# Matrix inequalities with applications to the theory of iterated kernels*

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**Abstract**

For an $m \times n$ matrix $A$ with nonnegative real entries, Atkinson, Moran and Watterson proved the inequality $s(A)^3 \leq mns(AA^tA)$, where $A^t$ is the transpose of $A$, and $s(\cdot)$ is the sum of the entries. We extend this result to finite products of the form $AA^tAA^t \ldots A$ or $AA^tAA^t \ldots A^t$ and give some applications to the theory of iterated kernels.

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## 1 Introduction

For any matrix $A$, let $s(A)$ denote the sum of its entries. For any integer $k \geq 1$, we define

$$A^{(2k)} = (AA^t)^k, \qquad A^{(2k+1)} = (AA^t)^k A,$$

where $A^t$ denotes the transpose of $A$. In Section 2, we prove the following sharp inequalities:

**Theorem 1.** *Let $A$ be an $m \times n$ matrix with nonnegative real entries. Then for every integer $k \geq 1$, the following matrix inequalities hold:*

$$s(A)^{2k} \leq m^{k-1} n^k \, s(A^{(2k)}), \qquad s(A)^{2k+1} \leq m^k n^k \, s(A^{(2k+1)}).$$

For the special case of symmetric matrices, this theorem was proved in 1959 by Mulholland and Smith [4], thus settling an earlier conjecture of Mandel and Hughes [3] that had been based on the study of certain genetical models. For arbitrary matrices (with nonnegative entries), Theorem 1 also generalizes the matrix inequality

$$s(A)^3 \leq mn \, s(AA^tA),$$

which was first proved in 1960 by Atkinson, Moran and Watterson [1] using methods of perturbation theory.

Theorem 1 has a graph theoretic interpretation when applied to matrices with entries in $\{0, 1\}$. Let $G$ be a graph with red vertices labeled $1, \ldots, m$ and blue vertices labeled $1, \ldots, n$ such that every edge connects only vertices of distinct colours: $G$ is a bipartite graph. Its reduced incidence matrix is an $m \times n$ matrix $A$ such that $a_{i,j} = 1$ if red vertex $i$ is adjacent to blue vertex $j$, and $a_{i,j} = 0$ otherwise. Then $s(A)$ is the size of $G$, while $s(A^{(\ell)})$ is the number of walks on $G$ of length $\ell$ starting from a red vertex, i.e., the number of sequences $(v_0, \ldots, v_\ell)$ such that $v_0$ is a red vertex and every pair $\{v_i, v_{i+1}\}$ is an edge in $G$. Theorem 1 then yields the optimal lower bound of the number of walks in terms of the size of $G$. We do not know of a corresponding lower bound for the number of trails (walks with no edge repeated) or paths (walks with no vertex repeated).

Recall that an $m \times n$ matrix $A$ is said to be *bistochastic* if every row sum of $A$ is equal to $s(A)/m$, and every column sum of $A$ is equal to $s(A)/n$. In Section 3 we prove the following asymptotic form of Theorem 1:

**Theorem 2.** *Let $A$ be an $m \times n$ matrix with nonnegative real entries. If $A$ is bistochastic, then for all $k \geq 1$,*

$$s(A)^{2k} = m^{k-1} n^k \, s(A^{(2k)}), \qquad s(A)^{2k+1} = m^k n^k \, s(A^{(2k+1)}).$$

*If $A$ is not bistochastic, then there exist constants $c > 0$ and $\gamma > 1$ (depending only on $A$) such that for all $\ell \geq 1$,*

$$s(A)^\ell < c \, \gamma^{-\ell} \, (mn)^{\ell/2} \, s(A^{(\ell)}).$$

As we show in Sections 2 and 3, both of the above theorems, though stated for arbitrary rectangular matrices with nonnegative entries, follow from the special case of *square* matrices.

Theorem 2 has an immediate application. Atkinson, Moran and Watterson [1] conjectured that for a nonnegative symmetric kernel function $K(x,y)$ that is Lebesgue integrable over the square $0 \leq x, y \leq a$, the inequality

$$\int_0^a \int_0^a K_\ell(x,y) \, dx \, dy \geq \frac{1}{a^{\ell-1}} \left( \int_0^a \int_0^a K(x,y) \, dx \, dy \right)^\ell \tag{1}$$

holds for all $\ell \geq 1$. Here $K_\ell(x,y)$ denotes the $\ell$*-th order iterate of* $K(x,y)$, which is defined recursively by

$$K_1(x,y) = K(x,y), \qquad K_\ell(x,y) = \int_0^a K_{\ell-1}(x,t) \, K(t,y) \, dt.$$

Beesack [2] showed that the Atkinson-Moran-Watterson conjecture follows from the matrix identities of Mulholland and Smith described above. Using Beesack's ideas together with Theorem 2, we prove in Section 4 the following asymptotic form of the Atkinson-Moran-Watterson inequality (1):

