On the Average Value of Divisor Sums in Arithmetic Progressions

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Abstract

We consider very short sums of the divisor function in arithmetic progressions prime to a fixed modulus and show that "on average" these sums are close to the expected value. We also give applications of our result to sums of the divisor function twisted with characters (both additive and multiplicative) taken on the values of various functions, such as rational and exponential functions; in particular, we obtain upper bounds for such twisted sums.

1 Introduction

Let $\tau(n)$ denote the classical *divisor function*, which is defined by

$$\tau(n) = \sum_{d \,|\, n} 1, \qquad \forall\, n \geq 1,$$

where the sum runs over all positive integral divisors d of n. For integers α and $q \geq 2$ with $(\alpha, q) = 1$, consider the *divisor sum* given by:

$$S(X, q, \alpha) = \sum_{\substack{n \leq X \\ n \equiv \alpha \ (\mathrm{mod}\ q)}} \tau(n).$$

In unpublished work, A. Selberg and C. Hooley independently discovered (see, for example, the discussion in [6]) that the Weil bound for Kloosterman sums implies that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$S(X, q, \alpha) = \frac{X P_q(\ln X)}{\varphi(q)} + O\left(\frac{X^{1-\delta}}{\varphi(q)}\right), \tag{1}$$

provided that $q < X^{2/3-\varepsilon}$, where P_q is the linear polynomial given by

$$
\begin{aligned}
P_q(\ln X) &= \operatorname*{res}_{s=1} \zeta^2(s) \prod_{p \,|\, q} (1 - p^{-s})^2 \frac{X^{s-1}}{s} \\
&= \frac{\varphi(q)^2}{q^2}(\ln X + 2\gamma - 1) + \frac{2\varphi(q)}{q} \sum_{d \,|\, q} \frac{\mu(d)\ln d}{d}.
\end{aligned}
$$

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Here $\zeta(s)$ denotes the Riemann zeta function, γ the Euler-Mascheroni constant, and $\varphi(k)$ and $\mu(k)$ the Euler and Möbius functions, respectively; for instance, see [3].

The *divisor problem for arithmetical progressions* (cf. [3]) asks whether the range of q for which (1) holds can be extended beyond $X^{2/3}$. This question appears to be quite difficult as it seems to require better uniform estimates for Kloosterman sums than those available from the Weil bound. Indeed, the exponent 2/3 has never been improved, and for $q > X^{2/3}$ it is not known whether $S(X, q, \alpha)$ lies close to its expected value (in the sense of (1)) for every α in the multiplicative group $\mathbb{Z}_q^* = (\mathbb{Z}/q\mathbb{Z})^*$.

In this paper, we show that the Weil-type bound for certain incomplete Kloosterman sums (which play an essential role in our arguments) can be sharpened "on average" as α runs over all of the residue classes in $\mathbb{Z}_q^*$; see Lemmas 3 and 4 below. These estimates for Kloosterman sums on average form the key technical ingredient of our approach, and using such estimates we show that (1) holds on average for all moduli q up to $X^{1-\varepsilon}$; more precisely, we show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{\alpha \in \mathbb{Z}_q^*} \left| S(X, q, \alpha) - \frac{X P_q(\ln X)}{\varphi(q)} \right| = O\left(X^{1-\delta}\right), \qquad \forall q < X^{1-\varepsilon}; \qquad (2)$$

see Theorem 1 in Section 3 below. From this it follows that for all moduli $q < X^{1-\varepsilon}$, the divisor sum $S(X, q, \alpha)$ lies close to its expected value for almost all $\alpha \in \mathbb{Z}_q^*$ in a suitable sense.

In Section 4, we give applications of Theorem 1 to other arithmetic sums involving the divisor function. In particular, we derive asymptotic formulas (or upper bounds) for sums of the divisor function twisted with characters (both additive and multiplicative) taken on the values of various functions, such as rational or exponential functions. We also show that the methods of Section 3 can be applied to certain twisted sums to obtain estimates for much shorter sums. We remark that these sums encode information about the uniformity of distribution of the values of $\tau(n)$ over numbers n from different residue classes modulo q.

Throughout the paper, the implied constants in symbols "O", "$\ll$" and "$\gg$" may occasionally, where obvious, depend on the small positive parameter ε and are absolute otherwise. We recall that the expressions $A \ll B$,

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$B \gg A$ and $A = O(B)$ are all equivalent to the statement that $|A| \leq cB$ for some constant c.

2 Preparations

In this section, we collect together a variety of estimates for use in the sequel.

Let H be an integer in the range $0 < H < q$, and put

$$\vartheta_H(\xi) = \min\left\{1, (H\|\xi\|)^{-1}\right\}, \qquad \forall\, \xi \in \mathbb{R},$$

where $\|\xi\|$ denotes the distance from ξ to the nearest integer.

As usual, we define $\mathbf{e}(z) = \exp(2\pi i z)$ for all $z \in \mathbb{R}$.

The following result is taken from [2, p. 341]:

Lemma 1 *For all $Y < Z$ and $\beta \in \mathbb{Z}$, we have*

$$\sum_{\substack{Y < m \leq Z \\ m \equiv \beta \pmod{q}}} 1 \;=\; \frac{Z - Y}{q} + \sum_{1 \leq |h| \leq H} c_{Y,Z}(h)\mathbf{e}(-h\beta/q)$$

$$+ O\left(\vartheta_H\left(\frac{Y - \beta}{q}\right) + \vartheta_H\left(\frac{Z - \beta}{q}\right)\right),$$

where

$$c_{Y,Z}(\xi) = \frac{1}{2\pi i \xi}\left(\mathbf{e}(Z\xi/q) - \mathbf{e}(Y\xi/q)\right) = \frac{1}{q}\int_Y^Z \mathbf{e}(y\xi/q)\, dy.$$

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Lemma 2 *Let* $(r, q) = 1$, $0 < H < q$, *and* $z \in \mathbb{R}$. *Then*

$$\sum_{\alpha \in \mathbb{Z}_q^*} \vartheta_H \left(\frac{z - r\alpha}{q} \right) \ll H^{-1} q^{1+\varepsilon} + H q^{-1}.$$

