New Examples of Noncommutative $\Lambda(p)$ Sets

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Abstract

In this paper, we introduce a certain combinatorial property $Z^\star(k)$, which is defined for every integer $k \geq 2$, and show that every set $E \subset \mathbb{Z}$ with the property $Z^\star(k)$ is necessarily a noncommutative $\Lambda(2k)$ set. In particular, using number theoretic results about the number of solutions to so-called "S-unit equations," we show that for any finite set Q of prime numbers, E_Q is noncommutative $\Lambda(p)$ for every real number $2 < p < \infty$, where E_Q is the set of natural numbers whose prime divisors all lie in the set Q.

1 Introduction

For any finite set Q of prime numbers, let $E_Q \subset \mathbb{N}$ denote the set of all natural numbers n such that every prime divisor of n lies in Q. If Q contains only a single prime q, then $E_Q = \{q^j \mid j \geq 0\}$ is a Hadamard set and therefore also a Sidon set; consequently, for every real number $2 < p < \infty$, the bound

$$\|f\|_{L^p} \leq C\|f\|_{L^2}$$

holds for every function $f \in L^p$ whose Fourier coefficients are supported on the set E_Q, where $C > 0$ is a constant depending only on p; in other words, the set E_Q is of type $\Lambda(p)$. When Q has cardinality $\#Q \geq 2$, the set E_Q is neither Hadamard nor Sidon; however, number theoretic results about solutions to so-called "S-unit equations" imply that E_Q is again a $\Lambda(p)$ set for $2 < p < \infty$.

In this paper, we show that for any finite set Q of prime numbers and any real number $2 < p < \infty$, the set E_Q satisfies a much stronger analytic property, namely the noncommutative $\Lambda(p)$ property; that is, E_Q is of type $\Lambda(p)_{cb}$. More precisely, we show that the bound

$$\|f\|_{L^p(S^p)} \leq C \max\left\{ \Big\|\Big(\sum_n \widehat{f}(n)^*\widehat{f}(n)\Big)^{1/2}\Big\|_{S^p}, \Big\|\Big(\sum_n \widehat{f}(n)\widehat{f}(n)^*\Big)^{1/2}\Big\|_{S^p} \right\}$$

holds for every function $f \in L^p(S^p)$ whose Fourier coefficients are supported on the set E_Q, where the constant $C > 0$ depends only on p and on the cardinality $\#Q$ of the set Q. Here S^p denotes the Schatten p-class over the Hilbert space ℓ_2; it is the Banach space of all compact operators $x : \ell_2 \to \ell_2$ with a finite norm given by

$$\|x\|_{S^p} = \Big(\mathrm{Tr}\,(x^*x)^{p/2}\Big)^{1/p},$$

where $\mathrm{Tr}(\,\cdot\,)$ denotes the usual trace. The Banach space $L^p(S^p)$ consists of all Bochner measurable S^p-valued functions defined on the unit circle, equipped with the norm

$$\|f\|_{L^p(S^p)} = \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(t)\|_{S^p}^p\,dt\right)^{1/p},$$

where dt is the Lebesgue measure.

To establish our results, we introduce a certain combinatorial property $Z^\star(k)$, defined for every integer $k \geq 2$, and show that every set E with the property $Z^\star(k)$ is necessarily of type $\Lambda(2k)_{cb}$. In particular, we observe that for any finite set Q of primes, the set E_Q satisfies $Z^\star(k)$ for every $k \geq 2$; this follows from the number theoretic results mentioned earlier. Note that the sets E_Q

with $\#Q \geq 2$, along with their translations, dilations, etc., provide the only currently known examples of sets that are of type $\Lambda(p)_{cb}$ for every $2 < p < \infty$ but are not Sidon sets.

The paper is organized as follows. Sections 2–7 are entirely expository in nature; there we review the definitions and results that are needed in the sequel. In Section 8, we show that the $Z^\star(k)$ property implies the $\Lambda(2k)_{cb}$ property. In Section 9, we observe that every set E_Q satisfies $Z^\star(k)$ for all $k \geq 2$, and that E_Q is not a Sidon set if $\#Q \geq 2$. In Section 10, we give some concluding remarks.

2 Khintchine inequalities

For every $n \in \mathbb{N}$, let $\varepsilon_n : \{\pm 1\}^{\mathbb{N}} \longrightarrow \{\pm 1\}$ denote the n-th coordinate projection, let ν be the uniform probability measure on $\{\pm 1\}^{\mathbb{N}}$, and let p be an arbitrary real number with $2 < p < \infty$.

The classical *Khintchine inequalities* show that there exists a constant $C > 0$, depending only on p, such that for all $m \geq 1$ and any sequence $x_1, x_2, \ldots, x_m$ in $\mathbb{C}$, one has

$$\left\| \sum_{n=1}^{m} \varepsilon_n x_n \right\|_{L^p(\{\pm 1\}^{\mathbb{N}},\, \nu,\, \mathbb{C})} \leq C \left(\sum_{n=1}^{m} |x_n|^2 \right)^{1/2}; \tag{2.1}$$

see [6], for example, for a proof of Khintchine inequalities in the general case $1 \leq p < \infty$. The inequalities (2.1) were later generalized to the noncommutative setting by Lust-Piquard [7], who showed that there exists a constant $C > 0$, depending only on p, such that for all $m \geq 1$ and any sequence of operators $x_1, x_2, \ldots, x_m$ in S^p, the following inequality holds:

$$\begin{aligned} &\left\| \sum_{n=1}^{m} \varepsilon_n x_n \right\|_{L^p(\{\pm 1\}^{\mathbb{N}},\, \nu,\, S^p)} \\ &\qquad \leq C \max \left\{ \left\| \left(\sum_{n=1}^{m} x_n^* x_n \right)^{1/2} \right\|_{S^p}, \left\| \left(\sum_{n=1}^{m} x_n x_n^* \right)^{1/2} \right\|_{S^p} \right\}; \end{aligned} \tag{2.2}$$

see [7] for a proof of the noncommutative Khintchine inequalities in the more general case where $1 < p < \infty$; see also [8] for the case $p = 1$.

