THE F-SIGNATURE AND STRONG F-REGULARITY

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Abstract. We show that the F-signature of a local ring of characteristic p, defined by Huneke and Leuschke, is positive if and only if the ring is strongly F-regular.

In [7], Huneke and Leuschke define the F-signature of an F-finite local ring of prime characteristic with perfect residue field. The F-signature, denoted $s(R)$, is an asymptotic measure of the proportion of R-free direct summands in a direct-sum decomposition of R^{1/p^e}, the ring of p^eth roots of R. This proportion seems to give subtle information on the nature of the singularity defining R. For example, the F-signature of any of the two-dimensional quotient singularities (A_n), (D_n), (E_6), (E_7), (E_8) is the reciprocal of the order of the group G defining the singularity [7, Example 18]. The main theorem of [7] on F-signatures is as follows.

Theorem 0.1. [7, Theorem 11] *Let $(R, \mathfrak{m}, k)$ be a reduced complete F-finite Cohen–Macaulay local ring containing a field of prime characteristic p. Assume that k is perfect. Then*

1. *If $s(R) > 0$, then R is weakly F-regular.*
2. *If in addition R is Gorenstein, then $s(R)$ exists, and is positive if and only if R is weakly F-regular.*

(See below for definitions of the F-signature and weak F-regularity.) In this note, we extend this theorem in two directions: we remove the assumption in (2) that R be Gorenstein, and we replace "weakly F-regular" by "strongly F-regular" throughout. Our main theorem is thus as follows.

Theorem 0.2. *Let $(R, \mathfrak{m}, k)$ be a reduced excellent F-finite local ring containing a field of characteristic p, and let $d = \dim R$. Then the following are equivalent:*

1. $\liminf \frac{a_q}{q^{d+\alpha(R)}} > 0$.
2. $\limsup \frac{a_q}{q^{d+\alpha(R)}} > 0$.
3. *R is strongly F-regular.*

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In particular, if the F-signature $s(R)$ is known to exist, then $s(R)$ is positive if and only if R is strongly F-regular.

We also extend the definition of the F-signature to the case of an imperfect residue field. This allows us to prove that $s(R)$ behaves well with respect to localization (Proposition 1.3).

Our results do not address the existence of the limit defining $s(R)$. Yao has shown that $s(R)$ exists whenever R is Gorenstein on the punctured spectrum [10].

1. The Main Result

Throughout what follows, $(R, \mathfrak{m}, k)$ is a reduced Noetherian local ring of dimension d, containing a field of positive characteristic p. We use q to denote a varying power of p. Set $d = \dim(R)$ and $\alpha(R) = \log_p[k : k^p]$. We assume throughout that R is F-finite, that is, the Frobenius endomorphism $F : R \longrightarrow R$ defined by $F(r) = r^p$ is a module-finite ring homomorphism. Equivalently, for each $q = p^e$, $R^{1/q} = \{r^{1/q} \mid r \in R\}$ is a finitely generated R-module. In particular, this implies that $\alpha(R) < \infty$, and that R is excellent [8, Propositions 1.1 and 2.5]. Also, when computing length over R, we have $\lambda(R/I^{[q]}R) = \lambda(R^{1/q}/IR^{1/q})/q^{\alpha(R)}$.

We first define the F-signature of R.

Definition 1.1. *Let $(R, \mathfrak{m}, k)$ be as above. For each $q = p^e$, decompose $R^{1/q}$ as a direct sum of finitely generated R-modules $R^{a_q} \oplus M_q$, where M_q has no nonzero free direct summands. The F-signature of R is*

$$s(R) = \lim_{q \longrightarrow \infty} \frac{a_q}{q^{d+\alpha(R)}},$$

provided the limit exists.

Our formulation differs slightly from the original definition in [7], where it is assumed that k is perfect, or equivalently that $\alpha(R) = 0$. This reformulation allows us to show that $s(R)$ cannot decrease upon localization. We use a lemma due to Kunz ([8]).