Proof: Taking into account that the function $\vartheta_H(\xi)$ is periodic with period 1 and changing variables if necessary, we can assume that $r = 1$. We have the Fourier series

$$\vartheta_H(z) = \widehat{\vartheta}_H(0) + \sum_{j \neq 0} \widehat{\vartheta}_H(j) \mathbf{e}(jz),$$

where (cf. [7, p. 29])

$$\begin{aligned} \widehat{\vartheta}_H(0) &\ll \frac{\ln(H + 2)}{H}, \\ \widehat{\vartheta}_H(j) &\ll \frac{H + 2}{j^2}, \qquad \forall j \neq 0. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{\alpha \in \mathbb{Z}_q^*} \vartheta_H \left(\frac{z - \alpha}{q} \right) &\leq \sum_{\alpha \in \mathbb{Z}_q} \vartheta_H \left(\frac{z - \alpha}{q} \right) \\ &= q \widehat{\vartheta}_H(0) + \sum_{j \neq 0} \widehat{\vartheta}_H(j) \mathbf{e}(jz/q) \sum_{\alpha \in \mathbb{Z}_q} \mathbf{e}(-j\alpha/q) \\ &= q \widehat{\vartheta}_H(0) + q \sum_{\substack{j \neq 0 \\ q \mid j}} \widehat{\vartheta}_H(j) \mathbf{e}(jz/q) \\ &= q \widehat{\vartheta}_H(0) + q \sum_{j \neq 0} \widehat{\vartheta}_H(qj) \mathbf{e}(jz) \\ &\ll \frac{q \ln(H + 2)}{H} + q \sum_{j \neq 0} \frac{H + 2}{q^2 j^2}, \end{aligned}$$

and the result follows. $\qquad\square$

For any b with $(b, q) = 1$, let $\overline{b}$ be a fixed multiplicative inverse of b modulo q; that is, $b\overline{b} \equiv 1 \pmod{q}$.

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Lemma 3 *Let*
$$Q = q \prod_{p \mid q} p^{-1}.$$

Then for all $Y < Z$ and $d, h \geq 1$ with $d \mid q$ and $(h, q) = 1$, we have

$$\sum_{\alpha \in \mathbb{Z}_q^*} \left| \sum_{\substack{Y < b \leq Z \\ (b,q)=1}} \mathbf{e}(\alpha d h \bar{b}/q) \right|$$
$$\ll \begin{cases} Vd + Lq^{1/2}d^{1/2} + L^{1/2}q + q & \text{if } Q \mid d, \\ Lq^{1/2}d^{1/2} + L^{1/2}q + q & \text{otherwise,} \end{cases}$$

where $V = q \lfloor (Z - Y)/q \rfloor$ and $L = q \{ (Z - Y)/q \}$.

Proof: First, we observe that for any $\beta \in \mathbb{Z}$, the Ramanujan sum

$$r_q(\beta) = \sum_{b \in \mathbb{Z}_q^*} \mathbf{e}(\beta \bar{b}/q)$$

can be evaluated explicitly (see, for example, Theorem 272 in [4]), and one has

$$r_q(\beta) = \varphi(q) \frac{\mu\big(q/(\beta, q)\big)}{\varphi\big(q/(\beta, q)\big)}.$$

Consequently,

$$\sum_{b \in \mathbb{Z}_q^*} \mathbf{e}(\alpha d h \bar{b}/q) = \varphi(q) \frac{\mu(q/d)}{\varphi(q/d)}$$

for every $\alpha \in \mathbb{Z}_q^*$. Observing that q/d is squarefree if and only if $Q \mid d$, and that $d\varphi(q/d) \geq \varphi(q)$, we have

$$\left| \sum_{b \in \mathbb{Z}_q^*} \mathbf{e}(\alpha d h \bar{b}/q) \right| \leq \begin{cases} d & \text{if } Q \mid d, \\ 0 & \text{otherwise.} \end{cases}$$

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Now let $V = q \lfloor (Z - Y)/q \rfloor$. We have

$$\sum_{\substack{\alpha \in \mathbb{Z}_q^*}} \left| \sum_{\substack{Y < b \leq Z \\ (b,q)=1}} \mathbf{e}(\alpha d h \overline{b}/q) \right|$$

$$= \sum_{\substack{\alpha \in \mathbb{Z}_q^*}} \left| \sum_{\substack{Y < b \leq Y+V \\ (b,q)=1}} \mathbf{e}(\alpha d h \overline{b}/q) \right| + \sum_{\substack{\alpha \in \mathbb{Z}_q^*}} \left| \sum_{\substack{Y+V < b \leq Z \\ (b,q)=1}} \mathbf{e}(\alpha d h \overline{b}/q) \right|.$$

Since V is a multiple of q, the interval $(Y, Y + V]$ contains precisely V/q copies of $\mathbb{Z}_q$; it follows that

$$\sum_{\substack{\alpha \in \mathbb{Z}_q^*}} \left| \sum_{\substack{Y < b \leq Y+V \\ (b,q)=1}} \mathbf{e}(\alpha d h \overline{b}/q) \right| \leq \begin{cases} Vd & \text{if } Q \,|\, d, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f = q/d$; then h is prime to f, and we have

$$\sum_{\substack{\alpha \in \mathbb{Z}_q^*}} \left| \sum_{\substack{Y+V < b \leq Z \\ (b,q)=1}} \mathbf{e}(\alpha d h \overline{b}/q) \right| \leq d \sum_{\substack{\alpha \in \mathbb{Z}_f}} \left| \sum_{\substack{Y+V < b \leq Z \\ (b,q)=1}} \mathbf{e}(\alpha h \overline{b}/f) \right|.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{\substack{\alpha \in \mathbb{Z}_f}} \left| \sum_{\substack{Y+V < b \leq Z \\ (b,q)=1}} \mathbf{e}(\alpha h \overline{b}/f) \right| \right)^2 \leq f \sum_{\substack{\alpha \in \mathbb{Z}_f}} \left| \sum_{\substack{Y+V < b \leq Z \\ (b,q)=1}} \mathbf{e}(\alpha h \overline{b}/f) \right|^2$$

$$= f \sum_{\substack{\alpha \in \mathbb{Z}_f}} \sum_{\substack{Y+V < b,c \leq Z \\ (bc,q)=1}} \mathbf{e}\left(\alpha h(\overline{b} - \overline{c})/f\right)$$

$$= f^2 \sum_{\substack{Y+V < b,c \leq Z \\ b \equiv c \pmod{f}}} 1$$

$$\leq f^2 (L+1)(L/f + 1)$$

$$\ll L^2 f + L f^2 + f^2.$$

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where $L = Z - (Y + V) = q\{(Z - Y)/q\}$. Consequently,

$$\sum_{\alpha \in \mathbb{Z}_q^*} \left| \sum_{\substack{Y+V<b\leq Z \\ (b,q)=1}} \mathbf{e}(\alpha h\bar{b}/f) \right| \quad \ll \quad Lq^{1/2}d^{1/2} + L^{1/2}q + q,$$

and the lemma follows. $\qquad\square$

Lemma 4 *For all $Y < Z$ and $\gamma \in \mathbb{Z}_q^*$, we have*

$$\sum_{\substack{Y<a\leq Z \\ (a,q)=1}} \mathbf{e}(\gamma\bar{a}/q) \ll (Z-Y)q^{-1} + q^{1/2+\varepsilon}.$$

Proof: Because $\mathbf{e}(\gamma\bar{a}/q)$ is a periodic function of a with period q, we obtain

$$\sum_{\substack{Y<a\leq Z \\ (a,q)=1}} \mathbf{e}(\gamma\bar{a}/q) = M \sum_{a\in\mathbb{Z}_q^*} \mathbf{e}(\gamma\bar{a}/q) + \sum_{\substack{Y<a\leq Y+K \\ (a,q)=1}} \mathbf{e}(\gamma\bar{a}/q).$$

where $M = \lfloor (Z-Y)/q \rfloor$, $K = q\{(Z-Y)/q\}$.

