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A COLLECTION OF GRAPHS TO ACCOMPANY CERTAIN TOPICS IN THE STUDY OF FUNCTION THEORY OF A REAL VARIABLE

by

Ruth Eversole, A. B.

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

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of the

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I N D E X .

P A R T I. - Rectangular Co-ordinates.

I N D E X. (Continued).

PART ONE — RECTANGULAR COORDINATES.

CHAPTER I. — SETS OF POINTS.

1. Introduction. - A group or collection of points is said to form a "set", if some law or criterion is known, whereby we may determine whether any given point is a member of that group.

An easy and fundamental illustration of the usefulness of the idea, exists in the Dedekind-cut definition of an irrational number. (Figure I) All the rational numbers are divided into two sets, A and B, so that any element in A is less than any element of B. It is assumed that there exists a number C, such that any A is less than C and any B is greater than C. This number C, is called an irrational number. In the figure 1, $\sqrt{2}$ furnishes the law for the construction of sets A and B.

Set A	1	1.4	1.41	1.414	. . . . .
Set B	2	1.5	1.42	1.415	. . . . .

2. Countability. - A set of points may be composed of a finite or an infinite number of elements. The points indicated in figure 2, form a finite set. If the mid-points of each indicated interval and of each successive interval be added to this set, the result will be an infinite set of points. Infinite sets are further subdivided into; (a) those having a countable number of elements (a different positive integer may be assigned to each element as a sub-script); (b) those having more than a countable number of elements. Figure 3 illustrates a

Fig. 1.

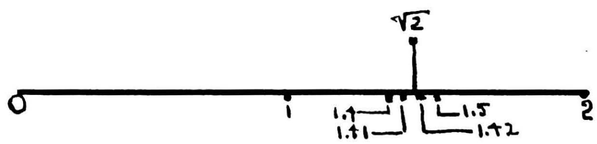


Fig 2

Fig. 3.

Fig 4

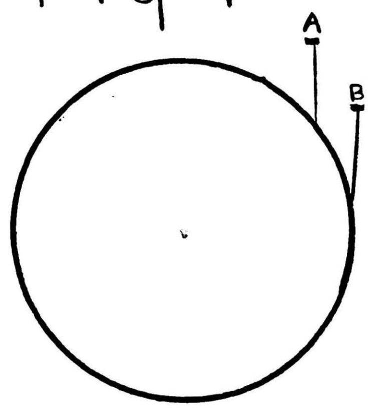


Fig. 5.

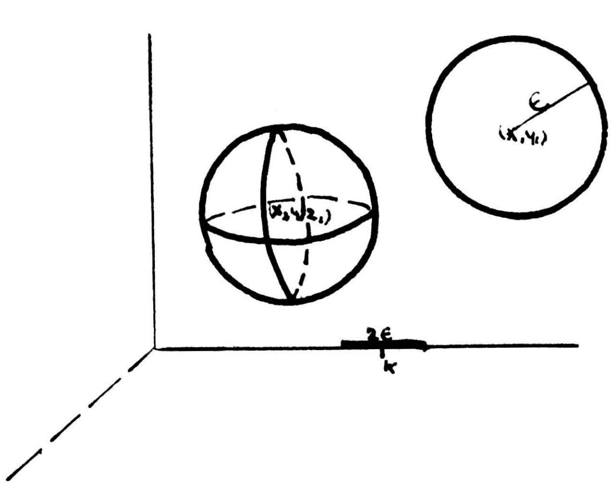
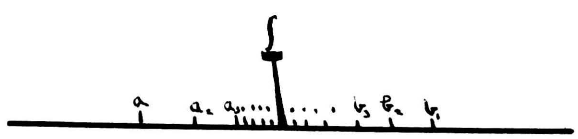
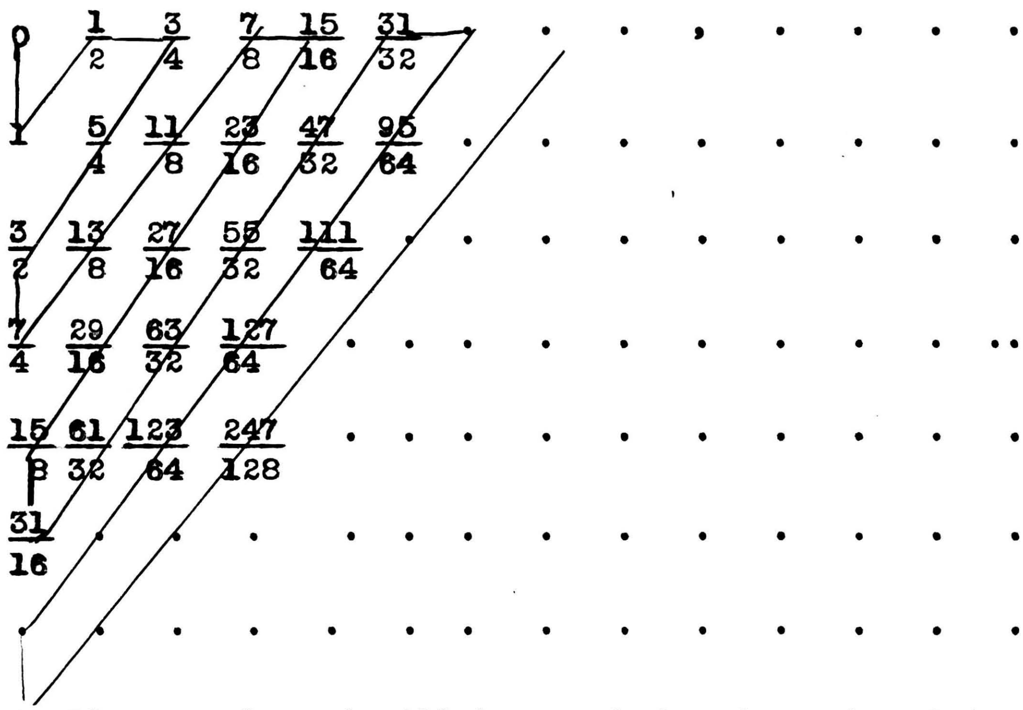


Fig. 6.



countable set, for the points may be arranged as in the following table, and the subscripts assigned to the points in the order in which they are reached by the connecting line.



Eventually any element will be reached and a subscript assigned to it.

Figure 4 illustrates a more than countable set of points, for if A and B are two points to which we desire to assign successive subscripts - between them lie an infinite number of points to which no subscript is assignable.

3. <u>Neighborhood</u>.—<u>Limit Point</u>.- The neighborhood of a given point is comprised of all those points of the set, whose distance from the given point is less than ϵ, (an arbitrarily small constant). Neighborhoods are one-dimensional, two-dimensional, three-dimensional, *n*-dimensional, according as the set of points lies on a line, in an area, a volume etc. Figure 5 illustrates the three kinds of neighborhoods which are capable of graphical construction.

This notion of neighborhood is necessary to the definition of a <u>limit-point</u>. A point is said to be a limit-point of

a set, if within its neighborhood, there lie an infinite number of points of the set. A limit point may or may not be an element of the set of which it is a limit point.

Stronger than a mere limit point is a <u>point of condensation</u>, within whose neighborhood there lie, more than a countable number of points of the set. From its definition it is evident that a finite set of points cannot have a limit-point, nor a countable set, a point of condensation. However a bounded infinite set of points must have at least one limit point. For let E be an infinite set of points lying in the interval 0 L (Figure 6). Bisect the interval 0 L and retain the half which contains an infinite number of points. Repeat this process. The interval (a_i, b_i), containing an infinite number of points, can be made as small as we please. Then this interval will lie within the neighborhood of some point 1 - hence 1 is a limiting point of the set.

In exactly the same manner a bounded set containing more than a countable number of points, can be shown to have at least one point of condensation.

4. <u>Derived Sets</u>. - The first derived set of a given set is the set, composed of the limit points of the original set. The second derived set is the first derived set of the first derived set etc. The points, $(1, \frac{3}{2}\ \frac{7}{4}\ \frac{15}{8}\ \frac{31}{16}\ .\ .\ .\ .\ 2)$, form the first derived set of the set pictured in figure 3. The second derived set is the single point 2. There are two useful theorems on derived sets, which I shall merely quote here, with references. (a) Each derived set contains all succeding derived sets. (b) In passing from one derived set to the next, at most, a countable number of points is dropped (Borel's "Leçons sur la Theorie des Fonctions", pp. 35-36).

A closed set of points is a set which contains all its first derived set - and hence contains all its derived sets (see preceeding paragraph). A set of points, which is identical with its first derivative is called a perfect set of points.

5. Properties of Perfect Sets. - As a simple illustration of a perfect set of points, all the points on a line segment of unit length may be used(Both end-points are included). The interior of an interval may be dropped from this segment without disturbing the perfect character of the set (Figure 7). To drop the interiors of a countable number of non-overlapping, non-abutting intervals from a line segment, is the characteristic manner in which all perfect sets are formed (For a proof of this statement see Baire's "Fonctions discontinues" pp. 56-57).

An interesting perfect set of points is Cantor's Ternary Set (Figure 8). The middle third of the interval (0,1) is dropped from the straight line segment; likewise the middle thirds of the remaining subintervals (End-points are always retained). This process repeated infinitely yields a perfect set - a line segment with the interiors of a countable number of intervals removed. This set can be put into one-to-one correspondence with the points on a unit line-segment - a property called, "the power of continuum". Since this set was formed in the characteristic manner of a perfect set, and since it is nowhere dense, this property may be assigned to any perfect set. Any perfect set has the power of continuum. (Complete proofs of the theorems suggested in this paragraph may be found in Baire's "Leçons sur les Fonctions Discontinues" pp. 54,55).

6. Application to Function Theory. - The fundamental importance of the point set introduction to function theory, lies

Fig.7.

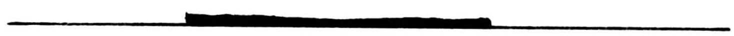


Fig 8

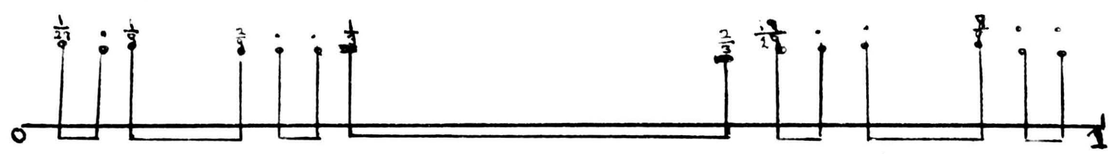


Fig.9.

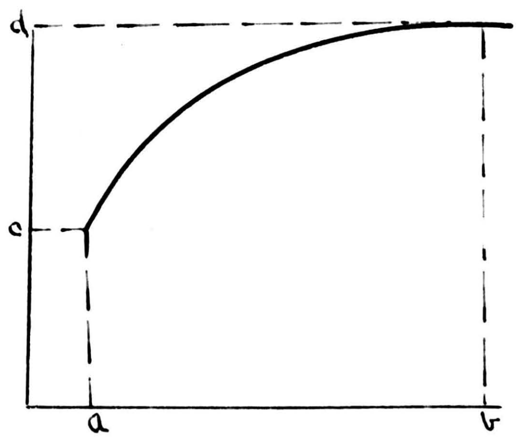
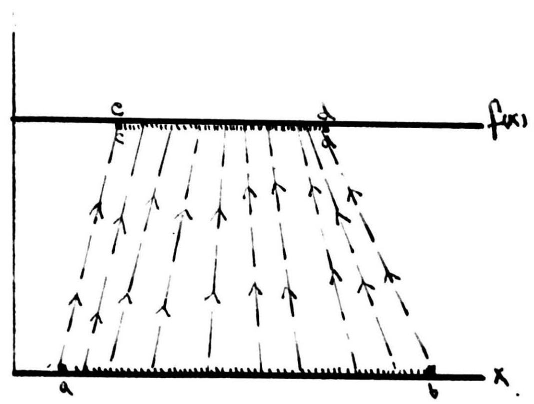


Fig.10.



in the new view-point of a function, to which it leads us. A function is always defined for certain definite values of the independent variable. The set of values for which the function is defined constitutes the first set of points. Each value of the independent variable determines one or more values of the function itself. These values constitute point set number two. Now between these two sets of points, there is a correspondence of some sort - the particular function determines the kind.

7. Single-valued Functions. The simplest possible correspondence (one-to-one) is illustrated by a monotone increasing function. (Figures 9 and 10). Here any two corresponding values have the same relative order in the two sets. In order that a one-to-one correspondence may exist, the function must be a reciprocally single-valued one. Figures 11 and 12, illustrate the only possibility other than a monotone function. Here the relative order is not preserved.

8. Multiple-valued Functions. If we have a double-valued function defined over the points indicated on the x-axis (Figure 13), the correspondence would be as indicated. Point C is an extremum point, hence the one-to-one correspondence at that point. The analogy is obvious when the function is *n*-valued, (*n* being any positive integer). Figure 14, $f(x) = \text{constant}$, illustrates the case, when *n* is infinite.

An interesting correspondence is that, to which the function - $f(x) = \frac{1}{n}$ when $x = \frac{m}{n}$ (m and n are prime) $f(x) = 0$ otherwise - gives rise.

To the point $f(x) = 0$, there correspond more than a countable number of x-points, all the irrational points between zero and one (Figure 15). To the points $f(x) = 1$ and $f(x) = \frac{1}{2}$,

Fig.11.

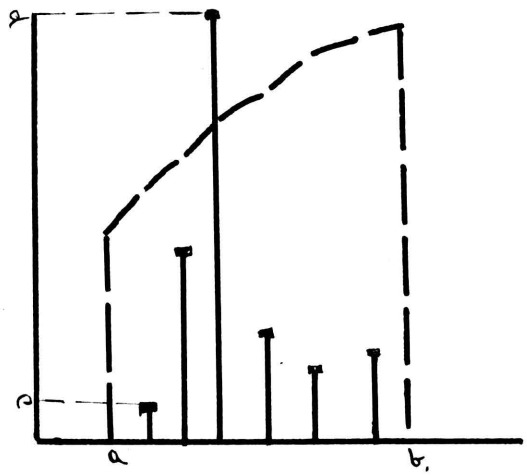


Fig.12.

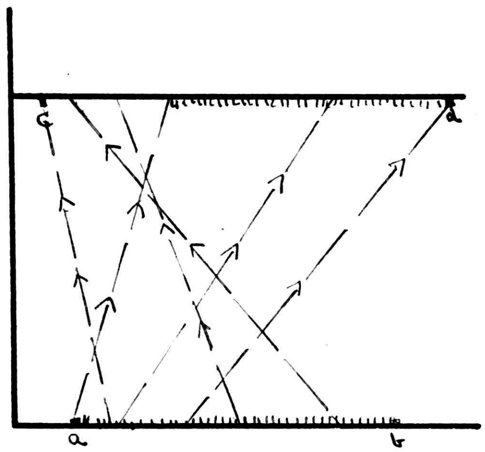


Fig.13.

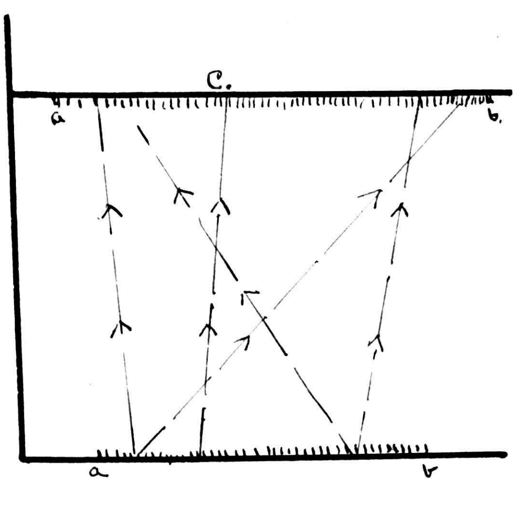


Fig.14.

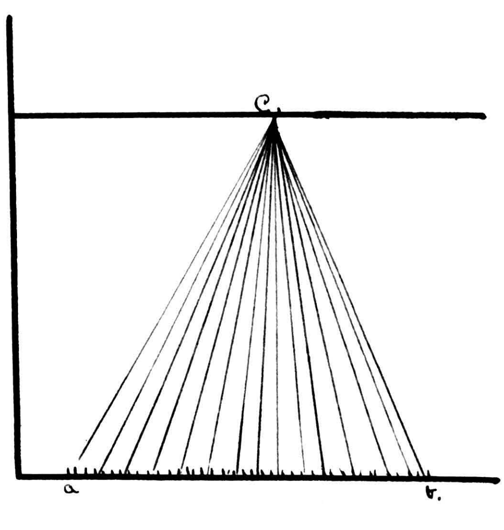


Fig. 15.

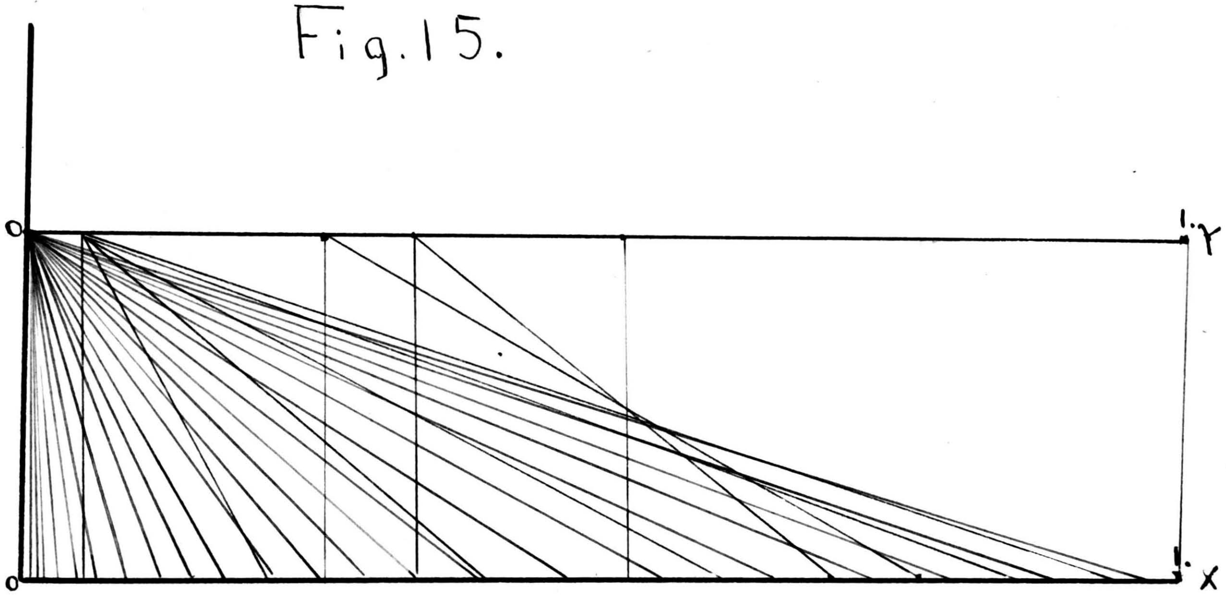
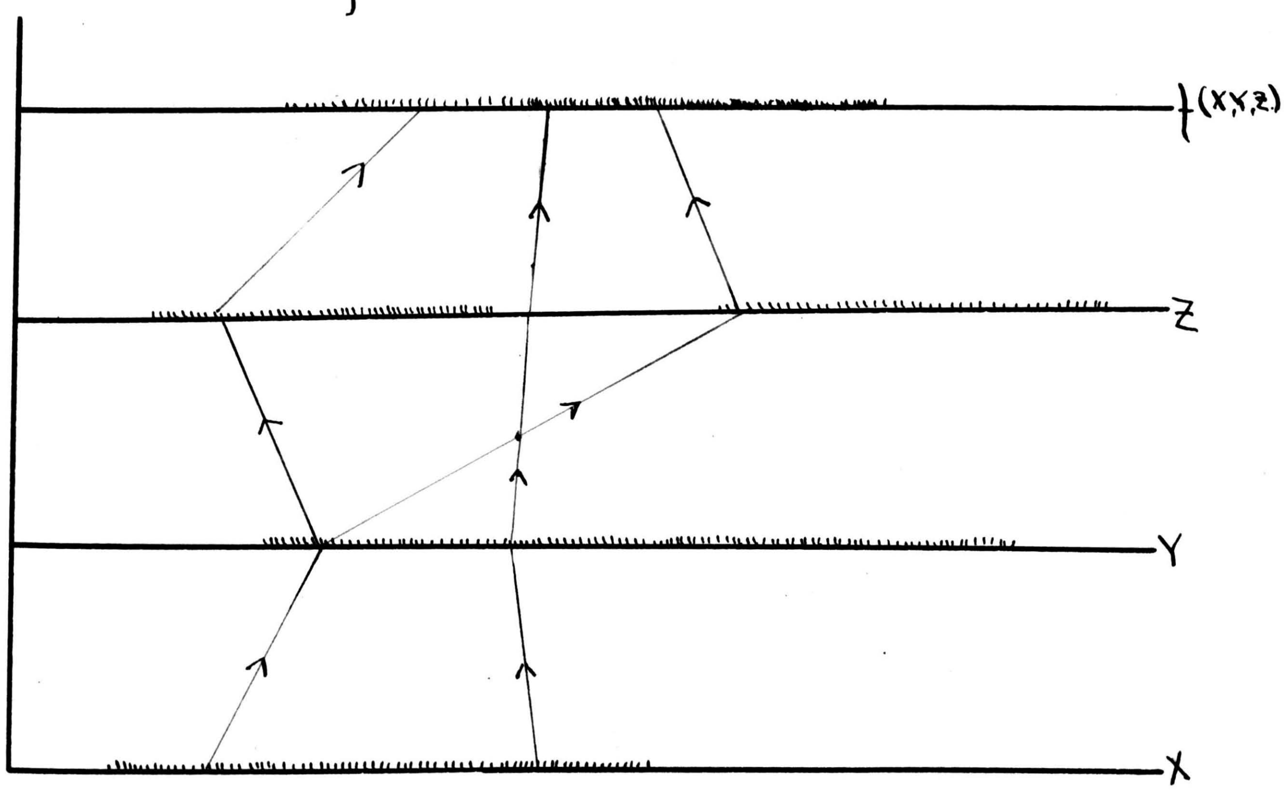


Fig. 16.



there correspond one x-point each. To the points $f(x) = \frac{1}{3}$ and $f(x) = \frac{1}{4}$, there correspond two x-points each etc. We can always find a point, other than zero, on the f(x) axis which corresponds to more than k points on the x-axis (k may be any positive integer). But f(x) = 0 is the only point, to which there correspond an infinite number of points on the x-axis.