3 $\Lambda(p)$ sets

The notion of a $\Lambda(p)$ set was first introduced in [16] and studied extensively by Rudin and many others. In this paper, we restrict ourselves to the case where $2 < p < \infty$, for simplicity. For any set $E \subset \mathbb{Z}$, let

$$L_E^p = \big\{ f \in L^p \,\big|\, \widehat{f} \text{ is supported on } E \big\},$$

where $\widehat{f}$ denotes the Fourier transform of f. Then E is said to be of type $\Lambda(p)$, or E has the $\Lambda(p)$ property, if there exists a constant $C > 0$, depending only on p and E, such that for every function in L_E^p, the following bound holds:

$$\big\| f \big\|_{L^p} \leq C \Big(\sum_{n \in E} \big| \widehat{f}(n) \big|^2 \Big)^{1/2}.$$

We denote by $\lambda_p(E)$ the smallest constant C for which this inequality holds for all $f \in L_E^p$.

Using convexity, one sees that every $\Lambda(p)$ set is also a $\Lambda(q)$ set for any real number $2 < q < p$.

We also recall that, as shown in [16], there is a natural size limitation for the intersection of any $\Lambda(p)$ set with a fixed arithmetic progression. More precisely, if $2 < p < \infty$ is fixed, and E is a $\Lambda(p)$ set, then

$$\#\big(E \cap \{ a + b, a + 2b, \ldots, a + Nb \} \big) \leq 4 \left(\lambda_p(E) \right)^2 N^{2/p} \tag{3.3}$$

for all integers a, b, N with $N \geq 1$. This result is optimal. Indeed, given $2 < p < \infty$, there is a subset E_N of $\{1, \ldots, N\}$ for each integer N, satisfying $\# E_N \geq N^{2/p}$ and $\lambda_p(E_N) \leq C_p$ where the constant C_p depends only on p. This result was first shown by Rudin for even integers (see [16]), then later by Bourgain for arbitrary real numbers (see [2], and also [19]). It follows that for every $2 < p < \infty$, there exists a $\Lambda(p)$ set that is not a $\Lambda(q)$ set for any $q > p$.

In [16], a certain combinatorial property has been considered which is not only stronger but often easier to deal with than the analytic property $\Lambda(2k)$. Let $k \geq 1$ be a fixed integer. A set $E \subset \mathbb{Z}$ is called a $Z^+(k)$ set if there exists a constant $C > 0$, depending only on E, such that for all $\gamma \in \mathbb{Z}$,

$$\#\big\{ (n_1, n_2, \ldots, n_k) \in E^k \,\big|\, n_1 + n_2 + \ldots + n_k = \gamma \big\} \leq C.$$

It has been shown by Rudin [16] that every $Z^+(k)$ set is necessarily of type $\Lambda(2k)$. In particular, for any finite set Q of primes, the set E_Q satisfies $Z^+(k)$ for all $k \geq 1$ (see Section 9); hence it follows that E_Q is of type $\Lambda(p)$ for every $2 < p < \infty$.

4 Noncommutative $\Lambda(p)$ sets

The notion of noncommutative $\Lambda(p)$ sets was first introduced and studied in [5]. For $2 < p < \infty$ and $E \subset \mathbb{Z}$, let

$$L^p_E(S^p) = \left\{ f \in L^p(S^p) \,\middle|\, \widehat{f} \text{ is supported on } E \right\}.$$

The set E is called a noncommutative $\Lambda(p)$ set (or simply, a $\Lambda(p)_{cb}$ set) if there exists a constant $C > 0$, depending only on p and E, such that for every function f in $L^p_E(S^p)$, the bound

$$\|f\|_{L^p(S^p)} \leq C \, |||f|||_p \tag{4.4}$$

holds, where the triple norm $|||\cdot|||_p$ is defined by

$$|||f|||_p = \max\left\{ \left\| \left(\sum_{n\in\mathbb{Z}} \widehat{f}(n)^* \widehat{f}(n) \right)^{1/2} \right\|_{S^p}, \left\| \left(\sum_{n\in\mathbb{Z}} \widehat{f}(n) \widehat{f}(n)^* \right)^{1/2} \right\|_{S^p} \right\}.$$

We denote by $\lambda^{cb}_p(E)$ the smallest constant C for which the inequality (4.4) holds for all $f \in L^p_E(S^p)$. Note that, by convexity, the opposite inequality

$$|||f|||_p \leq \|f\|_{L^p(S^p)} \tag{4.5}$$

always holds for every $f \in L^p(S^p)$. We remark that the notation *cb* is an abbreviation for the words "completely bounded." Harcharras [5] showed that a given set E has the $\Lambda(p)_{cb}$ property if and only if every bounded sequence $(a_n)_{n\in E}$ can be extended to a completely bounded Fourier multiplier on the operator space L^p when the latter is endowed with its canonical operator space structure as defined by Pisier [13].

It is clear from the definition that every $\Lambda(p)_{cb}$ set is necessarily a $\Lambda(p)$ set, therefore the size restriction (3.3) applies. On the other hand, it has been shown in [5] that there exist sets with the $\Lambda(p)$ property for every p which do not have the $\Lambda(p)_{cb}$ property for any p; thus, the $\Lambda(p)_{cb}$ property is much stronger than the $\Lambda(p)$ property in general.

Note that, by convexity, a $\Lambda(p)_{cb}$ set is also a $\Lambda(q)_{cb}$ set if $2 < q < p < \infty$. Building on the work of Rudin [16], it has been shown in [5] that for every even integer $p = 2k > 2$, there exists a $\Lambda(p)_{cb}$ set that does not have the $\Lambda(q)$ property for any $q > p$; the general case is still open.

In [5], a combinatorial property has been considered which is stronger and easier to deal with than the analytic property $\Lambda(2k)_{cb}$. Let $k \geq 1$ be a fixed integer. A set $E \subset \mathbb{Z}$ is called a $Z(k)$ set if there exists a constant $C > 0$, depending only on k and E, such that for all $\gamma \in \mathbb{Z}$,

$$\#\left\{(n_1, n_2, \ldots, n_k) \in E^k \,\middle|\, n_i \neq n_j \text{ if } i \neq j, \text{ and } \sum_{j=1}^{k} (-1)^{j+1} n_j = \gamma\right\} \leq C.$$

It has been shown in [5] that an arbitrary $Z^+(k)$ set need not possess the $\Lambda(2k)_{cb}$ property even though it is a $\Lambda(2k)$ set as mentioned earlier. However, any set with the $Z(k)$ property is necessarily of type $\Lambda(2k)_{cb}$.

