EXTENSION OF WEAKLY AND STRONGLY F-REGULAR RINGS BY FLAT MAPS

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§1. Introduction

Throughout this paper all rings will be Noetherian of positive characteristic p. Hence tight closure theory [HH1–4] takes a prominent place (see §2 for tight closure definitions and terminology). The purpose of this note is to help answer the following question: if R is weakly (resp. strongly) F-regular and $\phi : R \to S$ is a flat map then under what conditions on the fibers is S weakly (resp. strongly) F-regular. This question (among many others) is raised in [HH4] in section 7. It is shown there that if ϕ is a flat map of local rings, S is excellent and the generic and closed fibers are regular then weak F-regularity of R implies that of S (Theorem 7.24). One of our main results weakens the hypotheses considerably.

Theorem 3.4. *Let $\phi : (R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat map. Assume that $S/\mathbf{m}S$ is Gorenstein and R is weakly F-regular and Cohen-Macaulay. Suppose that either*

(1) *$c \in R^\circ$ is a common test element for R and S, and $S/\mathbf{m}S$ is F-injective, or*
(2) *$c \in S - \mathbf{m}S$ is a test element for S and $S/\mathbf{m}S$ is F-rational, or*
(3) *R is excellent and $S/\mathbf{m}S$ is F-rational.*

Then S is weakly F-regular.

We note that the Gorenstein assumption on the fiber is essential, even if R is regular. Even weakening the assumption on the fiber to $\mathbb{Q}$-Gorenstein is not strong enough to give a good theorem, as Singh [Si] gives an example of $R \to S$ flat, where R is a discrete valuation domain, $S/\mathbf{m}S$ is $\mathbb{Q}$-Gorenstein and strongly F-regular, yet S is not weakly F-regular!

We also prove a corresponding result for strong F-regularity.

Theorem 3.6. *Let $(R, \mathbf{m}, K) \to (S, \mathbf{n}, L)$ be a flat map of F-finite reduced rings with Gorenstein closed fiber. Assume that R is strongly F-regular. If $S/\mathbf{m}S$ is F-rational then S is strongly F-regular.*

In order to prove the first of these theorems we investigate how flat maps $\phi : (R, \mathbf{m}) \to (S, \mathbf{n})$ with Gorenstein closed fibers affect tight closure for $I \subseteq R$ such that $l(R/I) < \infty$

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and I is irreducible in R. In general these results do *not* depend on the relationship of $R/\mathbf{m} \to S/\mathbf{n}$ (e.g., separability or finiteness).

While not directly relevant to this paper, we note that other authors have recently investigated tight closure properties under good flat maps. For instance Enescu [En] and Hashimoto [Ha] have recently shown that for a flat map with F-rational base and F-rational closed fiber, the target is F-rational (in the presence of a common test element).

§2. Background for tight closure

Let R be a Noetherian ring of characteristic $p > 0$. We use $q = p^e$ for a varying power of p and for an ideal $I \subseteq R$ we let $I^{[q]} = (i^q : i \in I)$. Also let R° be the complement in R of the union of the minimal primes of R. Then x is in the *tight closure* of I if and only if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$. If $I^* = I$ then I is said to be *tightly closed*. We will say that I is *Frobenius closed* if $x^q \in I^{[q]}$ for some q always implies that $x \in I$.

There is a tight closure operation for a submodule $N \subseteq M$, but we will not discuss this case in general. It is however useful to discuss tight closure in the case of a particular type of direct limit. Suppose that $M = \varinjlim_t R/I_t$ for a sequence of ideals $\{I_t\}$. Let $u \in M$ be an element which is given by $\{u_t\}$ where in the direct limit system $u_t \mapsto u_{t+1}$. We will say that $u \in 0^*_M$ if there exists $c \in R^\circ$ and a sequence t_q such that for all $q \gg 0$, $cu^q_{t_q} \in I^{[q]}_{t_q}$. We will say that u is in the *finitistic tight closure* of 0 in M, 0^{*fg}_M, if there exists $c \in R^\circ$ and $t > 0$ such that $cu^q_t \in I^{[q]}_t$ for all q. This definition of finitistic tight closure agrees with that in [HH2] for this case. Clearly $0^{*fg}_M \subseteq 0^*_M$.