Lemma 1.2. *Let R be an F-finite Noetherian ring of characteristic p. Then for any prime ideals $P \subseteq Q$ of R, $[k(P) : k(P)^p] = [k(Q) : k(Q)^p]p^{\dim R_Q/PR_Q}$. In other words, $\alpha(R_P) = \alpha(R_Q) + \operatorname{ht} Q/P$.*

Proposition 1.3. *Let $(R, \mathfrak{m})$ be an F-finite local ring and P a prime ideal. For $q = p^e$, let a_q be the number of nonzero R-free direct summands in $R^{1/q}$, and let b_q be the corresponding quantity for R_P. Then*

$$\frac{b_q}{q^{\dim(R_P)+\alpha(R_P)}} \geq \frac{a_q}{q^{\dim(R)+\alpha(R)}}.$$

In particular, if both $s(R)$ and $s(R_P)$ exist, then $s(R_P) \geq s(R)$.

Proof. We have $(R_P)^{1/q} \cong (R^{1/q})_P$, so the number of R_P-free direct summands in $(R_P)^{1/q}$ is at least the number of R-free summands in $R^{1/q}$. A straightforward computation using Lemma 1.2 now gives the result. $\square$

We now begin to work toward showing that $s(R)$ is positive if and only if R is strongly F-regular. We refer the reader to [6] for basic notions concerning the theory of tight closure, including *finitistic tight closure*, but review briefly the ideas used in the proof.

A Noetherian ring R of characteristic p is said to be *weakly F-regular* provided every ideal of R is tightly closed. Equivalently, the zero module is finitistically tightly closed in $E = E_R(k)$, the injective hull of the residue field of R. In other symbols, $0_E^{*fg} = 0$. We say that R is *strongly F-regular* if for every $c \in R$ not in any minimal prime of R, the inclusion $Rc^{1/q} \subset R^{1/q}$ splits for $q \gg 0$. Equivalently, the zero module is tightly closed in E, that is, $0_E^* = 0$. Weak and strong F-regularity are conjecturally equivalent, but this is known only in low dimension and in some special cases.

A *test element* for R is an element c, not in any minimal prime of R, such that $cI^* \subseteq I$ for every ideal I of R, and the *test ideal*, denoted $\tau(R)$, is the ideal generated by all test elements. For a reduced local ring R, $\tau(R) = \operatorname{Ann}_R 0_E^{*fg}$ by [5, Theorem 8.23]. Thus R is weakly F-regular if and only if $\tau(R) = R$. On the other hand, the *CS test ideal*, *cf.* [9] and [2], is the ideal $\tilde{\tau}(R) = \operatorname{Ann}_R 0_E^*$. By work of [9] and [2], the CS test ideal behaves well under localization, so defines the non-strongly F-regular locus of $\operatorname{Spec}(R)$. In particular, R is strongly F-regular if and only if $\tilde{\tau}(R) = R$.

It is known that a weakly F-regular ring is *F-pure*, that is, the Frobenius morphism is a pure homomorphism, and that for an F-pure ring both $\tau(R)$ and $\tilde{\tau}(R)$ are radical ideals.

A local ring $(R, \mathfrak{m}, k)$ is said to be *approximately Gorenstein* provided there is a sequence $\{I_t\}$ of $\mathfrak{m}$-primary irreducible ideals cofinal with the powers of $\mathfrak{m}$. When R is Cohen–Macaulay and has a canonical ideal J (so is Gorenstein at all associated primes), such a family can be obtained as follows: Let $x_1, \ldots, x_d$ be a system of parameters such that $x_1 \in J$ and $x_2, \ldots, x_d$ form a system of parameters for R/J. Then $I_t := (x_1^{t-1}J, x_2^t, \ldots, x_d^t)R$, for $t \geq 1$, gives the required family. Furthermore, the direct limit $\varinjlim_t R/I_t$, where the maps in the direct system are $R/I_t \xrightarrow{x_1 \cdots x_d} R/I_{t+1}$, is isomorphic to $E_R(k)$. If $u_1 \in R$ is a representative for the socle generator of R/I_1, then $u_t := (x_1 \cdots x_d)^{t-1} u_1$ generates the socle of R/I_t, and each u_t maps in the limit to u, the socle element of $E_R(k)$.

More generally ([4, Thm. 1.7]), if R is any locally excellent Noetherian ring that is locally Gorenstein at associated primes, then R is approximately Gorenstein.