Our review of the combinatorial property $Z(k)$ has been intended primarily to motivate our consideration of the new property $Z^\star(k)$ introduced in Section 8. In many situations, it is useful to have combinatorial criteria like $Z(k)$ and $Z^\star(k)$ which imply the (albeit weaker) analytic property $\Lambda(2k)_{cb}$. For the purposes of this paper, the $Z(k)$ property alone is insufficient, since for an arbitrary finite set Q of primes, the set E_Q need not be of type $Z(k)$. For example, taking $Q = \{2, 3\}$, the relation

$$2^{i+3}3^j - 2^i 3^{j+2} + 2^i 3^j = 0, \qquad \forall\, i, j \geq 0,$$

implies that E_Q is not of type $Z(3)$ even though it is of type $Z^\star(k)$ for all $k \geq 2$ and therefore of type $\Lambda(p)_{cb}$ for every $2 < p < \infty$ (see Section 9).

5 Sidon sets

A set $E \subset \mathbb{Z}$ is called a Sidon set if there exists a constant $C > 0$, depending only on E, such that for all functions $f \in L^\infty_E$, the following bound holds:

$$\sum_{n \in E} \left|\widehat{f}(n)\right| \leq C \left\|f\right\|_{L^\infty}. \tag{5.6}$$

We denote by $\lambda_\infty(E)$ the smallest constant C for which this inequality holds for all $f \in L^\infty_E$.

It is well known that a Sidon set is a $\Lambda(p)$ set for every $2 < p < \infty$. In fact, it is a $\Lambda(p)_{cb}$ set for every $2 < p < \infty$ as shown in [5]. On the other hand, there is a natural size limitation for the intersections of any Sidon set with a fixed arithmetic progression. It has been shown in [16] that there exists an absolute constant $C > 0$ such that for every Sidon set E,

$$\#\big(E \cap \{a + b, a + 2b, \ldots, a + Nb\}\big) \leq C\big(\lambda_\infty(E)\big)^2 \log N$$

for all integers a, b, N with $N \geq 1$.

6 Pisier's Rademacherization principle

In this section, we describe a result of [14] that can be used to determine nontrivial upper bounds for the norm of certain sums of products of operators in which various repetitions of the indices occur.

Given two partitions $\mathcal{P} = \{\mathcal{P}_j\}$ and $\mathcal{Q} = \{\mathcal{Q}_i\}$ of the set $\{1, 2, \ldots, k\}$, write $\mathcal{P} \leq \mathcal{Q}$ if for every j, $\mathcal{P}_j \subset \mathcal{Q}_i$ for some i, and write $\mathcal{P} < \mathcal{Q}$ whenever $\mathcal{P} \leq \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$. It is easily verified that the relation $\leq$ provides a partial order on the set of all partitions of $\{1, 2, \ldots, k\}$; the unique minimal element with respect to $\leq$ is the partition $\mathcal{P}_{\min} = \big\{\{1\}, \{2\}, \ldots, \{k\}\big\}$, while $\mathcal{P}_{\max} = \big\{\{1, 2, \ldots, k\}\big\}$ is the unique maximal element.

Given a k-tuple $n = (n_1, n_2, \ldots, n_k) \in E^k$, where E is an arbitrary set, let $\mathcal{P}_n = \{\mathcal{P}_{n,j}\}$ denote the canonical partition attached to n; that is, for all $1 \leq i, \ell \leq k$, both i and ℓ belong to the same set $\mathcal{P}_{n,j}$ if and only if $n_i = n_\ell$.

Proposition 1. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\{\varepsilon_n\}_{n \in E}$ a family of independent random variables with*

$$\mathbb{P}\big(\{\varepsilon_n = 1\}\big) = \mathbb{P}\big(\{\varepsilon_n = -1\}\big) = 1/2, \qquad \forall\, n \in E.$$

Let $k \geq 2$ be an arbitrary integer. For $1 \leq j \leq k$, let X_j be a Banach space, and $f_j : E \longrightarrow X_j$ a finitely supported function. Let

$$\varphi : X_1 \times X_2 \times \ldots \times X_k \longrightarrow X$$

be a k-linear map of norm at most 1, where X is a given Banach space. Finally, for any partition $\mathcal{P}$ of the set $\{1, 2, \ldots, k\}$, put

$$A_{\mathcal{P}} = \big\{ j \in \{1, 2, \ldots, k\} \,\big|\, \{j\} \in \mathcal{P} \big\}.$$

Then the following inequality holds:

$$\Big\| \sum_{\substack{n=(n_1,\ldots,n_k)\in E^k \\ \mathcal{P}_n \geq \mathcal{P}}} \varphi\big(f_1(n_1), \ldots, f_k(n_k)\big) \Big\|_X$$

$$\leq \prod_{j \in A_{\mathcal{P}}} \Big\| \sum_{n \in E} f_j(n) \Big\|_{X_j} \prod_{\substack{1 \leq j \leq k \\ j \notin A_{\mathcal{P}}}} \left(\int_\Omega \Big\| \sum_{n \in E} \varepsilon_n f_j(n) \Big\|_{X_j}^k \, d\,\mathbb{P} \right)^{1/k}.$$

7 Some operator norm inequalities

Proposition 2. *Let $1 \leq p \leq \infty$, $a, b > 1$ with $a^{-1} + b^{-1} = 1$, y a positive operator in S^{ap}, and $x_1, x_2, \ldots, x_m$ a sequence of operators each in S^{2bp}. Then the following inequality holds:*

$$\Big\| \sum_{n=1}^{m} x_n^* y x_n \Big\|_{S^p} \leq \|y\|_{S^{ap}} \max \Big\{ \Big\| \sum_{n=1}^{m} x_n^* x_n \Big\|_{S^{bp}}, \Big\| \sum_{n=1}^{m} x_n x_n^* \Big\|_{S^{bp}} \Big\}.$$

This proposition first appears in [7] when $x_1, x_2, \ldots x_n$ is a sequence of self-adjoint operators. The general case requires only the three line lemma and can be found in [15].