9. Functions of Several Variables. - The functions just considered, have been functions of one variable only. Functions of n-variables can be analogously represented as the correspondence between $n + 1$ sets of points. Figure 6 illustrates the case, when n = 3. The order of procedure is (a) choose some point on x for which the function is defined, (b) connect with any point on y for which the function is defined, (c) connect with any point on z for which the function is defined, (d) the three previous choices determine the corresponding point, or points, on f(x,y,z).

CHAPTER II. - SERIES.

1. Introduction. - A fundamental assumption in the study of series, which may be taken as an axiom and which we shall call the telescoped-interval axiom, is the following. Given, the series of intervals, a_1b_1, a_2b_2, a_3b_3, . . . a_nb_n . . . . (each interior to the preceding), there is at least one point common to all the intervals (Fig.17). This axiom is assumed in the Dedekind-cut definition of an irrational number (Fig. 1). From this it is easy to prove that, if the lengths of the intervals approach zero, there is only one point common to all the intervals. (Borel's "Leçons sur la Theorie des Fonctions" p.25).

2. Series of Constant Terms. - Convergence. - The following is the notation used throughout the chapter. Given the infinite series $a_0 + a_1 + a_2 + \ldots\ldots$, $S_0 = a_0$, $S_1 = a_0 + a_1$, $S_2 = a_0 + a_1 + a_2$, etc; that is S_n is the sum of the first n terms of the series. This is equivalent to replacing the infinite series by the infinite sequence S_0, S_1, S_2, ... S_n . . . . Evidently if the sum of the series, S, exists, it is the $\lim_{n \doteq \infty} S_n$. A series is said to converge if S exists, and to diverge otherwise. Cauchy's test for the convergence of a series, is that $|S_n - S_{n+p}| < \epsilon$ when $n > N$ #. Figures 18-19 are the characteristic convergence graphs. In figure 18, we have a series of intervals S_1S, S_2S, S_3S, . . . S_nS, each interior to the preceeding. Since the length of the interval $|S_n - S_{n+p}|$ must approach zero,—by the telescoped interval

For a proof of this see Goursat-Hedrick Mathematical Analysis, Vol.I, pp. 330-331.

Fig.17.

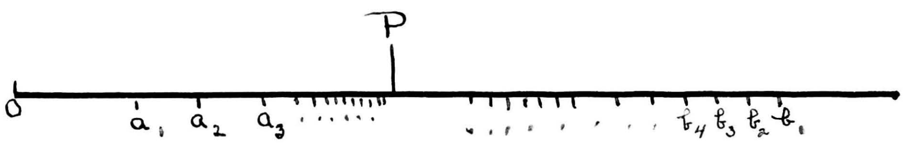


Fig.18.

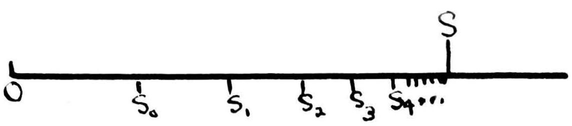


Fig.19.

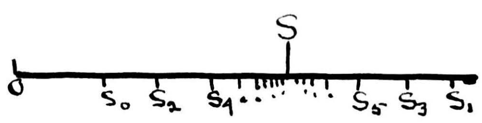


Fig.20.

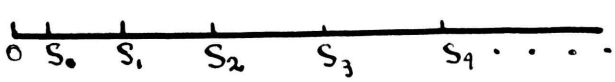
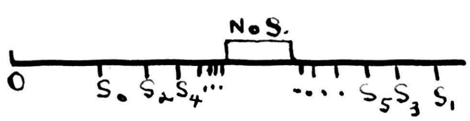


Fig.21.



axiom there is but one point, namely S common to all the intervals. Hence S is the $\lim_{n \to \infty} S_n$. Figure 19 represents a convergent series whose terms alternate in sign. The series of telescoped intervals is, $S_1 - S_2$, $S_3 - S_2$, ... $S_n - S_{n+1}$ . . . . By Cauchy's test, these intervals approach zero. Hence by the extension of the telescoped interval axiom there is but one point common to all these intervals, namely S.

Figures 20 and 21 illustrate the two types of divergent series. In 20, the terms have like signs, in 21 they alternate in sign. In each case the series has been replaced by the sequence, and an application of the telescoped interval axiom, combined with the Cauchy test, is all that is necessary to prove the existence of S

3. <u>Series of Functions</u>. - A series of functions of a variable x, say $f_1(x) + f_2(x) + \ldots . f_n(x) + \ldots .$ may be said to converge, for a given value of x, if the series of constant terms formed by substituting the value of x in the series converges. A series of functions is said to converge <u>in an interval</u>, or on a <u>set of points</u>, if it converges for every value of x, in the interval or on the set of points. The values to which the series converges correspond to the respective values of the independent variable, hence the series represents or defines a function.

If the terms of the series are functions of more than one variable, then the series will define a function of the variables involved for those values of the variables for which the series converges.

4. <u>Uniform Convergence</u>. - A series, convergent in a given interval, ($a \leq x \leq b$) is said to converge uniformly if $|S - S_n| < \overset{(1)}{\epsilon}$ when $n > \overset{(2)}{N}$, for any value of $\overset{(3)}{x}$ in the interval (1,2,3, designate the order of choice) Figures 22 and 23 illustrate the difference between convergence and uniform convergence. The intervals

Fig.22.

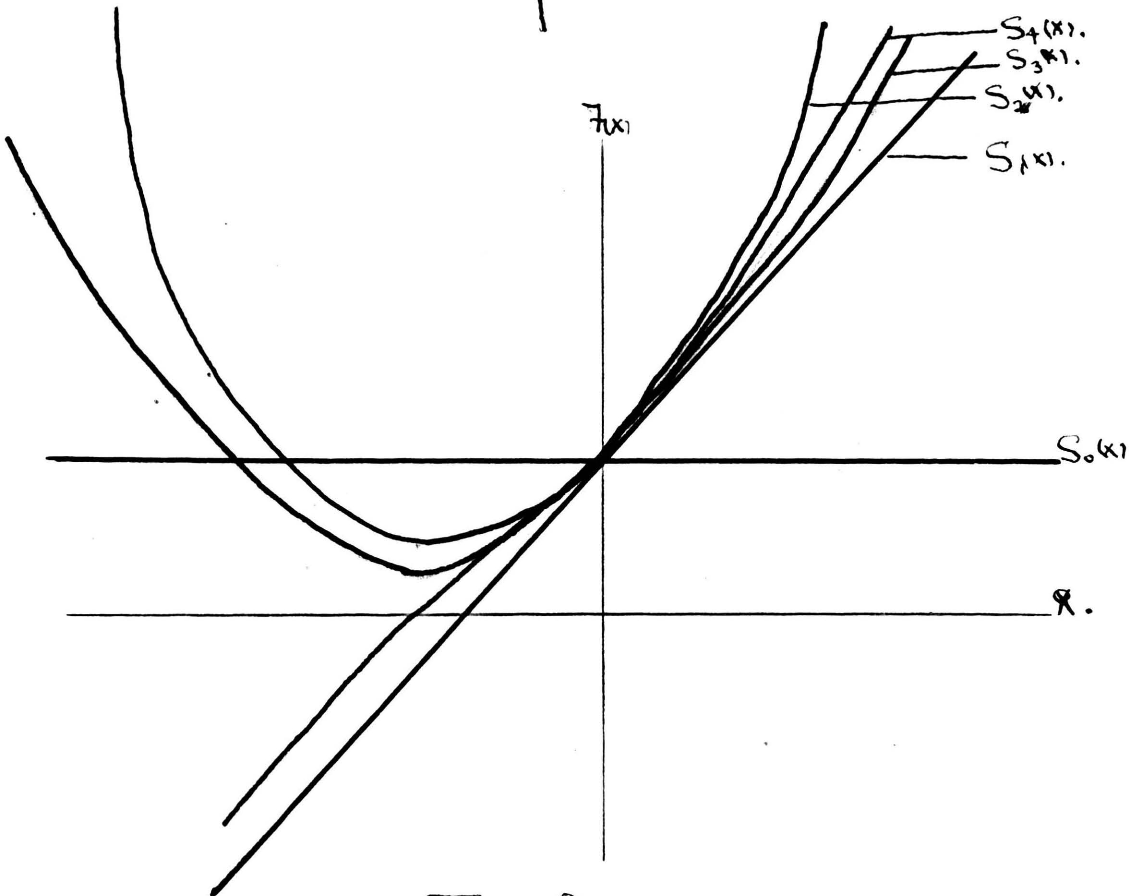
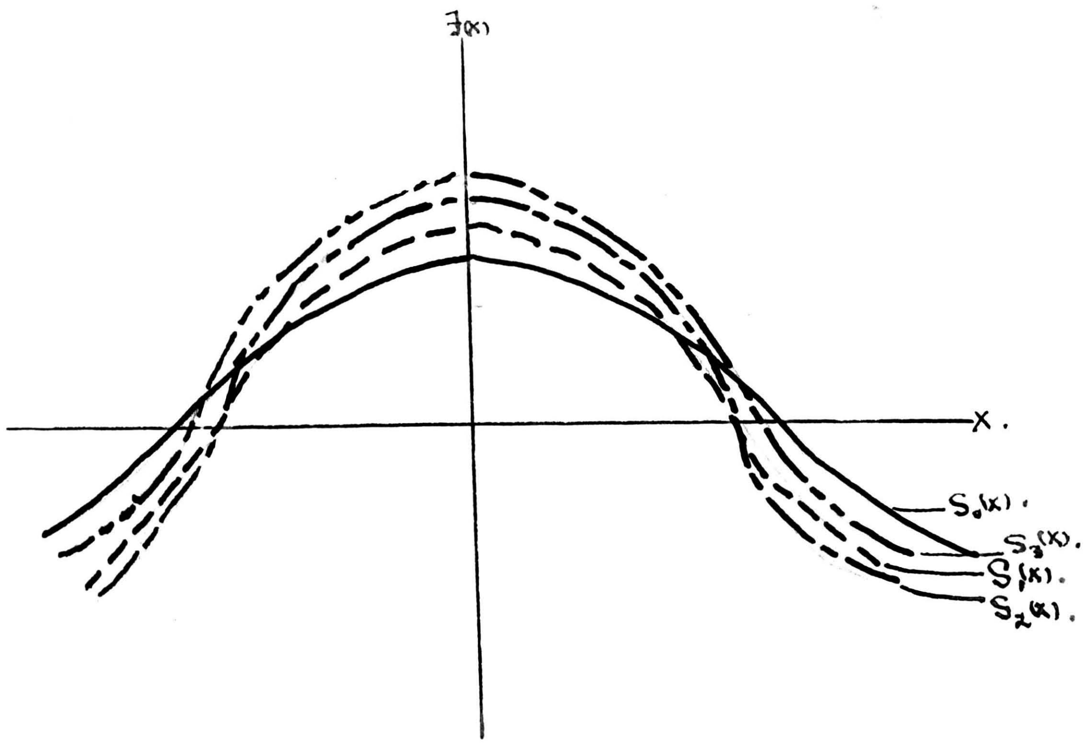


Fig.23.



in both cases are the entire infinite plane. Figure 22 represents the series $e^x = 1 + x + x^2/2! + x^3/3! + \ldots x^n/n! \ldots\ldots$, which is uniformly convergent in any finite interval, but not in the infinite interval. Figure 23 is the uniformly convergent series $S(x) = \cos x + \frac{\cos 2x}{2!} + \ldots \frac{\cos nx}{n!} + \ldots$ #

Graphically interpreted uniform convergence means the following proceedure: (1) choose a strip of uniform width ϵ; (2) choose n , so large that if this strip be laid on S , any succeeding S_{n+p} after S_n will lie on the surface underlying this strip.

5. <u>The Series a Function of "n"</u>. - The subscript "n" may be considered as one of the independent variables of the series (the series, of course, is defined only for positive integral values of n). S(x), then becomes S(x,n) and hence has a three dimensional representation. The advantages of this method of representation will become apparent in the following figures. Figure 24 illustrates $\frac{1}{x^2-1} = \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ldots\ldots \frac{1}{x^{2n-2}} \ldots\ldots$ in three dimensions. The transformation $N = \frac{3}{n}$ is made before the series is plotted. N takes on only the special values (3, 1-1/2, 3/4 . . .), however S(x,N) may be defined for all values of N between 0 and 3; in general, by connecting successive curves with straight lines. This would form a continuous surface, and the particular curves which represent the series are the cross sections of the surface made by N = (3, 1-1/2, 3/4 . . .). The limit curve S(x,o,) forms the boundary of the surface of the variable.

For method of determining whether a given series is uniformly convergent, see Goursat-Hedrick's Mathematical Analysis, Vol. I, pp. 363-364.

Fig.24.

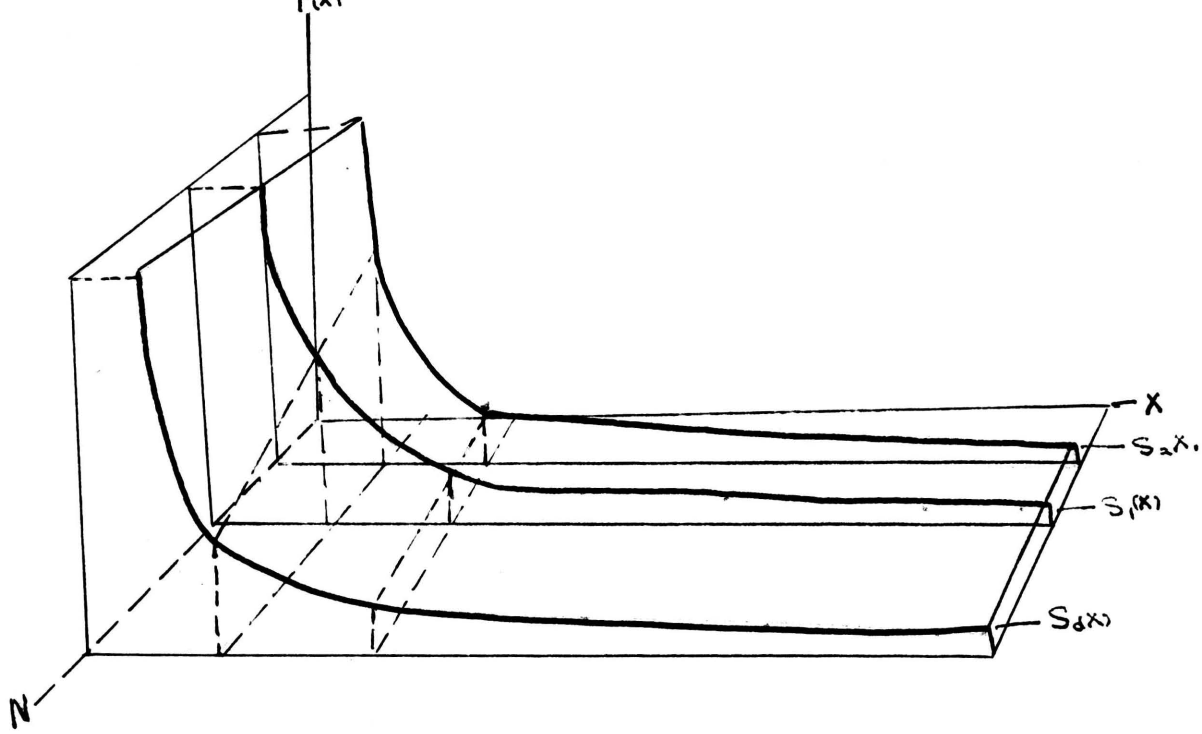
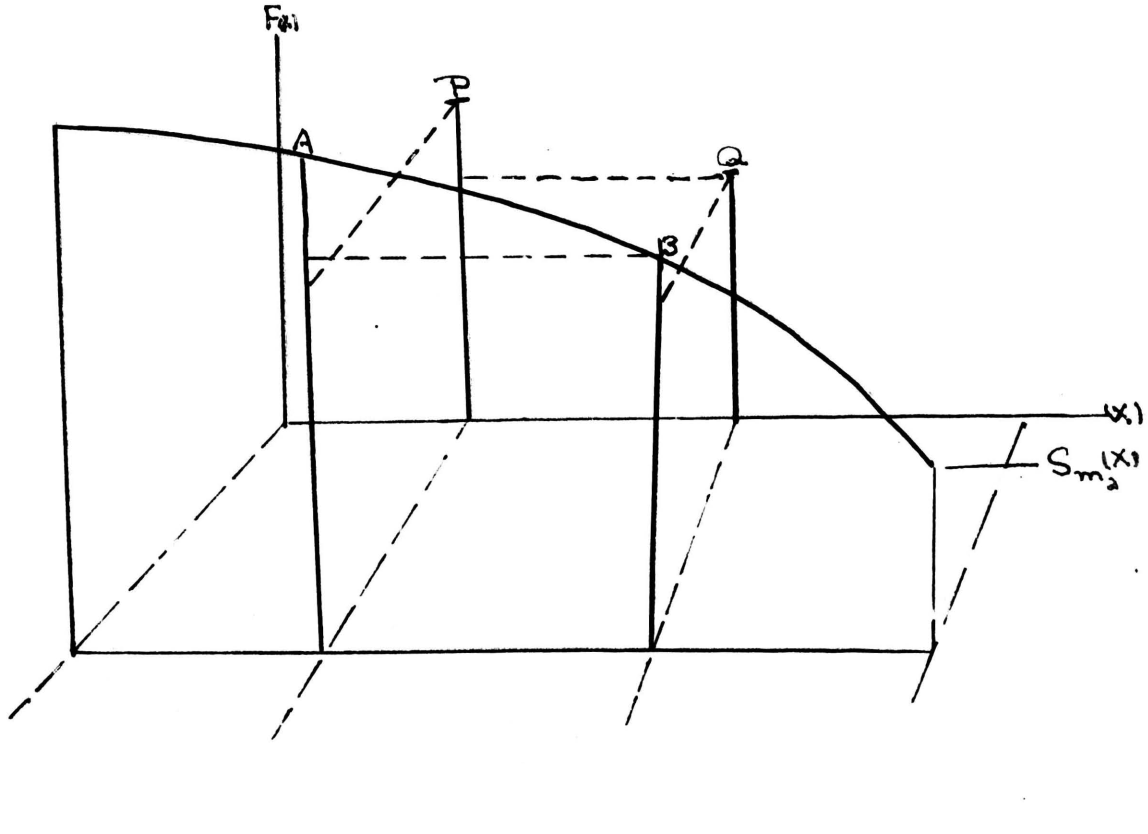


Fig.25.



7. A Uniformly Convergent Series of Continuous Functions. - If each term of a given series is continuous, and the series converges uniformly in a given interval, then the function represented by the series is continuous. In figure 25, $S_n(x)$ is a continuous function of x (the sum of a finite number of continuous functions is a continuous function).

$$|\text{height of A} - \text{height of P}| < \frac{\epsilon}{3} \quad \text{when} \quad n > m \quad ① ②$$

$$|\text{height of B} - \text{height of Q}| < \frac{\epsilon}{3} \quad \text{when} \quad n > m_2 \quad ③ \qquad m_2 > m$$

Because the series is uniformly convergent

$$|\text{height of A} - \text{height of B}| < \frac{\epsilon}{3} \quad \text{when} \quad (a - b) < \delta \quad ④$$

Because $S_{m_2}(x)$ is continuous.

Combining the three inequalities gives the result

$$|\text{height of P} - \text{height of Q}| < \epsilon \quad \text{when} \quad (a - b) < \delta,$$

which is the definition of continuity.

8. Hankel's Principle. - If all of the terms of a series except one, are continuous, and the series is otherwise uniformly convergent, then the function which the series represents, will have the same discontinuities as that one discontinuous term. Figure 26 illustrates this principle. If any number of the terms have discontinuities, but no two of them at the same point (lest they cancel each other), then the limiting curve will have all these discontinuities. Hankel's principle is an excellent scheme for building a badly discontinuous function, for example the function

$$F(x) = \sin\frac{1}{(x - 1/2)} + \frac{1}{2!}\sin\frac{1}{(x - 1/3)} + \frac{1}{3!}\sin\frac{1}{(x-2/3)} + \cdots \frac{1}{n}\sin\frac{1}{x - n\text{th proper fraction}} + \cdots\cdots$$

F(x) is discontinuous at every rational point in the interval zero to one#.