The following result of Hochster, together with its corollary below, explains our interest in approximately Gorenstein rings. It can be thought of as a generalization of [7, Lemma 12].

Proposition 1.4. [4, Theorem 2.6] *Let $(R, \mathfrak{m})$ be an approximately Gorenstein local ring and let $\{I_t\}$ be a sequence of irreducible ideals cofinal with the powers of $\mathfrak{m}$. Let $f : R \longrightarrow M$ be a homomorphism of finitely generated R-modules. Then f is a split injection if and only if $f \otimes_R R/I_t$ is injective for every t.*

Proposition 1.5. *Let $(R, \mathfrak{m})$ be an approximately Gorenstein local ring with a family of irreducible ideals $\{I_t\}$ as above, and let $u_t \in R$ represent a socle generator for R/I_t. Let $f : R \longrightarrow M$ be a homomorphism of finitely generated R-modules. If M has no free summands, then there exists $t_0 > 0$ such that $u_t M \subseteq I_t M$ for all $t \geq t_0$.*

Proof. By Proposition 1.4, $f \otimes R/I_t$ fails to be injective for some t. Since u_t is the unique socle element of R/I_t, we have $f(u_t) \in I_t M$, that is, $u_t M \subseteq I_t M$. This continues to hold for all $t' \geq t$, since there is an injection $R/I_t \longrightarrow R/I_{t'}$ with $u_t \mapsto u_{t'}$. □

We also use a result of Aberbach, which says that, in some sense, elements not in tight closures are very far from being in Frobenius powers.

Theorem 1.6. [1, Prop. 2.4] *Let $(R, \mathfrak{m})$ be an excellent local domain such that the completion is also a domain. Let $N = \varinjlim_t R/J_t$ be a direct limit system of cyclic modules. Fix $u \notin 0^*_N$. Then there exists q_0 such that*

$$\bigcup_t (J_t^{[q]} : u_t^q) \subseteq \mathfrak{m}^{[q/q_0]}$$

for all $q \gg 0$ (where the sequence $\{u_t\}$ represents $u \in N$ and $u_t \mapsto u_{t+1}$).

Proof of Theorem 0.2. The Cohen-Macaulayness of R is forced by the assumptions ([7, Theorem 11] and [5]), so we may assume throughout that R is Cohen-Macaulay.

That (1) implies (2) is trivial. So assume that (2) holds. We proceed by induction on the dimension d, the case $d = 0$ being trivial. If $d > 0$, then Proposition 1.3 shows that we may assume by induction on d that R is strongly F-regular on the punctured spectrum. We will show that $0^*_E = 0$, where as above $E = E_R(k)$ is the injective hull of the residue field of R.

Since $\tilde{\tau}(R) = \operatorname{Ann}_R 0^*_E$ is a radical ideal and is known to define the non-strongly F-regular locus of R (see [2]), and R is strongly F-regular on the punctured spectrum, $\operatorname{Ann}_R 0^*_E$ contains the maximal ideal $\mathfrak{m}$. If $\tilde{\tau}(R) = R$, then we are done, so we assume $\tilde{\tau}(R) = \mathfrak{m}$. Then $0^*_E = \operatorname{soc}(E)$.

As in the discussion above, $E = E_R(k) \cong \varinjlim_t R/I_t$ for a family of irreducible ideals I_t. Let u be a socle generator for E and $\{u_t\} \subseteq R$ a sequence of representatives for the socle generators of R/I_t, converging to u.

Fix a power q of the characteristic, and decompose $R^{1/q} \cong R^{a_q} \oplus M_q$, where M_q has no nonzero free summands. Then for each t, we have