The following corollary follows from Proposition 2 by a simple inductive argument.

Corollary 3. *Let $1 \leq p \leq \infty$ and $k \geq 1$ be fixed. For each $1 \leq j \leq k$, let E_j be a finite set of indices, let $a_j > 1$, and let $\{x_{j,n}\}_{n \in E_j}$ be a family of operators in $S^{2a_j p}$. Finally, suppose that $\sum_{j=1}^{k} a_j^{-1} = 1$. Then the following inequality holds:*

$$\Big\| \sum_{n_j \in E_j,\ 1 \leq j \leq k} x_{k,n_k}^* \ldots x_{2,n_2}^* x_{1,n_1}^* \, x_{1,n_1} x_{2,n_2} \ldots x_{k,n_k} \Big\|_{S^p}$$
$$\leq \prod_{j=1}^{k} \max \Big\{ \Big\| \sum_{n \in E_j} x_{j,n}^* x_{j,n} \Big\|_{S^{a_j p}}, \Big\| \sum_{n \in E_j} x_{j,n} x_{j,n}^* \Big\|_{S^{a_j p}} \Big\}.$$

8 Main results

Throughout this section, let k be a fixed integer with $k \geq 2$. Here we introduce a new combinatorial property for sets $E \subset \mathbb{Z}$, similar to the $Z(k)$ property described in Section 4.

We say that a set $E \subset \mathbb{Z}$ has the property $Z^\star(k)$ if there is a constant $C > 0$, depending only on E and k, such that:

(i) For every nonzero $\gamma \in \mathbb{Z}$, the conditions

$$n_1 - n_2 + \ldots + (-1)^{k+1} n_k = \gamma \tag{8.7}$$

and

$$\sum_{j\in\mathcal{J}}(-1)^{j+1}n_j \neq 0 \qquad \text{for all } \emptyset \neq \mathcal{J} \subsetneq \{1,\ldots,k\} \tag{8.8}$$

are satisfied for at most C elements $(n_1, n_2, \ldots, n_k) \in E^k$.

(*ii*) For every $\emptyset \neq \mathcal{J} \subset \{1,\ldots,k\}$, there are at most C vectors $\mathrm{v}_\ell \in \mathbb{Q}^{\#\mathcal{J}}$ such that if the vector $n = (n_j)_{j\in\mathcal{J}} \in E^{\#\mathcal{J}}$ satisfies the conditions

$$\sum_{j\in\mathcal{J}}(-1)^{j+1}n_j = 0 \tag{8.9}$$

and

$$\sum_{j\in\mathcal{J}'}(-1)^{j+1}n_j \neq 0 \qquad \text{for all } \emptyset \neq \mathcal{J}' \subsetneq \mathcal{J}, \tag{8.10}$$

then $n = \eta\mathrm{v}_\ell$ for some $\eta \in E$ and some $1 \leq \ell \leq C$.

Theorem 4. *If a set $E \subset \mathbb{Z}$ has the property $Z^\star(k)$, then E is a $\Lambda(2k)_{cb}$ set.*

Proof. Without loss of generality, one can assume that $E \subset \mathbb{N}$. Throughout the proof, the letter C is used to denote any positive constant that occurs and depends only on k and or E; its precise meaning might change from line to line.

For every $n = (n_1, n_2, \ldots, n_k) \in E^k$, let R_n denote the collection of all subsets $\emptyset \neq \mathcal{J} \subsetneq \{1,\ldots,k\}$ such that (8.9) and (8.10) hold, and let $\mathcal{R}$ be the set of all collections obtained in this way; that is,

$$\mathcal{R} = \{R \mid R = R_n \text{ for some } n \in E^k\}.$$

For $R, R' \in \mathcal{R}$, write $R' < R$ or $R > R'$ whenever $\emptyset \neq R' \subsetneq R$. Then the relation $<$ defines a partial order on $\mathcal{R}$. We also put

$$d_0 = \max\{\#R \mid R \in \mathcal{R}\},$$

and for $0 \leq d \leq d_0$, let

$$\mathcal{R}(d) = \{R \in \mathcal{R} \mid \#R = d\}.$$

Then $\mathcal{R}$ is the disjoint union $\mathcal{R} = \bigcup_{d=0}^{d_0}\mathcal{R}(d)$.

Now let $f = \sum_{n\in E} x_n e^{\mathrm{i}nt}$ be fixed; note that $x_n = \widehat{f}(n) \in S^{2k}$ for every $n \in E$. For simplicity, we assume that the Fourier transform $\widehat{f}$ is finitely supported.

For every k-tuple $n = (n_1, n_2, \ldots, n_k) \in E^k$, let

$$\widetilde{x}_n = x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \ldots x_{n_k}^{\mu_k} \in S^2,$$

where $\mu_j = 1$ if j is odd, and $\mu_j = *$ if j is even. Then

$$\|f\|_{L^{2k}(S^{2k})}^{2k} = \|f^{\mu_1} f^{\mu_2} \ldots f^{\mu_k}\|_{L^2(S^2)}^2 = \left\| \sum_{\gamma \in \mathbb{Z}} e^{\mathrm{i}\gamma t} \sum_{n \in E^k(\gamma)} \widetilde{x}_n \right\|_{L^2(S^2)}^2,$$

where for each $\gamma \in \mathbb{Z}$,

$$E^k(\gamma) = \{n = (n_1, n_2, \ldots, n_k) \in E^k \,|\, n_1 - n_2 + \ldots + (-1)^{k+1} n_k = \gamma\}.$$

It follows that

$$\|f\|_{L^{2k}(S^{2k})}^{2k} = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{n \in E^k(\gamma)} \widetilde{x}_n \right\|_{S^2}^2 = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{0 \le d \le d_0 \\ R \in \mathcal{R}(d)}} \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \widetilde{x}_n \right\|_{S^2}^2.$$