For proofs of the theorems suggested in this paragraph see Haukel's Mathematische Annalen, Vol. XX, pp. 77-81.

Fig 26

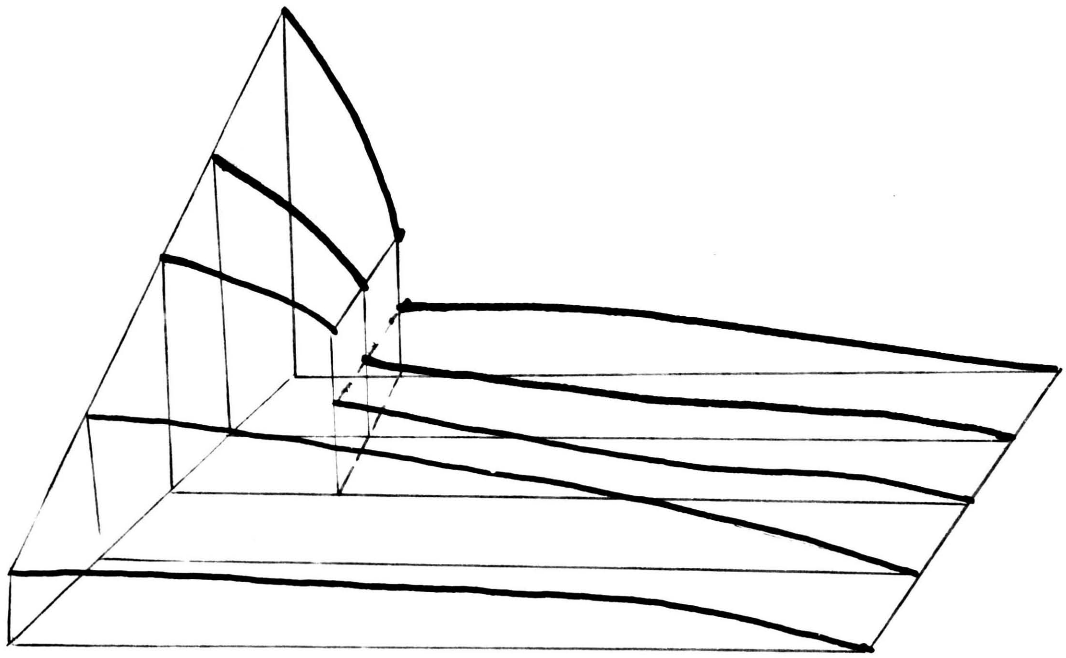
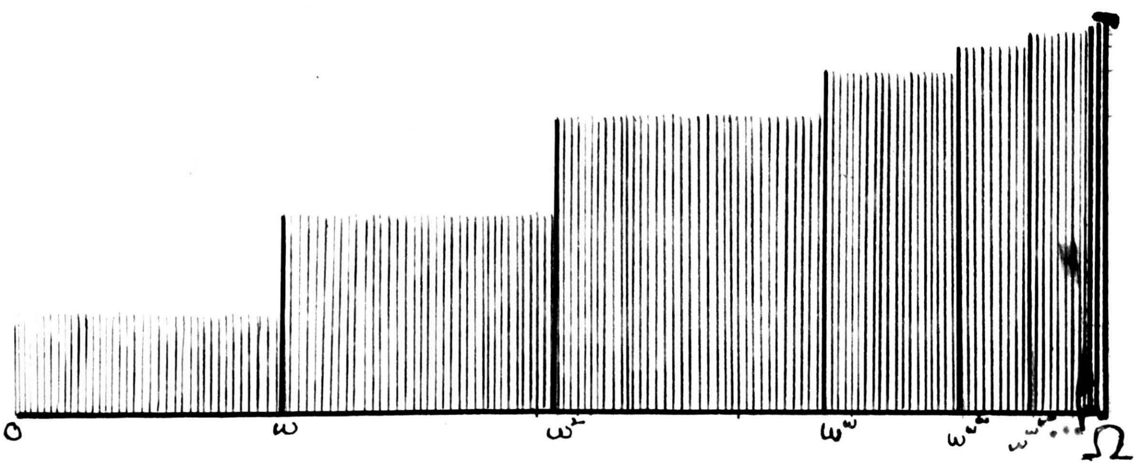


Fig 27.



9. Transfinite Numbers. - Let us make the function described in the preceding paragraph periodic in the following manner. Repeat the portion in the (0,1) interval, dinished 1/2 in size, in the interval (1,1-1/2). The (0,1-1/2) interval is diminished in size and reproduced in the interval (1-1/2, 1-3/4). This process is repeated infinitely. The set of points at which the function is discontinuous in the interval (0,2) is represented in figure 27.

The positive integers may be assigned to the points in the interval (0,1) as subscripts. That leaves no subscripts for the rest of the points, so we give point 1 the subscript ω. To the points in the interval (1,1-1/2) we assign the subscripts $\omega + 1$, $\omega + 2$, ... $\omega + n$ . . . . Point 1-1/2 is 2ω, point 1-3/4 is ω^2, etc. Point 2 is $\omega^{\omega^{\omega^{\cdot}}}$ or Ω. These ω's are the transfinite numbers, and as they are more than countable in number, a function may have more than a countable number of discontinuities in a finite interval. (See Baire's "Lecons sur les Fonctions Discontinues" pp.43-45).

Figure 28 shows that a merely convergent series of continuous functions does not represent a continuous function.

10. Term-wise Integration. - Without any very detailed knowledge of the subject of integration, (which will be taken up later), it can be easily shown that a uniformly convergent series can be integrated term-wise. However a uniformly convergent series of continuous terms cannot always be differentiated term-wise. Figure 29 represents the uniformly convergent series

$$S(x) = \frac{1}{x} + \sin x - \left(\sin x - \frac{\sin 2x}{2}\right) - \left(\frac{\sin 2x}{2} - \frac{\sin 3x}{3}\right) - \ . \ . \ .$$

$$- \left(\frac{\sin nx}{n} - \frac{\sin (n+1)x}{(n+1)}\right) \ . \ . \ . \ .$$

Fig. 28.

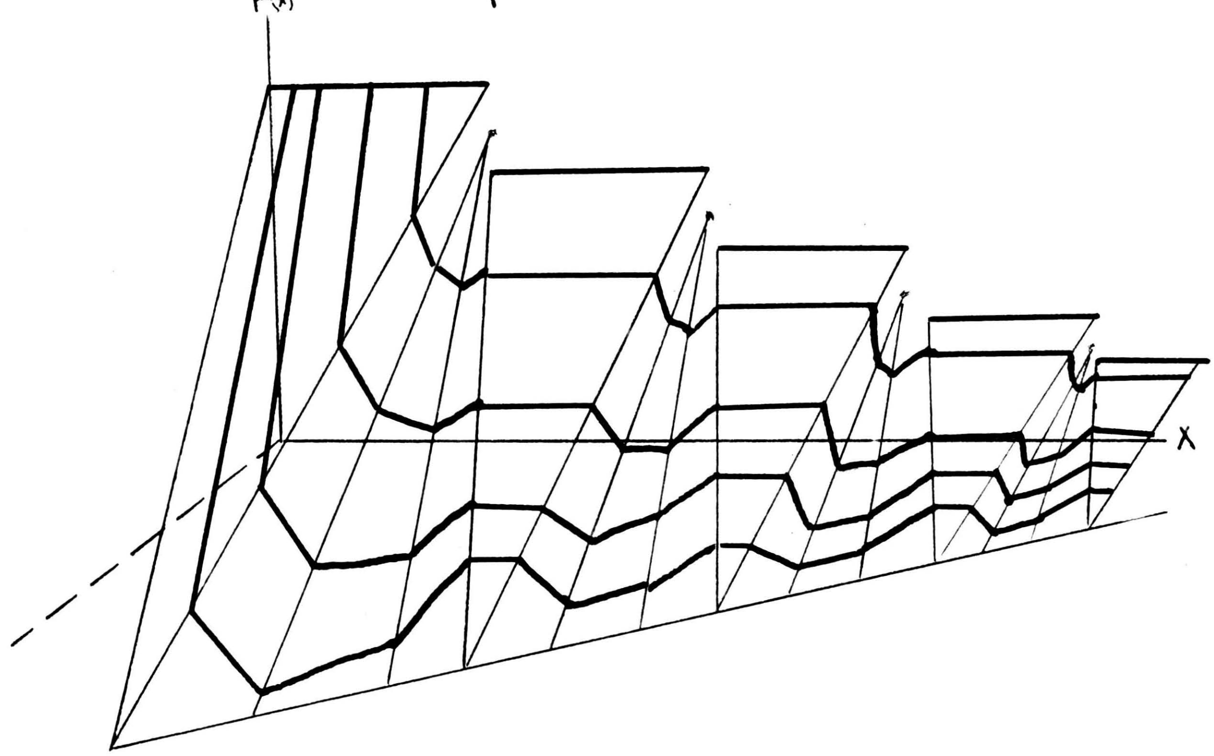


Fig. 29.

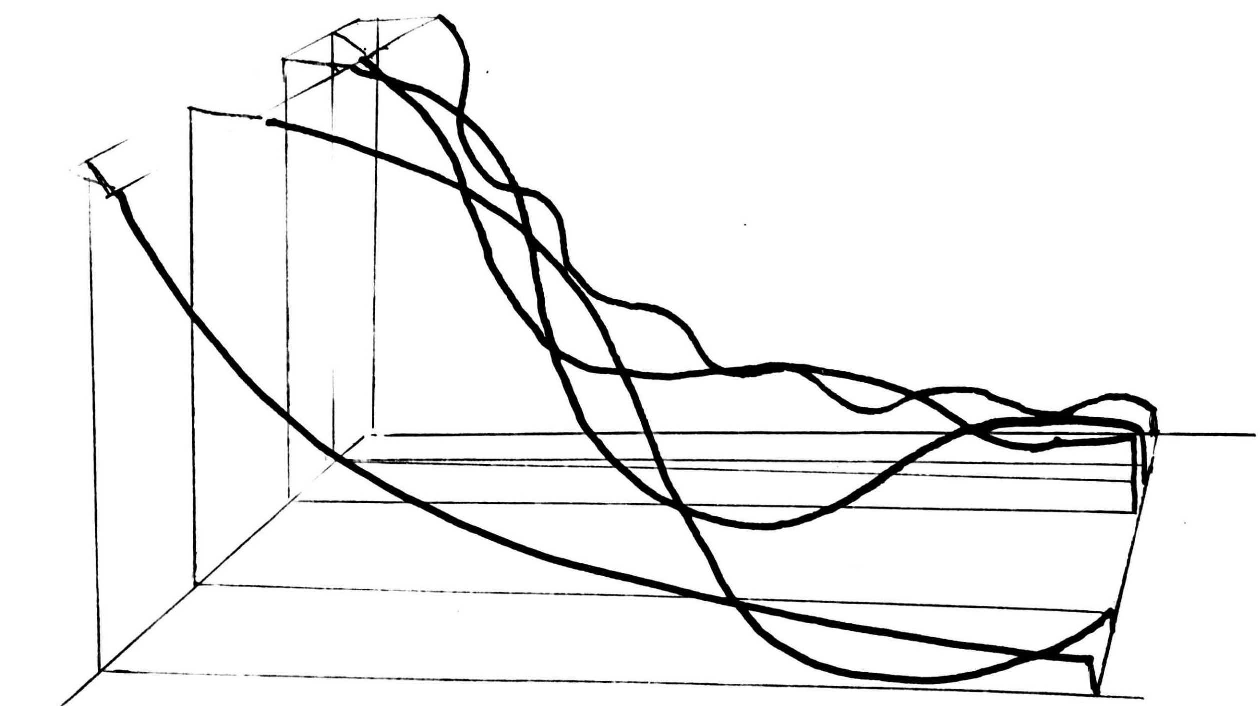


Fig.30.

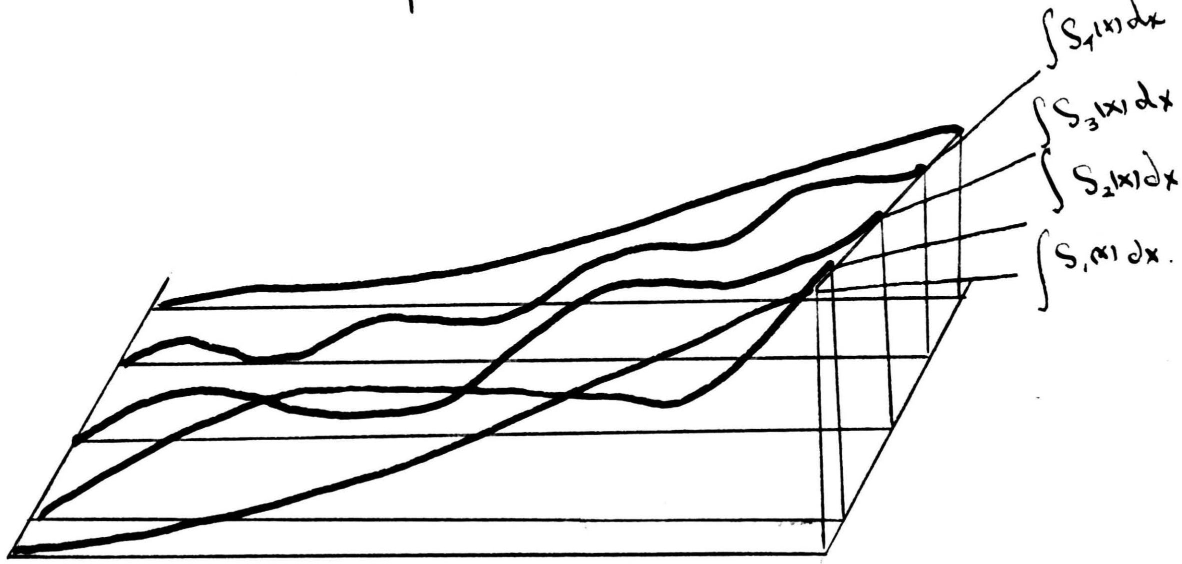


Fig.31.

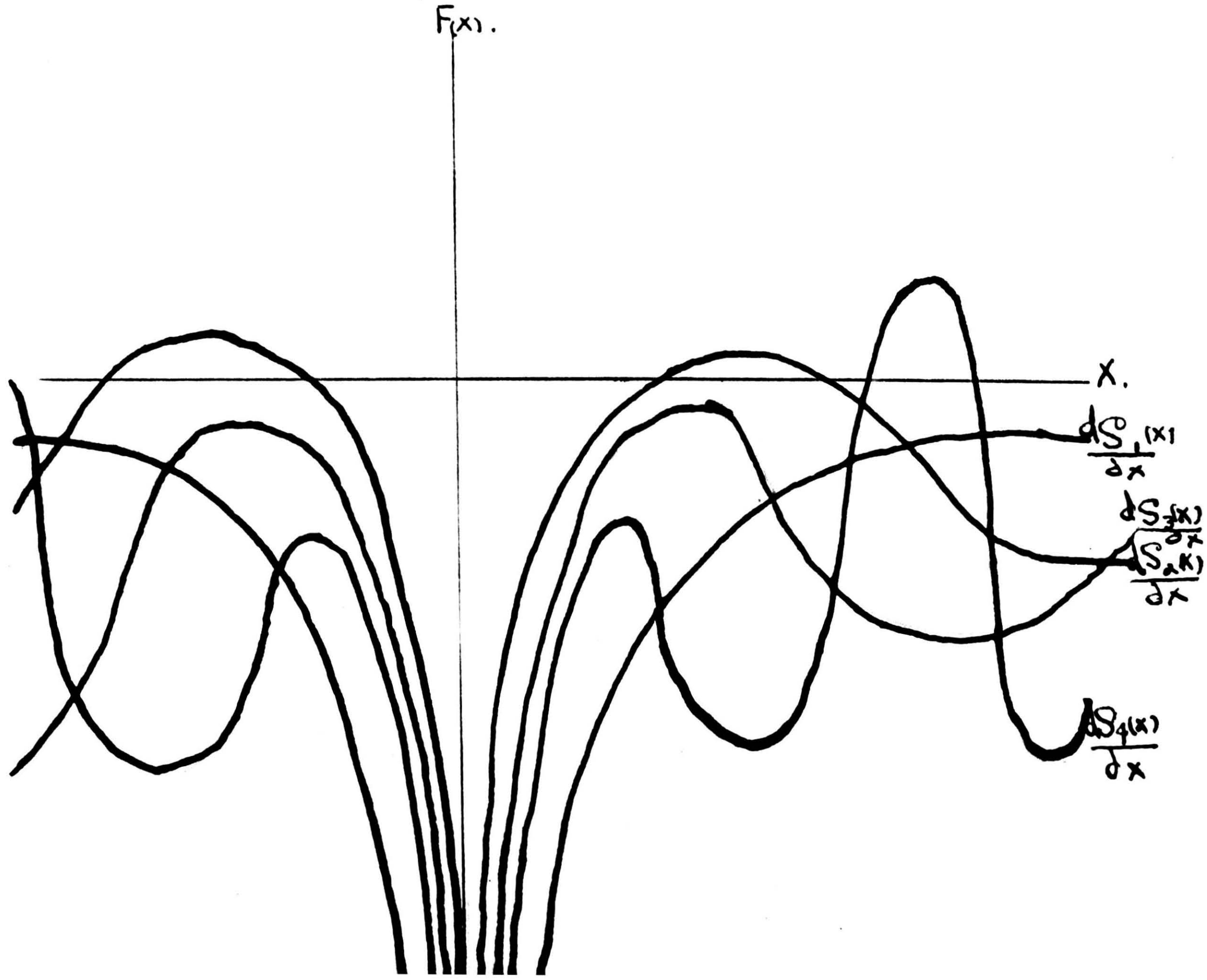


Figure 30 represents its term-wise integration and figure 31 its term-wise differentiation. Figure 30 is a uniformly convergent series, but figure 31 is a divergent series. In order to differentiate term-wise, we must first differentiate, then test the resulting series - - if it is uniformly convergent the differentiation is valid, otherwise not[#].

Other definitions of convergence, than the one I have adopted here, might be constructed. For example a series might be said to converge if $\int_0^x (S_n^2(x) - S_{n+1}^2(x))\,dx < \overset{①}{\epsilon} \quad n > \overset{②}{m}$ (This is the well-known "least-squares" approximation). Another definition and one often used in Calculus of Variations is

$$\int_0^x [S_n(x) - S_{n+1}(x)]^2 + [S_n'(x) - S_{n+1}'(x)]^2\,dx < \overset{①}{\epsilon} \quad n > \overset{②}{m} .$$

If this definition of convergence were adopted then the Series described in the preceeding paragraph would not converge.

11. __The Existence of $\lim_{n=\infty} \int S_n(x)\,dx$ Does Not Mean Term-wise Integration.__ - If a series is integrable term-wise, that means $\int_{x_0}^x S(x)\,dx = \lim_{n=\infty} \int_{x_0}^x S_n(x)\,dx$. Paragraph 10 shows this is true in the case of a uniformly convergent series of continuous functions. Sometimes the $\lim_{n=\infty} \int S_n(x)\,dx$ exists when the series has no integral or may not even converge and represent a function at all. Figure 32 illustrates this condition of affairs. $S_1(x)$ contains one triangle of dimensions 4" X 1/2". $S_2(x)$ contains two triangles of dimensions 1" X 1". $S_n(x)$ contains 2n triangles of $1/4^{n-2}$ X 2n". The bases of the triangles approach the xN plane as $n = \infty$: The $\lim_{n=\infty} \int_0^x S(x)$ = 2 sq. in. This series does

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\# For proofs of these theorems see "Goursat-Hedrick's Mathematical Analysis," pp.364-370.

