Some unusual identities for special values of the Riemann zeta function†

by William D. Banks

Abstract: *In this paper, we use elementary methods to derive some new identities for special values of the Riemann zeta function.*

§1. Introduction

In the region $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$, the Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $\mathbb{N}$ denotes the set of natural numbers, and the product is over all prime numbers p. It is well-known that the values of $\zeta(s)$ at positive even integers can be expressed in terms of the Bernoulli numbers $\{B_k \in \mathbb{Q} \mid k \geq 0\}$ by the formula

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2\,(2k)!} B_{2k}. \tag{1}$$

However, other special values of $\zeta(s)$ remain mysterious. Indeed, it is still an open problem to show that $\zeta(2k+1)$ is transcendental for any $k \geq 1$.

In this paper, we use elementary techniques to derive some new identities for special values of the Riemann zeta function. For example, we will show that for *every* integer $k \geq 2$, the following identity holds:

$$\zeta(k) = 2^{k-\frac{1}{2}} \pi^k \sqrt{\frac{|B_{2k}|}{(2k)!} \sum_{n \in \mathbb{N}} \left(\frac{1}{n} \sum_{m \,(\mathrm{mod}\ n)} \cos\left(2\pi m\,(m,n)^2/n\right) \right)^k}, \tag{2}$$

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where (m,n) denotes the greatest common divisor of m and n. To obtain identities of this type, we study the arithmetical functions $\{a_j \,|\, j \in \mathbb{N}\}$ defined by

$$(3) \qquad a_j(n) = \sum_{m \pmod n} \cos\big(2\pi m\,(m,n)^{j-1}/n\big), \qquad n \in \mathbb{N}.$$

Our main result is the following:

Theorem. *If $x, y, z \in \mathbb{R}$ with $x, y, z > 0$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, then*

$$\sum_{n\in\mathbb{N}} \frac{a_2(n)^x}{n^s} = \zeta(2s-x),$$

$$\sum_{n\in\mathbb{N}} \frac{a_3(n)^x}{n^s} = \frac{\zeta(2s-x)\zeta(3s-2x)}{\zeta(4s-2x)},$$

$$\sum_{n\in\mathbb{N}} \frac{a_2(n)^x a_3(n)^y}{n^s} = \frac{\zeta(2s-x-y)\zeta(6s-3x-4y)}{\zeta(4s-2x-2y)},$$

$$\sum_{n\in\mathbb{N}} \frac{a_2(n)^x a_4(n)^y}{n^s} = \frac{\zeta(2s-x-y)\zeta(4s-2x-3y)}{\zeta(4s-2x-2y)},$$

$$\sum_{n\in\mathbb{N}} \frac{a_2(n)^x a_3(n)^y a_6(n)^z}{n^s} = \frac{\zeta(2s-x-y-z)\zeta(6s-3x-4y-5z)}{\zeta(4s-2x-2y-2z)}.$$

From the proof of this theorem (see §2), it is clear that each identity is valid whenever $s \in \mathbb{C}$ lies in the region of absolute convergence for the Dirichlet series that occur on the right side of each expression. In §2, we also state a corollary that gives a list of identities generalizing (2).

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§2. Proof of the Theorem

To prove the theorem, we first need to evaluate the arithmetical function $a_j(n)$ defined by (3). This is accomplished by the following proposition.