**Theorem 3.** *Let $K(x, y)$ be a nonnegative symmetric kernel function that is Lebesgue integrable over the square $0 \leq x, y \leq a$, and consider the function $f(x) = \int_0^a K(x, y)\, dy$ defined on the interval $0 \leq x \leq a$. If $f(x)$ is constant almost everywhere, then for all $\ell \geq 1$*

$$\int_0^a \int_0^a K_\ell(x, y)\, dx\, dy = \frac{1}{a^{\ell-1}} \left( \int_0^a \int_0^a K(x, y)\, dx\, dy \right)^\ell .$$

*If not, there exist constants $c > 0$ and $\gamma > 1$ (depending only on $K$) such that for all $\ell \geq 1$*

$$\int_0^a \int_0^a K_\ell(x, y)\, dx\, dy > \frac{c\gamma^\ell}{a^{\ell-1}} \left( \int_0^a \int_0^a K(x, y)\, dx\, dy \right)^\ell .$$

**Remark:** Using an approximation argument as in the proof of Theorem 3, Theorem 1 can be also applied to establish an analogue to inequalities (1) and Theorem 3 in the case of nonsymmetric kernel functions. Let $K(x, y)$ be any nonnegative kernel function that is Lebesgue integrable over the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ and let $K_\ell$ be the $\ell$-th order iterate of $K$ defined by $K_1(x, y) = K(x, y)$ and for each integer $k \geq 1$,

$$K_{2k}(x, x') = \int_0^b K_{2k-1}(x, y) K(x', y)\, dy, \quad K_{2k+1}(x, y) = \int_0^a K_{2k}(x, x') K(x', y)\, dx'.$$

In this case, inequalities (1) become

$$\int_0^a \int_0^b K_{2k+1}(x, y)\, dx\, dy \geq \frac{1}{a^k b^k} \left( \int_0^a \int_0^b K(x, y)\, dx\, dy \right)^{2k+1}$$

$$\int_0^a \int_0^a K_{2k}(x, x')\, dx\, dx' \geq \frac{1}{a^{k-1} b^k} \left( \int_0^a \int_0^b K(x, y)\, dx\, dy \right)^{2k} .$$

The analogue of Theorem 3 is then obvious.

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## 2 Matrix inequality

Given a matrix $A = (a_{i,j})$ and an integer $\ell \ge 0$, we denote by $a_{i,j}^{(\ell)}$ the $(i,j)$-th entry of $A^{(\ell)}$, so that $A^{(\ell)} = (a_{i,j}^{(\ell)})$. This notation will be used often in the sequel.

**Lemma.** *Let $B = (b_{i,j})$ be a $d \times d$ matrix with nonnegative real entries. For any two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ of nonnegative real numbers, the following inequality holds:*

$$(I_2') : \qquad \sum_{i,j=1}^{d} \alpha_i\, \beta_i\, b_{i,j} \le d^{\frac{1}{2}} \left( \sum_{i,j=1}^{d} \alpha_i^2\, \beta_j^2\, b_{i,j}^{(2)} \right)^{\frac{1}{2}}.$$

**Proof.** To prove the lemma, we apply the Cauchy-Schwarz inequality twice as follows:

$$\sum_{i,j=1}^{d} \alpha_i\, \beta_i\, b_{i,j} = \sum_{i,k=1}^{d} \alpha_i\, \beta_i\, b_{i,k} \le d^{\frac{1}{2}} \left( \sum_{k=1}^{d} \left( \sum_{i=1}^{d} \alpha_i\, \beta_i\, b_{i,k} \right)^2 \right)^{\frac{1}{2}}. \tag{2}$$

$$\begin{aligned}
\sum_{i,j=1}^{d} \alpha_i\, \beta_i\, b_{i,j} &\le d^{\frac{1}{2}} \left( \sum_{i,j,k=1}^{d} \alpha_i\, \alpha_j\, \beta_i\, \beta_j\, b_{i,k}\, b_{j,k} \right)^{\frac{1}{2}} \\
&= d^{\frac{1}{2}} \left( \sum_{i,j=1}^{d} \alpha_i\, \alpha_j\, \beta_i\, \beta_j\, b_{i,j}^{(2)} \right)^{\frac{1}{2}} \\
&= d^{\frac{1}{2}} \left( \sum_{i,j=1}^{d} \alpha_i\, \beta_j\, (b_{i,j}^{(2)})^{\frac{1}{2}} \cdot \alpha_j\, \beta_i\, (b_{j,i}^{(2)})^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
&\le d^{\frac{1}{2}} \left( \sum_{i,j=1}^{d} \alpha_i^2\, \beta_j^2\, b_{i,j}^{(2)} \right)^{\frac{1}{2}}.
\end{aligned}$$