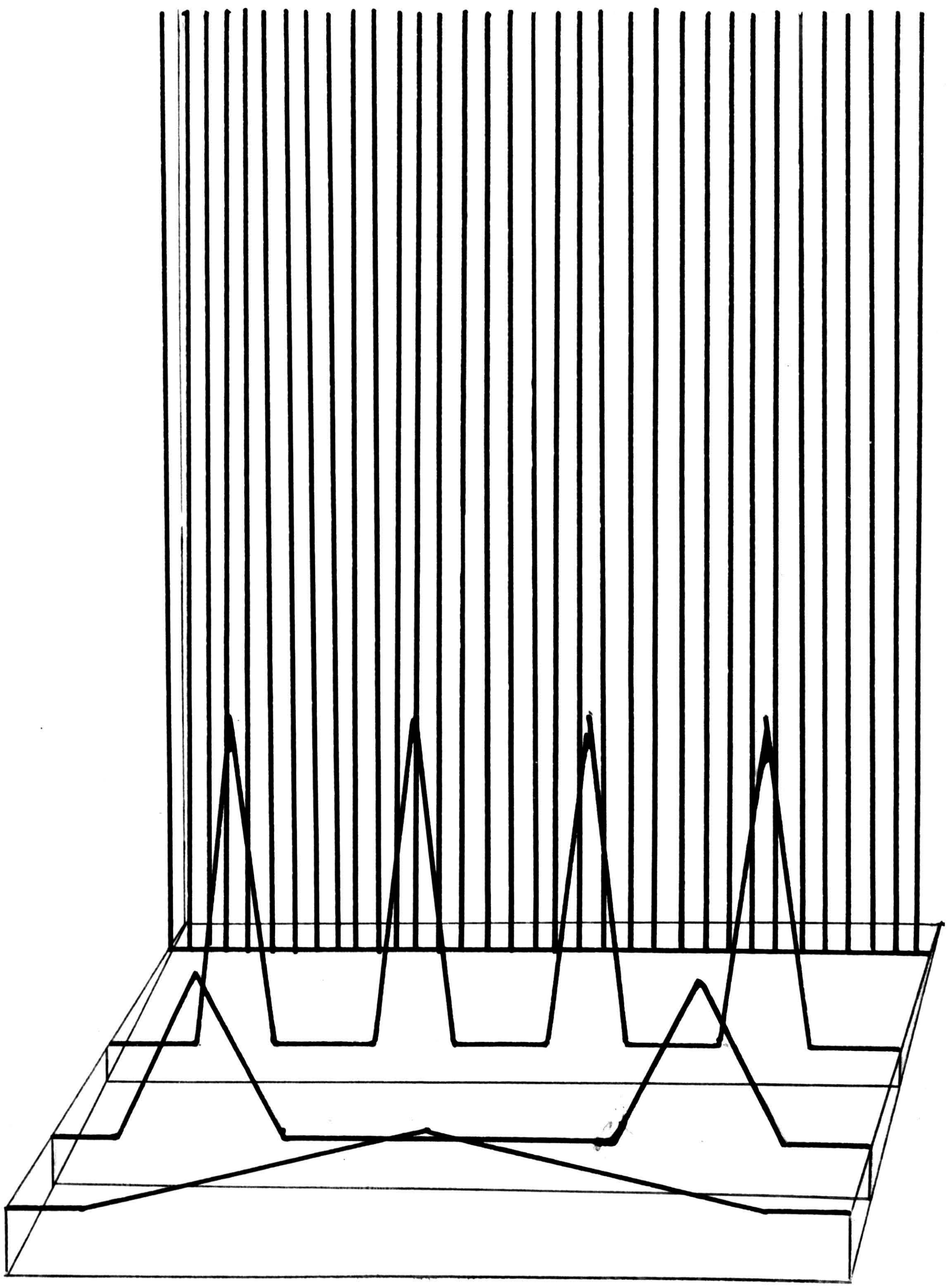


Fig.32.

Fig33.

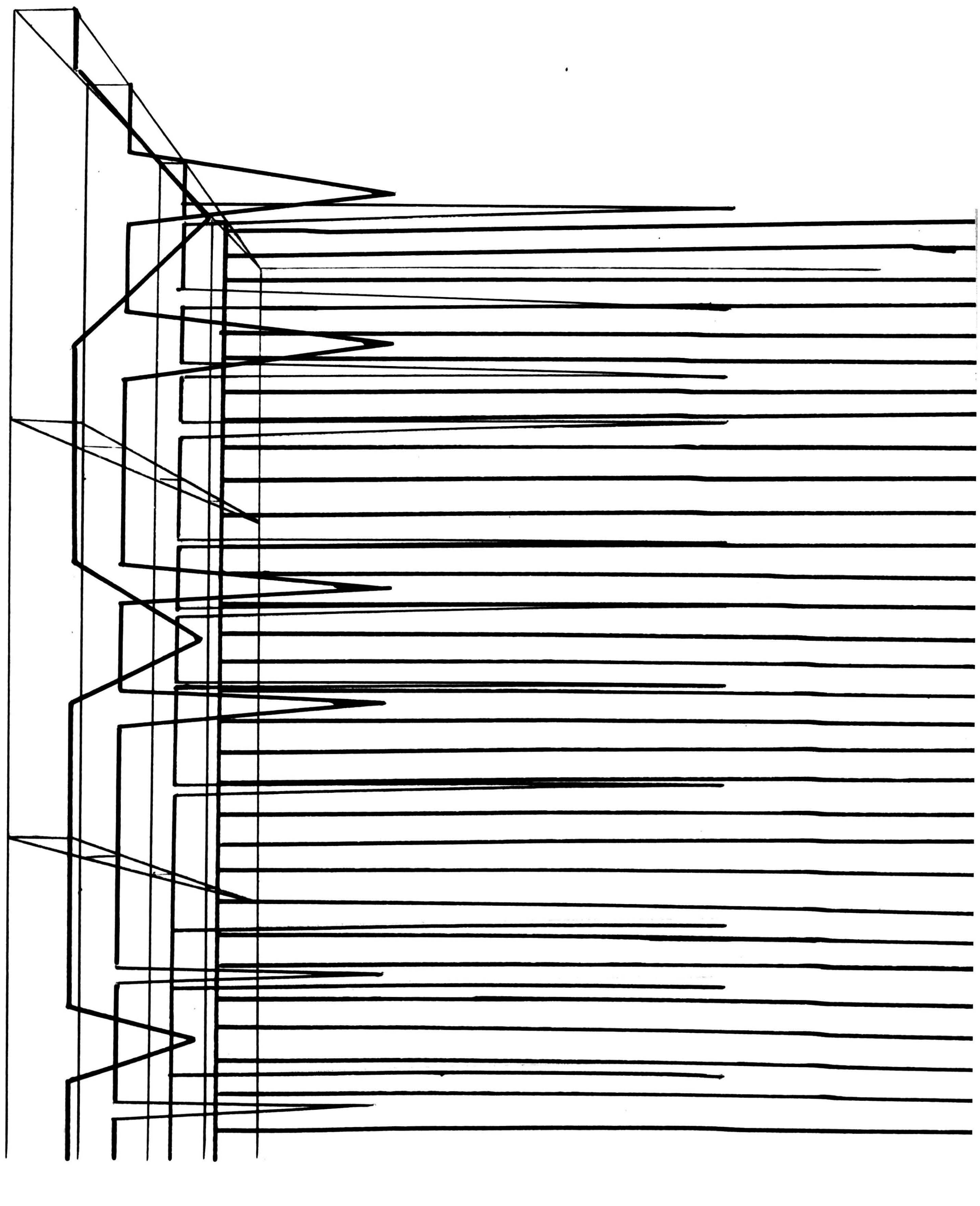
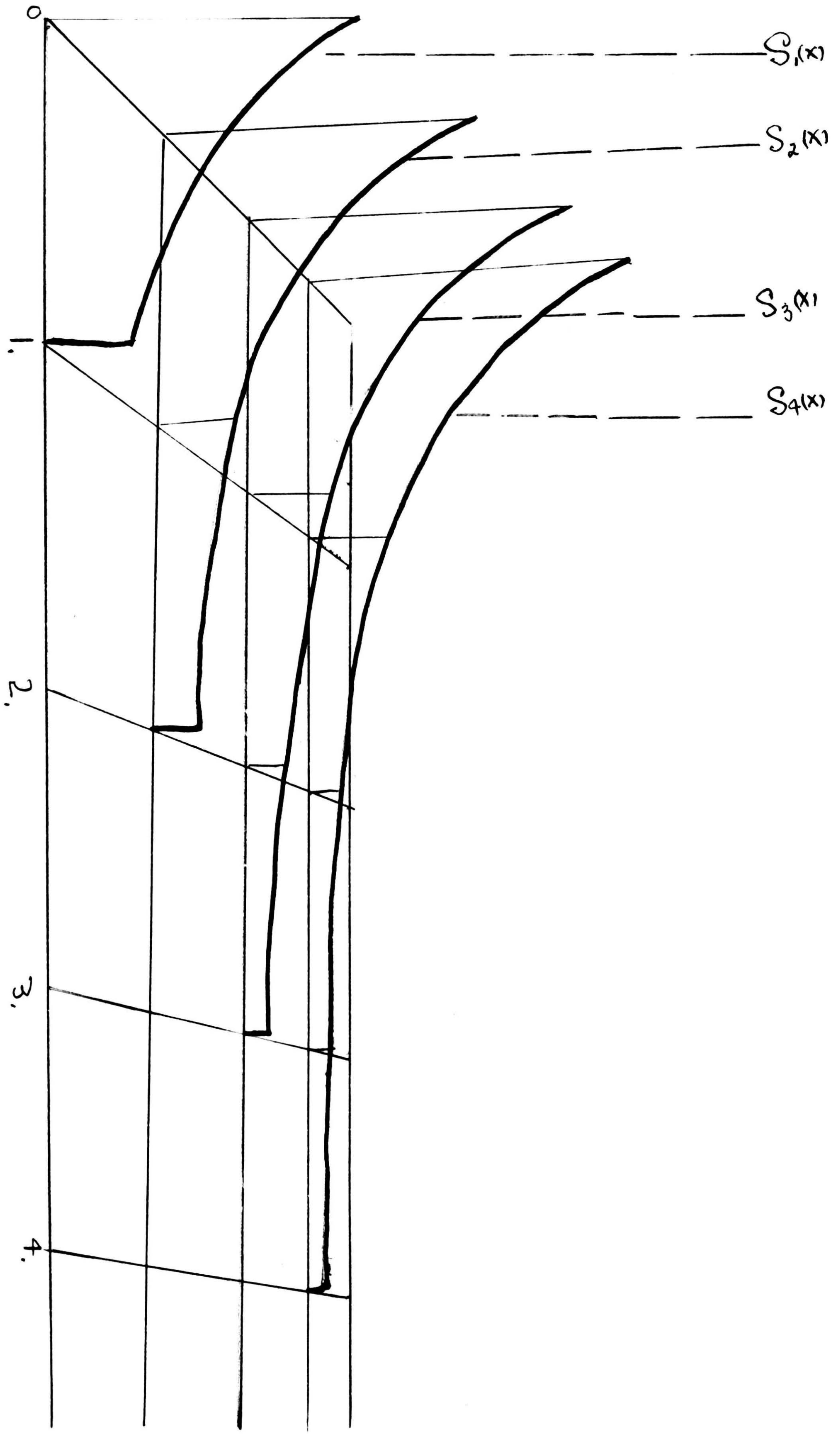


Fig. 37.



not converge at all, hence the integral of the sum of the series does not exist. Figure 33 shows how the same effect may be obtained in an infinite interval.

12. <u>A Possible Interpretation of an Improper Integral</u>. - Evaluating the integral of a function in an infinite interval may be considered graphically, as finding the limit of the integral of the following series as illustrated in Figure 34.

$$\int_0^\infty F(x)\,dx = \int_0^1 F(x)\,dx + \int_1^2 F(x)\,dx + \int_2^3 F x\,dx + \cdots \int_n^{n+1} F(x)\,dx + \cdots\cdots$$

CHAPTER III.

CONTINUITY AND ALLIED CONCEPTS.

1. Definition of Continuity. - The Algebraic definition of continuity is as follows: given f(x) at the point $x = x_0$, f(x) is continuous at that point if $|f(x_0 + h) - f(x_0)| < \epsilon$ [1] when $h < \delta$ [2]. The geometric analagon is shown in figure 35. If ϵ is chosen as the altitude of the rectangle then it must be possible to choose the base of the rectangle small enough (and extending on both sides of x_0), so that the function will remain within the rectangle. δ is not a function of ϵ, in the strict sense of the definition of a function, for if ϵ is known, δ is not uniquely defined. However, a particular value of ϵ does determine an upper limit of δ, and in this sense δ depends on ϵ. A function is said to be continuous in a given interval, if it is continuous at every point within the interval.

Continuity at the "point at infinity", is a slightly more complex notion. A rectangle might be constructed having but three sides, and the approach to the point could be but from one side. At present, we will say f(x) is continuous at the "point at infinity" if the three-sided rectangle in figure 36 can be constructed when $x > k$ #.

A graphical representation which throws the infinite portion of the plane into the finite portion, makes this perfectly tangible, hence a further discussion will be reserved till later, when we take up other than rectangular coordinates .

Fig.35.

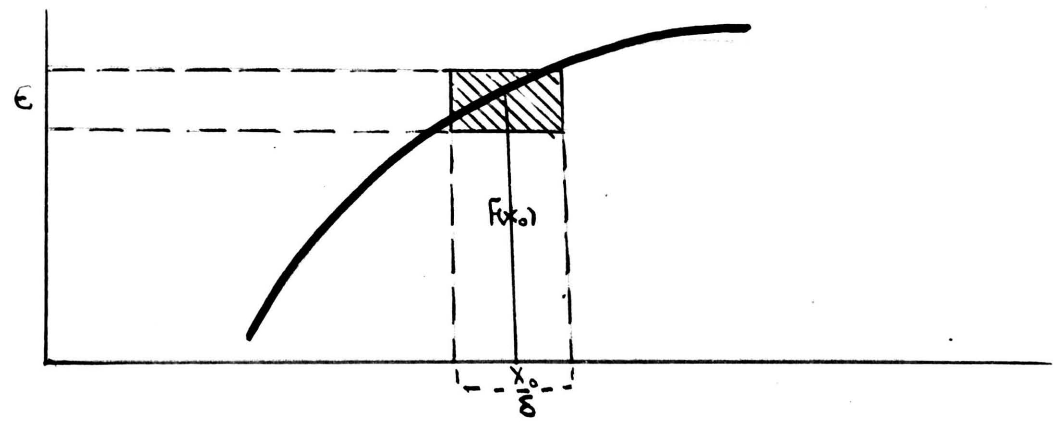


Fig.36.

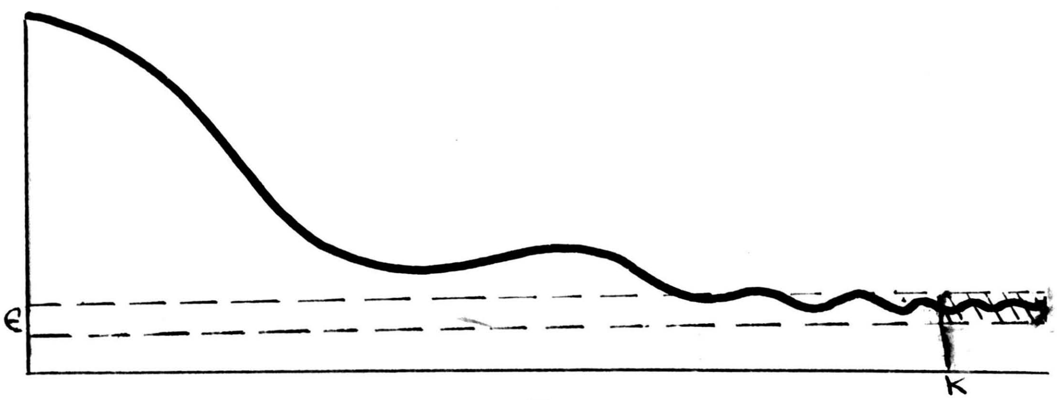
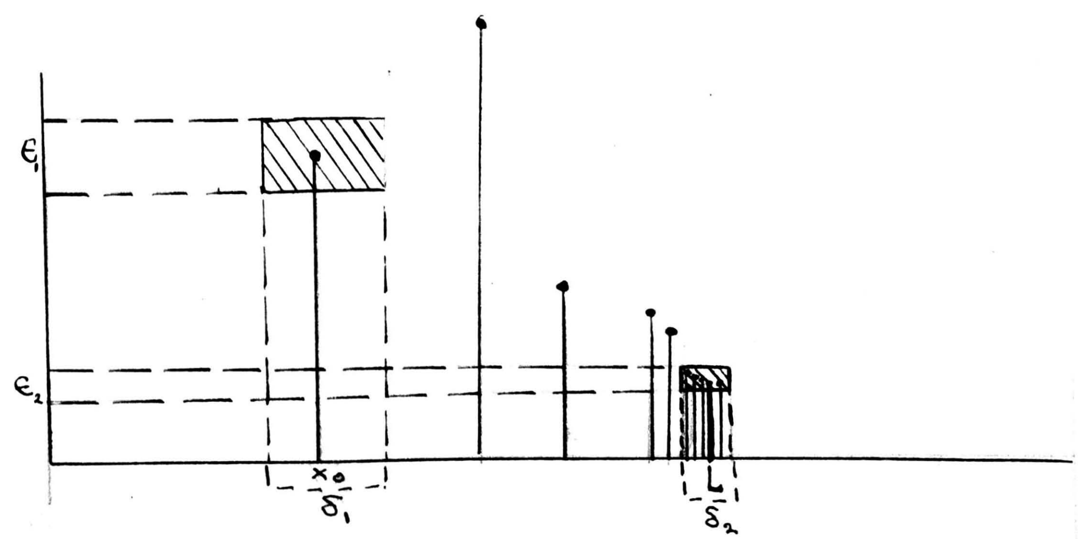


Fig.37.



If the function is defined over a set of points, the definition of continuity is necessarily different as is shown in figure 37. The rectangle definition will hold for the limit points, but at an isolated point, we must adopt the convention, that the rectangle around the isolated point contains no other values of the function, hence the function is continuous at every isolated point.

2. Uniform Continuity. - A function, $f(x)$, is said to be uniformly continuous in an interval $a \leq x \leq b$, if $|f(x_1)-f(x_0)| < \epsilon$, when $|x_1 - x_0| < \delta$, $a \leq x_1 \leq b$. Figure 42 illustrates the property of uniform continuity. The smallest Δx_i is chosen as the value of δ. Any function, continuous in a closed interval, is uniformly continuous in that interval (Goursat-Hedrick's "Mathematical Analysis", Vol. I, pp. 143-144).

3. Intermediate Property. - Closely akin to the property of continuity is the "intermediate property". Its Algebraic definition is as follows: if $f(x_0) = A$ and $f(x_1) = B$, then $f(c) = k$ where $A \leq k \leq B$ and $x_0 \leq c \leq x_1$. The difference between continuity and this last named property is clearly demonstrated in figure 38. In the interval (a,b) the function has the intermediate property, but it is not continuous in the interval, as it has a vibratory discontinuity at the point $x = c$.

Any derivative function has the intermediate property, for let $F(x)$ be a continuous function whose derivative exists at every point in the interval $(a \leq x \leq b)$. Suppose $F'(a) = A$ and $F'(b) = B$. Then $F'(c) = k$ when $A \leq k \leq B$ and $a \leq c \leq b$.

Proof: $$\Lambda(x, h) = \frac{F(x+h) - F(x)}{b}$$

Choose h_0 so small that $\Lambda(a, h_0)$ lies between A and k and

Fig.38.

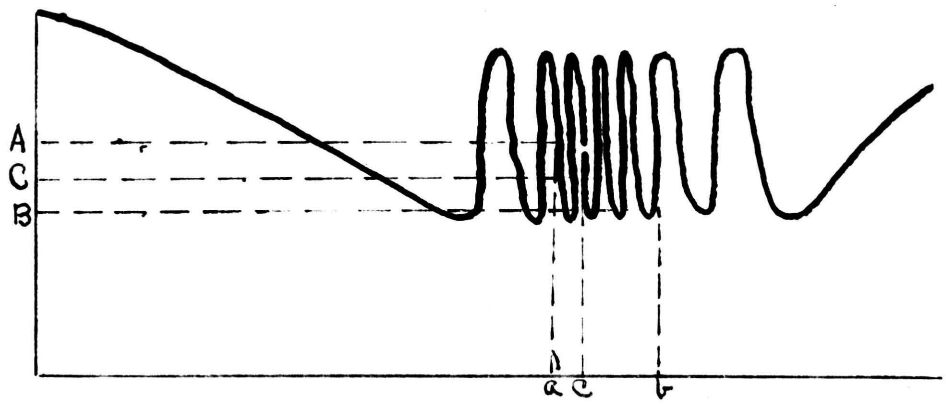
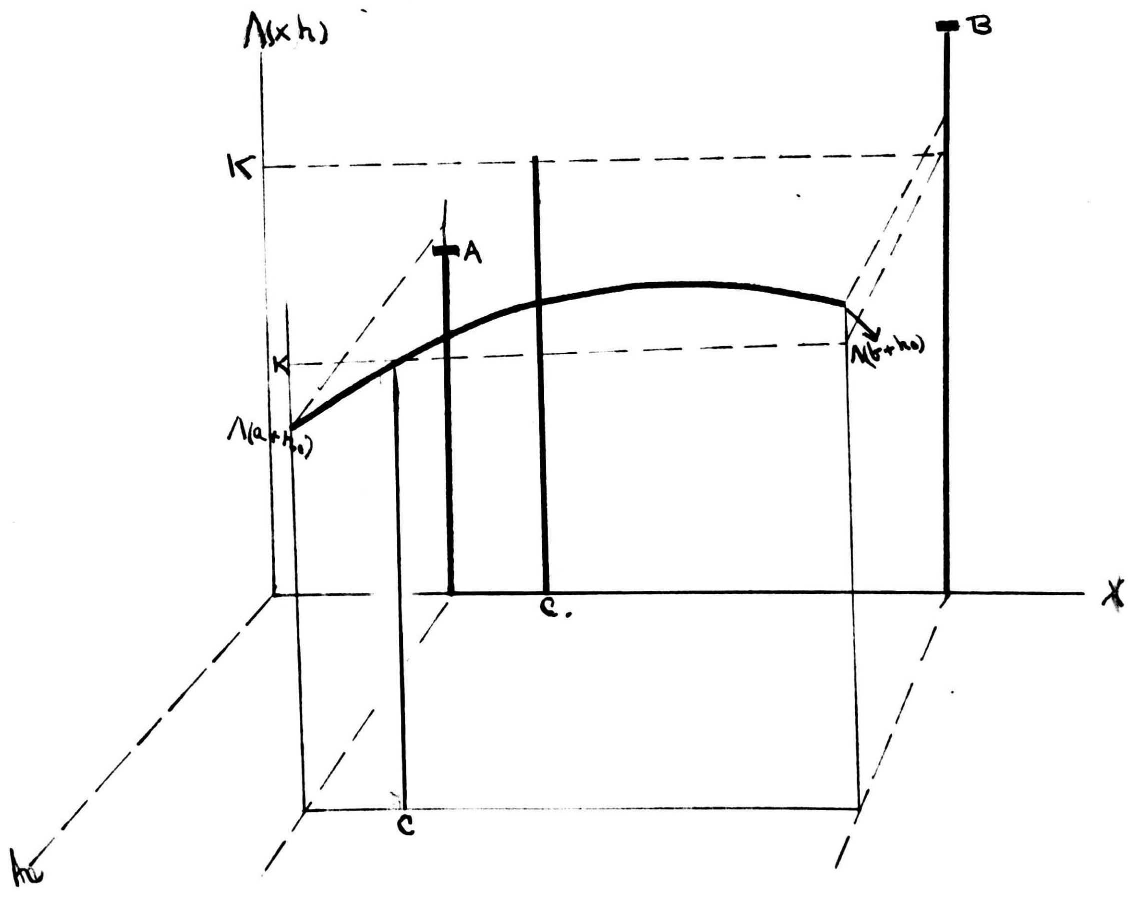


Fig.39.