Proposition. *For each $j \in \mathbb{N}$, the function $a_j(n)$ defined by (3) is multiplicative. Its values on prime powers are given by*

$$a_j(p^\alpha) = \begin{cases} p^{b(j-1)+r} & \text{if } \alpha - 1 = bj + r \text{ with } 0 < r < j, \\ 0 & \text{if } \alpha \equiv 1 \,(\mathrm{mod}\ j). \end{cases}$$

Proof: Collecting together the terms in (3) with a fixed value $d = (m, n)$, we can rewrite that equation as

$$a_j(n) = \sum_{d|n} \sum_{m \in (\mathbb{Z}/\frac{n}{d}\mathbb{Z})^\times} \cos(2\pi m d^j/n) = \sum_{d|n} r_{n/d}(d^{j-1}),$$

where for all $n, k \in \mathbb{N}$, $r_n(k)$ is the classical Ramanujan sum defined by

$$r_n(k) = \sum_{\substack{m \,(\mathrm{mod}\ n) \\ (m,n)=1}} \cos(2\pi m k/n) = \sum_{m \in (\mathbb{Z}/n\mathbb{Z})^\times} e^{2\pi i m k/n}.$$

Using the well-known (and easily proved) formula

$$r_n(k) = \sum_{e|(n,k)} e\, \mu(n/e),$$

we now find that

$$a_j(n) = \sum_{d|n} \sum_{e|(\frac{n}{d}, d^{j-1})} e\, \mu\left(\frac{n}{de}\right) = \sum_{g|n} \mu(n/g) \sum_{e^j | g^{j-1}} e, \tag{4}$$

where in the second formula we have set $g = de$ (note that $e^j | g^{j-1}$ implies $e|g$, hence $g = de$ with $e|d^{j-1}$). With this formula, $a_j(n)$ is expressed as a sum over divisors of n of multiplicative functions, so $a_j(n)$ is multiplicative as well. It remains to calculate $a_j(n)$ for prime powers. If $n = p^\alpha$ with p prime and $\alpha \geq 1$, then the coefficient $\mu(n/g)$ in (4) is $+1$ for $g = p^\alpha$, -1 for $g = p^{\alpha-1}$, and 0 otherwise, and e must have the form p^i for some $i \geq 0$. Thus

$$a_j(p^\alpha) = \sum_{0 \leq i \leq \frac{j-1}{j}\alpha} p^i - \sum_{0 \leq i \leq \frac{j-1}{j}(\alpha-1)} p^i = \sum_{\frac{j-1}{j}(\alpha-1) < i \leq \frac{j-1}{j}\alpha} p^i.$$

Writing $\alpha-1$ as $bj+r$, $0 \le r < j$, we see that the interval $\left(\frac{j-1}{j}(\alpha-1), \frac{j-1}{j}\alpha\right]$ contains the unique integer $i = b(j-1)+r$ if $r>0$ and no integer at all if $r=0$. The proposition follows. □

According to the proposition, we have

$$(5) \qquad a_j(n) = \begin{cases} \prod_{i=1}^{k} p_i^{[\alpha_i(j-1)/j]} & \text{if } n = p_1^{\alpha_1} \dots p_k^{\alpha_k} \text{ with each } \alpha_i \not\equiv 1 \bmod j, \\ 0 & \text{otherwise,} \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to x.

Proof of the theorem: We will prove only the last identity stated in the theorem. By (5), we see that $0 \le a_j(n) \le n^{(j-1)/j}$, hence the Dirichlet series defined by

$$D(s) = \sum_{n \in \mathbb{N}} \frac{a_2(n)^x a_3(n)^y a_6(n)^z}{n^s}$$

converges absolutely if $\mathrm{Re}(s)$ is sufficiently large. For such s, the multiplicativity of $a_j(n)$ gives rise to an Euler product expansion of the form

$$D(s) = \prod_p D_p(s),$$

where for each prime p, the local factor $D_p(s)$ is given by

$$D_p(s) = \sum_{\alpha=0}^{\infty} \frac{a_2(p^\alpha)^x a_3(p^\alpha)^y a_6(p^\alpha)^z}{p^{\alpha s}}.$$