As we have seen in the proof of Lemma 3,

$$\left| \sum_{a\in\mathbb{Z}_q^*} \mathbf{e}(\gamma\bar{a}/q) \right| = \left| r_q(\gamma) \right| \leq 1.$$

Applying the Weil bound and using the standard reduction from complete exponential sums to incomplete ones (see, for example, Lemma 4 of Chapter 2

of [7]), we derive that

$$
\left| \sum_{\substack{Y < a \leq Y+K \\ (a,q)=1}} \mathbf{e}(\gamma\bar{a}/q) \right| = \left| \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \mathbf{e}(\gamma\bar{a}/q) | \sum_{Y < b \leq Y+K} \frac{1}{q} \sum_{\lambda=0}^{q-1} \mathbf{e}(\lambda(a-b)/q) \right|
$$

$$
\leq \frac{1}{q} \sum_{\lambda=0}^{q-1} \left| \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \mathbf{e}((\gamma\bar{a} + \lambda a)/q) \right| \left| \sum_{Y < b \leq Y+K} \mathbf{e}(-\lambda b)q) \right|
$$

$$
\ll \frac{1}{q^{1/2}} \sum_{\lambda=0}^{q-1} \left| \sum_{Y < b \leq Y+K} \mathbf{e}(-\lambda b)q) \right|
$$

$$
\ll q^{1/2} \log q,
$$

and the result follows. $\qquad\square$

Lemma 5 *The following estimate holds:*

$$
\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n) = X P_q(\ln X) + O(X^{1/2} q^\varepsilon),
$$

where

$$
P_q(\ln X) = \frac{\varphi(q)^2}{q^2}(\ln X + 2\gamma - 1) + \frac{2\varphi(q)}{q} \sum_{d\,|\,q} \frac{\mu(d)\ln d}{d}.
$$

Proof: In what follows, $\sum^*$ indicates that the sum is restricted to integers relatively prime to q. To simplify the notation, we write

$$
c_q = \sum_{d\,|\,q} \frac{\mu(d)\ln d}{d}.
$$

The following estimates can be easily obtained through the use of standard

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sieve techniques:

$$\sum_{n \le X}^{*} 1 = \frac{\varphi(q)}{q} X + O(q^{\varepsilon}),$$

$$\sum_{n \le X}^{*} n = \frac{\varphi(q)}{q} \frac{X^2}{2} + O(X q^{\varepsilon}),$$

$$\sum_{n \le X}^{*} \frac{1}{n} = \frac{\varphi(q)}{q} (\ln X + \gamma) + c_q + O(X^{-1} q^{\varepsilon}).$$

Thus,

$$\sum_{n \le X}^{*} \tau(n) = \sum_{ab \le X}^{*} 1 = 2 \sum_{\substack{ab \le X \\ a \le b}}^{*} 1 - \sum_{a^2 \le X}^{*} 1$$

$$= 2 \sum_{a \le X^{1/2}}^{*} \sum_{a \le b \le X/a}^{*} 1 + O(X^{1/2}).$$

Since

$$\sum_{a \le b \le X/a}^{*} 1 = \frac{\varphi(q)}{q} \left(\frac{X}{a} - a \right) + O(q^{\varepsilon}),$$

we have

$$\sum_{a \le X^{1/2}}^{*} \sum_{a \le b \le X/a}^{*} 1 = \frac{\varphi(q)}{q} \sum_{a \le X^{1/2}}^{*} \left(\frac{X}{a} - a \right) + O(X^{1/2} q^{\varepsilon})$$

$$= \frac{\varphi(q)}{q} X \left(\frac{\varphi(q)}{q} (\ln X^{1/2} + \gamma) + c_q + O(X^{-1/2} q^{\varepsilon}) \right)$$

$$- \frac{\varphi(q)}{q} \left(\frac{\varphi(q)}{q} \frac{X}{2} + O(X^{1/2} q^{\varepsilon}) \right) + O(X^{1/2} q^{\varepsilon}),$$

and the lemma follows. $\qquad \square$

Finally, we recall that the Euler function $\varphi(k)$ and the divisor function $\tau(k)$ satisfy the inequalities

$$\varphi(k) \gg \frac{k}{\ln \ln(k+2)} \qquad \text{and} \qquad \tau(k) \ll k^{\varepsilon};$$

see Theorem 5.1 and Theorem 5.2 in Chapter 1 of [19].

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3   Main Result

We are now prepared to prove our main result.

Let us denote

$$W(X, q) = \sum_{\alpha \in \mathbb{Z}_q^*} \left| S(X, q, \alpha) - \frac{X P_q(\ln X)}{\varphi(q)} \right|.$$

Then $W(X, q)/\varphi(q)$ is the average difference (in absolute value) between $S(X, q, \alpha)$ and its expected value.

Let us also denote

$$\mathcal{E}(X, q) = \begin{cases} q^{1/5} X^{4/5+\varepsilon} & \text{if } q > X^{1/2}, \\ q^{2/5} X^{7/10+\varepsilon} & \text{if } X^{1/3} < q \leq X^{1/2}, \\ q^{1/2} X^{2/3+\varepsilon} & \text{if } X^{1/6} < q \leq X^{1/3}, \\ X^{3/4+\varepsilon} & \text{if } q \leq X^{1/6}. \end{cases}$$

Theorem 1 *For every $\varepsilon > 0$, the following bound holds:*

$$W(X, q) \ll \mathcal{E}(X, q),$$

where the implied constant depends only on ε.

Proof: Put

$$T(X, q) = \sum_{\alpha \in \mathbb{Z}_q^*} \left| S(X, q, \alpha) - \frac{S^*(X, q)}{\varphi(q)} \right|,$$

where

$$S^*(X, q) = \sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n).$$

For arbitrary Δ in the range $0 < \Delta < 1/2$, let

$$M = \left\{ \tfrac{1}{2}(1 + \Delta)^j \,|\, 0 \leq j \leq R \right\},$$

where

$$R = \left\lfloor \frac{\ln(2X)}{\ln(1 + \Delta)} \right\rfloor \ll \Delta^{-1} \ln X.$$

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Then

$$S(X, q, \alpha) = \sum_{\substack{n \leq X \\ n \equiv \alpha \pmod{q}}} \tau(n) = \sum_{\substack{ab \leq X \\ ab \equiv \alpha \pmod{q}}} 1 = \sum_{A,B \in M} D_\alpha(A, B),$$

where

$$D_\alpha(A, B) = D_\alpha(B, A) = \sum_{\substack{A < a \leq A(1+\Delta) \\ B < b \leq B(1+\Delta) \\ ab \leq X \\ ab \equiv \alpha \pmod{q}}} 1.$$