Thus,

$$\|f\|_{L^{2k}(S^{2k})}^{2k} \le C \sum_{\substack{0 \le d \le d_0 \\ R \in \mathcal{R}(d)}} \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \widetilde{x}_n \right\|_{S^2}^2 = C \sum_{\substack{0 \le d \le d_0 \\ R \in \mathcal{R}(d)}} \mathcal{S}(R),$$

where we have set

$$\mathcal{S}(R) = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \widetilde{x}_n \right\|_{S^2}^2.$$

For each collection R with $0 < \#R < d_0$, one has

$$\begin{aligned} \mathcal{S}(R) &\le 2 \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \ge R}} \widetilde{x}_n \right\|_{S^2}^2 + 2 \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n > R}} \widetilde{x}_n \right\|_{S^2}^2 \\ &\le 2\, \widetilde{\mathcal{S}}(R) + C \sum_{\substack{d < d' \le d_0 \\ R' \in \mathcal{R}(d')}} \mathcal{S}(R'), \end{aligned}$$

where we have set

$$\widetilde{\mathcal{S}}(R) = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \ge R}} \widetilde{x}_n \right\|_{S^2}^2;$$

when $\#R = 0$ or d_0, it is clear that $\mathcal{S}(R) = \widetilde{\mathcal{S}}(R)$. Consequently,

$$\|f\|_{L^{2k}(S^{2k})}^{2k} \leq C \sum_{\substack{0 \leq d \leq d_0 \\ R \in \mathcal{R}(d)}} \widetilde{\mathcal{S}}(R). \tag{8.11}$$

Step 1. We start by showing that the inequality $\widetilde{\mathcal{S}}(\emptyset) \leq C |\!|\!| f |\!|\!|_{2k}^{2k}$ holds for some constant $C > 0$. Indeed,

$$\widetilde{\mathcal{S}}(\emptyset) = \mathcal{S}(\emptyset) = \sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 + \Big\| \sum_{\substack{n \in E^k(0) \\ R_n = \emptyset}} \widetilde{x}_n \Big\|_{S^2}^2. \tag{8.12}$$

Since E has the property $Z^\star(k)$, for every $\gamma \neq 0$ the equation (8.7) has at most C solutions $n \in E^k$ such that (8.8) also holds. Thus,

$$\begin{aligned}
\sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 &\leq C \sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \|\widetilde{x}_n\|_{S^2}^2 \\
&\leq C \sum_{n \in E^k} \|\widetilde{x}_n^* \widetilde{x}_n\|_{S^1} = C \Big\| \sum_{n \in E^k} \widetilde{x}_n^* \widetilde{x}_n \Big\|_{S^1} \\
&= C \Big\| \sum_{n_1, n_2, \ldots, n_k \in E} (x_{n_k}^{\mu_k})^* \ldots (x_{n_2}^{\mu_2})^* (x_{n_1}^{\mu_1})^* x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \ldots x_{n_k}^{\mu_k} \Big\|_{S^1} \\
&\leq C \prod_{j=1}^{k} \max\Big\{ \Big\| \sum_{n_j \in E} x_{n_j}^* x_{n_j} \Big\|_{S^k}, \Big\| \sum_{n_j \in E} x_{n_j} x_{n_j}^* \Big\|_{S^k} \Big\}
\end{aligned}$$

where for the last inequality, we have applied Corollary 3. It follows that

$$\sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 \leq C \, |\!|\!| f |\!|\!|_{2k}^{2k}.$$

For every n occurring in the second term of (8.12), since $\gamma = 0$, we see that the equation (8.9) holds with $\mathcal{J} = \{1, 2, \ldots, k\}$; since $R_n = \emptyset$, the condition (8.10) also applies. Hence, since E has the property $Z^\star(k)$, there are at most C vectors $\mathrm{v}_\ell \in \mathbb{Q}^k$ such that for each n occurring in the second term of (8.12), $n = \eta \mathrm{v}_\ell$ for some $\eta \in E$ and $1 \leq \ell \leq C$. Using Cauchy-Schwarz's inequality, we see that

$$\Big\| \sum_{\substack{n \in E^k(0) \\ R_n = \emptyset}} \widetilde{x}_n \Big\|_{S^2}^2 \leq C \sum_{1 \leq \ell \leq C} \Big\| \sum_{\eta \in E} x_{\eta \mathrm{v}_{\ell,1}}^{\mu_1} x_{\eta \mathrm{v}_{\ell,2}}^{\mu_2} \ldots x_{\eta \mathrm{v}_{\ell,k}}^{\mu_k} \Big\|_{S^2}^2.$$

Note that here and elsewhere in the proof, we write $x_z = 0$ if $z \in \mathbb{Q}$, $z \notin E$.

At this point, fix $1 \leq \ell \leq C$. We apply Proposition 1 with the following choices: Ω is $\{\pm 1\}^{\mathbb{N}}$ equipped with the counting probability; $\{\varepsilon_\eta\}_{\eta \in E}$ is a family of coordinate projections, where ε_η is the m_η-th projection on Ω, for some enumeration $\{m_\eta \,|\, \eta \in E\}$ of the set $\mathbb{N}$; $\mathcal{P}$ is the maximal partition $\mathcal{P}_{\max}$; φ is the k-linear contractive map that is simply the k-fold product from $S^{2k} \times S^{2k} \times \ldots \times S^{2k}$ into S^2; the functions $f_j : E \longrightarrow S^{2k}$ are defined by mapping $\eta \in E$ to $f_j(\eta) = x^{\mu_j}_{\eta \mathrm{v}_{\ell,j}}$ in S^{2k}, for each $1 \leq j \leq k$. By the proposition, it follows that

$$\Big\| \sum_{\eta \in E} x^{\mu_1}_{\eta \mathrm{v}_{\ell,1}} x^{\mu_2}_{\eta \mathrm{v}_{\ell,2}} \ldots x^{\mu_k}_{\eta \mathrm{v}_{\ell,k}} \Big\|_{S^2} \leq \prod_{j=1}^{k} \left(\int_\Omega \Big\| \sum_{\eta \in E} \varepsilon_\eta \, x_{\eta \mathrm{v}_{\ell,j}} \Big\|^k_{S^{2k}} \, d\,\mathbb{P} \right)^{1/k}.$$