A ring R in which every ideal is tightly closed is called *weakly F-regular*. If every localization of R is weakly F-regular then R is *F-regular*. When R is reduced then $R^{1/p}$ denotes the ring of pth roots of elements of R. More generally, $R^{1/q}$ is the ring of qth roots. Clearly $R \subseteq R^{1/q}$. If R is F-finite and reduced ($R^{1/p}$ is a finite R-module) then R is called *strongly F-regular* if for all $c \in R^\circ$, there exists a q such that the inclusion $Rc^{1/q} \subseteq R^{1/q}$ splits over R. If R is F-finite and R_c is strongly F-regular for some $c \in R^\circ$, then R is strongly F-regular if and only if there exists q such that $Rc^{1/q} \subseteq R^{1/q}$ splits over R [HH1, Theorem 3.3]. Strongly F-regular rings are F-regular, and weakly F-regular rings are normal and under mild conditions (e.g., excellent) are Cohen-Macaulay.

The equivalence of the three conditions is an important open question. Let $(R, \mathbf{m})$ be an excellent reduced local ring and let E be an injective hull of the residue field of R. Then E can be written as a direct limit of the form above since R is approximately Gorenstein. Weak F-regularity of R is equivalent to $0^{*fg}_E = 0$ [HH2, Theorem 8.23], while strong F-regularity is equivalent to (F-finiteness and) $0^*_E = 0$ [LS, Proposition 2.9].

By a *parameter ideal* in $(R, \mathbf{m})$ we mean an ideal generated by part of a system of parameters. We say that $(R, \mathbf{m})$ is *F-rational* if every parameter ideal is tightly closed, and *F-injective* if every parameter ideal is Frobenius closed (this is a slightly different notion of F-injectivity from that in [FW], but is equivalent for CM rings). F-rational rings are normal and under mild conditions are Cohen-Macaulay. In a Gorenstein ring, F-rationality is equivalent to all forms of F-regularity.

A priori, the multiplier element c in the definition of tight closure depends on both I and x. If c works for every tight closure test then we say that c is a *test element* for R.

If c works for every tight closure test for every completion of every localization of R then we say that c is a *completely stable test element*. It is shown in [HH4] that if $(R, \mathbf{m})$ is a reduced excellent domain, $c \in R^\circ$, and R_c is Gorenstein and weakly F-regular then c has a power which is a completely stable test element for R.

In [HH2, HH3] it is shown that the multiplier c in the definition of tight closure need not remain constant. Let R be a domain. One may have a sequence of elements c_q such that $c_q x^q \in I^{[q]}$ where c_q must have "small order." We can obtain a notion of order, denoted ord, by taking a $\mathbb{Z}$-valued valuation on R which is non-negative on R and positive on $\mathbf{m}$. Let R^+ be the integral closure of R in an algebraic closure of the fraction field of R (R^+ has many wonderful properties, such as being a big Cohen-Macaulay algebra for R when R is excellent [HH5]). The valuation then extends to a function on R^+ which takes values in $\mathbb{Q}$. In particular, $\operatorname{ord}(c^{1/q}) = \operatorname{ord}(c)/q$. We will need to use the following theorem [HH3, Theorem 3.1].

Theorem 2.1. *Let $(R, \mathbf{m})$ be a complete local domain of characteristic p, let $x \in R$ and let $I \subseteq R$. Then $x \in I^*$ if and only if there exists a sequence of elements $\epsilon_n \in (R^+)^\circ$ such that $\operatorname{ord}(\epsilon_n) \to 0$ as $n \to \infty$ and $\epsilon_n x \in IR^+$.*

In fact we would like to strengthen this theorem in order to apply it to tight closure calculations for non finitely generated modules which are defined by a direct limit system of ideals. The proof we give is just an altered version of the proof of Theorem 3.1 given in [HH3]. The key component is [HH3, Theorem 3.3]:

Theorem 2.2. *Let $(R, \mathbf{m}, k)$ be a complete local domain. Let ord be a $\mathbb{Q}$-valued valuation on R^+ nonnegative on R (and hence on R^+) and positive on $\mathbf{m}$ (and, hence, on $\mathbf{m}^+$). Then there exists a fixed real number $\nu > 0$ and a fixed positive integer r such that for every element u of R^+ of order $< \nu$ there is an R-linear map $\phi : R^+ \to R$ such that $\phi(u) \notin \mathbf{m}^r$.*

The generalization of Theorem 2.1 is given below.