Here we have used the fact that $B^{(2)} = BB^t$ is a symmetric matrix. ■

**Theorem 1′.** *Let $B = (b_{i,j})$ be a square $d \times d$ matrix with nonnegative real entries, and let $\{\alpha_i\}$ be any sequence of nonnegative real numbers. Then for each integer $\ell \ge 1$, we have*

$$(I_\ell) : \qquad \sum_{i,j=1}^{d} \alpha_i\, b_{i,j} \le d^{\frac{\ell-1}{\ell}} \left( \sum_{i,j=1}^{d} \alpha_i^{\ell}\, b_{i,j}^{(\ell)} \right)^{\frac{1}{\ell}}.$$

**Proof of Theorem 1′.** The case $\ell = 1$ is trivial while the case $\ell = 2$ is a consequence of the lemma above. We prove the general case by induction. Suppose that $p \geq 2$, and the inequalities $(I_1), (I_2), \ldots, (I_p)$ hold for all square matrices with nonnegative real entries. If $p = 2k-1$ is an odd integer, then the inequality $(I_{p+1})$ follows immediately from $(I_2)$ and $(I_k)$. Indeed, since $B^{(2k)} = B^{(2)(k)}$, we have

$$\sum_{i,j=1}^{d} \alpha_i \, b_{i,j} \leq d^{\frac{1}{2}} \left( \sum_{i,j=1}^{d} \alpha_i^2 \, b_{i,j}^{(2)} \right)^{\frac{1}{2}} \leq d^{\frac{1}{2}} \left( d^{\frac{k-1}{k}} \left( \sum_{i,j=1}^{d} \alpha_i^{2k} \, b_{i,j}^{(2)(k)} \right)^{\frac{1}{k}} \right)^{\frac{1}{2}}. \tag{3}$$

Thus

$$\sum_{i,j=1}^{d} \alpha_i \, b_{i,j} \leq d^{\frac{2k-1}{2k}} \left( \sum_{i,j=1}^{d} \alpha_i^{2k} \, b_{i,j}^{(2k)} \right)^{\frac{1}{2k}}.$$

If $p = 2k$ is an even integer, then the inequality $(I_{p+1})$ follows from Hölder's inequality, and the inequalities $(I_k)$ and $(I_2')$. Indeed, by Hölder's inequality, we have

$$\sum_{i,j=1}^{d} \alpha_i \, b_{i,j} \leq d^{\frac{1}{2k+1}} \left( \sum_{i=1}^{d} \alpha_i^{\frac{2k+1}{2k}} \left( \sum_{j=1}^{d} b_{i,j} \right)^{\frac{2k+1}{2k}} \right)^{\frac{2k}{2k+1}}. \tag{4}$$

Let $\mathcal{I}$ denote the term between parentheses, and set $\beta_i = \sum_{j=1}^{d} b_{i,j}$ for each $i$. Then

$$\mathcal{I} = \sum_{i=1}^{d} \alpha_i^{\frac{2k+1}{2k}} \left( \sum_{j=1}^{d} b_{i,j} \right)^{\frac{2k+1}{2k}} = \sum_{i,j=1}^{d} \alpha_i^{\frac{2k+1}{2k}} \, \beta_i^{\frac{1}{2k}} \, b_{i,j}.$$

Applying $(I_k)$, it follows that

$$\mathcal{I} \leq d^{\frac{k-1}{k}} \left( \sum_{i,j=1}^{d} \alpha_i^{\frac{2k+1}{2}} \, \beta_i^{\frac{1}{2}} \, b_{i,j}^{(k)} \right)^{\frac{1}{k}}.$$

Applying the lemma to the sequences $\{\alpha_i^{\frac{2k+1}{2}}\}$ and $\{\beta_i^{\frac{1}{2}}\}$, and using the fact $B^{(k)(2)} = B^{(2k)}$, we see that

$$\mathcal{I} \leq d^{\frac{k-1}{k}} \left( d^{\frac{1}{2}} \left( \sum_{i,j=1}^{d} \alpha_i^{2k+1} \, \beta_j \, b_{i,j}^{(k)(2)} \right)^{\frac{1}{2}} \right)^{\frac{1}{k}} = d^{\frac{2k-1}{2k}} \left( \sum_{i,j=1}^{d} \alpha_i^{2k+1} \, \beta_j \, b_{i,j}^{(2k)} \right)^{\frac{1}{2k}}.$$

Putting everything together, we have therefore shown that

$$\sum_{i,j=1}^{d} \alpha_i \, b_{i,j} \leq d^{\frac{2k}{2k+1}} \left( \sum_{i,j=1}^{d} \alpha_i^{2k+1} \, \beta_j \, b_{i,j}^{(2k)} \right)^{\frac{1}{2k+1}}.$$

