(t) + v(t) \diamond dS(t) \\ &:= u(t)dA(t) + \mu v(t) \diamond S(t)dt + \sigma v(t) \diamond S(t)dB_H(t) \\ &+ v(t) \diamond S(t-)Y_t dN_t; \quad t \in [0,T]. \end{aligned} \quad (8.10)$$

Consider a predictable $\mathfrak{F}_t^{(H)}$-process ψ_t, such that $\int_0^t \psi_t \lambda_s ds < \infty$. Choose θ and

ψ_t such that

$$\mu + \sigma\theta + Y_t\psi_t\lambda_t = r \tag{8.11}$$

and

$$\psi_t \geq 0.$$

We see that

$$\theta = \sigma^{-1}(r - \mu - Y_t\psi_t\lambda_t) \tag{8.12}$$

where the choice of ψ_t is arbitrary. Define

$$L_t = \exp\left\{\int_0^t [(1-\psi_s)\lambda_s]ds + \int_0^t \ln\psi_s dN_s - \int_0^t K(s)dB_H(s) - \frac{1}{2}|K|_\phi^2\right\} \tag{8.13}$$

for $t \in [0, T]$ where $K(s) = K(T, s)$ is defined by the following properties: supp $K \subset [0, T]$ and

$$\int_0^T K(T, s)\phi(t, s)ds = -\theta, \quad \text{for } 0 \leq t \leq T. \tag{8.14}$$

and the Radon-Nikodym derivative is

$$d\hat{\mu}_\phi(\omega) = L_T d\mu_\phi(\omega). \tag{8.15}$$

Define

$$\hat{B}_H(t) := -\theta t + B_H(t). \tag{8.16}$$

Then we have

Theorem 8.4. (**Girsanov Formula**)

(a.)$\hat{B}_H(t)$ *defined by* (8.16) *is a fractional Brownian motion that has the hurst parameter* $H \in (1/2, 1)$ *with respect to the measure* $\hat{\mu}_\phi$.

(b.) N_t is a Poisson process with intensity $\lambda_t\psi_t$ with respect to the measure $\hat{\mu}_\phi$.

Proof. (a.)For any $f \in \mathcal{S}(\mathbb{R})$, $\text{supp} f \subset [0,T]$ we have

$$\begin{aligned}
& \mathbb{E}_{\hat{\mu}_\phi} \exp\left(\int_0^T f(t)d(-\theta t + B_H(t))\right) \\
= & \mathbb{E}_{\mu_\phi}\Bigg[\exp\left(\int_0^T -f(t)\theta dt + \int_0^T f(t)dB_H(t)\right) \\
& \times \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds + \int_0^T \ln\psi_s dN_s - \int_0^T K(s)dB_H(s) - \frac{1}{2}|K|_\phi^2\right)\Bigg] \\
= & \mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds + \int_0^T \ln\psi_s dN_s\right)\exp\left(\int_0^T -f(t)\theta dt\right) \\
& \times \mathbb{E}_{\hat{\mu}_\phi} \exp\left(\int_0^T f(t)d(-\theta t + B_H(t))\right)\exp\left(-\frac{1}{2}|K|_\phi^2\right).
\end{aligned}$$

For $\mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds + \int_0^T \ln\psi_s dN_s\right)$, and we have

$$\begin{aligned}
& \mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds + \int_0^T \ln\psi_s dN_s\right) \\
= & \mathbb{E}_{\mu_\phi} \exp\left(\int_0^T \ln\psi_s dN_s\right)\mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds\right) \\
= & \mathbb{E}_{\mu_\phi} \prod_{n=1}^{N_T} \psi_n \mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds\right) \\
= & \sum_{k=0}^{\infty} \mathbb{E}_{\mu_\phi} \psi_n^k \mathbb{P}(N_T = k)\mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds\right) \\
= & \exp\left(\int_0^T -\lambda_s ds\right)\sum_{k=0}^{\infty} \mathbb{E}_{\mu_\phi} \frac{\left(\int_0^T -\lambda_s\psi_s ds\right)^k}{k!}\mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds\right) \\
= & \exp\left(\int_0^T -\lambda_s ds\right)\mathbb{E}_{\mu_\phi} \exp\left(\int_0^T \psi_s\lambda_s ds\right)\mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds\right) \\
= & 1.
\end{aligned}$$

