THE VANISHING OF $\mathrm{Tor}_1^R(R^+,k)$ IMPLIES THAT R IS REGULAR

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Abstract. Let (R,m,k) be an excellent local ring of positive prime characteristic. We show that if $\mathrm{Tor}_1^R(R^+,k)=0$ then R is regular. This improves a result of Schoutens, in which the additional hypothesis that R was an isolated singularity was required for the proof.

Let R be an integral domain. Then we denote by R^+ the integral closure of R in an algebraic closure of the fraction field of R. Under the assumption that R is a local excellent domain with positive prime characteristic p, the ring R^+ is a balanced big Cohen-Macaulay algebra [3]. We assume for the rest of this paper that R is a commutative ring with positive prime characteristic p. Let $F : R\longrightarrow R$ be the Frobenius endomorphism given by $r \mapsto r^p$. It is a theorem of Kunz [6] that R is regular if and only if F is a flat map. From this theorem it is not difficult to show that R is regular if and only if R^+ is flat over R. The more general question of whether $\mathrm{Tor}_1^R(R^+,k)=0$ implies that R is regular for a local ring $(R,\mathfrak{m},k)$ of positive characteristic is posed in the exercises in section 8 of [5] (when $\mathrm{Tor}_1^R(S,k)=0$ for a module-finite extension then Nakayama's lemma shows that S is flat over R, however, R^+ is far from finitely generated over R). Schoutens has shown that for an excellent local ring the condition $\mathrm{Tor}_1^R(R^+,k)=0$ implies that R is weakly F-regular, and if R has an isolated singularity then R is regular ([8], Theorems 1.3 and 1.1). We show here that, in fact, the vanishing of $\mathrm{Tor}_1^R(R^+,k)$ suffices to imply regularity for excellent rings of positive prime characteristic.

Assume that $(R,\mathfrak{m},k)$ is a reduced excellent local ring. R is then approximately Gorenstein, so there is a sequence of irreducible $\mathfrak{m}$-primary ideals $\{I_t\}$ cofinal with the powers of $\mathfrak{m}$ (see [2]). By taking a subsequence we may assume that the sequence is non-increasing. Let u_t be an element of R representing the socle modulo I_t. Then the injective hull of the residue field is $E = E_R(k) = \varinjlim_t R/I_t$ and the image of u_t in E is the socle element u of E for all t. Moreover, because the sequence is non-decreasing we may assume that for all t there is an injection $R/I_t \hookrightarrow R/I_{t+1}$ sending $u_t + I_t \mapsto u_{t+1} + I_{t+1}$.

Recall that a ring R of positive prime characteristic is called *F-finite* if the Frobenius endomorphism is module-finite. Such rings are excellent [7], so if in addition R is reduced then it is approximately Gorenstein. Whenever R is reduced there is a well-defined ring of

Date: February 1, 2008.
1991 *Mathematics Subject Classification.* Primary: 13A35; Secondary: 13H05.
The author was partially supported by the National Security Agency. He also wishes to thank the referee for a careful reading of this paper and several corrections.

qth roots of R, denoted $R^{1/q}$, which is a finitely generated R-module for some (equivalently, all) q precisely when R is F-finite. In this case we will write $R^{1/q} \cong R^{a_q} \oplus_R M_q$, where M_q is a module with no free R summands.

The characterization of the injective hull given above is very helpful in proving the next Lemma, which shows how to compute the values of a_q in a special case. By $I^{[q]}$ we mean the ideal $(i^q : i \in I)$.

Lemma 1. *Let $(R, \mathfrak{m}, k)$ be a reduced, F-finite ring with perfect residue field k. Then $a_q = \lambda_R(R/(I_t^{[q]} : u_t^q))$ for all $t \gg 0$.*

Proof. This result is a special case of Corollary 2.8 of [1]. However, we give a proof here for the benefit of the reader. We will use the fact that over an approximately Gorenstein ring, a homomorphism $f : R \longrightarrow M$, where M is finitely generated, has a splitting over R if and only if for all t, $f(u_t) \notin I_t M$ (see [2]).

Fix q, and write $R^{1/q} \cong R^{a_q} \oplus_R M_q$ as above. We first claim that for $t \gg 0$, $u_t M_q \subseteq I_t M_q$, since for any minimal generator of M_q, the map $Rx \longrightarrow M_q$ does not split, and hence, $xu_t \in I_t M_q$. The claim follows since M_q is a finitely generated R-module. We will also use the fact that if I is an $\mathfrak{m}$-primary ideal then $\lambda_R(R/I^{[q]}) = \lambda_R(R^{1/q}/IR^{1/q})$, since k is perfect.