Finally, note that

$$\sum_{j=1}^{d} \beta_j \, b_{i,j}^{(2k)} = \sum_{\ell=1}^{d} b_{i,\ell}^{(2k)} \, \beta_\ell = \sum_{j,\ell=1}^{d} b_{i,\ell}^{(2k)} \, b_{\ell,j} = \sum_{j=1}^{d} b_{i,j}^{(2k+1)}$$

since $B^{(2k+1)} = B^{(2k)} B$. Consequently,

$$\sum_{i,j=1}^{d} \alpha_i \, b_{i,j} \leq d^{\frac{2k}{2k+1}} \left( \sum_{i,j=1}^{d} \alpha_i^{2k+1} \, b_{i,j}^{(2k+1)} \right)^{\frac{1}{2k+1}} \tag{5}$$

and $(I_{p+1})$ holds for the case $p = 2k$. Theorem $1'$ now follows by induction. ■

**Theorem 1.** *Let $A$ be an $m \times n$ matrix with nonnegative real entries. Then for every integer $k \geq 1$, the following matrix inequalities hold:*

$$s(A)^{2k} \leq m^{k-1} n^k \, s(A^{(2k)}), \qquad s(A)^{2k+1} \leq m^k n^k \, s(A^{(2k+1)}).$$

**Proof of Theorem 1.** For the case of square matrices, Theorem 1 follows immediately from Theorem $1'$. Indeed, taking $\alpha_i = 1$ for each $i$, the inequality $(I_\ell)$ yields the corresponding inequality in Theorem 1.

Now, let $A$ be an $m \times n$ matrix with nonnegative real entries, put $d = mn$, and let $B$ be the $d \times d$ matrix with nonnegative real entries defined as the tensor product $B = A \otimes \mathbb{1}_{n,m}$, where $\mathbb{1}_{n,m}$ is the $n \times m$ matrix with every entry equal to 1. For any integers $\ell, k \geq 0$, the relations

$$B^{(\ell)} = A^{(\ell)} \otimes \mathbb{1}_{n,m}^{(\ell)}, \quad s\big(B^{(\ell)}\big) = s\big(A^{(\ell)}\big) \, s\big(\mathbb{1}_{n,m}^{(\ell)}\big),$$

$$s\big(\mathbb{1}_{n,m}^{(2k)}\big) = m^k n^{k+1}, \quad s\big(\mathbb{1}_{n,m}^{(2k+1)}\big) = m^{k+1} n^{k+1}.$$

are easily checked. In particular, $s(B) = mn \, s(A)$. Applying Theorem 1 to the matrix $B$ and using these identities, the inequalities of Theorem 1 follow for the matrix $A$. ■

# 3 Asymptotic matrix inequality

As will be shown below, Theorem 2 is a consequence of the following more precise theorem for square matrices:

**Theorem 2′.** *Let $B$ be a square $d\times d$ matrix with nonnegative real entries and $s(B)\neq 0$. Let $\lambda$ be the largest eigenvalue of $B^{(2)}=BB^t$, and put $\gamma=\lambda\, d^2/s(B)^2$. Then $\gamma\geq 1$, and there exists a constant $c>0$ (depending only on $B$) such that for all integers $\ell\geq 0$,*

$$s(B)^\ell < c\,\gamma^{-\frac{\ell}{2}}\,d^{\ell-1}\,s(B^{(\ell)}). \tag{6}$$

*Moreover, the following assertions are equivalent:*

*(a)* $\gamma=1$,

*(b)* $s(B)^\ell = d^{\ell-1}\,s(B^{(\ell)})$ *for every integer* $\ell\geq 0$,

*(c)* $s(B)^\ell = d^{\ell-1}\,s(B^{(\ell)})$ *for some integer* $\ell\geq 3$,

*(d)* $B$ *is bistochastic.*

**Proof.** We express $B^{(2)}=BB^t$ in the form $B^{(2)}=U^tDU$, where $U=(u_{i,j})$ is an orthogonal matrix, and $D$ is a diagonal matrix $\mathrm{diag}(\lambda_1,\ldots,\lambda_d)$ with $\lambda_1\geq\ldots\geq\lambda_d\geq 0$. Here $\lambda=\lambda_1$. For each $\nu=1,\ldots,d$, let $E_\nu$ be the projection matrix whose $(\nu,\nu)$-th entry is 1, and all other entries are equal to 0. Put $A_\nu=U^tE_\nu U$ for each $\nu$. Then for all integers $k\geq 0$,

$$B^{(2k)}=\sum_{\nu=1}^d \lambda_\nu^k\,A_\nu,\qquad B^{(2k+1)}=\sum_{\nu=1}^d \lambda_\nu^k\,A_\nu B.$$

By a straightforward calculation, we see that for each $\nu$

$$s(A_\nu)=\Big(\sum_{i=1}^d u_{\nu,i}\Big)^2,\qquad s(A_\nu B)=\Bigg(\sum_{i=1}^d u_{\nu,i}\Bigg)\Big(\sum_{j,k=1}^d u_{\nu,k}\,b_{k,j}\Big). \tag{7}$$