On the other hand, since $\text{supp} f \subset [0,T]$ and $\text{supp} K \subset [0,T]$, we have

$$\int_0^T -f(t)\theta dt = \int_0^T f(t)\int_0^t K(t,s)\phi(t,s)dsdt = \langle K, f\rangle_\phi^2.$$

Moreover, we have

$$\mathbb{E}_{\hat{\mu}_\phi} \exp\left(\int_0^T f(t)dB_H(t) - \int_0^T K(s)dB_H(s)\right) = \exp\left(\frac{1}{2}|f-K|_\phi\right).$$

Thus, we have

$$\begin{aligned}
& \mathbb{E}_{\hat{\mu}_\phi} \exp\left(\int_0^T f(t)d(-\theta t + B_H(t))\right) = \exp\left(\frac{1}{2}|f-K|_\phi^2 - \frac{1}{2}|K|_\phi^2 + \langle K, f\rangle_\phi\right) \\
= & \exp\left(\frac{1}{2}(|f|_\phi^2 + |K|_\phi^2 - 2\langle K, f\rangle_\phi) - \frac{1}{2}|K|_\phi^2 + \langle K, f\rangle_\phi\right) \\
= & \exp\left(\frac{1}{2}|f|_\phi^2\right) = \mathbb{E}_{\mu_\phi} \exp\left(\int_0^T f(t)dB_H(t)\right).
\end{aligned}$$

Thus, we have proved that $\hat{B}_H(t)$ defined by (8.16) is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ with respect to the measure $\hat{\mu}_\phi$.

(b.)Using the result of $\mathbb{E}_{\mu_\phi} \exp\left(\int_0^T [(1-\psi_s)\lambda_s]ds + \int_0^T \ln\psi_s dN_s\right)$ in part (a), for any nonnegative integer k, we have

$$\begin{aligned}
& \hat{\mu}_\phi(N_T = k) = \mathbb{E}_{\hat{\mu}_\phi} \mathbf{1}_{N_T=k} = \mathbb{E}_{\mu_\phi} \mathbf{1}_{N_T=k} L_T \\
= & \mathbb{E}_{\mu_\phi} \mathbf{1}_{N_T=k} \exp\left(\int_0^t [(1-\psi_s)\lambda_s]ds + \int_0^t \ln\psi_s dN_s\right) \\
& \times \mathbb{E}_{\mu_\phi} \exp\left(-\int_0^t K(s)dB_H(s) - \frac{1}{2}|K|_\phi^2\right) \\
= & \mathbb{E}_{\mu_\phi} \mathbf{1}_{N_T=k} \exp\left(\int_0^t [(1-\psi_s)\lambda_s]ds + \int_0^t \ln\psi_s dN_s\right) \\
= & \mathbf{1}_{N_T=k} \exp\left(\int_0^T -\lambda_s ds\right) \sum_{i=0}^{\infty} \mathbb{E}_{\mu_\phi} \frac{\left(\int_0^T -\lambda_s\psi_s ds\right)^i}{i!} \exp\left(\int_0^t [(1-\psi_s)\lambda_s]ds\right) \\
= & \exp\left(\int_0^T -\lambda_s ds\right) \mathbb{E}_{\mu_\phi} \frac{\left(\int_0^T -\lambda_s\psi_s ds\right)^k}{k!} \exp\left(\int_0^t [(1-\psi_s)\lambda_s]ds\right) \\
= & \exp\left(\int_0^T -\lambda_s\psi_s ds\right) \frac{\left(\int_0^T -\lambda_s\psi_s ds\right)^k}{k!}.
\end{aligned}$$

Thus, we have proved that N_t is a Poisson process with intensity $\lambda_t\psi_t$ with respect to the measure $\hat{\mu}_\phi$. $\square$