$$\begin{aligned}
\lambda\left(R/I_t^{[q]}\right) - \lambda\left(R/(I_t, u_t)^{[q]}\right) &= \frac{\lambda\left(R^{1/q}/I_t R^{1/q}\right)}{q^{\alpha(R)}} - \frac{\lambda\left(R^{1/q}/(I_t, u_t)R^{1/q}\right)}{q^{\alpha(R)}} \\
&= \frac{a_q\lambda(R/I_t) + \lambda(M_q/I_t M_q)}{q^{\alpha(R)}} \\
&\qquad - \frac{a_q\lambda\left(R/(I_t, u_t)\right) - \lambda\left(M_q/(I_t, u_t)M_q\right)}{q^{\alpha(R)}} \\
&= \frac{a_q\lambda(R/I_t) - a_q\lambda(R/(I_t, u_t))}{q^{\alpha(R)}} \\
&\qquad + \frac{\lambda(M_q/I_t M_q) - \lambda(M_q/(I_t, u_t)M_q)}{q^{\alpha(R)}} \\
&= \frac{a_q + c_{t,q}}{q^{\alpha(R)}},
\end{aligned}$$

for some $c_{t,q} \geq 0$. By Proposition 1.5, there exists $t_0 > 0$ such that $u_t M_q \subseteq I_t M_q$ for $t \geq t_0$, that is, $c_{t,q} = 0$ for $t \geq t_0$. On the other hand, $\lambda(R/I_t^{[q]}) - \lambda(R/(I_t, u_t)^{[q]}) = \lambda(R/(I_t^{[q]} : u_t^q))$ is equal to 1 for large t since $(I_t^{[q]} : u_t^q) = \mathfrak{m}$ for large t. Thus, for large t,

$$\lim_{q \longrightarrow \infty} \frac{a_q + c_{t,q}}{q^{d+\alpha(R)}} = \lim_{q \longrightarrow \infty} \frac{1}{q^{d+\alpha(R)}} = 0,$$

a contradiction.

Lastly, assume that R is strongly F-regular and keep the same notation. We then have $0^*_E = 0$, so $u \notin 0^*_E$. By Theorem 1.6, then, there exists q_0 such that

$$(I_t^{[q]} :_R u_t^q) \subseteq \mathfrak{m}^{[q/q_0]}$$

for all $q \geq q_0$. Fix $q \geq q_0$. Then there exists t_0 such that for all $t \geq t_0$ we have

$$\begin{aligned}
\frac{a_q}{q^{\alpha(R)}} &= \lambda\left(R/I_t^{[q]}\right) - \lambda\left(R/(I_t^{[q]}, u_t^q)\right) \\
&= \lambda\left(I_t^{[q]} : u_t^q\right) \\
&\geq \lambda\left(R/\mathfrak{m}^{[q/q_0]}\right).
\end{aligned}$$

Divide by q^d and pass to the limit; we see that $\liminf \frac{a_q}{q^{d+\alpha(R)}} \geq \mathrm{e}_{HK}(\mathfrak{m}, R)/q_0^d > 0$. Thus (1) holds.

The last statement is immediate if there is a limit. □

The F-signature suggests a form of dimension that we may attach to an F-finite reduced local ring. Let $s_j = \lim_{q\longrightarrow\infty} \dfrac{a_q}{q^{j+\alpha(R)}}$ for $0 \le j \le d = \dim(R)$ and set $s_{-1} = 1$. Then we can define the s-dimension of R as $\operatorname{sdim}(R) = \max\{j \ge -1 | s_j > 0\}$. A ring which is F-pure then has non-negative s-dimension, and Theorem 0.2 says that R is strongly F-regular if and only if $\operatorname{sdim}(R) = \dim(R)$.

References

1. I. Aberbach, *Extension of weakly and strongly F-regular rings by flat maps*, J. Algebra **241** (2001), 799–807.
2. I. Aberbach and F. Enescu, *Test ideals and base change problems in tight closure theory*, Trans. Amer. Math. Soc. (to appear).
3. M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973), 25–43.
4. ______, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977), no. 2, 463–488.
5. M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.
6. C. Huneke, *Tight Closure and its Applications*, CBMS Lecture notes in Mathematics, vol. 88, American Math. Soc., 1996.
7. C. Huneke and G. Leuschke, *Two theorems about maximal Cohen–Macaulay modules*, Math. Annalen (2002), 1–2, to appear.
8. E. Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), no. 4, 999–1013.
9. G. Lyubeznik and K. E. Smith, *On the commutation of the test ideal with localization and completion*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3149–3180 (electronic).
10. Y. Yao, *Observations on the F-signature of local rings of characteristic p*, preprint.

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