In particular, $s(A_\nu)\geq 0$. By Theorem 1′, it follows that

$$\frac{s(B)^2}{d}\leq s(B^{(2)})=\sum_{\nu=1}^d\lambda_\nu\,s(A_\nu)\leq\lambda\sum_{\nu=1}^d s(A_\nu)=\lambda\,d. \tag{8}$$

Therefore, $\gamma=\dfrac{\lambda\,d^2}{s(B)^2}\geq 1$. Now, from the definition of $\gamma$, we have

$$\frac{\gamma^{\frac{\ell}{2}}s(B)^\ell}{d^{\ell-1}s(B^{(\ell)})}=d\,\frac{\lambda^{\frac{\ell}{2}}}{s(B^{(\ell)})}\cdot$$

Then, in order to show inequality (6), we will show that the $\lambda^{\frac{\ell}{2}}/s(B^{(\ell)})$ are bounded above by a constant that is independent of $\ell$. Indeed, let $C_\ell = B^{(\ell)}/s(B^{(\ell)})$ for every $\ell \geq 0$. Since each $C_\ell$ has nonnegative real entries, and $s(C_\ell) = 1$, the entries of $C_\ell$ all lie in the closed interval $[0, 1]$. Thus the entries of the matrices $UC_{2k}U^t$ and $UC_{2k+1}B^tU^t$ are bounded by a constant that depends only on $B$. Noting that for each nonnegative integer $k$, we have

$$UC_{2k}U^t = \frac{D^k}{s\big(B^{(2k)}\big)}, \qquad UC_{2k+1}B^tU^t = \frac{D^{k+1}}{s\big(B^{(2k+1)}\big)},$$

and on examining the $(1,1)$-th entry for each of these matrices, we see that $\lambda^k/s(B^{(2k)})$ and $\lambda^{k+1}/s(B^{(2k+1)})$ are both bounded above by a constant that is independent of $k$. Consequently, inequality (6) holds.

$(a) \Longrightarrow (b)$: If $\gamma = 1$, then $\lambda\, d = s(B)^2/d$, hence from (8) we see that $s(A_\nu) = 0$ whenever $\lambda_\nu \neq \lambda$. By (7), we also have that $s(A_\nu B) = 0$ whenever $\lambda_\nu \neq \lambda$. Thus

$$\begin{aligned} s(B^{(2k)}) &= \sum_{\nu=1}^{d} \lambda_\nu^k\, s(A_\nu) = \lambda^k \sum_{\nu\,:\,\lambda_\nu=\lambda} s(A_\nu) = \lambda^k \sum_{\nu=1}^{d} s(A_\nu) = \lambda^k\, d = \frac{s(B)^{2k}}{d^{2k-1}}, \\ s(B^{(2k+1)}) &= \sum_{\nu=1}^{d} \lambda_\nu^k\, s(A_\nu B) = \lambda^k \sum_{\nu\,:\,\lambda_\nu=\lambda} s(A_\nu B) = \lambda^k \sum_{\nu=1}^{d} s(A_\nu B) = \lambda^k\, s(B) = \frac{s(B)^{2k+1}}{d^{2k}}. \end{aligned}$$

$(b) \Longrightarrow (a)$: If $(b)$ holds, then inequality (6) implies $1 < c\,\gamma^{-\frac{\ell}{2}}$ for some $\gamma \geq 1$ and all integers $\ell \geq 0$. This forces $\gamma = 1$.

$(b) \Longrightarrow (c)$: Trivial.

$(c) \Longrightarrow (d)$: Suppose that $\ell = 2k+1 \geq 3$ is an odd integer such that $s(B)^\ell = d^{\ell-1}\, s(B^{(\ell)})$. Taking every $\alpha_i = 1$ in the proof of Theorem $1'$, our hypothesis means that equality holds in (5), hence (4) must also hold with equality:

$$\sum_{i,j=1}^{d} b_{i,j} = d^{\frac{1}{2k+1}} \Bigg( \sum_{i=1}^{d} \Big( \sum_{j=1}^{d} b_{i,j} \Big)^{\frac{2k+1}{2k}} \Bigg)^{\frac{2k}{2k+1}}.$$

By Hölder's inequality, this is only possible if all of the row sums of $B$ are equal. Since $\ell$ is odd and $s$ is transpose-invariant, we also have

$$s(B^t)^\ell = d^{\ell-1}\, s\big((B^{(\ell)})^t\big) = d^{\ell-1}\, s\big((B^t)^{(\ell)}\big).$$

Thus all of the row sums of $B^t$ are equal as well, and $B$ is bistochastic.