Assume that $\theta = (u, v)$ is self-financing. Then by (8.9), we have

$$u(t) = \frac{Z^\theta(t) - v(t) \diamond S(t)}{A(t)} \tag{8.17}$$

which, substituted into (8.10) gives

$$\begin{aligned} dZ^\theta(t) &= \frac{Z^\theta(t) - v(t) \diamond S(t)}{A(t)} dA(t) + \mu v(t) \diamond S(t)dt + \sigma v(t) \diamond S(t) dB_H(t) \\ &+ v(t)S(t-)Y_t dN_t \\ &= rZ^\theta(t) - rv(t) \diamond S(t)dt + \mu v(t) \diamond S(t)dt + \sigma v(t) \diamond S(t)dB_H(t) \\ &+ v(t) \diamond S(t-)Y_t dN_t \\ &= rZ^\theta(t) + \sigma v(t) \diamond S(t)\left[dB_H(t) - \theta dt\right] + v(t) \diamond S(t)Y_t(dN_t - \psi_t \lambda_t dt). \end{aligned} \tag{8.18}$$

Let $\hat{L}_\phi^{1,2}(\mathbb{R})$ denote the completion of the set of all $\mathfrak{F}_t^{(H)}$-adapted processes $f(t) = f(t, \omega)$ such that

$$||f||_{\hat{L}_\phi^{1,2}(\mathbb{R})} := \mathbb{E}_{\hat{\mu}_\phi}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)f(t)\phi(s,t)dsdt\right] + \mathbb{E}_{\hat{\mu}_\phi}\left[\left(\int_{\mathbb{R}} D_s^\phi f(s)ds\right)^2\right] < \infty.$$

Definition 8.5. A portfolio is called **admissible** if it is self-financing and $v \diamond S \in \hat{L}_\phi^{1,2}(\mathbb{R})$.

Definition 8.6. An admissible portfolio θ is called an **arbitrage** for the market in $t \in [0, T]$ if

$$Z^\theta(0) \leq 0, Z^\theta(T) \geq 0 \quad \text{a.s. and}$$
$$\mu_\phi\left(\omega : Z^\theta(T, \omega) > 0\right) > 0.$$

From (8.18), we see that

$$\mathbb{E}_{\hat{\mu}_\phi}\left[e^{-rT}Z^\theta(T)\right] = Z^\theta(0), \tag{8.19}$$

thus, no arbitrage exists.

Definition 8.7. The market $(A(t), S(t)); t \in [0, T]$ is called **complete** if for every $\mathfrak{F}_T^{(H)}$-measurable bounded random variable $F(\omega)$ there exists $z \in \mathbb{R}$ and portfolio $\theta = (u, v)$ such that

$$F(\omega) = Z^{\theta,z}(T, \omega). \tag{8.20}$$

Proposition 8.8. *The fractional Black-Scholes market with jumps is not complete.*

Proof. $\hat{\mu}_\phi$ is not unique since we could choose different ψ_s in (8.13). Thus, it is without loss of generality to assume $\hat{\mu}_{\phi,1}$ and $\hat{\mu}_{\phi,2}$ as two distinguished measures on probability space $(\Omega, \mathfrak{F})$.

If the fractional Black-Scholes market with jumps is complete, then for every $\mathfrak{F}_T^{(H)}$-measurable bounded random variable $F(\omega)$ there exist $z \in \mathbb{R}$ and portfolio $\theta = (u, v)$ such that

$$F(\omega) = Z^{\theta,z}(T, \omega).$$

By (8.18), we see

$$\mathbb{E}_{\hat{\mu}_{\phi,1}} e^{-rT} F(\omega) = \mathbb{E}_{\hat{\mu}_{\phi,2}} e^{-rT} F(\omega). \tag{8.21}$$

This contradicts our assumption that the $\hat{\mu}_{\phi,1}$ and $\hat{\mu}_{\phi,2}$ are distinct measures on the probability space $(\Omega, \mathfrak{F})$. Therefore, The fractional Black-Scholes market with jumps is not complete. ☐

8.1.1 Esscher transform

From our previous section we know that the Radon-Nikodym measure transform (8.15) is not unique. The Esscher Transform technique in [30] provides us a unique risk-neutral transform. Here we apply the Esscher transform on our fractional Brownian Markets with jumps model.