Thus, for any $t \gg 0$, we have

$$\begin{aligned}\lambda(R/(I_t^{[q]} : u_t^q)) &= \lambda(R/I_t^{[q]}) - \lambda(R/(I_t, u_t)^{[q]}) = \lambda(R^{1/q}/I_t R^{1/q}) - \lambda(R^{1/q}/(I_t, u_t)R^{1/q}) \\ &= \lambda(R^{a_q}/I_t R^{a_q}) + \lambda(M_q/I_t M_q) - (\lambda(R^{a_q}/(I_t, u_t)R^{a_q}) + \lambda(M_q/(I_t, u_t)M_q)) \\ &= a_q \cdot 1 + \lambda(M_q/I_t M_q) - \lambda(M_q/(I_t, u_t)M_q) = a_q,\end{aligned}$$

since $(I_t, u_t)M_q = I_t M_q$ (for $t \gg 0$). □

We will need to pass to a Γ construction as described in [4], Section 6. We refer the reader to [4] for details. What we need to know is as follows. Let $(R, \mathfrak{m}, k)$ be a complete ring of characteristic p. Then $R \longrightarrow R^\Gamma$ is a faithfully flat, purely inseparable extension, the maximal ideal of R^Γ is $\mathfrak{m}R^\Gamma$, and R^Γ is F-finite. Note that if $I \subseteq R$ is an irreducible $\mathfrak{m}$-primary ideal of R then IR^Γ is is also an irreducible $\mathfrak{m}R^\Gamma$-primary ideal of R^Γ. Moreover, if $E_R(R/\mathfrak{m}) = \varinjlim_t R/I_t$, then $E_{R^\Gamma}(R^\Gamma/\mathfrak{m}R^\Gamma) = E_R(R/\mathfrak{m}) \otimes_R R^\Gamma = \varinjlim_t R^\Gamma/I_t R^\Gamma$.

Our main theorem is

Theorem 2. *Let $(R, \mathfrak{m}, k)$ be an excellent local domain of positive prime characterstic. Suppose that $\operatorname{Tor}_1(R^+, k) = 0$. Then R is regular.*

Proof. By [8], Theorem 1.2, the ring R is weakly F-regular, therefore a Cohen-Macaulay, normal domain. In particular, R is approximately Gorenstein. Also $R \longrightarrow R^+$ is cyclically pure. The assumption that $\operatorname{Tor}_1(R^+, k) = 0$ and an induction on length shows that for any $\mathfrak{m}$-primary ideal $I \subseteq R$ and element x we have $IR^+ :_{R^+} x = (I :_R x)R^+$.

We first claim that for all q and all t, $I_t^{[q]} :_R u_t^q \subseteq \mathfrak{m}^{[q]}$. To see this suppose that $vu_t^q \in I_t^{[q]}$. Taking qth roots shows that $v^{1/q} \in I_t R^+ :_{R^+} u_t = \mathfrak{m}R^+$, and hence that $v \in (\mathfrak{m}^{[q]})^+ = \mathfrak{m}^{[q]}$ (by cyclic purity of R in R^+). This shows that for all q and for all t, $\lambda(R/(I_t^{[q]} : u_t^q)) \geq \lambda(R/\mathfrak{m}^{[q]})$, which is greater than or equal to q^d ([6]).

We consider $R \longrightarrow \widehat{R} \longrightarrow (\widehat{R})^\Gamma = S$ for any Gamma extension of $\widehat{R}$. In particular we may take Γ to be the empty set, in which case the residue field of S is perfect. Then by faithful flatness and the fact that the maximal ideal of S is $\mathfrak{m}S$, $\lambda_R(R/(I_t^{[q]} :_R u_t^q)) = \lambda_S(S/(I_t S^{[q]} :_S u_t^q))$. Since $u_t^q \notin I_t S^{[q]}$ for all t, the ring S is F-pure, and hence reduced. Thus by Lemma 1, for large enough t (depending on q), $\lambda_R(R/(I_t^{[q]} :_R u_t^q)) = a_q(S)$ is the number of S-free summands in $S^{1/q}$. Since S has perfect residue field, the rank of $S^{1/q}$ as an S-module is precisely q^d, hence $a_q(S) \leq q^d$. We have now shown that $q^d \geq \lambda(R/(I_t^{[q]} : u_t^q)) \geq \lambda(R/\mathfrak{m}^{[q]}) \geq q^d$ Thus $\lambda(R/\mathfrak{m}^{[q]}) = q^d$ and R is regular [6]. □

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