Now suppose that $\ell = 2k \geq 4$ is an even integer such that $s(B)^\ell = d^{\ell-1}\, s(B^{(\ell)})$. By taking every $\alpha_i = 1$ in (3), we see that $s(B)^2 = d\, s(B^{(2)})$. Then, taking every $\alpha_i = \beta_i = 1$ in the proof of the lemma, we see that equality holds in (2) which is only possible if all of the column sums of $B$ are equal. Therefore $s(BA) = \beta\, s(A)$ for every $d \times d$ matrix $A$, where $\beta = s(B)/d$ is the sum of each column of $B$. In particular,

$$s(B)^\ell = d^{\ell-1}\, s\big(B^{(\ell)}\big) = d^{\ell-1}\,\beta\, s\big((B^t)^{(\ell-1)}\big) = d^{\ell-1}\,\beta\, s\big((B^{(\ell-1)})^t\big) = d^{\ell-1}\,\beta\, s(B^{(\ell-1)}),$$

thus $s(B)^{\ell-1} = d^{\ell-2}\, s(B^{(\ell-1)})$. Since $\ell - 1$ is odd, we can apply the previous result to conclude that $B$ is bistochastic.

$(d) \Longrightarrow (b)$: Suppose $B$ is bistochastic, with every row or column sum equal to $\beta = s(B)/d$. For any $d \times d$ matrix $A$, one has $s(AB) = \beta\, s(A)$ and $s(AB^t) = \beta\, s(A)$. In particular, $s(B^{(2k+1)}) = \beta\, s(B^{(2k)})$ and $s(B^{(2k+2)}) = \beta\, s(B^{(2k+1)})$ for all $k \geq 0$. Consequently,

$$s(B^{(\ell)}) = \beta^{\ell-1}\, s(B) = \frac{s(B)^\ell}{d^{\ell-1}}, \qquad \ell \geq 0.$$

This completes the proof. ∎

**Corollary.** *Let $B$ be a square $d \times d$ matrix with nonnegative real entries and $s(B) \neq 0$. Let $\beta_j$ be the $j$-th column sum of $B$ for each $j$, and put*

$$\delta = 1 + \frac{1}{2\, s(B)^2} \sum_{i,j=1}^{d} (\beta_i - \beta_j)^2.$$

*Then there exists a constant $c > 0$ (depending only on $B$) such that for all $\ell \geq 0$, we have*

$$s(B)^\ell < c\, \delta^{-\frac{\ell}{2}}\, d^{\ell-1}\, s(B^{(\ell)}).$$

**Proof.** Note first that for *any* $d \times d$ matrix $B$, if $\beta_j$ denotes the $j$-th column sum of $B$, then it is easily seen that

$$s(B^{(2)}) = \frac{s(B)^2}{d} + \frac{1}{2d} \sum_{i,j=1}^{d} (\beta_i - \beta_j)^2. \tag{9}$$

Using the notation of Theorem $2'$ and applying the relations (8) and (9) , we have

$$\gamma = \frac{\lambda\, d^2}{s(B)^2} \ge \frac{d\, s(B^{(2)})}{s(B)^2} = 1 + \frac{1}{2\, s(B)^2} \sum_{i,j=1}^{d} (\beta_i - \beta_j)^2 = \delta.$$

The corollary therefore follows from (6). ■

**Theorem 2.** *Let $A$ be an $m \times n$ matrix with nonnegative real entries. If $A$ is bistochastic, then for all $k \ge 1$,*

$$s(A)^{2k} = m^{k-1} n^k\, s(A^{(2k)}), \qquad s(A)^{2k+1} = m^k n^k\, s(A^{(2k+1)}).$$

*If $A$ is not bistochastic, then there exist constants $c > 0$ and $\gamma > 1$ (depending only on $A$) such that for all $\ell \ge 1$,*

$$s(A)^\ell < c\, \gamma^{-\ell}\, (mn)^{\ell/2}\, s(A^{(\ell)}).$$

**Proof of Theorem 2.** Given an $m \times n$ matrix $A$ with nonnegative real entries, we proceed as in the proof of Theorem 1: put $d = mn$, and let $B = A \otimes 1\!\!1_{n,m}$. Note that $A$ is bistochastic if and only if $B$ is bistochastic. Applying the corollary above to $B$, Theorem 2 follows immediately for the matrix $A$. The details are left to the reader. ■

# 4 Asymptotic kernel inequality

**Theorem 3.** *Let $K(x,y)$ be a nonnegative symmetric kernel function that is Lebesgue integrable over the square $0 \le x, y \le a$, and consider the function $f(x) = \int_0^a K(x,y)\, dy$ defined on the interval $0 \le x \le a$. If $f(x)$ is constant almost everywhere, then for all $\ell \ge 1$*

$$\int_0^a \int_0^a K_\ell(x,y)\, dx\, dy = \frac{1}{a^{\ell-1}} \left( \int_0^a \int_0^a K(x,y)\, dx\, dy \right)^\ell.$$

*If not, there exist constants $c > 0$ and $\gamma > 1$ (depending only on $K$) such that for all $\ell \ge 1$*

$$\int_0^a \int_0^a K_\ell(x,y)\, dx\, dy > \frac{c\, \gamma^\ell}{a^{\ell-1}} \left( \int_0^a \int_0^a K(x,y)\, dx\, dy \right)^\ell.$$

**Proof of Theorem 3.** By changing variables if necessary, we can assume that $a = 1$. For simplicity, we will also assume that $K(x, y)$ is continuous. Consider the function $f(x)$ defined by

$$f(x) = \int_0^1 K(x, y)\, dy, \qquad x \in [0, 1].$$

If $f(x)$ is a constant function, then since $K(x, y)$ is symmetric, the equality

$$\int_0^1 \int_0^1 K_\ell(x, y)\, dx\, dy = \Big( \int_0^1 \int_0^1 K(x, y)\, dx\, dy \Big)^\ell$$

for all $\ell \geq 1$ follows from an easy inductive argument.