Denote

$$X(t) = \sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H} + \sum_{n=1}^{N_t} \ln(1 + Y_n)$$

and with density $f(x,t)$. Then the stock price can be expressed as $S_t = S_0 \exp[X(t)]$. By the Esscher transform, the density function of X_t is: (refer [30])

$$f(x,t;h) = \frac{e^{hx} f(x,t)}{\int_{-\infty}^{\infty} e^{hy} f(y,t)dy} = \frac{e^{hx} f(x,t)}{M(h,t)}, \tag{8.22}$$

where $M(h,t) := \int_{-\infty}^{\infty} e^{hy} f(y,t)dy$ is the generating function. Denote by $M(z,t;h)$ the moment generating function of $X(t)$. From reference [30] we have

$$M(z,t;h) = \frac{M(z+h,t)}{M(h,t)}, \tag{8.23}$$

and

$$M(z,t;h) = [M(z,1;h)]^t. \tag{8.24}$$

As in [30], we define the risk-neutral Esscher transform as follows:

Definition 8.9. The **risk-neutral Esscher transform** is the Esscher transform with the parameter $h = h^*$ and denote by μ_ϕ^* the correspondent probability measure, such that

$$S(0) = \mathbb{E}_{\mu_\phi^*}[e^{-rt} S(t)] \tag{8.25}$$

[30] deduces that

$$e^r = M(1,1;h^*). \tag{8.26}$$

On the other hand, we have

$$\begin{aligned}
M(z,t) &= \mathbb{E}_{\mu_\phi}\left[e^{zX(t)}\right] \\
&= \mathbb{E}_{\mu_\phi}\left[e^{z(\sigma B_H(t)+\mu t-\frac{1}{2}\sigma^2 t^{2H}+\sum_{n=1}^{N_t}\ln(1+Y_n))}\right] \\
&= e^{z+\mu t}\mathbb{E}_{\mu_\phi}\left[e^{(\sum_{n=1}^{N_t}\ln(1+Y_n))}\right] \\
&= e^{z+\mu t}\mathbb{E}_{\mu_\phi}\prod_{n=1}^{N_t}(1+Y_n) \\
&= e^{z+\mu t}\sum_{k=1}^{\infty}\frac{e^{-\int_0^t\lambda_s ds}(\int_0^t\lambda_s ds)^k}{k!}\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+Y_n)\right].
\end{aligned}$$

8.2 CPPI in fractional Black-Scholes market with jumps

Recall that V_t represents the portfolio value, $F_t = rF_t dt$, $F_T = G$ is the floor, $C_t = V_t - F_t$ is the cushion, m is the multiplier and $e_t = mC_t$ is the exposure.

Proposition 8.10. *The portfolio value of CPPI under the fractional Black-Scholes model with jumps in continuous time trading is*

$$\begin{aligned}
V_t =&(V_0 - F_0)\exp\left[(m\mu - r(m-1))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right] \\
&\times\left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t.
\end{aligned} \tag{8.27}$$

Proof. With $\theta(t) = (u(t), v(t)$ as the trading strategy, we have:

$$V_t = u_t A_t + v_t \diamond S_t, \tag{8.28}$$

$$dV_t = u_t dA_t + v_t \diamond dS_t, \tag{8.29}$$

and

$$v_t \diamond S_t = m(V_t - F_t). \tag{8.30}$$

By (8.18), we have

$$\begin{aligned} dV_t =& rV_t dt - rv(t) \diamond S(t)dt + \mu v(t) \diamond S(t)dt \\ &+ \sigma v(t) \diamond S(t)dB_H(t) + v(t) \diamond S(t-)Y_t dN_t. \end{aligned} \tag{8.31}$$