Similarly,

$$S^*(X, q) = \sum_{A,B \in M} D^*(A, B),$$

where

$$D^*(A, B) = D^*(B, A) = \sum_{\substack{A < a \leq A(1+\Delta) \\ B < b \leq B(1+\Delta) \\ ab \leq X \\ (ab,q)=1}} 1.$$

Following [2], we call the pair (A, B) *good* if $AB \leq X(1+\Delta)^{-2}$, otherwise we say that (A, B) is *bad*. It is easy to see that, for good pairs, the condition $ab \leq X$ in the definitions of $D_\alpha(A, B)$ and $D^*(A, B)$ is redundant. As in [2], we have

$$\sum_{(A,B) \text{ bad}} D_\alpha(A, B) \ll (\Delta X q^{-1} + 1) X^\varepsilon,$$

$$\sum_{(A,B) \text{ bad}} D^*(A, B) \ll (\Delta X + 1) X^\varepsilon.$$

Consequently,

$$T(X, q) \ll (\Delta X + q) X^\varepsilon + \sum_{\substack{(A,B) \text{ good} \\ A \geq B}} \sum_{\alpha \in \mathbb{Z}_q^*} \left| D_\alpha(A, B) - \frac{D^*(A, B)}{\varphi(q)} \right|. \quad (3)$$

For any good pair (A, B) with $A \geq B$, Lemma 1 implies that

$$D_\alpha(A, B) - \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \Delta A q^{-1} \ll E_1(A, B, \alpha) + E_2(A, B, \alpha),$$

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where

$$E_1(A, B, \alpha) = \left| \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \sum_{1 \leq |h| \leq H} c_A(h)\mathbf{e}(-h\bar{b}\alpha/q) \right|,$$

$$E_2(A, B, \alpha) = \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \left(\vartheta_H\left(\frac{A - \bar{b}\alpha}{q}\right) + \vartheta_H\left(\frac{A(1+\Delta) - \bar{b}\alpha}{q}\right) \right),$$

and

$$c_A(\xi) = \frac{1}{2\pi i \xi} \left(\mathbf{e}\left(A(1+\Delta)\xi/q\right) - \mathbf{e}(A\xi/q) \right) = q^{-1} \int_A^{A(1+\Delta)} \mathbf{e}(y\xi/q) \, dy.$$

Note that we have the trivial bound

$$c_A(\xi) \ll \min\left\{|\xi|^{-1}, \Delta A q^{-1}\right\}. \tag{4}$$

Now put $H = q - 1$ and $J = \lfloor \log(\Delta A) \rfloor - 1$. We have

$$E_1(A, B, \alpha) \leq \sum_{j=0}^{J+1} E_{1,j}(A, B, \alpha)$$

where

$$E_{1,j}(A, B, \alpha) = \left| \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \sum_{h \in \mathcal{H}_j} c_A(h)\mathbf{e}(-h\bar{b}\alpha/q) \right|,$$

and the summation is taken over the sets

$$\begin{aligned}
\mathcal{H}_0 &= \{h \mid 1 \leq |h| \leq q/\Delta A\}, \\
\mathcal{H}_j &= \{h \mid e^j q/\Delta A < |h| \leq e^{j+1}q/\Delta A\}, \qquad j = 1, \ldots, J, \\
\mathcal{H}_{J+1} &= \{h \mid e^{J+1}q/\Delta A < |h| \leq q - 1\}.
\end{aligned}$$

By the Cauchy inequality, we derive that

$$\left(\sum_{\alpha \in \mathbb{Z}_q^*} E_1(A, B, \alpha) \right)^2 \ll q \log X \sum_{j=0}^{J+1} \sum_{\alpha \in \mathbb{Z}_q^*} E_{1,j}(A, B, \alpha)^2.$$

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For each j, we have

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_{1,j}(A, B, \alpha)^2$$

$$= \sum_{\alpha \in \mathbb{Z}_q^*} \sum_{\substack{B < b_1, b_2 \leq B(1+\Delta) \\ (b_1 b_2, q) = 1}} \sum_{h_1, h_2 \in \mathcal{H}_j} c_A(h_1) c_A(h_2) \mathbf{e}(\alpha(h_1 \overline{b}_1 - h_2 \overline{b}_2)/q)$$

$$\leq q \sum_{\substack{B < b_1, b_2 \leq B(1+\Delta) \\ (b_1 b_2, q) = 1}} \sum_{\substack{h_1, h_2 \in \mathcal{H}_j \\ h_1 \overline{b}_1 \equiv h_2 \overline{b}_2 \pmod{q}}} c_A(h_1) c_A(h_2).$$

Using (4), we see that $c_A(h) \ll e^{-j} \Delta A / q$ for $h \in \mathcal{H}_j$. Therefore

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_{1,j}(A, B, \alpha)^2 \ll e^{-2j} \frac{\Delta^2 A^2}{q} T_j(B)$$

where $T_j(B)$ is number of solutions (h_1, h_2, b_1, b_2) of the congruence

$$h_1 \overline{b}_1 \equiv h_2 \overline{b}_2 \pmod{q}$$

with $B < b_1, b_2 \leq B(1 + \Delta)$, $(b_1 b_2, q) = 1$ and $h_1, h_2 \in \mathcal{H}_j$. Rewriting this congruence as $h_1 b_2 \equiv h_2 b_1 \pmod{q}$, we see that for given h_1, b_2, the values of h_2 and b_1 are such that $s = h_2 b_1 \ll e^j B q / \Delta A$ belongs to a prescribed residue class modulo q. Thus there are at most $O(1 + e^j B / \Delta A)$ possibilities for s, each of which gives rise to at most $\tau(s) \ll X^{\varepsilon/2}$ values of h_2 and b_1. Therefore,

$$T_j(B) \ll (1 + e^j q / \Delta A)(1 + \Delta B)(1 + e^j B / \Delta A) X^{\varepsilon/2}$$

and we derive that

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_{1,j}(A, B, \alpha)^2 \quad \ll \quad q^{-1}(1 + \Delta B)(e^{-j} \Delta A + q)(e^{-j} \Delta A + B) X^{\varepsilon/2}$$

$$\ll \quad q^{-1}(1 + \Delta B)(\Delta A + q)(\Delta A + B) X^{\varepsilon/2}.$$

Consequently,

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_1(A, B, \alpha) \ll (1 + \Delta B)^{1/2}(\Delta A + q)^{1/2}(\Delta A + B)^{1/2} X^{\varepsilon/2}. \qquad (5)$$

Next, we estimate the sum on the left side of (5) in a different way. With $H = q - 1$, we have