Now suppose that $f(x)$ is not constant, and let $m$ and $M$ denote respectively the minimum and maximum value of $f(x)$ on $[0, 1]$. Choose $\varepsilon > 0$ such that $4\varepsilon < M - m$. For every integer $d \geq 1$, let $\mathcal{U}_i^{[d]}$ be the open interval

$$\mathcal{U}_i^{[d]} = \Big( \frac{i-1}{d}, \frac{i}{d} \Big), \qquad 1 \leq i \leq d,$$

and let $\mathcal{U}_{i,j}^{[d]}$ be the rectangle $\mathcal{U}_i^{[d]} \times \mathcal{U}_j^{[d]}$ for $1 \leq i, j \leq d$. Let $K^{[d]}(x, y)$ be the function that is defined on $[0, 1] \times [0, 1]$ as follows:

$$K^{[d]}(x, y) = \begin{cases} \min \Big\{ K(s, t) \,\big|\, (s, t) \in \overline{\mathcal{U}_{i,j}^{[d]}} \Big\} & \text{if } (x, y) \in \mathcal{U}_{i,j}^{[d]} \text{ for some } 1 \leq i, j \leq d \\ K(x, y) & \text{otherwise.} \end{cases}$$

Here $\overline{\mathcal{U}_{i,j}^{[d]}}$ denotes the closure of $\mathcal{U}_{i,j}^{[d]}$. Noting that $K^{[d]}(x, y)$ is constant on each rectangle $\mathcal{U}_{i,j}^{[d]}$, let $B_{[d]}$ be the $d \times d$ matrix whose $(i, j)$-th entry is equal to $K^{[d]}(\mathcal{U}_{i,j}^{[d]})$. Let $K_\ell^{[d]}(x, y)$ denote the $\ell$-th order iterate of $K^{[d]}(x, y)$ for each $\ell \geq 1$. Then

$$K_\ell^{[d]}(x, y) = \int_0^1 K_{\ell-1}^{[d]}(x, t)\, K^{[d]}(t, y)\, dt = \sum_{k=1}^{d} \int_{\mathcal{U}_k^{[d]}} K_{\ell-1}^{[d]}(x, t)\, K^{[d]}(t, y)\, dt.$$

It follows by induction that $K_\ell^{[d]}(x, y)$ is also constant on each rectangle $\mathcal{U}_{i,j}^{[d]}$, and

$$K_\ell^{[d]}(\mathcal{U}_{i,j}^{[d]}) = \frac{1}{d} \sum_{k=1}^{d} K_{\ell-1}^{[d]}(\mathcal{U}_{i,k}^{[d]})\, K^{[d]}(\mathcal{U}_{k,j}^{[d]});$$

by induction, this is the $(i,j)$-th entry of the matrix $\dfrac{1}{d^{\ell-1}} B_{[d]}^{(\ell)}$. In other words,

$$\left(K_\ell^{[d]}(\mathcal{U}_{i,j}^{[d]})\right) = \frac{1}{d^{\ell-1}} B_{[d]}^{(\ell)}, \qquad \text{for all } \ell, d \geq 1. \tag{10}$$

Now since $f(x)$ is continuous, we can choose $d$ sufficiently large such that for some integers $1 \leq i_m, i_M \leq d$, we have

$$f(x) < m + \varepsilon, \qquad \text{for all } x \in \mathcal{U}_{i_m}^{[d]},$$

$$f(x) > M - \varepsilon, \qquad \text{for all } x \in \mathcal{U}_{i_M}^{[d]}.$$

Taking $d$ larger if necessary, we can further assume that $0 \leq K(x,y) - K^{[d]}(x,y) < \varepsilon$ for all $0 \leq x, y \leq 1$. Fixing this value of $d$, we define

$$\gamma = 1 + \frac{\varepsilon^2}{2d^2 \left( \int_0^1 \int_0^1 K(x,y)\, dx\, dy \right)^2}.$$

Finally, since $\gamma^{-\frac{1}{4}} < 1$, we can choose $e$ sufficiently large so that $K^{[de]}(x,y) > \gamma^{-\frac{1}{4}} K(x,y)$ for all $0 \leq x, y \leq 1$. For this value of $e$, we therefore have

$$\int_0^1 \int_0^1 K^{[de]}(x,y)\, dx\, dy > \gamma^{-\frac{1}{4}} \int_0^1 \int_0^1 K(x,y)\, dx\, dy.$$