Substitute (8.30) into (8.31), we obtain,

$$\begin{aligned} dV_t =& rV_t dt - rm(V_t - F_t)dt + \mu m(V_t - F_t)dt \\ &+ \sigma m(V_t - F_t)dB_H(t) + m(V_{t-} - F_t)Y_t dN_t. \end{aligned} \tag{8.32}$$

Since $C_t = V_t - F_t$ and $dF_t = rF_t dt$, we have

$$\begin{aligned} d(V_t - F_t) =& - r(m-1)(V_t - F_t)dt + \mu m(V_t - F_t)dt \\ &+ \sigma m(V_t - F_t)dB_H(t) + m(V_{t-} - F_t)Y_t dN_t. \end{aligned} \tag{8.33}$$

Thus,

$$dC_t = -r(m-1)C_t dt + \mu m C_t dt + \sigma m C_t dB_H(t) + mC_{t-}Y_t dN_t, \tag{8.34}$$

then

$$\begin{aligned} C_t =& C_0 \exp\left[(m\mu - r(m-1))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right] \\ &\times \left[\prod_{n=1}^{N_t}(1 + mY_n)\right]. \end{aligned} \tag{8.35}$$

Therefore, we have (8.27). □

Proposition 8.11. *The expected CPPI portfolio value at time t under the fractional Black-Scholes model with jumps is*

$$\mathbb{E}_{\mu_\phi}[V_t] = C_0 \exp\{(r + m(\mu - r))t\} \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \times \mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1 + mY_n)\right] + F_t.$$

Proof. Since

$$\begin{aligned}
& \mu_\phi\left[\prod_{n=1}^{N_t}(1 + mY_n) \le x\right] = \mu_\phi\left[\bigcup_{k=1}^{\infty}\left[\prod_{n=1}^{N_t}(1 + mY_n) \le x, N_t = k\right]\right] \\
= & \sum_{k=1}^{\infty} \mu_\phi\left[\prod_{n=1}^{N_t}(1 + mY_n) \le x | N_t = k\right] \mu_\phi[N_t = k] \\
= & \sum_{k=1}^{\infty} \mu_\phi\left[\prod_{n=1}^{k}(1 + mY_n) \le x | N_t = k\right] \mu_\phi[N_t = k] \\
= & \sum_{k=1}^{\infty} \mu_\phi \frac{\left[\prod_{n=1}^{k}(1 + mY_n) \le x, N_t = k\right]}{\mu_\phi[N_t = k]} \mu_\phi[N_t = k] \\
= & \sum_{k=1}^{\infty} \mu_\phi \frac{\left[\prod_{n=1}^{k}(1 + mY_n) \le x\right] \mu_\phi[N_t = k]}{\mu_\phi[N_t = k]} \mu_\phi[N_t = k] \\
= & \sum_{k=1}^{\infty} \mu_\phi\left[\prod_{n=1}^{k}(1 + mY_n) \le x\right] \mu_\phi[N_t = k] \\
= & \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mu_\phi\left[\prod_{n=1}^{k}(1 + mY_n) \le x\right],
\end{aligned}$$

we get,

$$\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{N_t}(1 + mY_n)\right] = \sum_{k=1}^{\infty} \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1 + mY_n)\right].$$

From (3.50) in [35], we obtain

$$\mathbb{E}_{\mu_\phi}[V_t] = C_0 \mathbb{E}_{\mu_\phi}\left[\exp\left\{(m\mu - r(m-1))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right\}\right]$$

$$
\begin{aligned}
&\times \mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{N_t}(1+mY_n)\right] + F_t \\
= \;& C_0 \exp\{(r+m(\mu-r))t\}\sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \\
&\times \mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right] + F_t.
\end{aligned}
$$

□

Proposition 8.12. *The variance of the CPPI portfolio value at time t under the fractional Balck-Scholes model with jumps is*

$$
\begin{aligned}
&C_0^2 \exp\left\{2((r+m(\mu-r)))t + m^2\sigma^2 t^{2H}\right\} \\
&\times \sum_{k=1}^{\infty}\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right]^2 \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \\
&-C_0^2\left[\exp\{(r+m(\mu-r))t\sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2.
\end{aligned}
$$