Now, apply Jensen's inequality followed by the noncommutative version of Khintchine inequalities (2.2) as follows:

$$\begin{aligned}\Big\| \sum_{\eta \in E} x^{\mu_1}_{\eta \mathrm{v}_{\ell,1}} x^{\mu_2}_{\eta \mathrm{v}_{\ell,2}} \ldots x^{\mu_k}_{\eta \mathrm{v}_{\ell,k}} \Big\|_{S^2} \quad &\leq \quad \prod_{j=1}^{k} \left(\int_\Omega \Big\| \sum_{\eta \in E} \varepsilon_\eta \, x_{\eta \mathrm{v}_{\ell,j}} \Big\|^{2k}_{S^{2k}} \, d\,\mathbb{P} \right)^{1/(2k)} \\ &\leq \quad C \prod_{j=1}^{k} \Big\| \Big\| \sum_{\eta \in E} x_{\eta \mathrm{v}_{\ell,j}} \, e^{\mathrm{i}\eta \mathrm{v}_{\ell,j} t} \Big\| \Big\|_{2k} \leq C \; \| \| f \| \|^k_{2k},\end{aligned}$$

since for each $1 \leq j \leq C$,

$$\Big\| \Big\| \sum_{\eta \in E} x_{\eta \mathrm{v}_{\ell,j}} \, e^{\mathrm{i}\eta \mathrm{v}_{\ell,j} t} \Big\| \Big\|_{2k} \leq \| \| f \| \|_{2k}.$$

It follows that

$$\Big\| \sum_{\substack{n \in E^k(0) \\ R_n = \emptyset}} \widetilde{x}_n \Big\|^2_{S^2} \leq C \| \| f \| \|^{2k}_{2k},$$

which completes Step 1.

Step 2. Next, we show that the inequality $\widetilde{\mathcal{S}}(R) \leq C \big\| f \big\|^{2k-2}_{L^{2k}(S^{2k})} \, \| \| f \| \|^2_{2k}$ holds for every $1 \leq d \leq d_0$ and every $R \in \mathcal{R}(d)$.

For this aim, fix $1 \leq d \leq d_0$ and $R \in \mathcal{R}(d)$. There is a canonical equivalence relation $\mathcal{P}_R$ induced by the collection R on the set $\{1, 2, \ldots, k\}$, defined as follows. Write $j \equiv \ell \pmod{\mathcal{P}_R}$ if and only if there exists a positive integer $t = t(j, \ell)$ and sets $\mathcal{J}_1, \ldots, \mathcal{J}_t$ in the collection R such that:

(*i*) $j \in \mathcal{J}_1$ and $\ell \in \mathcal{J}_t$,

(*ii*) $\mathcal{J}_j \cap \mathcal{J}_{j+1} \neq \emptyset$ for $1 \leq j < t$.

Let $\mathcal{P}_R$ also denote the corresponding partition of $\{1, 2, \ldots, k\}$, and let a_R denote the number of singleton sets in $\mathcal{P}_R$. Below we show the following more precise inequality:

$$\widetilde{\mathcal{S}}(R) = \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \widetilde{x}_n \Big\|_{S^2}^2 \leq C \|f\|_{L^{2k}(S^{2k})}^{2a_R} \, \|\!|f|\!\|_{2k}^{2k-2a_R}.$$

Combining (*ii*) in property $Z^\star(k)$ with condition (*ii*) in our definition of the equivalence relation $\mathcal{P}_R$ above, it is not hard to see that there are at most C vectors $\mathrm{v}_\ell = (\mathrm{v}_{\ell,j})_{j=1}^k \in \mathbb{Q}^k$ with the properties:

(*i*) $\mathrm{v}_{\ell,j} = 1$ if $\{j\} \in \mathcal{P}_R$,

(*ii*) For every $n = (n_1, n_2, \ldots, n_k) \in E^k$, the inequality $R_n \geq R$ holds if and only if for some $1 \leq \ell \leq C$ and some $\eta = (\eta_1, \eta_2, \ldots, \eta_k) \in E^k$ with $\eta_i = \eta_\ell$ whenever $i \equiv \ell \pmod{\mathcal{P}_R}$, $n_j = \eta_j \mathrm{v}_{\ell,j}$ for all $1 \leq j \leq k$.

Consequently,

$$\begin{aligned}
\widetilde{\mathcal{S}}(R) &= \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \widetilde{x}_n \Big\|_{S^2}^2 = \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{n = (n_1, \ldots, n_k) \in E^k(\gamma) \\ R_n \geq R}} x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \cdots x_{n_k}^{\mu_k} \Big\|_{S^2}^2 \\
&= \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{1 \leq \ell \leq C} \sum_{\substack{\eta = (\eta_1, \ldots, \eta_k) \in E^k, \ \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 \mathrm{v}_{\ell,1} - \eta_2 \mathrm{v}_{\ell,2} + \ldots + (-1)^{k+1} \eta_k \mathrm{v}_{\ell,k} = \gamma}} x_{\eta_1 \mathrm{v}_{\ell,1}}^{\mu_1} x_{\eta_2 \mathrm{v}_{\ell,2}}^{\mu_2} \cdots x_{\eta_k \mathrm{v}_{\ell,k}}^{\mu_k} \Big\|_{S^2}^2 \\
&\leq C \sum_{1 \leq \ell \leq C} \sum_{\gamma \in \mathbb{Z}} \Big\| \sum_{\substack{\eta = (\eta_1, \ldots, \eta_k) \in E^k, \ \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 \mathrm{v}_{\ell,1} - \eta_2 \mathrm{v}_{\ell,2} + \ldots + (-1)^{k+1} \eta_k \mathrm{v}_{\ell,k} = \gamma}} x_{\eta_1 \mathrm{v}_{\ell,1}}^{\mu_1} x_{\eta_2 \mathrm{v}_{\ell,2}}^{\mu_2} \cdots x_{\eta_k \mathrm{v}_{\ell,k}}^{\mu_k} \Big\|_{S^2}^2 \\
&= C \sum_{1 \leq \ell \leq C} \widetilde{\mathcal{S}}_\ell(R),
\end{aligned}$$

where $\widetilde{\mathcal{S}}_\ell(R)$ denotes the inner summation for each ℓ.