According to the proposition, $a_2(p^\alpha)a_3(p^\alpha)a_6(p^\alpha) = 0$ unless $\alpha \equiv 0$ or $2 \pmod 6$, thus $D_p(s)$ can be expressed as the sum of

$$\sum_{\beta=0}^{\infty} \frac{a_2(p^{6\beta})^x a_3(p^{6\beta})^y a_6(p^{6\beta})^z}{p^{6\beta s}} = \sum_{\beta=0}^{\infty} p^{3\beta x + 4\beta y + 5\beta z - 6\beta s}$$

and

$$\sum_{\beta=0}^{\infty} \frac{a_2(p^{6\beta+2})^x a_3(p^{6\beta+2})^y a_6(p^{6\beta+2})^z}{p^{(6\beta+2)s}} = \sum_{\beta=0}^{\infty} p^{(3\beta+1)x + (4\beta+1)y + (5\beta+1)z - (6\beta+2)s}.$$

On summing the two geometrical series, we find that

$$D_p(s) = \frac{1 + p^{x+y+z-2s}}{1 - p^{3x+4y+5z-6s}} = \frac{1 - p^{2x+2y+2z-4s}}{\left(1 - p^{x+y+z-2s}\right)\left(1 - p^{3x+4y+5z-6s}\right)},$$

and therefore

$$D(s) = \prod_p D_p(s) = \frac{\zeta(2s - x - y - z)\,\zeta(6s - 3x - 4y - 5z)}{\zeta(4s - 2x - 2y - 2z)}.$$

The other identities listed in the theorem can be proved in a similar way. □

Corollary 1. *If* $x, y \in \mathbb{R}$ *with* $x, y > 0$, *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) \gg 0$, *then*

$$\sum_{n\in\mathbb{N}} \frac{a_3(n^2)^x}{n^s} = \frac{\zeta(s-x-1)\zeta(3s-4x-3)}{\zeta(2s-2x-2)},$$

$$\sum_{n\in\mathbb{N}} \frac{a_4(n^2)^x}{n^s} = \frac{\zeta(s-x-1)\zeta(2s-3x-2)}{\zeta(2s-2x-2)},$$

$$\sum_{n\in\mathbb{N}} \frac{a_3(n^2)^x a_6(n^2)^y}{n^s} = \frac{\zeta(s-x-y-1)\zeta(3s-4x-5y-3)}{\zeta(2s-2x-2y-2)}.$$

Proof: This follows immediately from the theorem, since by (5), $a_2(n) = \sqrt{n}$ if n is a perfect square, and $a_2(n) = 0$ otherwise. □

Corollary 2. *For every integer* $k \geq 2$, *we have*

$$\begin{aligned}
\zeta(k) &= 2^{k-\frac{1}{2}}\pi^k \sqrt{\frac{|B_{2k}|}{(2k)!}\sum_{n\in\mathbb{N}} \frac{a_3(n)^k}{n^k}} \\
&= 2^{k-\frac{1}{2}}\pi^k \sqrt{\frac{|B_{2k}|}{(2k)!}\sum_{n\in\mathbb{N}} \frac{a_3(n^2)^{2k}}{n^{3k+1}}} \\
&= 2^{k-\frac{1}{2}}\pi^k \sqrt{\frac{|B_{2k}|}{(2k)!}\sum_{n\in\mathbb{N}} \frac{a_4(n^2)^k}{n^{2k+1}}} \\
&= 2^{k-\frac{1}{2}}\pi^k \sqrt{\frac{|B_{2k}|}{(2k)!}\sum_{n\in\mathbb{N}} \frac{a_3(n^2)^{2k/3}a_6(n^2)^{2k/3}}{n^{(7k+3)/3}}}.
\end{aligned}$$

Proof: Using the relation (1), these identities follow easily as special cases of the theorem and the preceding corollary. □

References

[1] E. C. Titchmarsh, The theory of the Riemann zeta function, 2nd edition, Revised and edited by D. R. Heath-Brown, Clarendon Press, Oxford (1986).

William D. Banks
Department of Mathematics
University of Missouri–Columbia
202 Mathematical Sciences Bldg.
Columbia, MO 65211 USA
bbanks@math.missouri.edu