$\Lambda(b_1, h_0)$ lies between B and k. Since Λ is a continuous function of x, when h is finite, this choice is possible

$$\Lambda(a_1, h_0) < k \quad \text{and} \quad (b_1, h\ ) > k$$

$\therefore \Lambda(x_0 h_0) = k$ (A continuous function has the intermediate property).

$\therefore \dfrac{F(x_0 + h_0) - F(x_0)}{h_0} = k$ and by the law of the mean,

$F'(c) = \dfrac{F(x_0 + h_0) - F(x_0)}{h_0} = k$ (Figure 39 belongs with this proof, due to Lebesque).

From this property of a derivative function it is very evident that such a function, can have only vibratory discontinuities. The kind of discontinuity illustrated in figure 40 would be impossible for a derivative function. If it has any discontinuities at all, they must be of the nature illustrated in figure 41.

4. Circle-fitting. - Some of the figures I have drawn (figures 28,32,33) have a great many sharp-corners. These corners are not discontinuities of the function itself but they do represent discontinuities of the first derivative of the function. Now without altering the nature of these curves, or their particular properties which interested us at the time, these sharp corners may be smoothed off and the first derivative made continuous. Figure 44 shows one of these corners before the circle is fitted into it $(f_1(x))$ and also after the circle has been fitted in $(f_2(x))$. The second derivative may also be made continuous by graphing the first derivative, fitting circles into its corners and adding the area between the two curves to the original function (figure 42). In this manner , these rectilinear figures may be replaced by, so-called "analytic functions", and that without the loss of any of the characteristic properties of the curves.

Fig. 40.

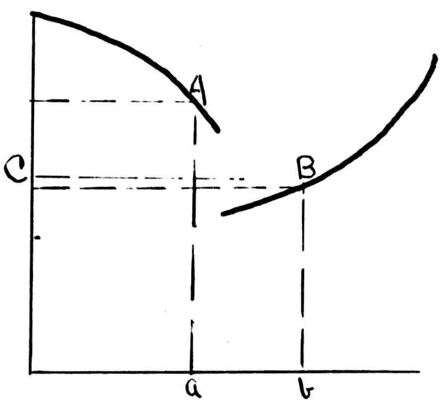


Fig. 41.

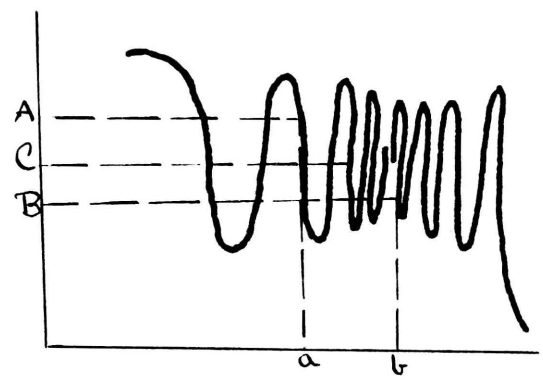


Fig 42.

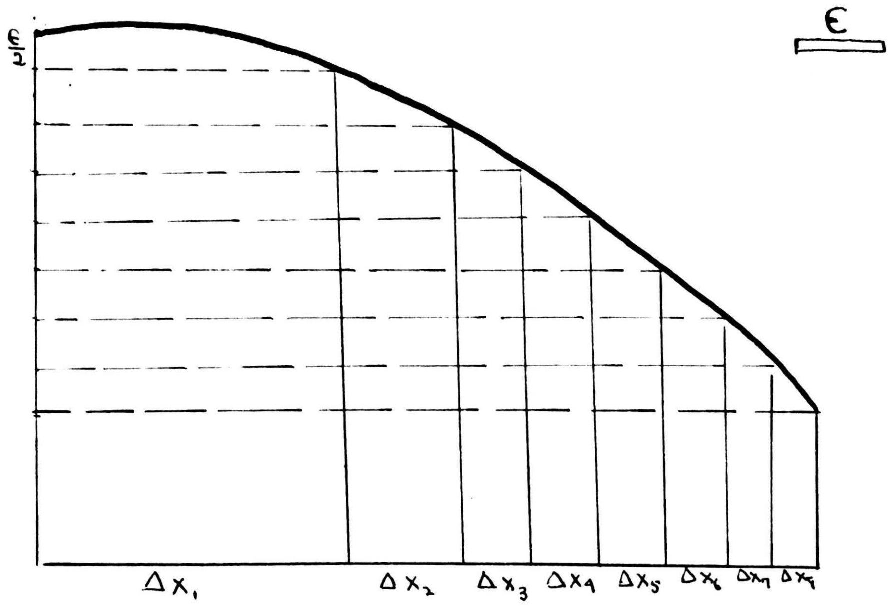


Fig.43.

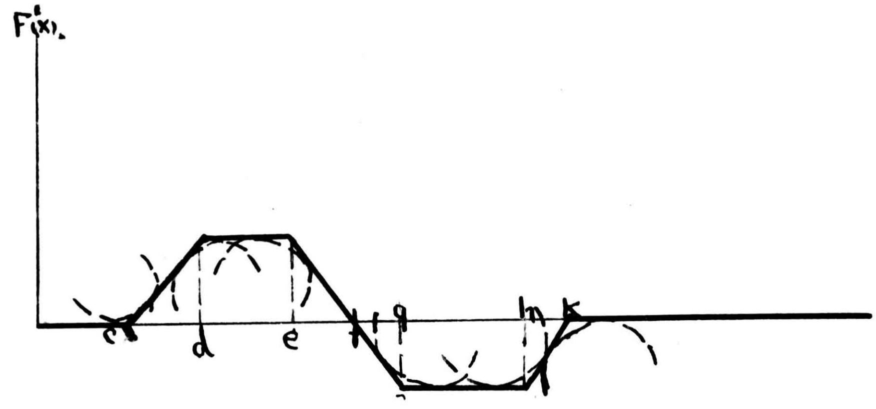
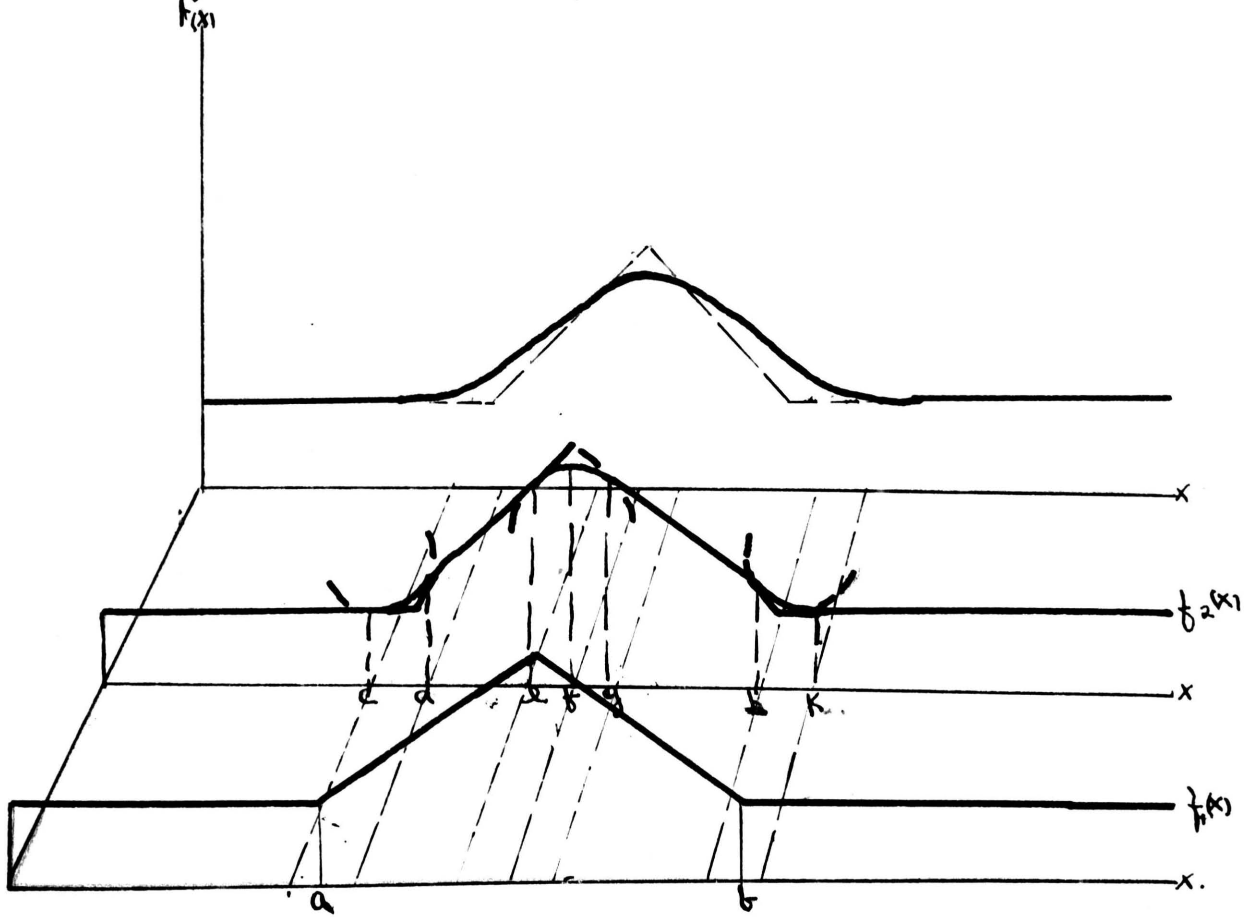


Fig.44.



Polynomial Approximation. - By a very important theorem due to Weierstrass (Goursat-Hedrick's "Mathematical Analysis"p.422) any continuous curve can be approached uniformly by polynomials, so that the error is less than any preassigned quantity. Hence the Algebraic analogon to any of these curves can be obtained and the result is a finite polynomial, the simplest of all functions.

5. The Extrema in an Interval. - The maximum of a function in a given interval (a,b) is the highest value which the function assumes in that interval. The minimum in a given interval is the lowest value which the function assumes in that interval. The maximum or minimum may be an end-point or it may be some interior point (figure 43a,points M and m respectively). The function may not even assume its maximum and minimum values in the interval as in the interval $a \leqq x < b$, where the maximum of the interval is f(b). The function may assume its maximum or minimum values more than once in an interval as M in figure 44a.

6. The Extrema at a Given Point. - The maximum in a given interval is the upper limit of the function in that interval. If the end-points of the interval be made to approach each other, the value of the maximum either remains fixed or diminishes. Similarly since the minimum is the lower limit of the function in a given interval, as the end-points approach each other the minimum either increases or remains fixed.

Suppose that the interval (a,b) is made to shrink onto the point c, so as to obtain a series of "telescoped intervals" whose width approaches zero. The limit, which the maximum of f(x), in these intervals, approaches is called the maximum value of the function at the point c. That is the maximum value of f(x) at x = c is the $\underset{\delta x \doteq 0}{\text{lower limit}}$ maximum value of f(x) in the interval

Fig. 43a.

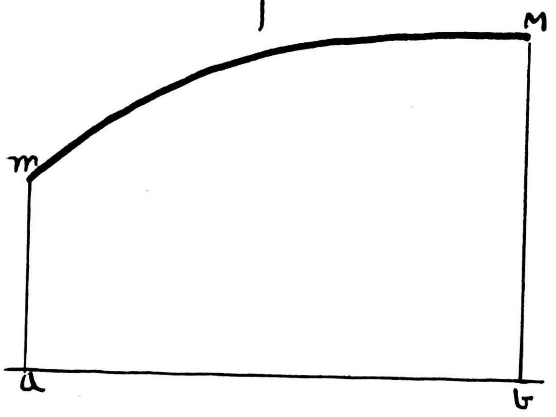


Fig. 44a.

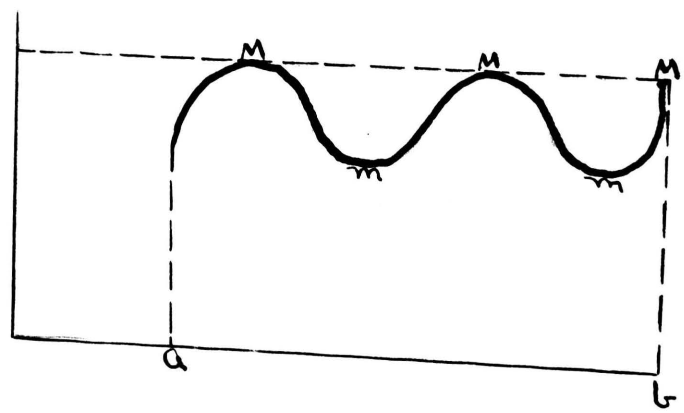
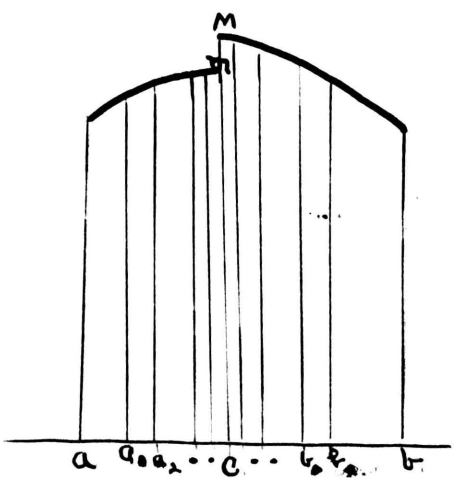


Fig. 45.



$(c - \delta x,\ c \,/\, \delta x)$ and we denote it by M(x). Analgously the minimum value of f(x) at the point c is the $\underset{\delta x \doteq 0}{\text{upper limit}}$ of the minimum value of f(x) in the interval $(c \pm \delta x)$, and it is denoted by m(x).

7. Oscillation. - Oscillation is the difference between the maximum value of the function and the minimum value of the function. It is oscillation and its close connection with continuity which binds the questions of maxima and minima, and continuity together and makes it logically possible for the two to be treated in the same chapter.

If the oscillation of a function is greater than zero at $(x = c)$, that means the construction of the continuity rectangle is impossible (Figure 46). The set of points at which the oscillation of a function is greater than zero, then, is identical with the set of points at which the function is discontinuous.

8. Extremum Points of a Function. - A function f(x) is said to have a maximum at the point x = c, if $f(c) \geqq f(c \,/\, \delta x)$. It is said to have a minimum at a given point x = b, if $f(b) \leqq f(b \,/\, \delta x)$. If the equality sign holds at the point in question, then that maximum or minimum is called weak. If the equality sign is unnecessary then the opposite adjective strong is applied to it. A comparison of figure 47 (a strong maximum) and figure 48 (a weak maximum) illustrates the difference between these two kinds of extremum points.

9. Semi-continuity. - By comparing figures 43,46 and 47, it is evident that if a function has a maximum at a given point, that value is also the maximum value of the function at that point. The converse is rarely ever true.

A function may assume its maximum value at a given point or it may assume its minimum value or any intermediate value.

Fig.46.

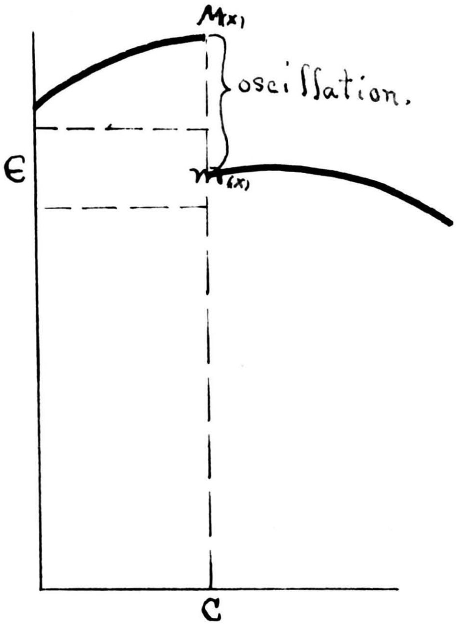


Fig.47.

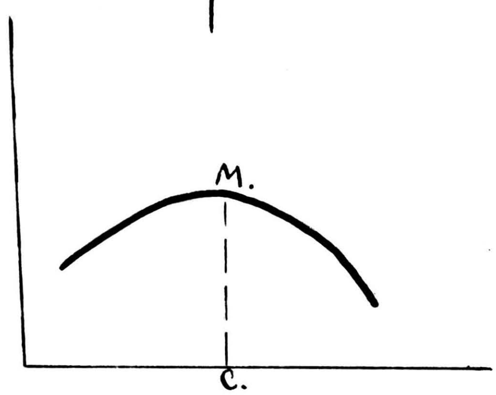


Fig.48

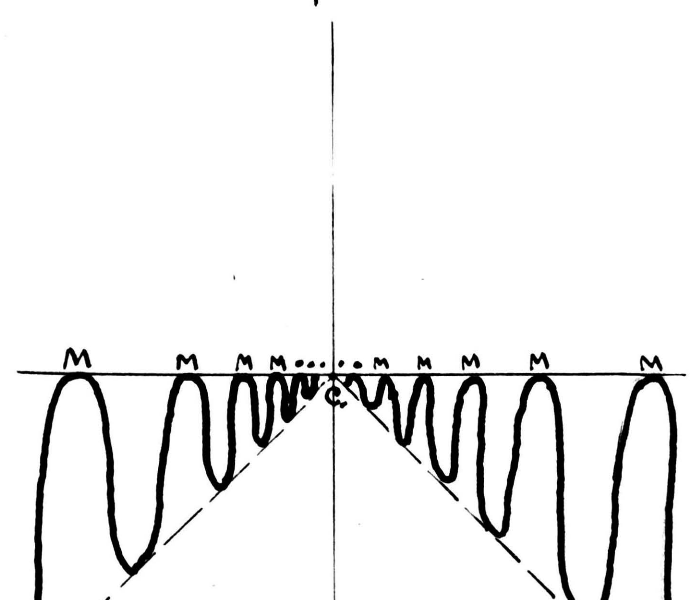
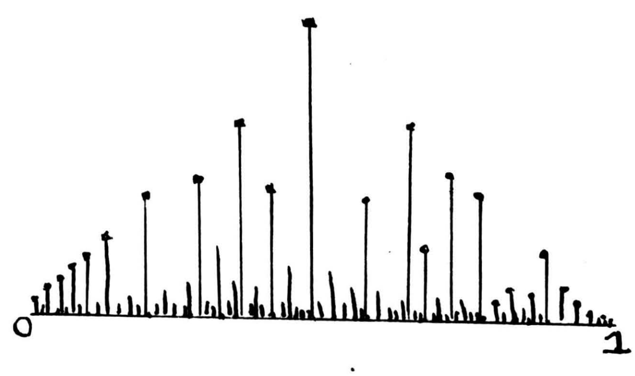


Fig.49.



If, at a given point, it assumes its maximum value that is if $F(x) = M(x)$ it is said to be semi-continuous above. If it assumes its minimum value it is said to be semi-continuous below.

The function, $F(x) = 1/n$ when $m/n = X$ and $F(x) = 0$ otherwise; (figure 49) illustrates a function which is continuous above at every rational point and continuous at every irrational point.

CHAPTER IV.

MEASURE, CONTENT and INTEGRABILITY.

1. Measure. - The definition of measure, as applied to a set of points, corresponds to our general intuitive notion of measurement.

(a). If a given set of points E_1, completely fills a line segment, its measure is the length of that segment.

