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SQUARES FROM PRODUCTS OF INTEGERS

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To Jean–Louis Nicolas in celebration of his sixtieth birthday

1. Introduction

Notice that $1\cdot2\cdot3\cdot4+1=5^2$, $2\cdot3\cdot4\cdot5+1=11^2$, $3\cdot4\cdot5\cdot6+1=19^2$, $\ldots$. Indeed, it is well known that the product of any four consecutive integers always differs by one from a perfect square. However, a little experimentation readily leads one to guess that there is no integer n, other than four, so that the product of any n consecutive integers always differs from a perfect square by some fixed integer $c=c(n)$ depending only on n.

The two issues that are present here can be readily dealt with. The apparently special status of the number four arises from the fact that any quadratic polynomial can be completed by a constant to become the square of a polynomial. Second, [5] provides an elegant proof that there is in fact no integer n larger than four with the property stated above.

In [5] one finds a reminder that a polynomial taking too many square values must be the square of a polynomial (see [4, Chapter VIII.114 and .190], and [2]). One might therefore ask whether there are polynomials other than integer multiples of $x(x+1)(x+2)(x+3)$ and $4x(x+1)$, with integer zeros and differing by a nonzero constant from the square of a polynomial. We will show that this is quite a good question in that it has a nontrivial answer, inter alia giving new insight into the results of [5]. As a new example, the reader might check that

$$1\cdot2\cdot3\cdot5\cdot6\cdot7+36=4^2\cdot9^2 \qquad 2\cdot3\cdot4\cdot6\cdot7\cdot8+36=5^2\cdot18^2$$
$$3\cdot4\cdot5\cdot7\cdot8\cdot9+36=6^2\cdot29^2 \qquad 4\cdot5\cdot6\cdot8\cdot9\cdot10+36=7^2\cdot42^2 \qquad \ldots\,.$$

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2. Squares from Products of a Set of Integers

We study polynomials $P_S(x)=\prod_{s\in S}(x+s)$ and find all nonempty sets S of integers with the property that for some rational number c, P_S+c is the square of a polynomial.

Call that polynomial $a(x)=a_{S,c}(x)$. Then we have

$$P=a^2-c=(a+\sqrt{c}\,)(a-\sqrt{c}\,).$$

It follows there is a partition $S=R\cup T$ of S so that

$$a(x)+\sqrt{c}=\prod\nolimits_{r\in R}(x+r) \quad \text{and} \quad a(x)-\sqrt{c}=\prod\nolimits_{t\in T}(x+t).$$

Because $S\subset\mathbb{Z}$, it follows that $c=k^2$ for some rational k.

Since $a(x)+\sqrt{c}$ and $a(x)+\sqrt{c}$ have the same degree, we see that R and T have the same cardinality, m say, and S has cardinality $2m$. Because the polynomials $a(x)\pm\sqrt{c}$ differ by a constant, it follows that the respective elementary symmetric functions in the integers $r\in R$ and the integers $t\in T$, other than those of order m, coincide. Equivalently, but more strikingly, we have for $j=0,\ 1,\ \ldots,\ m-1$, the identity

$$\sum\nolimits_{r\in R} r^j=\sum\nolimits_{t\in T} t^j\,. \tag{1}$$

This follows immediately from Newton's formulas whereby if

$$(x+x_1)(x+x_2)\cdots(x+x_n)=x^n+\sigma_1x^{n-1}+\cdots+\sigma_{n-1}x+\sigma_n,$$

then for $h=0,\ 1,\ 2,\ \ldots,$

$$\sigma_0s_h+\sigma_1s_{h-1}+\cdots+\sigma_{n-1}s_1+s_0\sigma_h=0\,,$$

where the s_j are the power sums $x_1^j+x_2^j+\cdots+x_n^j$ and, of course, $\sigma_0=1$ while $\sigma_k=0$ for $k>n$. In particular, the ring $\mathbb{Z}[\sigma_1,\sigma_2,\ldots,\sigma_n]$ coincides with the ring $\mathbb{Z}[s_1,s_2,s_3,\ldots]$.

Moreover, one sees that the case $m=1$ is trivial, and the case $m=2$ is nearly trivial. Indeed, for $m=1$ the conditions (1) are essentially empty, and for $m=2$ it is plain that one may select any three of the integers r_1, r_2, t_1, t_2, and obtain an integer for the fourth; in that case, incidentally, one has $c=(r_1r_2-t_1t_2)^2/4$.

3. The Prouhet–Tarry–Escott Problem

Seeing (1), one recalls that the Tarry–Escott problem is precisely the issue of finding distinct sets of integers $r_1,\ r_2,\ \ldots,\ r_n$ and $t_1,\ t_2,\ \ldots,\ t_n$ with

$$r_1^j+r_2^j+\cdots+r_n^j=t_1^j+t_2^j+\cdots+t_n^j$$

for $j=1,\ 2,\ \ldots,\ j=m$. A solution is said to be *ideal* if $m=n-1$. The critical reference is the observation by Wright [7] that the question of Tarry and Escott had already been dealt with by Prouhet [6].

Clearly, our remarks above amount to the following theorem.

Theorem 1. *Let S be a finite set of integers and set $P_S(x)=\prod_{s\in S}(x+s)$. Then P_S differs by a constant c from the square of a polynomial if and only if S is the disjoint union of sets R and T that provide an ideal solution to the Tarry-Escott problem.*

Thus [5] reminds us that there are no ideal solutions $R\cup T=S$ to the Tarry–Escott problem for which S is an arithmetic progression of more than four integers.

There is much activity in the matter of finding new solutions to the Tarry-Escott problem; it is best followed on the web, starting from [1] or [3]. The following sporadic examples come from there and other linked sources.

3.1. **The Opening Example.**

$$x(x+1)(x+2)(x+4)(x+5)(x+6)+36=(x+3)^2(x^2+6x+2)^2.$$

3.2. **From Tarry's Ideal Symmetric Solution of 1912.**

$$x(x+1)(x+2)(x+5)(x+6)(x+10)(x+12)(x+16)(x+17)(x+20)(x+21)(x+22)+2540160000$$
$$=(x^6+66x^5+1633x^4+18612x^3+95764x^2+179520x+50400)^2.$$

3.3. **From Escott's Ideal Symmetric Solution of 1910.**

$$x(x+1)(x+13)(x+18)(x+27)(x+38)(x+44)(x+58)(x+64)(x+75)(x+84)(x+89)(x+101)(x+102)+c$$
$$=(x+51)^2(x^6+306x^5+34801x^4+1793364x^3+40430980x^2+315284448x+136936800)^2;$$

one readily confirms that here $c=48773138392218240000$.

3.4. **Shifting by Primes.**

$$(x+7)(x+11)(x+13)(x+19)(x+29)(x+31)+82944=(x^3+55x^2+887x+4145)^2;$$
$$(x+11)(x+13)(x+19)(x+23)(x+29)(x+31)+25600=(x+21)^2(x^2+42x+357)^2.$$

3.5. **Shifting by Squares.**

$$(x+1^2)(x+5^2)(x+6^2)(x+9^2)(x+10^2)(x+11^2)+50400^2=(x^3+182x^2+8281x+58500)^2.$$

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