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_1(A, B, \alpha) = \sum_{\alpha \in \mathbb{Z}_q^*} \left| \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \sum_{1 \leq |h| < q} c_A(h) \mathbf{e}(-h\overline{b}\alpha/q) \right|$$

$$\leq \sum_{1 \leq |h| < q} |c_A(h)| \sum_{\alpha \in \mathbb{Z}_q^*} \left| \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \mathbf{e}(-h\overline{b}\alpha/q) \right|$$

$$= \sum_{\substack{d \mid q \\ d < q}} \sum_{\substack{1 \leq |h| < q \\ (h,q)=d}} |c_A(h)| \sum_{\alpha \in \mathbb{Z}_q^*} \left| \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \mathbf{e}(-h\overline{b}\alpha/q) \right|$$

$$= \sum_{\substack{d \mid q \\ d < q}} \sum_{\substack{1 \leq |h| < q/d \\ (h,q)=1}} |c_A(dh)| \sum_{\alpha \in \mathbb{Z}_q^*} \left| \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \mathbf{e}(-dh\overline{b}\alpha/q) \right|.$$

If $\Delta B < q$, Lemma 3 and the bound (4) together imply that

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_1(A, B, \alpha) \ll \sum_{\substack{d \mid q \\ d < q}} \sum_{\substack{1 \leq |h| < q/d \\ (h,q)=1}} \frac{1}{|dh|} (\Delta B q^{1/2} d^{1/2} + \Delta^{1/2} B^{1/2} q + q)$$

$$\ll (\Delta B q^{1/2} + \Delta^{1/2} B^{1/2} q + q) X^{\varepsilon/2};$$

that is,

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_1(A, B, \alpha) \ll (\Delta^{1/2} B^{1/2} q + q) X^{\varepsilon/2}. \tag{6}$$

If $\Delta B \geq q$, then Lemma 3 and (4) give

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_1(A, B, \alpha) \ll \sum_{\substack{d \mid q \\ d < q}} \sum_{\substack{1 \leq |h| < q/d \\ (h,q)=1}} |c_A(dh)| \sum_{\alpha \in \mathbb{Z}_q^*} \left| \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \mathbf{e}(-dh\overline{b}\alpha/q) \right|$$

$$\ll \sum_{\substack{d \mid q \\ d < q}} \sum_{\substack{1 \leq |h| < q/d \\ (h,q)=1}} \frac{1}{|dh|} \left(\Delta B d + q^{3/2} d^{1/2} \right),$$

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and it follows that

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_1(A, B, \alpha) \ll (\Delta B + q^{3/2}) X^{\varepsilon/2}. \tag{7}$$

To estimate the sum of the second error term $E_2(A, B, \alpha)$ over all $\alpha \in \mathbb{Z}_q^*$, we apply Lemma 2 (with $H = q - 1$), which gives

$$\sum_{\alpha \in \mathbb{Z}_q^*} E_2(A, B, \alpha) \ll (1 + \Delta B) X^{\varepsilon/2}.$$

Suppose first that $q > X^{1/2}$. Note that $B < q$ for each good pair (A, B) (since $AB \leq X(1 + \Delta)^{-2}$; as $A \geq B$, this gives $B < X^{1/2} < q$), and the inequality $\Delta B \geq q$ does not occur. Combining this estimate with (5) and (6) we obtain that

$$\sum_{\alpha \in \mathbb{Z}_q^*} \left| D_\alpha(A, B) - \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \Delta A q^{-1} \right| \ll E(A, B) X^{\varepsilon/2}, \tag{8}$$

where

$$E(A, B) = \begin{cases} (1 + \Delta^{1/2} B^{1/2})(\Delta^{1/2} A^{1/2} + B^{1/2}) q^{1/2} & \text{if } \max\{\Delta A, B\} < q, \\ (1 + \Delta^{1/2} B^{1/2}) q & \text{otherwise.} \end{cases}$$

Since

$$D^*(A, B) = \sum_{\alpha \in \mathbb{Z}_q^*} D_\alpha(A, B),$$

we also have that

$$\sum_{\alpha \in \mathbb{Z}_q^*} \left| \frac{1}{\varphi(q)} D^*(A, B) - \sum_{\substack{B < b \leq B(1+\Delta) \\ (b,q)=1}} \Delta A q^{-1} \right| \ll E(A, B) X^{\varepsilon/2}. \tag{9}$$

Combining (8) and (9), and substituting into (3), we now see that

$$T(X, q) \ll (\Delta X + q) X^\varepsilon + \sum_{\substack{(A,B) \text{ good} \\ A \geq B}} E(A, B) X^{\varepsilon/2}. \tag{10}$$

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Then

$$\sum_{\substack{(A,B)\text{ good}\\ A\geq B}} E(A,B) \leq \sum_{\substack{B\in M\\ B\leq X^{1/2}}} \sum_{\substack{A\in M\\ B\leq A\leq X/B}} E(A,B) \ll U_1 + U_2 + U_3,$$

where

$$
\begin{aligned}
U_1 &= \sum_{\substack{B\in M\\ B\leq \Delta X/q}} \sum_{\substack{A\in M\\ A<\Delta^{-1}q}} (1+\Delta^{1/2}B^{1/2})(\Delta^{1/2}A^{1/2}+B^{1/2})q^{1/2},\\
U_2 &= \sum_{\substack{B\in M\\ B\leq \Delta X/q}} \sum_{\substack{A\in M\\ \Delta^{-1}q\leq A\leq X/B}} (1+\Delta^{1/2}B^{1/2})q,\\
U_3 &= \sum_{\substack{B\in M\\ \Delta X/q<B\leq X^{1/2}}} \sum_{\substack{A\in M\\ A\leq X/B}} (1+\Delta^{1/2}B^{1/2})(\Delta^{1/2}A^{1/2}+B^{1/2})q^{1/2}.
\end{aligned}
$$

To estimate these expressions, we use the fact that M is a geometric series, hence we have the trivial estimates:

$$\sum_{\substack{C\in M\\ C\leq Y}} C \ll \Delta^{-1}Y, \qquad \sum_{\substack{C\in M\\ C\leq Y}} C^{1/2} \ll \Delta^{-1}Y^{1/2}, \qquad \sum_{\substack{C\in M\\ C\leq Y}} 1 \ll \Delta^{-1}X^{\varepsilon/2}. \quad (11)$$

Therefore, recalling that $B<q$, we deduce

$$
\begin{aligned}
U_1 &\ll q^{1/2} \sum_{\substack{B\in M\\ B\leq \Delta X/q}} (1+\Delta^{1/2}B^{1/2}) \sum_{\substack{A\in M\\ A<\Delta^{-1}q}} (\Delta^{1/2}A^{1/2}+B^{1/2})\\
&\ll q^{1/2} \sum_{\substack{B\in M\\ B\leq \Delta X/q}} (1+\Delta^{1/2}B^{1/2})(\Delta^{-1}q^{1/2}+\Delta^{-1}B^{1/2})X^{\varepsilon/2}\\
&\ll \Delta^{-1}qX^{\varepsilon/2} \sum_{\substack{B\in M\\ B\leq \Delta X/q}} (1+\Delta^{1/2}B^{1/2})\\
&\ll \left(\Delta^{-1}q^{1/2}X^{1/2}+\Delta^{-2}q\right) X^{\varepsilon/2}.
\end{aligned}
$$