(b). If a given set E, whose measure is S, contains all the points of a second set E_1, then the measure of E_1 is $\leqq$ S. In figure 50, E_1 is composed of all the points on the line - - segment (A,B), and E is the set indicated. Then by (a) and (b), the measure of E $\leqq$ length AB. The converse is also true. If E is a set of points, whose measure is S, and which is completely contained in E_1, then the measure of $E_1 \geqq$ S. In figure 51, E is the set of points composing the line segment (A B), E_1 is the set composing the line segment (A C). The measure of $E_1 \geqq$ the measure of E. Also the measure of the set $[E_1 - E]$ = measure of E_1 - measure of E .

(c). The measure of the set formed by uniting a countable number of mutually exclusive sets of points, whose measures are respectively $(S_1, S_2, S_3, \ldots S_n, \ldots)$ is $(S_1 + S_2 + S_3 + S_4 + \ldots S_n + \ldots + \ldots)$.

2. Existence of Measure. - Not all sets of points are measurable, for example this set, cited by Van Vleck in the "Transactions of the American Mathematical Society", volume 9. All the points in the interval zero to one, are separated into three sets

Fig 50

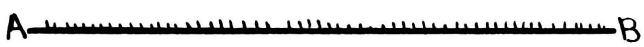


Fig 51

Fig. 52.

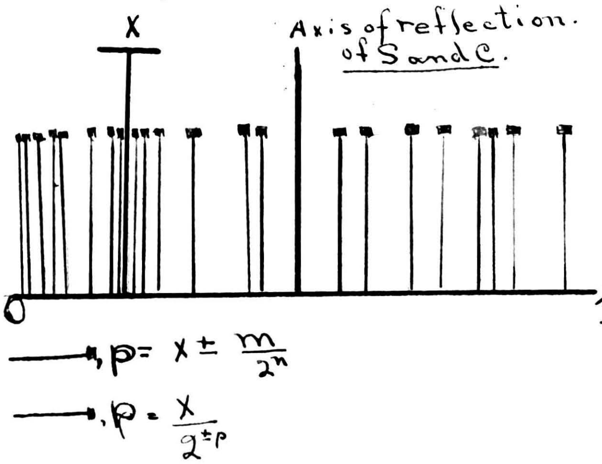


Fig. 53.

Fig. 54.

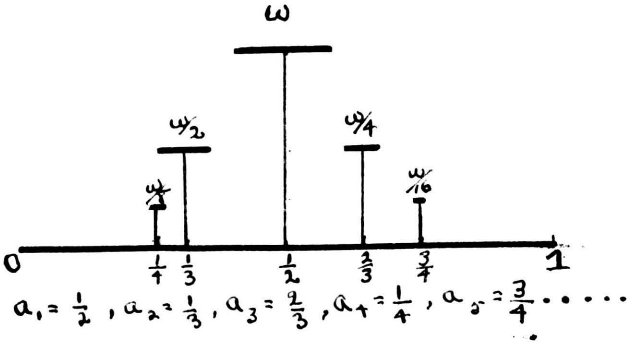


Fig. 55.

S, C, and K. $\underline{x}$ is a member of S and $y = 1 - x$, a member of C. With each x is associated the countable set $\{x \pm m/2^n\}$ and correspondingly with each y, $\{1 - (x \pm m/2^n)\}$. These two sets are distinct except where $x = m/2^n$. These points are assigned to K (they are countable in number). Further when x is assigned to S, the points x/2, x/4, x/8 . . . . . . are also assigned to S and conversely. This distribution will give rise to a conflict in S and C if and only if x is rational. Hence all the rational points are assigned to K.

$$S = \sum \left\{ \frac{x}{2^{\pm p}} \pm \frac{m}{2^n} \right\} ; \; C = \sum \left\{ 1 - \left(\frac{x}{2^{\pm p}} \pm \frac{m}{2^n} \right) \right\} ; \; m, n, p = 0, 1, 2, 3 \cdots$$

S can be easily shown to have 1 as an upper limit to its measure, and since C is merely a reflection of S, 1 is the upper limit of its measure. K, being countable, has the measure zero. Clearly C and S are not measurable for they violate that part of the definition of measure which says: "the measure of a set formed by uniting two or more sets equals the sum of their respective measures". This set is hard to illustrate graphically but figure 52 shows its construction in so far as that is possible.

<u>Any countable set has measure zero</u>. For, let $a_1, a_2 \; . \; . \; . \; . \; a_n \; . \; .$ be a countable set of points (figure 53). Draw an interval of width ω around a_1, an interval of width $\frac{\omega}{2}$ around a_2, an interval of width $\frac{\omega}{2^{n-1}}$ around a_n. Then $a_1, a_2, \ldots a_n, \ldots$ are enclosed in a series of interval whose total width (since ω is arbitrary) can be made as small as we please. By (b) of definition of measure, the measure of $a_1 + a_2 + \ldots$ is less than the sum of the intervals. Hence the measure of the set $< 2\omega$, and therefore equal to zero. Figure 54 shows the measure of the rational numbers from 0 to 1, equals zero.

The measure of a perfect set of points always exists. The perfect set of points, P, in figure 55, is formed in the characteristic manner, by the removal of the interiors of non-abutting intervals from a line segment. The remaining portion of the line segment comprises the complementary set C(P). C(P) is composed of a countable number of sets whose measures are l_1, l_2, l_3, ... l_n, ; . Then the measure of C(P) is (l_1, + l_2 + l_3 + l_4 + ...) - endpoints (By part (c) of definition of measure). The end-points form a countable set, hence their measure is zero. The measure of P = length of the line segment — the measure of C(P), by part (b) of the definition of measure. Therefore the measure of P exists and equals AB - (l_1 + l_2 + ...).

3. Content. - Content is also a scheme for the measurement of a set of points, but right there its analogy to "measure" ends. Content is based upon the Riemann subdivision of an interval.

As illustrated in figure 56 an interval A B is divided into subintervals finite in number, but whose maximum length approaches zero. Let E be a set of points partially filling the line segment A B . Divide this interval by the Riemann method into subintervals. These intervals will than fall into three classes

$I_f = \Sigma$ (intervals full of points in E.).

$I_e = \Sigma$ (intervals empty of points in E).

$I_{ef} = \Sigma$ (intervals partially filled).

Upper Content is defined as

$$\text{lower limit}_{\Delta x \doteq 0} \left[I_f + I_{ef} \right] . \qquad \text{(Figure 57).}$$

This is also equal to the lower limit (A B $- I_e$). The lower content of a set of points is the upper limit of I_f.

Fig.56.

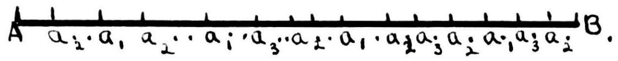


Fig.57.

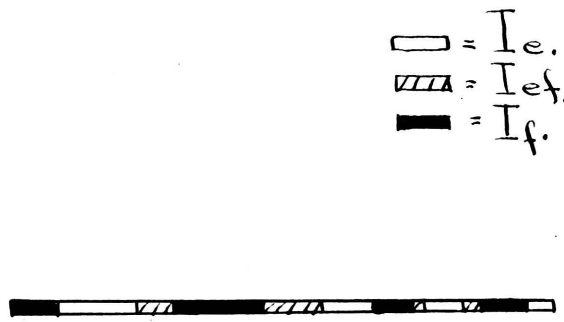


Fig58

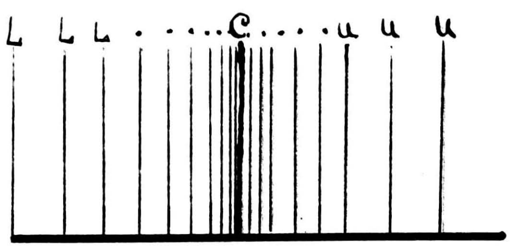


Fig59

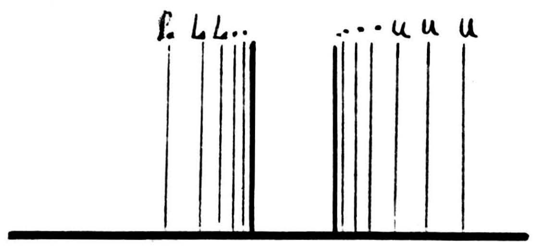
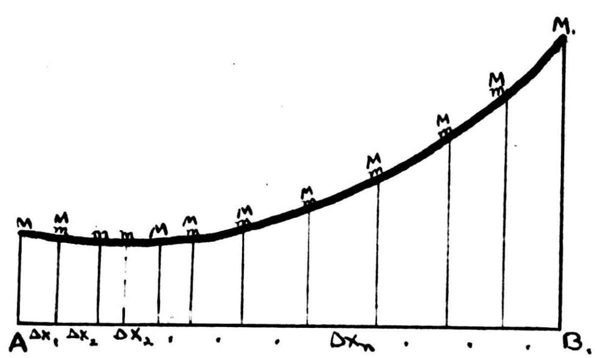


Fig.60.

Fig.61.



$\lim_{\Delta x \doteq 0} \overline{I}_f$ = A B - lower limit ($\overline{I}_e + \overline{I}_{ef}$). The set is said to have content, if both its upper and lower content exist and equal each other.

Successive Riemann divisions tend to increase $\overline{I}_f$ and $\overline{I}_e$ and to decrease $\overline{I}_{ef}$. Hence the first two will have upper limits, and the last named a lower limit.

Figure 58 illustrates the condition of affairs, when a set has content. L and U represent the lower and upper content. Figure 59 represents the case when the set has both upper and lower content and yet has no content.

An examination of the definitions of upper and lower content shows that, whether a given set has content depends upon the value of lower limit of $\overline{I}_{ef}$. If the lower limit of $\overline{I}_{ef}$ equals zero, the set will evidently have content, and otherwise not.

4. Comparison of Content and Measure. - A countable set of points does not necessarily have content zero, for take as an example the rational numbers between zero and one. Successive Riemann divisions give us always an $\overline{I}_{ef}$ equal to the entire line segment. Hence the content of this set does not exist at all. Its measure however is zero, for the measure of any countable set is zero.

The content of a perfect set of points always exists. For in the characteristic method of forming a perfect set, the intervals dropped form $\overline{I}_e$. Any given portion of the line segment, eventually remains undisturbed forevermore, or else never reaches the undisturbed condition. In the first case, these undisturbed portions belong to $\overline{I}_f$. In the second case, the disturbed portion

must decrease and approach zero. Hence the only opportunity for members belonging to I_{ef} to be formed, is in this portion which approaches zero. Therefore a perfect set of points has both measure (proved previously) and content (Figure 60).

5. Analogy of Content and Integrability. - An integral of a function taken over a set of points, or an interval is defined as follows. The interval is divided into Riemann divisions and the two sums S and s are formed (Figure 60).

$$S = M_1 \Delta x_1 + M_2 \Delta x_2 + \ldots M_n \Delta x_n$$

$$s = m_1 \Delta x_1 + m_2 \Delta x_2 + \ldots m_n \Delta x_n$$

The upper integral $\overline{\int}$ is defined as the lower limit of S, and the lower integral $\underline{\int}$ as the upper limit of s. The integral is said to exist, if the $\overline{\int} = \underline{\int}$, and to equal either one of them. Both concepts are based on the Riemann subdivision scheme and the correspondence between the definitions of content and integration is very close.

6. Proper Integrals. - To begin with, we shall consider the function as bounded and the limits of integration as finite - - - that is we will consider only proper integrals. This insures us the existence of M and m and thus simplifies to a considerable degree the question of integrability.

Sufficient conditions for integration are easily found, and I shall give these; gradually broadening them until the necessary and sufficient condition appears.

A function F(x), continuous in an interval (a,b) is integrable (Figure 61). Since the function is continuous at every point, M and m approach the same limit, hence S and s approach the same limit and the integral exists (Figure 62).

Fig 62

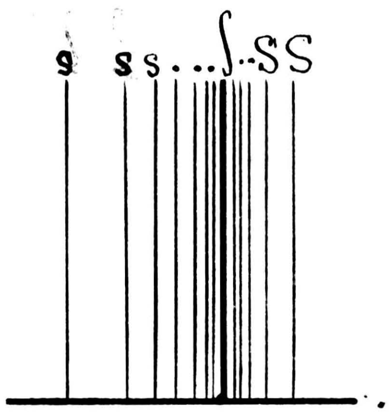


Fig.63.

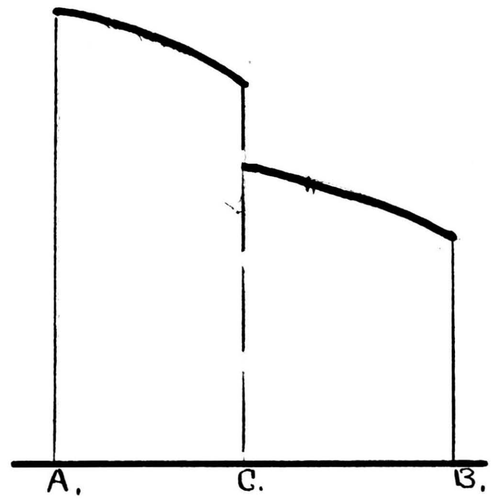


Fig.64.

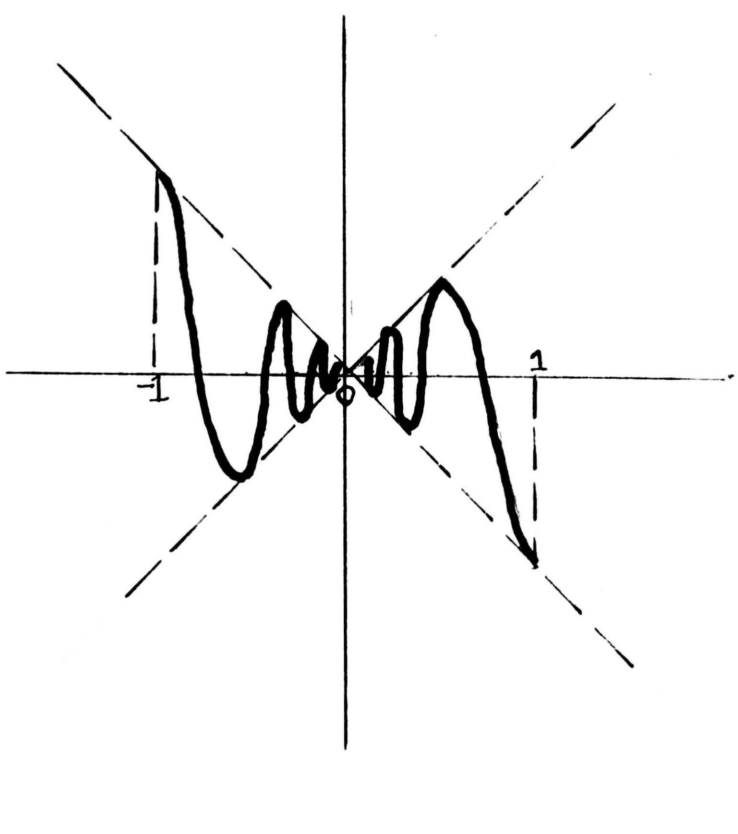


Fig.65.

Figure 63 shows that the integral of a function having one discontinuity exists and may be evaluated by dividing the interval at the discontinuous point and evaluating the two integrals separately.

In an exactly analagous manner a function having a finite number of discontinuities is integrable. Extending the same reasoning a little farther, a function having a countable infinity of discontinuous points is integrable.

7. A Function whose Total Variation is Bounded. - If a function is defined in an interval (a,b), and this interval divided into partial intervals in any manner whatever, and the sum of the oscillation in these intervals is calculated - - the function is said to have bounded total variation if this sum has an upper limit.

A function of bounded total variation is integrable. (Lebesgue's "Lecons sur l' Integration" page 50). The function $f(x) = x \sin \pi/x$ in the interval (-1, -1) is an illustration (Figure 64). This is the last of the sufficient condition and leads us to the necessary and sufficient condition, of du Bois Reymond, of which these preceeding examples are readily seen to be special cases.

8. Du Bois-Reymond Theorems. - Du Bois Reymond developed the following necessary and sufficient condition for integrability that whatever $\epsilon > 0$ may be, the points where the oscillation is greater than ϵ form an integrable set. An assemblage of points situated on a straight line form an integrable set, if they can be enclosed in intervals whose total length may be made as small as we please.

This theorem has been revised into the more convenient

form: In order that a bounded function $f(x)$ be integrable, it is necessary and sufficient that its points of discontinuity form a set whose measure is zero. The developments and proofs of these du Bois-Reymond theorems are given in Lebesgue's "Lecons sur l' Integration et la Recherche des Fonctions Primitives" pp.23-30.

Riemann's statement of the same theorem is: In order that a bounded function be integrable in (a,b), it is necessary and sufficient that (a,b) may be divided into partial intervals, so that the sum of the lengths of the intervals in which the oscillation of the function is greater than ϵ (ϵ is any number greater than zero) is as small as we please. A comparison of this statement with the preceeding brings out the equivalence of discontinuity at a point, and oscillation greater than zero at a point.

Figure 65 illustrates the function $f(x) = 1/n$ when $x = m/n$, where m is less than n and prime to it. $f(x) = 0$ for all other values of x. This function is discontinuous at every rational point but continuous at every irrational point. Its discontinuities then, form a set whose measure is zero. Hence the function is integrable in the interval $0 \leqq x \leqq 1$.

PART II. - Representations on Cylinder and Anchor-ring.

CHAPTER V.

1. Introduction. - Part I, which confined itself to ordinary rectangular co-ordinates, slighted several points, which I shall take up more fully in this Chapter. The question of the extension of the "rectangle definition" of Continuity to the point at infinity; graphical representation of transfinite numbers; and improper integrals;—all these demand some more adequate scheme of representation than the ordinary co-ordinate systems. The difficulty lies in the fact that the "infinite portion of the plane" is not represented at all, by these elementary methods. Projection of the entire infinite plane onto some finite surface is necessary to make the "point at infinity" something tangible and capable of graphical representation.

In what is to follow, I shall suggest a very simple scheme for doing this and develope it far enough so that its characteristics and advantages become apparent.

2. Transformation of the Entire Plane onto a Finite Square. - By means of the equations of transformation,

$$x = \frac{1}{a-m} - \frac{1}{a+m} \qquad \begin{pmatrix} x = \infty & y = \infty \\ n = a & s = a \end{pmatrix}$$

$$y = \frac{1}{a-s} - \frac{1}{a+s} \qquad \begin{pmatrix} x = -\infty & y = -\infty \\ m = -a & s = -a \end{pmatrix}$$

Fig 66

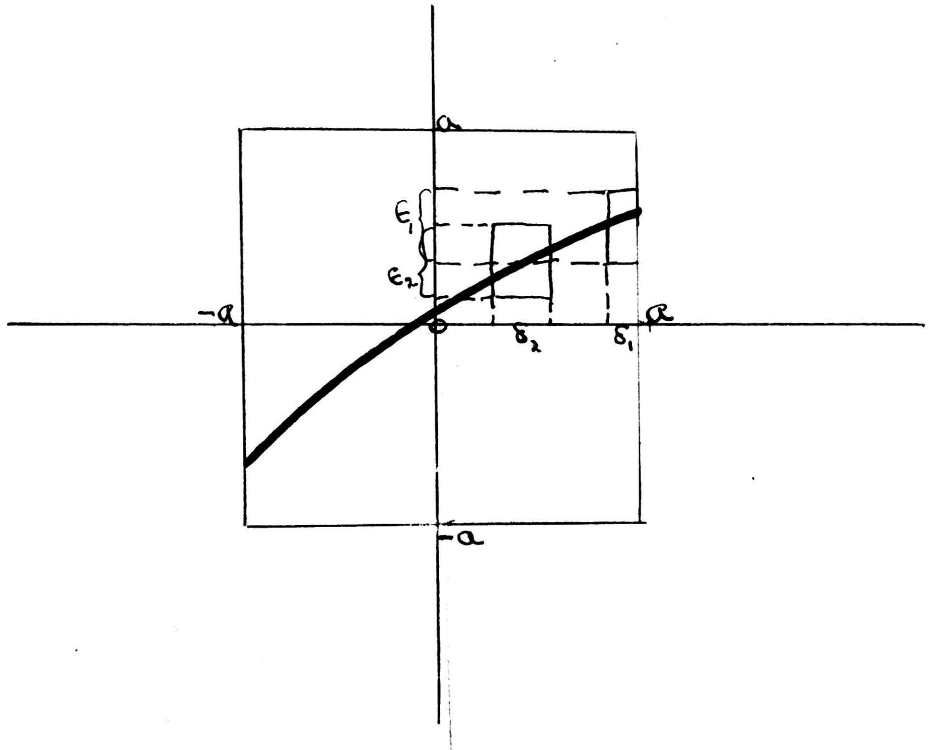
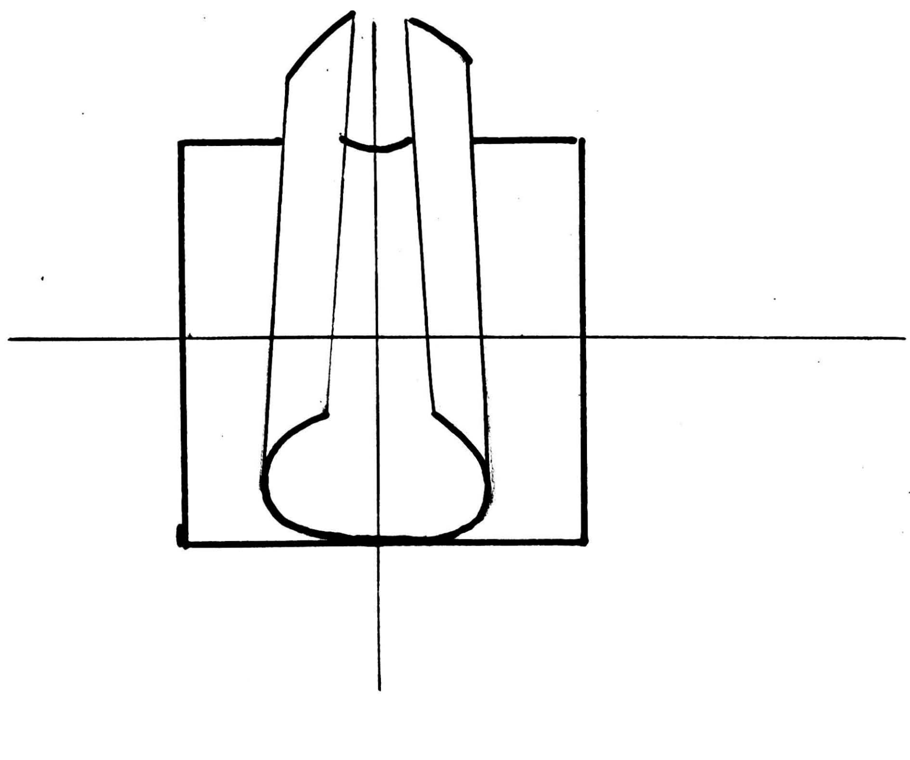


Fig. 67.



the entire infinite plane becomes a square of dimension 2a. Figure 66 illustrates this transformation. The function $y = f(x)$ is graphed on the square.