By the corollary to Theorem $2'$ applied to the matrix $B_{[de]}$, there exists a constant $c > 0$, which is independent of $\ell$, such that

$$s\left(B_{[de]}\right)^\ell < c\, \delta^{-\frac{\ell}{2}}\, (de)^{\ell-1}\, s\left(B_{[de]}^{(\ell)}\right)$$

for all integers $\ell \geq 0$, where

$$\delta = 1 + \frac{1}{2\, s\left(B_{[de]}\right)^2} \sum_{i,j=1}^{de} \left(\beta_{[de],i} - \beta_{[de],j}\right)^2.$$

Here $\beta_{[de],j}$ denotes the $j$-th column sum of $B_{[de]}$ for each $j$. We now claim that $\delta > \gamma$.

Granting this fact for the moment, we apply (10) to $K^{[de]}(x,y)$ and obtain:

$$\int_0^1\int_0^1 K_\ell(x,y)\,dx\,dy \ge \int_0^1\int_0^1 K_\ell^{[de]}(x,y)\,dx\,dy = \frac{1}{(de)^2}\sum_{i,j=1}^{de} K_\ell^{[de]}\big(\mathcal{U}_{i,j}^{[de]}\big)$$
$$= \frac{1}{(de)^{\ell+1}}\; s\big(B_{[de]}^{(\ell)}\big) > c^{-1}\,\delta^{\frac{\ell}{2}}\,(de)^{-2\ell}\,s\big(B_{[de]}\big)^{\ell}$$
$$= c^{-1}\,\delta^{\frac{\ell}{2}}\left(\frac{1}{(de)^2}\sum_{i,j=1}^{de} K^{[de]}\big(\mathcal{U}_{i,j}^{[de]}\big)\right)^{\ell} = c^{-1}\,\delta^{\frac{\ell}{2}}\left(\int_0^1\int_0^1 K^{[de]}(x,y)\,dx\,dy\right)^{\ell}$$
$$> c^{-1}\,\delta^{\frac{\ell}{2}}\gamma^{-\frac{\ell}{4}}\left(\int_0^1\int_0^1 K(x,y)\,dx\,dy\right)^{\ell} > c^{-1}\,\gamma^{\frac{\ell}{4}}\left(\int_0^1\int_0^1 K(x,y)\,dx\,dy\right)^{\ell}.$$

This completes the proof of the theorem modulo our claim that $\delta > \gamma$. To see this, let $\mathcal{V}$ be any interval of the form $\mathcal{U}_i^{[de]}$ such that $\mathcal{V} \subset \mathcal{U}_{i_m}^{[d]}$. Note that there are $e$ such intervals. Since $B^{[de]}$ is a symmetric matrix, the column sum $\beta_{[de],\mathcal{V}}$ of $B_{[de]}$ corresponding to the interval $\mathcal{V}$ is equal to the "$\mathcal{V}$-th" row sum, which can be bounded as follows:

$$\beta_{[de],\mathcal{V}} = \sum_{j=1}^{de} K^{[de]}\big(\mathcal{V},\mathcal{U}_j^{[de]}\big) = (de)^2\int_{\mathcal{V}}\int_0^1 K^{[de]}(x,y)\,dy\,dx \le (de)^2\int_{\mathcal{V}}\int_0^1 K(x,y)\,dy\,dx$$
$$= (de)^2\int_{\mathcal{V}} f(x)\,dx < de(m+\varepsilon).$$

Similarly, let $\mathcal{W}$ be any interval of the form $\mathcal{U}_i^{[de]}$ such that $\mathcal{W} \subset \mathcal{U}_{i_M}^{[d]}$. Again, there are $e$ such intervals, and by a similar calculation, the column sum $\beta_{[de],\mathcal{W}}$ satisfies the bound

$$\beta_{[de],\mathcal{W}} = \sum_{j=1}^{de} K^{[de]}\big(\mathcal{W},\mathcal{U}_j^{[de]}\big) > de(M-2\varepsilon).$$

Thus

$$\sum_{i,j=1}^{de}\big(\beta_{[de],i}-\beta_{[de],j}\big)^2 \ge \sum_{\mathcal{V},\mathcal{W}}\big(\beta_{[de],\mathcal{W}}-\beta_{[de],\mathcal{V}}\big)^2 > d^2e^4(M-m-3\varepsilon)^2 > d^2e^4\varepsilon^2.$$

On the other hand, we have

$$s\big(B_{[de]}\big) = (de)^2\int_0^1\int_0^1 K^{[de]}(x,y)\,dx\,dy \le (de)^2\int_0^1\int_0^1 K(x,y)\,dx\,dy,$$

and the claim follows. ■

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