Similarly, we derive that

$$
\begin{aligned}
U_2 &\ll \left(\Delta^{-1}q^{1/2}X^{1/2}+\Delta^{-2}q\right) X^{\varepsilon/2},\\
U_3 &\ll \left(\Delta^{-3/2}q^{1/2}X^{1/2}+\Delta^{-2}q^{1/2}X^{1/4}\right) X^{\varepsilon/2}.
\end{aligned}
$$

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Therefore, since $q > X^{1/2}$, it follows from (10) that

$$T(X, q) \ll \left(\Delta X + \Delta^{-3/2} q^{1/2} X^{1/2} + \Delta^{-2} q \right) X^\varepsilon.$$

Choosing $\Delta = q^{1/5} X^{-1/5}$, we obtain the estimate

$$T(X, q) \ll q^{1/5} X^{4/5 + \varepsilon}$$

in this case.

Now suppose that $q \leq X^{1/2}$. Simple (but rather tedious) calculations show that, in this case, use of the bound (5) does not lead to a sharper overall estimate; thus, in this case we use only (6) and (7). Accordingly, instead of (10), we now have

$$T(X, q) \ll (\Delta X + q) X^\varepsilon + \sum_{\substack{(A, B) \text{ good} \\ A \geq B}} F(A, B) X^{\varepsilon/2}.$$

where

$$F(A, B) = \begin{cases} (1 + \Delta^{1/2} B^{1/2}) q & \text{if } \Delta B < q, \\ \Delta B + q^{3/2} & \text{if } \Delta B \geq q, \end{cases}$$

As before, we remark that if (A, B) is good and $A \geq B$, then $B < X^{1/2}$; thus, by (11) (with $X^{\varepsilon/2}$ replaced by $X^{\varepsilon/4}$), we derive that

$$
\begin{aligned}
\sum_{\substack{(A, B) \text{ good} \\ A \geq B}} F(A, B) \ \leq \ & q \sum_{A \in M} \sum_{\substack{B \in M \\ \Delta B < \min\{q, \Delta X^{1/2}\}}} (1 + \Delta^{1/2} B^{1/2}) \\
& + \sum_{A \in M} \sum_{\substack{B \in M \\ q \leq \Delta B < \Delta X^{1/2}}} (\Delta B + q^{3/2}) \\
\ll \ & q \Delta^{-1} X^{\varepsilon/4} \sum_{\substack{B \in M \\ \Delta B < \min\{q, \Delta X^{1/2}\}}} (1 + \Delta^{1/2} B^{1/2}) \\
& + \Delta^{-1} X^{\varepsilon/4} \sum_{\substack{B \in M \\ q \leq \Delta B < \Delta X^{1/2}}} (\Delta B + q^{3/2}).
\end{aligned}
$$

Therefore, if $q \geq \Delta X^{1/2}$, it follows that

$$T(X, q) \ll (\Delta X + \Delta^{-2} q + \Delta^{-3/2} q X^{1/4}) X^\varepsilon,$$

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while for $q < \Delta X^{1/2}$, we have

$$T(X, q) \ll (\Delta X + \Delta^{-2} q^{3/2} + \Delta^{-1} X^{1/2}) X^\varepsilon.$$

Choosing

$$\Delta = \begin{cases} q^{2/5} X^{-3/10} & \text{for } q \geq X^{1/3}, \\ q^{1/2} X^{-1/3} & \text{for } X^{1/6} \leq q \leq X^{1/3}, \\ X^{-1/4} & \text{for } q \leq X^{1/6}, \end{cases}$$

we obtain the estimate $T(X, q) \ll \mathcal{E}(X, q)$ in every case.

Recalling Lemma 5, the desired result follows immediately. $\qquad \square$

In particular, we see that the bound (2) holds.

4 Twisted Sums

Here we consider "twisted" sums of the form

$$T_\psi(X) = \sum_{n \leq X} \tau(n) \psi(n),$$

where ψ is a complex-valued function. Sums of this type have been considered in a number of works (see [8, 9, 10, 11, 14]); however, our results cover a much wider class of functions $\psi(n)$.

Theorem 2 *Let $\psi(n)$ be periodic function of period p where p is prime, and let*

$$\Psi = \sum_{n \in \mathbb{Z}_p^*} \psi(n), \qquad \psi_0 = \max_{n \in \mathbb{Z}_p^*} |\psi(n)| .$$

Then

$$T_\psi(X) = X Q_{p,\psi}(\ln X) + \begin{cases} O\left(\mathcal{E}(X, p)\right) & \text{if } \psi_0 \neq 0, \\ O(X^{1/2} p^\varepsilon) & \text{if } \psi_0 = 0, \end{cases}$$

where

$$Q_{p,\psi}(\ln X) = \frac{(p-1)\Psi + \psi(0)(2p-1)}{p^2} \ln X$$

$$+ \frac{((p-1)\Psi + \psi(0)(2p-1))(2\gamma - 1) - 2(\Psi + \psi(0))(p-1)\ln p}{p^2}.$$

Proof: We have

$$T_\psi(X) = \sum_{\alpha \in \mathbb{Z}_p} \psi(\alpha) \sum_{\substack{n \leq X \\ n \equiv \alpha \pmod{p}}} \tau(n)$$

$$= \sum_{\alpha \in \mathbb{Z}_p^*} \psi(\alpha) S(X, p, \alpha) + \psi(0) \sum_{n \leq X/p} \tau(pn)$$

From the well known asymptotic formula (see Chapter 1 of [19])

$$\sum_{n \leq X} \tau(n) = XP(\ln X) + O\left(X^{1/2}\right),$$

where $P(\ln X) = \ln X + 2\gamma - 1$, and using Lemma 5, we obtain

$$\sum_{n \leq X/p} \tau(pn) = \sum_{n \leq X} \tau(n) - \sum_{\substack{n \leq X \\ (n,p)=1}} \tau(n)$$

$$= X\left(P(\ln X) - P_p(\ln X)\right) + O\left(X^{1/2}p^\varepsilon\right).$$

Thus

$$T_\psi(X) = \frac{XP_p(\ln X)}{p-1} \sum_{\alpha \in \mathbb{Z}_p^*} \psi(\alpha) + \psi(0) X \left(P(\ln X) - P_p(\ln X)\right)$$

$$+ O\left(\psi_0 W(X, p) + X^{1/2}p^\varepsilon\right).$$

The result now follows from Theorem 1. $\square$

For example, if $\psi(n) = \chi_p(n)$ is a non-principal character modulo p, then we have $\Psi = \psi(0) = 0$ and $\psi_0 = 1$, thus

$$\sum_{n \leq X} \tau(n)\chi_p(n) \ll \mathcal{E}(X, p). \tag{12}$$

More generally, we obtain

$$\sum_{n \leq X} \tau(n)\chi_p(n+a) = \chi_p(a) \cdot \frac{X(\ln X + 2\gamma - 1)}{p} + O\left(\mathcal{E}(X, p)\right) \tag{13}$$

for all $a \in \mathbb{Z}_p$.