Let $1 \leq \ell \leq C$ be fixed; then we can estimate $\widetilde{\mathcal{S}}_\ell(R)$ as follows:

$$\begin{aligned}
\widetilde{\mathcal{S}}_\ell(R) &= \Bigg\| \sum_{\gamma\in\mathbb{Z}} e^{\mathrm{i}\gamma t} \sum_{\substack{\eta=(\eta_1,\ldots,\eta_k)\in E^k,\ \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 \mathrm{v}_{\ell,1} - \eta_2 \mathrm{v}_{\ell,2} + \ldots + (-1)^{k+1}\eta_k \mathrm{v}_{\ell,k} = \gamma}} x^{\mu_1}_{\eta_1 \mathrm{v}_{\ell,1}} x^{\mu_2}_{\eta_2 \mathrm{v}_{\ell,2}} \ldots x^{\mu_k}_{\eta_k \mathrm{v}_{\ell,k}} \Bigg\|^2_{L^2(S^2)} \\
&= \Bigg\| \sum_{\substack{\eta=(\eta_1,\ldots,\eta_k)\in E^k \\ \mathcal{P}_\eta \geq \mathcal{P}_R}} \left(x_{\eta_1 \mathrm{v}_{\ell,1}} e^{\mathrm{i}\eta_1 \mathrm{v}_{\ell,1}} \right)^{\mu_1} \ldots \left(x_{\eta_k \mathrm{v}_{\ell,k}} e^{\mathrm{i}\eta_k \mathrm{v}_{\ell,k}} \right)^{\mu_k} \Bigg\|^2_{L^2(S^2)}.
\end{aligned}$$

We apply Proposition 1 with the following choices: Ω is $\{\pm 1\}^{\mathbb{N}}$ equipped with the counting probability; $\{\varepsilon_\eta\}_{\eta\in E}$ is a family of coordinate projections, where ε_η is the m_η-th projection on Ω, for some enumeration $\{m_\eta \,|\, \eta \in E\}$ of the set $\mathbb{N}$; $\mathcal{P}$ is the partition $\mathcal{P}_R$; φ is the k-linear contractive map that is simply the k-fold product from $L^{2k}(S^{2k}) \times L^{2k}(S^{2k}) \times \ldots \times L^{2k}(S^{2k})$ into $L^2(S^2)$; the functions $f_j : E \longrightarrow L^{2k}(S^{2k})$ are defined by mapping $\eta \in E$ to

$$f_j(\eta) : t \mapsto \left(x_{\eta \mathrm{v}_{\ell,j}} e^{\mathrm{i}\eta \mathrm{v}_{\ell,j} t} \right)^{\mu_j}$$

for each $1 \leq j \leq k$. Note that each $f_j \in L^{2k}(S^{2k})$. By the proposition, it follows that

$$\begin{aligned}
\widetilde{\mathcal{S}}_\ell(R)^{1/2} &\leq \prod_{\substack{1\leq j\leq k \\ \{j\}\in\mathcal{P}_R}} \Big\| \sum_{\eta\in E} f_j(\eta) \Big\|_{L^{2k}(S^{2k})} \prod_{\substack{1\leq j\leq k \\ \{j\}\notin\mathcal{P}_R}} \left(\int_\Omega \Big\| \sum_{\eta\in E} \varepsilon_\eta f_j(\eta) \Big\|^k_{L^{2k}(S^{2k})} d\,\mathbb{P} \right)^{1/k} \\
&= \Big\| \sum_{\eta\in E} x_\eta e^{\mathrm{i}\eta t} \Big\|^{a_R}_{L^{2k}(S^{2k})} \prod_{\substack{1\leq j\leq k \\ \{j\}\notin\mathcal{P}_R}} \left(\int_\Omega \Big\| \sum_{\eta\in E} \varepsilon_\eta\, x_{\eta \mathrm{v}_{\ell,j}} e^{\mathrm{i}\eta \mathrm{v}_{\ell,j} t} \Big\|^{2k}_{L^{2k}(S^{2k})} d\,\mathbb{P} \right)^{1/2k} \\
&\leq C \, \|f\|^{a_R}_{L^{2k}(S^{2k})} \prod_{\substack{1\leq j\leq k \\ \{j\}\notin\mathcal{P}_R}} \widetilde{\mathcal{S}}_j
\end{aligned}$$

where for the second inequality, we have used Jensen's inequality, and for the third one, we have used Fubini's Theorem followed by (2.2), and where

$$\widetilde{\mathcal{S}}_j = \max\left\{ \Big\| \Big(\sum_{\eta\in E} x_{\eta \mathrm{v}_{\ell,j}} x^*_{\eta \mathrm{v}_{\ell,j}} \Big)^{1/2} \Big\|_{S^{2k}}, \Big\| \Big(\sum_{\eta\in E} x^*_{\eta \mathrm{v}_{\ell,j}} x_{\eta \mathrm{v}_{\ell,j}} \Big)^{1/2} \Big\|_{S^{2k}} \right\} \leq |\!|\!| f |\!|\!|_{2k}$$

for every $1 \leq j \leq k$ with $\{j\} \notin \mathcal{P}_R$. Therefore, we have shown that for each $1 \leq \ell \leq C$,

$$\widetilde{\mathcal{S}}_\ell(R) \leq C \, \|f\|^{2a_R}_{L^{2k}(S^{2k})} |\!|\!| f |\!|\!|^{2k-2a_R}_{2k}.$$

It follows that

$$\widetilde{\mathcal{S}}(R) \leq C \left\|f\right\|_{L^{2k}(S^{2k})}^{2a_R} \left\|\left|f\right\|\right|_{2k}^{2k-2a_R} \leq C \left\|f\right\|_{L^{2k}(S^{2k})}^{2k-2} \left\|\left|f\right\|\right|_{2k}^{2},$$

where for the second inequality, we have used (4.5). This completes Step 2.