3. Continuity on the Square. - The rectangle definition of Continuity, given in Chapter 3 of Part I, holds for any point on the interior of this square. Consider the continuity at a point on the boundary, for example - is $\phi(m)$ continuous at the point $m = a$? (Figure 66). Evidently the continuity rectangle cannot be constructed around this point, for the function is not defined to the right of $m = a$.

4. Cylindrical Representation. - The problem is now, to so arrange the representation scheme, that the function is defined both to the right and left of the line $m = a$. Figure 67 illustrates the mechanical means of accomplishing this result. In order to simplify computations, I shall take the dimension of the square equal to 2π. The radius of a circular corss-section of the cylinder is one unit. The transformation equations are

$$x = \frac{1}{\pi - m} - \frac{1}{\pi + m} \qquad \text{and} \qquad y = \frac{1}{\pi - s} - \frac{1}{\pi + s}$$

5. Continuity on the Cylinder. - Figures 68 and 69 show that the rectangle definition of continuity is applicable to all points lying on the cylinder except those points on the circles $s = \pm\pi$. The reason is that the rectangle could not extend on all four sides of points lying on these circles. Figure 70 illustrates the method of overcoming this difficulty.

Fig 68

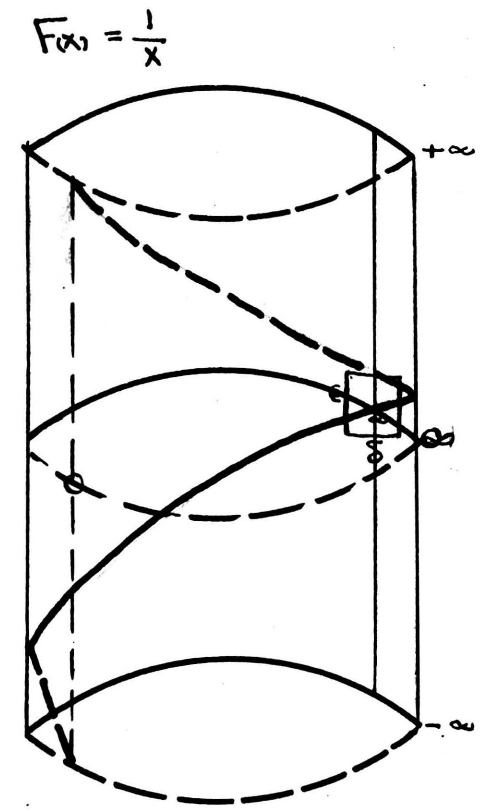


Fig.69.

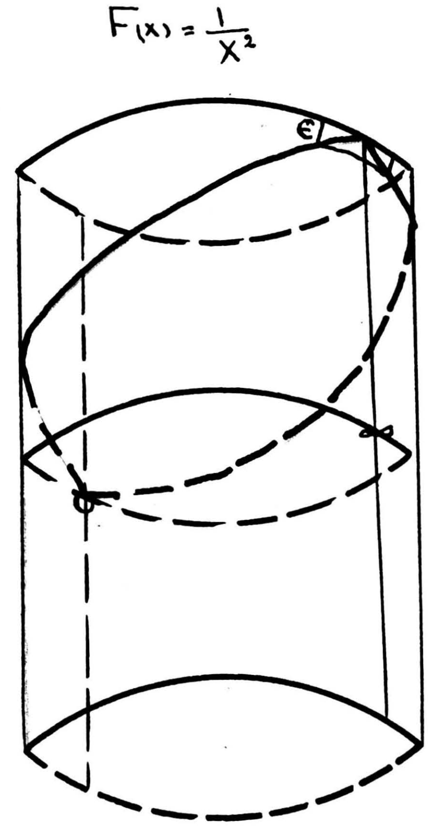
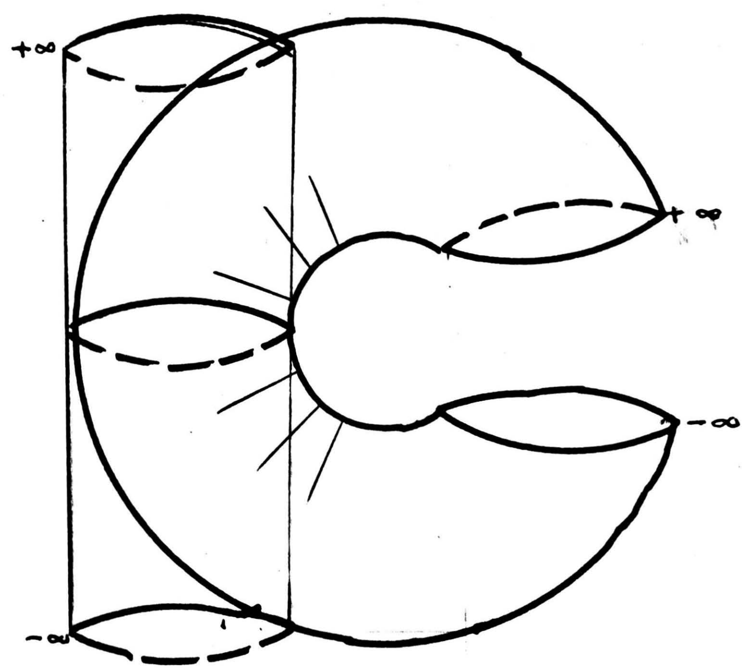


Fig.70.



6. *Representation on the Anchor-ring*. - This method is referred to as the anchor-ring method - the reason is obvious. On *this* surface the rectangle test of Continuity can be applied to *any point whatever*.

To determine whether a given function is continuous at the *point* at *infinity* it must be graphed on the anchor-ring and the rectangle-test applied to it. This means that a definition of continuity can be given which will apply equally well to *all* points *finite* and *infinite*.

7. *Continuity at the Point at Infinity*. - Continuity at the point at infinity admits of the three following definitions,

1. A function may be said to be continuous at the point at infinity if the rectangle illustrated in figure 66 can be constructed. This means

$$|f(x_1) - f(x_2)| < \overset{(1)}{\epsilon} \quad \text{when} \quad x_1 \text{ and } x_2 > \overset{(2)}{K}.$$

2. A function may be said to be continuous at the point at infinity if the rectangle on the cylinder, as in figure 68, can be constructed. This means the function is bounded as x approaches infinity, and, in addition, the limits of $f(x)$ as x approaches $+\infty$ and $-\infty$ are the same.

3. A function may be said to be continuous at the point at infinity, if the rectangle on the anchor-ring can be constructed. This means the function may become infinite as x becomes infinite, but it must become infinite as x approaches both $+\infty$ and $-\infty$, as shown in figure 71.

Fig.71.

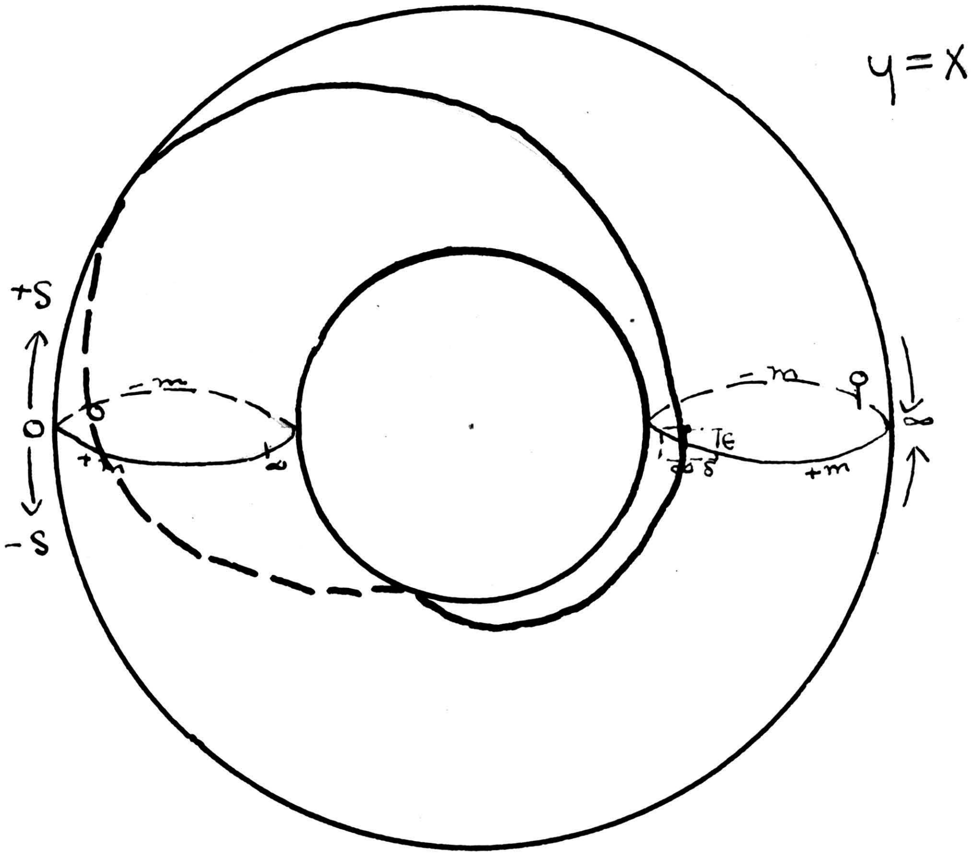
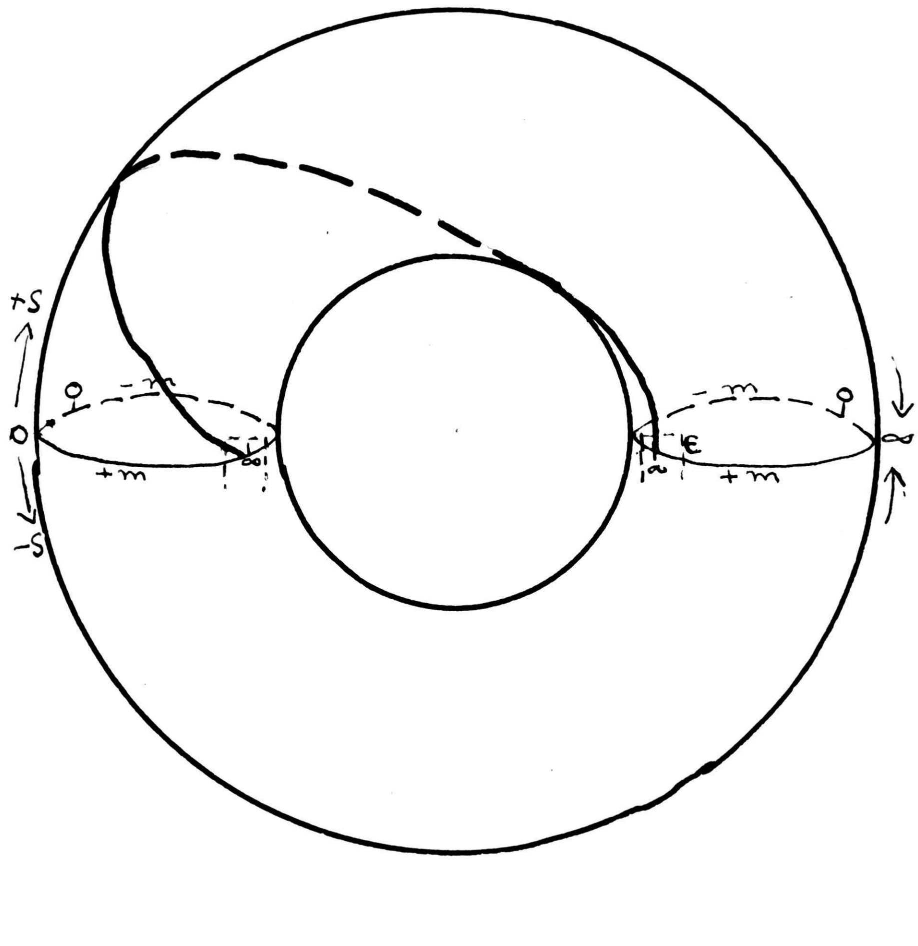


Fig.72.

$y=e^{-x}$



8. An Essential Discontinuity. - A function, continuous on the cylinder will also be continuous on the anchor-ring and in addition, other functions whose continuity on the cylinder is not apparent, are continuous on the anchor-ring. A function which is not continuous, even on the anchor-ring, at the point $x = \infty$, might be said to have an essential discontinuity. This is illustrated in figure 72.

9. The Relationship of Certain Functions Emphasized when those Functions are Represented on the Cylinder. - Figures 73 and 74 are the functions $y = x^2$ and $y = 1/x^2$. The second is merely the first, rotated through an angle of 180°. What happens to the first function in the neighborhood of $x = \infty$, happens to the second function in the neighborhood of $x = 0$, and conversely. These two are then essentially the same function. The same is true of $f(x) = 1/x^3$ and $f(x) = x^3$, and in general $f(x) = x^n$ and $f(x) = 1/x^n$. That is the transformation $x = 1/x$ means, on the cylinder, a rotation through an angle of 180°.

10. Uniformity on the Anchor-ring. - Any function, continuous over a bounded interval is uniformly continuous. On the Anchor-ring the entire plane is a bounded interval, hence any function continuous over the entire plane is uniformly continuous. From previous experience we know this last statement is incorrect. Figure 71 shows that uniform continuity must be differently defined on the Anchor-ring from its definition in ordinary coordinates.

By means of the transformation equations, I shall set up the exact analogon, on the anchor-ring, of

Fig.73.

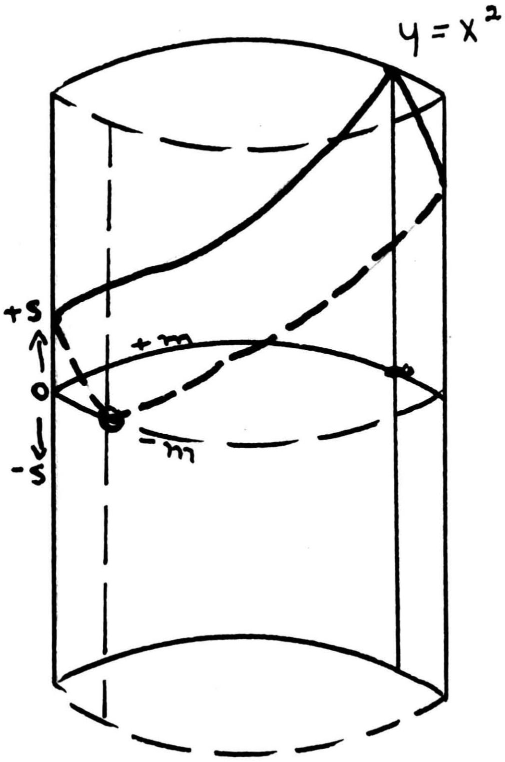


Fig.74.

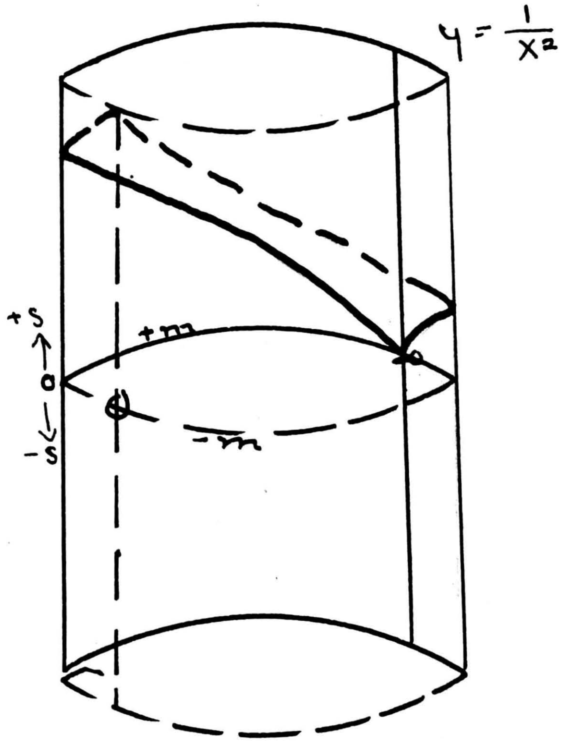


Fig.75.

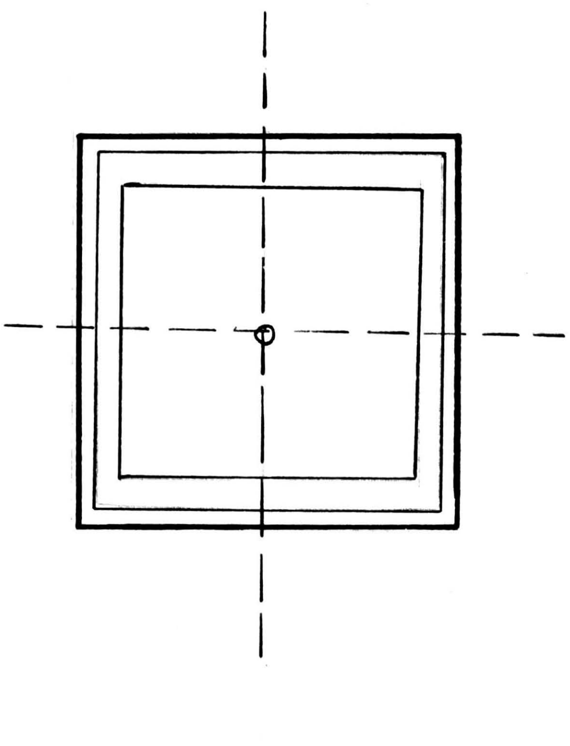
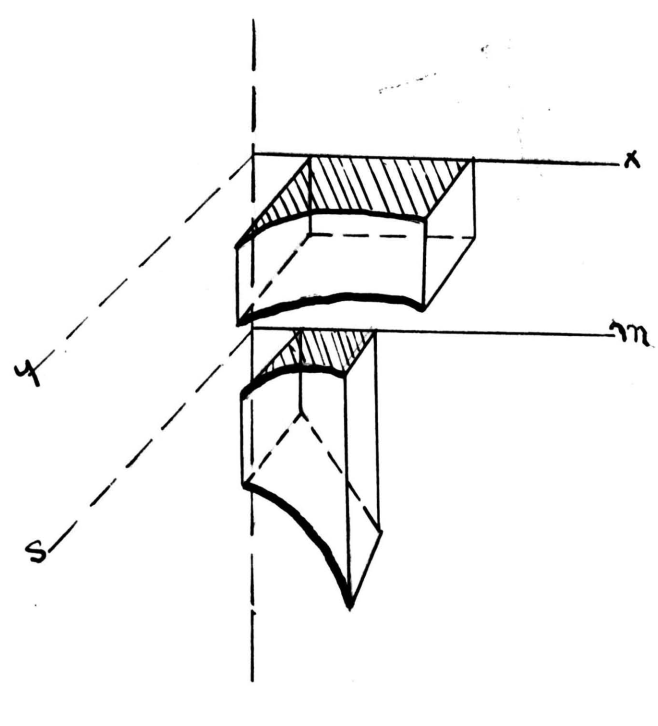


Fig.76.



uniformity in its ordinary interpretation. The definition of uniform continuity is: given $y = F(x)$, in the interval $a \leqq x \leqq b$ and $|F(x_1) - F(x_2)| < \overset{①}{\epsilon}$ when $|x_1 - x_2| < \overset{②}{\delta}$, when $a \leqq \overset{③}{x_1} \leqq b$.