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Similarly, if $\psi(n) = \chi_p(n+a)\overline{\chi}_p(n+b)$, where a and b are distinct modulo p, then $\Psi \ll 1$, and $|\psi(0)| \leq 1$, and $\psi_0 = 1$; thus we find that

$$\sum_{n \leq X} \tau(n)\chi_p(n+a)\overline{\chi}_p(n+b) \ll \frac{X \ln X}{p} + \mathcal{E}(X, p). \qquad (14)$$

For smaller values of X, nontrivial upper bounds for the sums involved in (13) and (14) are given in [10, 11, 14]. On the other hand, the method of those papers cannot be applied to the sum (12) (in fact, the possibility of finding a nontrivial upper bound on the sum (12) has been doubted in [10]). Moreover, our results imply that an analogue of (12) holds for characters $\chi_q(n)$ modulo a composite q as well.

One can also consider the function $\psi(n) = \chi_p(F(n)) \mathbf{e}(G(n)/p)$, where $F(X) = f_1(X)/f_2(X)$ and $G(X) = g_1(X)/g_2(X)$ are rational functions formed with polynomials $f_1(X), f_2(X), g_1(X), g_2(X)$ in $\mathbb{Z}[X]$ of degree at most k such that $\psi(n)$ is non-constant. From the Weil bound (cf. Chapter 7 of [20]), we see that Theorem 2 applies with $\Psi = O(kp^{1/2})$, and $|\psi(0)| \leq 1$, and $\psi_0 = 1$, yielding

$$\sum_{n \leq X} \tau(n)\chi_p(F(n)) \mathbf{e}(G(n)/p) \ll kp^{-1/2}X \ln X + \mathcal{E}(X, p).$$

Finally, we can apply the results of [16, 17, 18] to the function $\psi(n) = \mathbf{e}(ag^n/m)$, where $(ag, m) = 1$ and $g > 1$ is of prime multiplicative order p modulo m. For an arbitrary integer m, one can take $\Psi = O(m^{1/2})$, and $|\psi(0)| = \psi_0 = 1$ (cf. [17, 18]), thus

$$\sum_{n \leq X} \tau(n)\mathbf{e}(ag^n/m) \ll m^{1/2}p^{-1}X \ln X + \mathcal{E}(X, p).$$

If m is also prime, then stronger bounds on Ψ are available; see [1, 5, 15, 16].

We have already mentioned that an analogue of (12) holds modulo a composite number q. There are several other cases where one can estimate $T_\psi(X)$ nontrivially with functions ψ of composite period.

Sums $T_\psi(X)$ with $\psi(n) = \mathbf{e}(\gamma n/q)$ have been considered in [8]. Here we consider the case of the function $\psi(n) = \mathbf{e}(\gamma \overline{n}/q)$.

Theorem 3 *For any $\gamma \in \mathbb{Z}_q^*$,*

$$\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n)\mathbf{e}(\gamma \overline{n}/q) = \mu(q)\frac{XP_q(\ln X)}{\varphi(q)} + O\left(\mathcal{E}(X,q)\right).$$

Proof: For a fixed $\gamma \in \mathbb{Z}_q^*$, as in the proof of Theorem 2, we have

$$\begin{aligned}
\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n)\mathbf{e}(\gamma \overline{n}/q) &= \sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \overline{\alpha}/q) \sum_{\substack{n \leq X \\ n \equiv \alpha \pmod{q}}} \tau(n) \\
&= \sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \overline{\alpha}/q)S(X,q,\alpha) \\
&= \frac{XP_q(\ln X)}{\varphi(q)} \sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \overline{\alpha}/q) + O(W(X,q)).
\end{aligned}$$

Similarly

$$\begin{aligned}
\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n)\mathbf{e}(\gamma n/q) &= \sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \alpha/q) \sum_{\substack{n \leq X \\ n \equiv \alpha \pmod{q}}} \tau(n) \\
&= \sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \alpha/q)S(X,q,\alpha) \\
&= \frac{XP_q(\ln X)}{\varphi(q)} \sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \alpha/q) + O(W(X,q)).
\end{aligned}$$

Recalling that

$$\sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \overline{\alpha}/q) = \sum_{\alpha \in \mathbb{Z}_q^*} \mathbf{e}(\gamma \alpha/q) = \sum_{d|q} \mu(d) \sum_{\alpha \in \mathbb{Z}_{q/d}} \mathbf{e}(\gamma \alpha d/q) = \mu(q),$$

and using Theorem 1, we obtain the desired result. $\qquad\square$

We remark that if q is squarefree, then in both statements of Theorem 3, the main term exceeds the error term $\mathcal{E}(X,q)$ when $q \leq X^{2/9-\varepsilon}$. Thus we have asymptotic formulas for such values of q. For larger values of q, that is, when $X^{2/9-\varepsilon} \leq q \leq X^{1-\varepsilon}$, we have only (nontrivial) upper bounds.

Similarly, recalling that Gaussian sums can be explicitly evaluated, one can obtain asymptotic formulas for the sums

$$\sum_{\substack{n \leq X \\ (n,p)=1}} \tau(n) \mathbf{e}(\gamma \overline{n}^2/q), \qquad \text{and} \qquad \sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n) \mathbf{e}(\gamma n^2/q).$$

In the simplest case where $q = p \geq 3$ is prime, we have

$$\sum_{\substack{n \leq X \\ (n,p)=1}} \tau(n) \mathbf{e}(\gamma \overline{n}^2/p) = \left(\frac{\gamma}{p}\right) \eta(p) X P_p(\ln X) + O\left(\mathcal{E}(X,p)\right)$$

$$\sum_{\substack{n \leq X \\ (n,p)=1}} \tau(n) \mathbf{e}(\gamma n^2/p) = \left(\frac{\gamma}{p}\right) \eta(p) X P_p(\ln X) + O\left(\mathcal{E}(X,p)\right).$$

where (γ/p) is the Legendre symbol, and

$$\eta(p) = \begin{cases} (p^{1/2}+1)^{-1} & \text{if } p \equiv 1 \pmod 4, \\ (ip^{1/2}-1)/(p-1) & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Because $|\eta(p)| \sim p^{-1/2}$, the main term in the above formulas exceeds the error term $\mathcal{E}(X,p)$ when $p \leq X^{1/3-\varepsilon}$. Clearly $\overline{n}^2$ and n^2 can also be replaced by $\overline{n}^{2s}$ and n^{2s} for any s such that $(s, (p-1)/2) = 1$.