Step 3. Combining our estimates from Steps 1 and 2, we have by (8.11):

$$\left\|f\right\|_{L^{2k}(S^{2k})}^{2k} \leq C \left(\left\|\left|f\right\|\right|_{2k}^{2k} + \left\|f\right\|_{L^{2k}(S^{2k})}^{2k-2} \left\|\left|f\right\|\right|_{2k}^{2} \right),$$

which clearly implies that

$$\left\|f\right\|_{L^{2k}(S^{2k})} \leq C \left\|\left|f\right\|\right|_{2k}.$$

This completes the proof. □

9 S-unit equations

In this section, we use some known number theoretic results to show that for an arbitrary finite set Q of primes, the set E_Q is of type $\Lambda(p)_{cb}$ for $2 < p < \infty$.

Let K be an algebraic number field of degree d; that is, K is a finite extension of the rationals $\mathbb{Q}$, with $d = [K : \mathbb{Q}]$. Let S be a finite collection of places of K containing all of the archimedean places, and let $\mathcal{U}_S$ be the group of S-units inside the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in K. Given nonzero elements $a_1, \ldots, a_k \in K$, one is interested in counting the number of nondegenerate solutions to the S-unit equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_k x_k = 1, \qquad x_1, x_2, \ldots, x_k \in \mathcal{U}_S, \tag{9.13}$$

i.e., those where no proper subsum $a_{j_1} x_{j_1} + \ldots + a_{j_\ell} x_{j_\ell}$ vanishes.

Mahler [9] proved that for $k = 2$ and $K = \mathbb{Q}$, (9.13) has only finitely many solutions. Van der Poorten and Schlickewei [12] and Evertse [3] independently proved that for all $k \geq 2$ and every number field K, (9.13) has only finitely many solutions. This result was later extended by Evertse and Győry [4], who showed that the number of solutions is bounded by a constant which is independent of the coefficients $a_1, \ldots, a_k$. Later, Schlickewei showed that the constant depends only on k, on the cardinality $\#S$ of the set S, and on the degree d (see [17] for the case $K = \mathbb{Q}$, and [18] for the general case).

In particular, when $K = \mathbb{Q}$, for any finite set Q of primes, one can apply the results of [17] mentioned above with $S = Q \cup \{\infty\}$ to deduce that E_Q

satisfies both properties $Z^+(k)$ and $Z^\star(k)$ for all $k \geq 2$, where the constant $C > 0$ depends only on k and on the cardinality $\#Q$ of the set Q. In fact, our definition of property $Z^\star(k)$ was chosen with precisely these sets in mind. Applying now Theorem 4 together with our remarks from Section 4, we obtain the following:

Theorem 5. *Let Q be a nonempty finite set of prime numbers. Then the set E_Q is of type $\Lambda(p)_{cb}$ for every real number $2 < p < \infty$.*

We conclude this section by observing that E_Q is not a Sidon set whenever $\#Q \geq 2$. Indeed, let $s = \#Q$, and let $q_1 < q_2 < \ldots < q_s$ be the primes in Q. Then for all nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_s \leq (\log N)/(s \log q_s)$, the integer $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$ lies in E_Q and in $[1, N]$. Thus, if N is sufficiently large,

$$\#\big(E_Q \cap [1, N]\big) \geq C\,(\log N)^s$$

where the constant $C > 0$ depends only on Q. This contradicts (5.6) (with $a = 0$ and $b = 1$) whenever $s = \#Q \geq 2$.

10 Remarks

The notions of $\Lambda(p)$ and $\Lambda(p)_{cb}$ sets and the properties $Z^+(k)$ and $Z(k)$ can be naturally defined for an arbitrary discrete group G. In this more general context, it has been shown that any subset of G with the $Z^+(k)$ property is necessarily of type $\Lambda(2k)$. The argument is identical to that given by Rudin in the special case $G = \mathbb{Z}$; see [16]. It is also known that any subset of G with the $Z(k)$ property is necessarily of type $\Lambda(2k)_{cb}$ by the results of [5]. It would be interesting to find a suitable generalization of the property $Z^\star(k)$ for an arbitrary discrete group G and to show that any subset of G with the $Z^\star(k)$ property is necessarily of type $\Lambda(2k)_{cb}$. It would also be of interest to obtain explicit examples of $\Lambda(2k)_{cb}$ sets in G that are similar to the sets E_Q considered here.

Let G be any discrete group and $k \geq 2$ a fixed integer. If a set $E \subset G$ has the $Z(k)$ property, then it is of type $\Lambda(2k)_{cb}$ as we have just mentioned. Consequently, the union of any finite number of sets with the $Z(k)$ property is also of type $\Lambda(2k)_{cb}$. It is natural to ask whether the converse statement is also true; this question was originally raised by Pisier when $G = \mathbb{Z}$ and is still open.

Question 1. *Let G be a discrete group, and let $E \subset G$ be a set of type $\Lambda(2k)_{cb}$, where $k > 2$ is a fixed integer. Does there exist a finite collection*

$E_1, E_2, \ldots, E_c$ *of subsets of* G *such that each* E_j *has the* $Z(k)$ *property and such that* E *is the union of the* E_j*?*

Using Mihăilescu's recent proof of the Catalan conjecture (see [10], and also [1]), one can show that every set E_Q with $\#Q = 2$ can be decomposed into (at most) four sets, each with the $Z(3)$ property. In particular, this shows that E_Q is of type $\Lambda(6)_{cb}$ without using our Theorem 4. However, we do not see how to generalize this to an arbitrary set E_Q and an arbitrary integer $k \geq 2$, since the appropriate analogue to Mihăilescu's result is missing.

Finally, it has been shown in [5] that any noncommutative $\Lambda(p)$ set cannot contain the sum $A+A$ for any infinite set A. Neuwirth [11] later noticed that the arguments in [5] can be slightly modified to show that a noncommutative $\Lambda(p)$ set cannot contain the sum $A+B$ for any infinite sets A and B. By Theorem 4, this can therefore be applied to any set E with the property $Z^\star(k)$. For the special sets E_Q, stronger results are known: E_Q cannot contain the sum $A+B$ for any infinite set A and any set B with at least two elements. This follows, for example, from a fairly deep result due to Mahler: for any finite set of primes Q, the gaps between consecutive integers free of primes outside of Q tend to infinity. The authors wish to thank Carl Pomerance for bringing this to our attention.

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