Substituting $\left(x = \frac{1}{\pi - m} - \frac{1}{\pi + m} \right.$ and $\left. y = \frac{1}{\pi - S} - \frac{1}{\pi + S} \right)$ — in $y = F(x)$ and solving for S in terms of m gives the equation $S = \phi(m)$. (This can be done when $F(x)$ is continuous and the transformation equations are continuous #) The transformation equations are continuous in any interval $-\pi < m < +\pi$ and $-\pi < S < +\pi$.

$$|F(x_1) - F(x_2)| = \left| \frac{1}{\pi - \phi(m_1)} - \frac{1}{\pi + \phi(m_1)} - \frac{1}{\pi - \phi(m_2)} + \frac{1}{\pi + \phi(m_2)} \right| < \epsilon$$

Clearing of fractions and simplifying the above condition reduces to $|(\pi^2 + \phi_1\phi_2)(\phi_1 - \phi_2)| < \overset{①}{\epsilon} \cdot k$ when $|(\pi^2 + m_1 m_2)(m_1 - m_2)| < \overset{②}{\delta} \cdot e$. ; $m_a \leqq m_1 \leqq m_b$

$$a = \frac{1}{\pi - m_a} - \frac{1}{\pi + m_a}, \quad b = \frac{1}{\pi - m_b} - \frac{1}{\pi + m_b}.$$

In the definition of uniformity on the anchor-ring developed in the preceeding paragraph, the product $(\pi^2 + \phi_1\phi_2)(\phi_1 - \phi_2)$ takes the place of the difference $(F_1 - F_2)$. That is an element of area (represented by the product) replaces the element of length (represented by the difference). The product just mentioned might also be considered not as an area, but (to keep the analogy with what is to follow, complete) as a line with <u>varying density</u>. If the area is compressed into a single line, its second dimension becomes <u>density</u>.

Goursat-Hedrick "Mathematical Analysis", page #38.

The quantity $(\pi^2 + \phi_1 \phi_2)$ has an upper limit $2\pi^2$ hence it can be treated as a constant and ϕ (m) is uniformly continuous in the interval $-\pi < a \leq m \leq b < +\pi$ if $|2\pi^2(\phi_1 - \phi_2)| < \epsilon$ (1) when $|2\pi^2(m_1 - m_2)| < \delta$ (2) and $a \leq m_1 \leq b$ (3). Therefore the definition of uniformity on the anchor-ring is essentially (except for a constant multiple) the same, in any rectangle on the anchor-ring, as in ordinary rectangular co-ordinates. Figure 75 illustrates a series of such rectangles. Uniform continuity over the entire anchor-ring is as yet undefined. A function might be said to be uniformly continuous over the entire anchor-ring if, being given ϵ, it is possible to choose a δ which will do for any one of these rectangles.

11. Integration on the Anchor-ring. - The definition of a simple integral on the anchor-ring as the area under the curve, will surely not hold, for then there would be no improper integral which could not be evaluated. In the next few paragraphs I shall construct as graphically as possible the definitions of simple and double integrals on the anchor-ring. For convenience and clearness I shall picture the anchor-ring in the "unrolled" form, assuming that the reader will roll it up for himself.

12. A Simple Integral on the Anchor-ring. - Let us construct the analogon of a simple integral on the anchor-ring. In figure 76 an element of area under f(x) goes over into an element of area under $\phi(m)$. $\left[\phi(m) = f(x) \text{ when } x = \frac{1}{\pi - m} - \frac{1}{\pi + m}\right]$. Consider $\int_a^b f(x)\,dx$ not as the area under f(x) but as the volume of a solid of unit height whose upper face is the area under f(x) as in figure 76. This solid when transformed onto the

Fig77

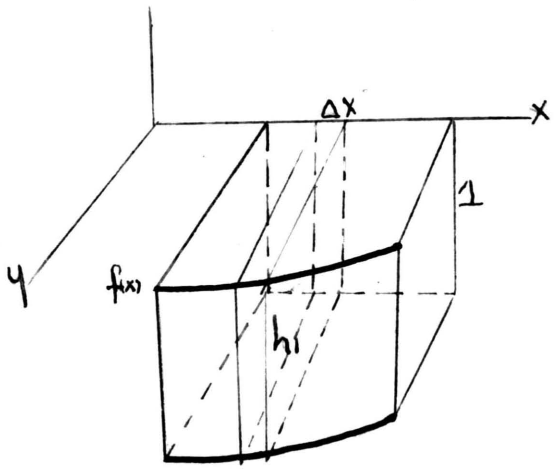


Fig.78.

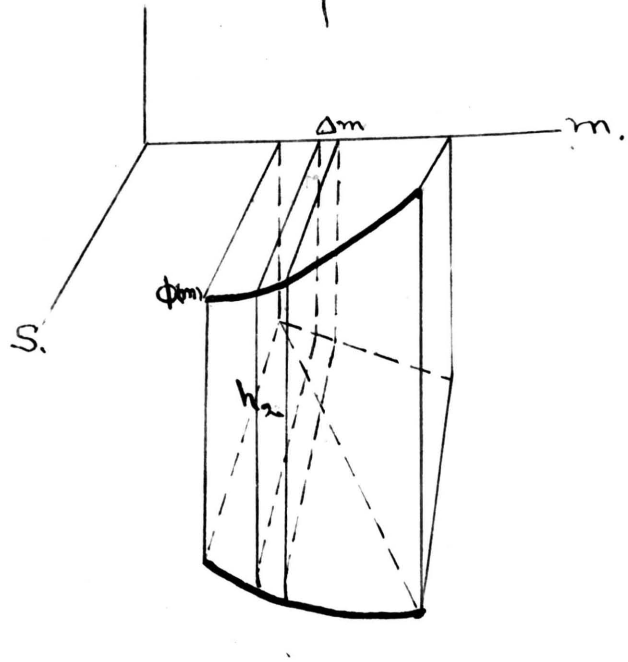


Fig79

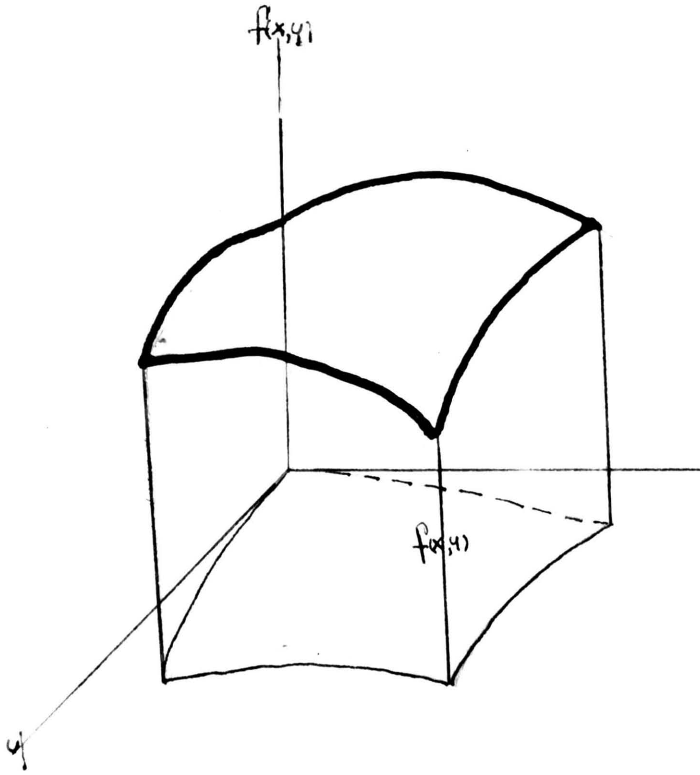
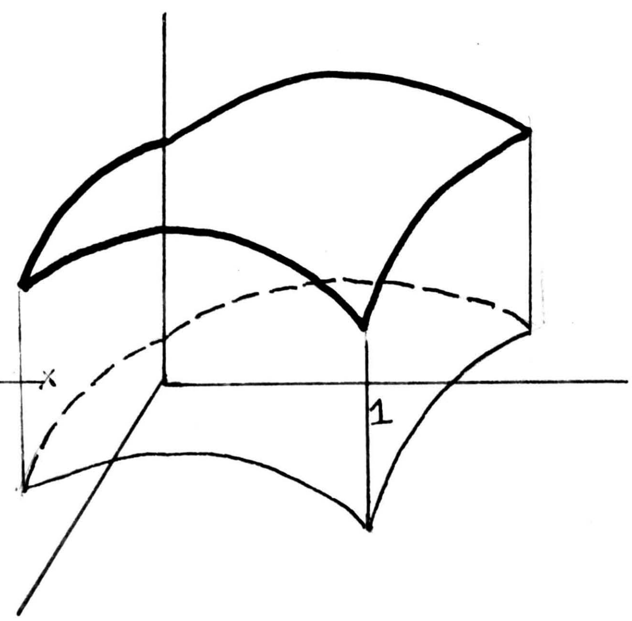


Fig.80.



anchor-ring has its base diminished in size, hence its height must be increased in order to maintain the equivalence of volume. From figures 77 and 78 we get the equations, -
$\Delta x \cdot f(x) \cdot 1 = \Delta m \cdot \phi m \cdot h_2$; $f(x) = \phi(m)$; $vol\,1 = vol\,2$; hence $h_2 = \frac{\Delta x}{\Delta m}$, and passing to the limit as Δx and Δm approach zero, $h_2 = \frac{\partial x}{\partial m}$.

Compress the solid, whose base is $\phi(m)$ and height $\frac{\partial x}{\partial m}$, back into the (S,m)plane. Its height becomes the density of the area under $\phi(m)$ and we obtain the equation -

$$\int_a^b f(x)\,dx = \int_{m_1}^{m_2} \phi(m)\,\frac{dx}{dm}\,dm .$$

In rectangular coordinates, the simple integral represents an area, whose density is equal to one. On the anchor-ring a simple integral represents an area of variable density, and in evaluating the area, it must be multiplied by the density at each point. The density is a function of m, namely $\frac{dx}{dm} = \frac{2(\pi^2+m^2)}{(\pi-m)^2(\pi+m)^2}$

13. The Region of Validity of the Preceding Result. - Assuming that F(x,y) is everywhere integrable, the result obtained in the preceding paragraph is valid for any interval within which $\frac{dx}{dm}$ is continuous#. $\frac{dx}{dm}$ is continuous in any region $-\pi < x < +\pi$. If the region of integration is extended to include the end-points $\pm\pi$, which correspond to $x = \pm\infty$, we no longer have a proper integral.

The result obtained coincides with the ordinary interpretation of transformation under the integral sign, and hence is valid under the same conditions. Goursat-Hedrick "Mathematical Analysis," Volume I, page 296.

14. Improper Integrals. - A simple integral is called an improper integral if either the independent or dependent variable become infinite within the interval of integration. I shall consider the type: $\int_a^\infty f(x)\,dx$.

$$\int_a^\infty f(x)\,dx = \lim_{n \doteq \infty}\int_a^n f(x)\,dx = \lim_{l \doteq \pi}\int_{m_a}^{l} \phi(m)\frac{dx}{dm}\,dm$$

$$\lim_{l \doteq \pi}\int_{m_a}^{l}\phi(m)\frac{dx}{dm}\,dm = \int_{m_a}^{l}\phi(m)\frac{dx}{dm}\,dm + \int_{l}^{\pi}\phi(m)\frac{dx}{dm}\,dm.$$

(If two variables are constantly equal, their limits, if they exist are also equal.

If $\left|\int_m^\pi \phi(m)\frac{dx}{dm}\,dm\right| < \overset{①}{\epsilon}$ when $|\pi - m| < \overset{②}{\delta}$, then the improper integral may be evaluated.

The fact that the integrand approaches zero is neither a necessary nor a sufficient condition. In figure 33 an example is given of an improper integral, which can be evaluated, yet the value of the integrand increases beyond all limit as $x \doteq \infty$. It is evident, however, that the following is a necessary condition: in order that an improper integral may be evaluated it is necessary that ϵ, being chosen first, it is possible to choose k, so large that $|f(x) - 0| < \epsilon$ when $x > k$, with the possible exception of a set of points whose measure is zero.

This necessary condition in the anchor-ring co-ordinates means, $\phi(m)\frac{dx}{dm}$ must approach zero as m approaches π, except for a set of points of measure zero.

$$\phi(m)\frac{dx}{dm} = \phi(m)\frac{2(\pi^2+m^2)}{(\pi-m)^2(\pi+m)^2}$$

Let m approach π through positive values, then

$$\lim_{m \doteq \pi} \frac{2(\pi^2+m^2)}{(\pi+m)^2} = \frac{4\pi^2}{4\pi^2} = 1.$$

Therefore $\phi(m) \cdot \frac{1}{(\pi-m)^2}$ must approach zero as $m \doteq \pi$. That is $\phi(m)$ must be an infinitesimal of higher order than the second with respect to $(\pi - m)$ in the neighborhood of $m = \pi$.

If m approaches $-\pi$ through negative values an exactly analagous process gives us the result that $\phi(m)$ must be an infinitesimal of higher than the second order with respect to $(\pi+m) = (\pi-(-m))$.

A sufficient condition for the existence of an improper integral is that the function in question eventually lies under the curve $y = x^{\mu-1}$ (with the possible exception of a set of points, whose measure is zero). $\mu < 0$ #. That means on the anchor-ring that $\phi(m)\frac{dx}{dm}$ must lie under the curve, $y = x^{\mu-1}$ or its analogon on the anchor-ring

$$S = \frac{-(\pi^2 - m^2)^{\mu-1} + \sqrt{\pi^2 (2m)^{3\mu-3} + (\pi^2 - m^2)^{2\mu-2}}}{(2m)^{\mu-1}}$$

15. Double Integrals. - The construction on the anchor-ring of an equivalent to the double integral is somewhat analagous to the construction of the simple integral. In figure 78, the $\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x\ y)\ dy\ dx$ is the volume of the solid lying under the surface f(x,y). That is the usual interpretation of the of the double integral. I would like to take a slightly

Goursat-Hedrick "Mathematical Analysis", page 177.

different one here. The $\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}$ f(x y) dy dx = the mass of the indicated solid, the density being equal to unity. Or the $\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}$ f(x y) dy dx is the mass of a solid whose height is is unity and whose density is f(x,y). Figure 70.

Transforming this last volume onto the anchor-ring, the base of figure 70 is diminished in size hence, to preserve the equivalence of mass (hence of volume), the height of the transformed solid must be increased. The ratio of increase is given by the formula:

$$\# \iint f(x\,y)\,dx\,dy = \iint f(\Phi,\Psi)\frac{D(\Phi,\Psi)}{D(u,v)}\,du\,dv \qquad \begin{cases}\Phi = x = \dfrac{1}{\pi - m} - \dfrac{1}{\pi + m}\\[2mm] \Psi = y = \dfrac{1}{\pi - s} - \dfrac{1}{\pi + s}\end{cases}$$

$$\frac{D(\Phi\,\Psi)}{D(u,v)} = \begin{vmatrix}\frac{\partial x}{\partial m}, & \frac{\partial y}{\partial m}\\ \frac{\partial x}{\partial s}, & \frac{\partial y}{\partial s}\end{vmatrix} = \frac{2(\pi^2+m^2)}{(\pi^2-m^2)^2}\cdot\frac{2(\pi^2+s^2)}{(\pi^2-s^2)^2} = \frac{4(\pi^2+m^2)(\pi^2+s^2)}{(\pi^2-m^2)^2(\pi^2-s^2)^2} \qquad u = m,\ v = s$$

The determinant used in the preceding paragraph is the well-known "Jacobian". It represents the ratio in height of the two solids we are considering.

The double integral, on the anchor-ring, represents, the mass of a solid whose density is a variable, a function of the m and s co-ordinates,- namely the product of the partial derivatives of x(with respect to m) and of y(with respect to s).

\# See Goursat-Hedrick "Mathematical Analysis", Vol. 1, p.302.

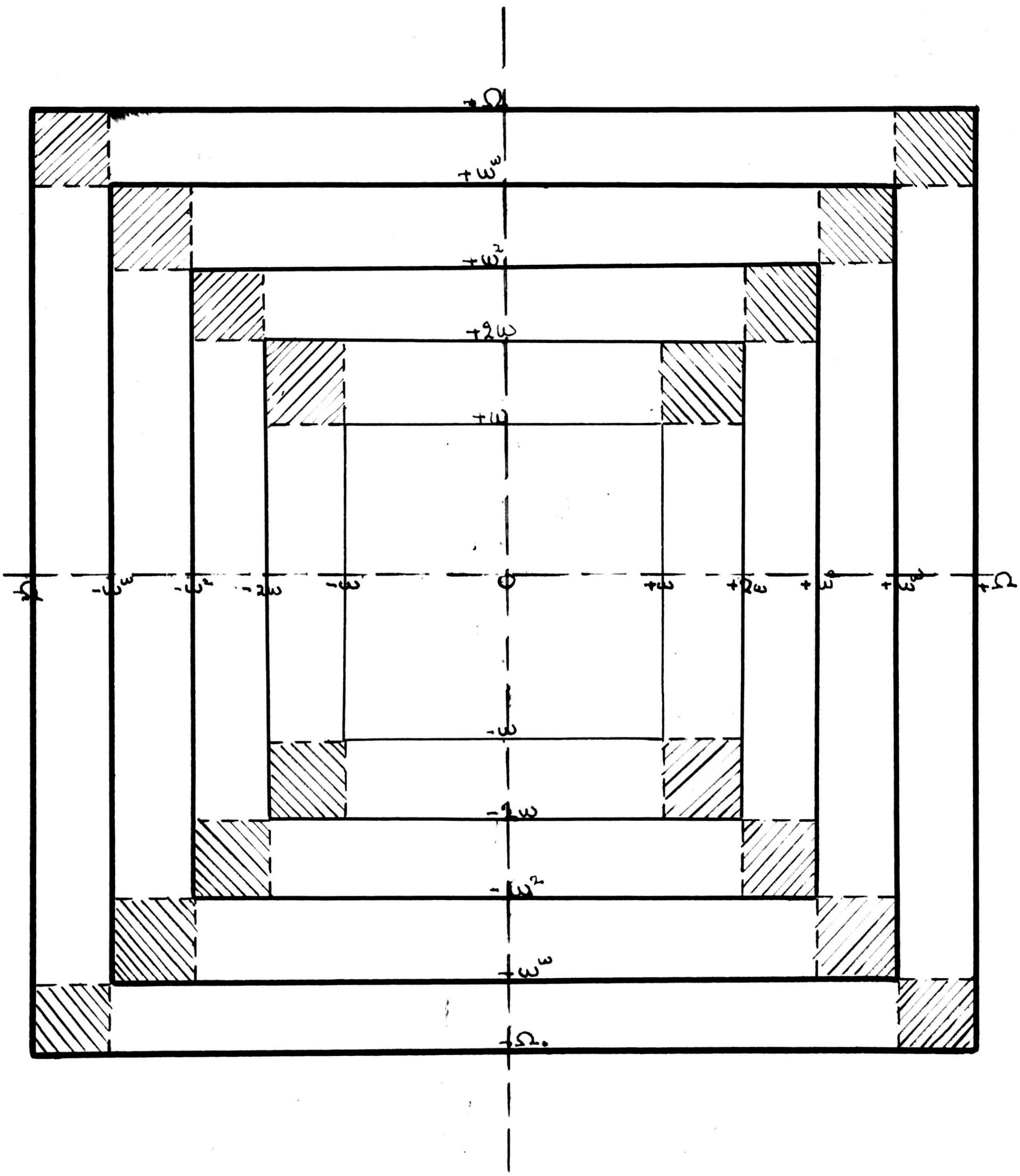
For conditions under which, and limits within which, these results are valid, see Goursat-Hedrick "Mathematical Analysis", page 301.

16. <u>General Remarks</u>. - The advantage of this anchor-ring method of representation of infinite points lies - not so much in the value of any especial formulae which may arise from it - but rather from the fact that the "point at infinity" is brought <u>within reach</u>, where it can be graphed, tested and treated exactly as any finite point - with certain limitations. We are thus, given the very vivid graphic point of view of the characteristic properties of the "point at infinity", as opposed to the rather hazy unreal one derived from a merely formal treatment, from an Algebraic standpoint.

17. <u>A Possible Graphical Representation of Transfinite Numbers</u>. - For definitions and a complete treatment of the subject of transfinite numbers, see Baire's "Lecons sur les Fonctions Discontinues", pages 43-45. I am, here, merely going to suggest a possible graphic interpretation of transfinite numbers or what might be termed transfinite space.

In figure 80 the entire infinite plane is represented in the innermost square. The boundary lines of the square are $x = \pm\omega$ and $y = \pm\omega$. The transfinite numbers from $|\omega|$ to $|2\omega|$ are represented in the "frame" immediately surrounding this square. In the next largest frame all numbers from $|2\omega|$ to $|\omega^2|$ are represented. The next in order, takes in all numbers from $|\omega^2|$ to $|\omega^\omega|$. The last frame contains all numbers from $|\omega^\omega|$ to $|\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}| = \Omega$.

Fig.81.



Thus inside this largest square we have represented all the transfinite numbers, and their negative analogons.

18. <u>Conclusion</u>. - In Part I of this paper, I have dealt with only well-known properties of functions - treating them from the graphic standpoint entirely and referring the reader, to the best authorities I could find, for the Algebraic treatment.

In Part II, I have referred to standard works for formulae, conditions, existence theorems etc., but I found it necessary to develope the Algebraic <u>side</u> in somewhat greater detail, because of the fact that the view-point adopted is slightly different from the ordinary one.