We now show that the *methods* of this paper (rather than the previously established results) can be used to obtain upper bounds on the sums above which remain nontrivial for significantly larger values of q. Although our methods apply to much more general sums, we demonstrate it here only for sums of the divisor function $\tau(n)$ twisted by $\mathbf{e}(\alpha \overline{n}/q)$, where the bound is stronger than in more general cases.

Theorem 4 *For any $\gamma \in \mathbb{Z}_q^*$,*

$$\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n) \mathbf{e}(\gamma \overline{n}/q) \ll \begin{cases} q^{1/4} X^{3/4+\varepsilon} & \text{if } X^{1/3} \leq q < X, \\ q^{-1/2} X & \text{if } q \leq X^{1/3}. \end{cases}$$

Proof: As in the proof of Theorem 1, take Δ in the range $0 < \Delta < 1/2$, and define

$$M = \left\{ \tfrac{1}{2}(1+\Delta)^j \,|\, 0 \leq j \leq R \right\},$$

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where
$$R = \left\lfloor \frac{\ln(2X)}{\ln(1+\Delta)} \right\rfloor \ll \Delta^{-1} \ln X.$$

Defining *good* and *bad* pairs as in the proof Theorem 1, we derive that

$$\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n) \mathbf{e}(\alpha \overline{n}/q) \ll \sum_{\substack{(A,B) \text{ good} \\ A \geq B}} |S(A,B)| + (\Delta X + 1) X^{\varepsilon},$$

where

$$S(A,B) = \sum_{\substack{A < a \leq A(1+\Delta) \\ B < b \leq B(1+\Delta) \\ (ab,q)=1}} \mathbf{e}(\gamma \overline{a}\overline{b}/q).$$

Applying Lemma 4 we obtain

$$S(A,B) \ll q^{\varepsilon/2}(\Delta B + 1)\left(\Delta A q^{-1} + q^{1/2}\right)$$

Therefore,

$$\sum_{\substack{(A,B) \text{ good} \\ A \geq B}} |S(A,B)|$$

$$\ll q^{\varepsilon/2} \sum_{A \in M} \left(\Delta A q^{-1} + q^{1/2}\right) \sum_{\substack{B \in M \\ B \leq \min\{A, X/A\}}} (\Delta B + 1)$$

$$\ll q^{\varepsilon/2} \sum_{A \in M} \left(\Delta A q^{-1} + q^{1/2}\right)\left(\min\{A, X/A\} + \Delta^{-1} X^{\varepsilon/4}\right)$$

$$\ll \Delta^{-1} q^{\varepsilon/2} X^{\varepsilon/4} \sum_{A \in M} \left(\Delta A q^{-1} + q^{1/2}\right)$$

$$+ q^{\varepsilon/2} \sum_{\substack{A \in M \\ A < X^{1/2}}} A \left(\Delta A q^{-1} + q^{1/2}\right)$$

$$+ q^{\varepsilon/2} X \sum_{\substack{A \in M \\ X^{1/2} \leq A \leq X}} A^{-1} \left(\Delta A q^{-1} + q^{1/2}\right).$$

As in the proof of Theorem 1, we see that the first sum is bounded by

$$\sum_{A \in M} \left(\Delta A q^{-1} + q^{1/2}\right) \ll q^{-1} X + \Delta^{-1} q^{1/2} X^{\varepsilon/4}.$$

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Similarly, the second sum can be bounded as follows:

$$\sum_{\substack{A \in M \\ A < X^{1/2}}} A \left(\Delta A q^{-1} + q^{1/2}\right) = \Delta q^{-1} \sum_{\substack{A \in M \\ A < X^{1/2}}} A^2 + q^{1/2} \sum_{\substack{A \in M \\ A < X^{1/2}}} A$$

$$\ll q^{-1} X + \Delta^{-1} q^{1/2} X^{1/2}.$$

Finally, for the third sum, we derive that

$$\sum_{\substack{A \in M \\ X^{1/2} \leq A \leq X}} A^{-1} \left(\Delta A q^{-1} + q^{1/2}\right) \leq \left(\Delta q^{-1} + q^{1/2} X^{-1/2}\right) \sum_{A \in M} 1$$

$$\ll \left(\Delta q^{-1} + q^{1/2} X^{-1/2}\right) \Delta^{-1} X^{\varepsilon/4}.$$

Thus, it follows that

$$\sum_{\substack{(A,B) \text{ good} \\ A \geq B}} |S(A,B)|$$

$$\ll X^{\varepsilon/2} q^{\varepsilon/2} \left(\Delta^{-1} q^{-1} X + \Delta^{-2} q^{1/2} + \Delta^{-1} q^{1/2} X^{1/2}\right).$$

Since for $q > X$ the bound is trivial, we obtain that

$$\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n) \mathbf{e}(\alpha \overline{n}/q) \ll X^{\varepsilon} \left(\Delta X + \Delta^{-1} q^{-1} X + \Delta^{-2} q^{1/2} + \Delta^{-1} q^{1/2} X^{1/2}\right).$$

Choosing

$$\Delta = \begin{cases} q^{1/4} X^{-1/4} & \text{for } X^{1/3} \leq q < X, \\ q^{-1/2} & \text{for } q \leq X^{1/3}, \end{cases}$$

the result follows. $\qquad\square$

Thus, combining Theorem 3 and Theorem 4 we obtain that for any $\gamma \in \mathbb{Z}_q^*$,

$$\sum_{\substack{n \leq X \\ (n,q)=1}} \tau(n) \mathbf{e}(\gamma \overline{n}/q)$$

$$= \begin{cases} \mu(q) X P_q(\ln X)/\varphi(q) + O\left(\mathcal{E}(X,q)\right) & \text{if } q \leq X^{1/3}, \\ O\left(q^{1/4} X^{3/4+\varepsilon}\right) & \text{if } q \geq X^{1/3}. \end{cases}$$

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5   Remarks

An analogue of Theorem 1 can also be obtained for the averaging over all residue classes modulo q. The error term remains the same, however the main terms must be adjusted according to the greatest common divisor (α, q) as we sum over the progression $n \equiv \alpha \pmod{q}$.

We also believe that one can use our method to study the sums of the form

$$\tau_{u,v}(n) = \sum_{d|n} u(d)v(n/d),$$

where u and v are complex valued functions that satisfy certain growth conditions. One can probably extend our approach to some other similar functions. In particular, the function $r(n)$ that counts the number of representations of n as a sum of two squares and the function $\tau_k(n)$ that counts the number of representations of n as a product of k integers exceeding 1 seem to be the most natural examples of such extensions.

It would be interesting to improve the bounds of Theorem 4, especially to obtain a nontrivial estimate for $q > X$. One possible approach is to use other known bounds for double Kloosterman sums (see [2, 12, 13]) instead of (or in combination with) Lemma 4 to improve the bounds of the sums $S(A, B)$ in the proof of Theorem 4.

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