Proof. Similar to the proof of the above proposition, we have

$$
\mathbb{E}_{\mu_\phi}\left[\left[\prod_{n=1}^{N_t}(1+mY_n)\right]^2\right] = \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}_{\mu_\phi}\left[\left[\prod_{n=1}^{k}(1+mY_n)\right]^2\right].
$$

Thus,

$$
\begin{aligned}
&\mathrm{Var}_{\mu_\phi}[V_t] = \mathrm{Var}_{\mu_\phi}[C_t] \\
= \;& C_0^2 \mathrm{Var}_{\mu_\phi}\left[\exp\left\{(r+m(\mu-r))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right] \\
= \;& C_0^2 \mathbb{E}_{\mu_\phi}\left[\exp\left\{(r+m(\mu-r))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right]^2 \\
&-C_0^2\left(\mathbb{E}_{\mu_\phi}\left[\exp\left\{(r+m(\mu-r))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right]\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= C_0^2 \mathbb{E}_{\mu_\phi}\left[\exp\left\{(r+m(\mu-r))t - \frac{1}{2}m^2\sigma^2 t^{2H} + m\sigma B_H(t)\right\}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]\right]^2 \\
&\quad -C_0^2\left[\exp\{(r+m(\mu-r))t\}\sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2 \\
&= C_0^2 \mathbb{E}_{\mu_\phi}\left[\exp\left\{2(r+m(\mu-r))t - m^2\sigma^2 t^{2H} + 2m\sigma B_H(t)\right\}\right]\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{N_t}(1+mY_n)\right]^2 \\
&\quad -C_0^2\left[\exp\{(r+m(\mu-r))t\}\sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2 \\
&= C_0^2 \exp\left\{2(r+m(\mu-r))t + m^2\sigma^2 t^{2H}\right\} \\
&\quad \times \sum_{k=1}^{\infty}\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right]^2 \frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!} \\
&\quad -C_0^2\left[\exp\{(r+m(\mu-r))t\}\sum_{k=1}^{\infty}\frac{e^{-\int_0^t \lambda_s ds}(\int_0^t \lambda_s ds)^k}{k!}\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right]\right]^2.
\end{aligned}
$$

□

The following lemma gives the density function of $1+mY_i$.

Lemma 8.13. *Let the density function of* $\ln(1+Y_n)$ *be* $f_Q(y)$. *Then the density funtion* f'_Q *of the random variable* $1+mY_i$ *is*

$$
f'_Q(z) = f_Q\left(\ln\left(1+\frac{z-1}{m}\right)\right)\frac{1}{m+z-1}.
$$

Proof. Since

$$
\mu_\phi(1+mY_i \le z) = \mu_\phi\left(\ln(1+Y_i) \le \ln\left(1+\frac{z-1}{m}\right)\right) = \int_{-\infty}^{\ln\left(1+\frac{z-1}{m}\right)} f_Q(y)dy,
$$

the density f'_Q of the random variable $1+mY_i$ is

$$
f'_Q(z) = \frac{d(\mu_\phi(1+mY_i \le z))}{dz} = f_Q\left(\ln\left(1+\frac{z-1}{m}\right)\right)\frac{1}{m+z-1}.
$$

□

Now we can calculate

$$\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)\right] = \mathbb{E}_{\mu_\phi}\left[\exp\left\{\sum_{n=1}^{k}\ln(1+mY_n)\right\}\right]$$
$$= \int_{\mathbb{R}} \exp\left\{\underbrace{f'_Q * f'_Q * ... * f'_Q(x)}_{\text{Convolved } k \text{ times}}\right\} dx$$

and

$$\mathbb{E}_{\mu_\phi}\left[\prod_{n=1}^{k}(1+mY_n)^2\right] = \mathbb{E}_{\mu_\phi}\left[\exp\left\{\sum_{n=1}^{k}2\ln(1+mY_n)\right\}\right]$$
$$= \int_{\mathbb{R}} \exp\left\{2\underbrace{f'_Q * f'_Q * ... * f'_Q(x)}_{\text{Convolved } k \text{ times}}\right\} dx.$$

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VITA

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