Theorem 2.3. *Let (R, m) be a complete local domain of characteristic p. Let $M = \varinjlim_t R/I_t$ be an R-module and let $x \in M$. Suppose that x comes from the sequence $\{x_t\}$ where $x_t \mapsto x_{t+1}$. Then $x \in 0^*_M$ if and only if there exists a sequence of elements $\epsilon_n \in (R^+)^0$ such that $\operatorname{ord}(\epsilon_n) \to 0$ as $n \to \infty$ and for each n there exists t such that $\epsilon_n x_t \in I_t R^+$.*

Proof. The "only if" part is trivial, as if $cx^q = 0$ for all $q \gg 0$ then we can take $\epsilon_q = c^{1/q}$.

To see the "if" direction, choose $\nu > 0$ and r as in Theorem 2.2. Fix $q = p^e > 0$. Choose n large enough that $\operatorname{ord}(\epsilon_n) < \nu/q$. Let $\epsilon = \epsilon_n^q$. Then there exists t such that $\epsilon x_t^q \in I_t^{[q]} R^+$ and $\operatorname{ord}(\epsilon) < \nu$. Applying an R linear map ϕ as in Theorem 2.2 we find that $c_q x_t^q \in I_t^{[q]} \subseteq (I_t^{[q]})^*$ with $c_q = \phi(\epsilon) \in R - \mathbf{m}^r$. Thus, setting $J_q = \cup_t (I_t^{[q]})^* :_R x_t^q$ we have $c_q \in J_q$ for all q.

The sequence J_q is nonincreasing. If for some t, $y x_t^{pq} \in (I_t^{[pq]})^*$ then $c'(y x_t^{pq})^{q'} \in (I_t^{[pq]})^{[q']} = (I_t^{[pqq']})$ for all $q' \gg 0$ where $c' \neq 0$. But then $c'(y x_t^q)^{pq'} \in (I_t^{[q]})^{[pq']}$ for all $q' \gg 0$ and hence $y x_t^q \in (I_t^{[q]})^*$, as required.

Since the sequence $\{J_q\}_q$ is nonincreasing, it cannot have intersection 0, or Chevalley's theorem would give $J_q \subseteq \mathbf{m}^r$ for $q \gg 0$. As $c_q \in J_q - \mathbf{m}^r$ for all q, we can choose a nonzero element $d \in \cap_q J_q$. Then for each q there exists t such that $dx_t^q \in (I_t^{[q]})^*$. If c is a test element for R then $cdx_t^q \in I_t^{[q]}$. Thus $x \in 0_M^*$. □

Proposition 2.4. *Let $(R, \mathbf{m})$ be an excellent local domain such that its completion is a domain. Let $M = \varinjlim_t R/I_t$ be a direct limit system. Fix $u \notin 0_M^*$. Then there exists q_0 such that $J_q = \cup_q (I_t^{[q]} : u_t^q) \subseteq \mathbf{m}^{[q/q_0]}$ for all $q \gg 0$ (where $\{u_t\}$ represents $u \in M$ and $u_t \mapsto u_{t+1}$). In particular if $I \subseteq R$ we may take $M = R/I$ where the limit system consists of equalities. Then $u \notin I^*$ implies that $(I^{[q]} : u^q) \subseteq \mathbf{m}^{[q/q_0]}$.*

Proof. Suppose that we can show that the proposition holds in $\widehat{R}$. Then $(I_t^{[q]} :_R u_t^q) \subseteq (I_t^{[q]} :_{\widehat{R}} u_t^q) \cap R \subseteq \mathbf{m}^{[q/q_0]}\widehat{R} \cap R \subseteq \mathbf{m}^{[q/q_0]}R$. Thus we may assume that R is complete.

For $x \in R$ let $f(x)$ be the largest power of $\mathbf{m}$ that x is in, and set $\mathbf{f}(x) = \lim_{n\to\infty} f(x^n)/n$. By the valuation theorem [Re, Theorem 4.16], there exist a finite number of $\mathbb{Z}$-valued valuations $v_1, \dots, v_k$ on R which are non-negative on R and positive on $\mathbf{m}$ and positive rational numbers $e_1, \dots, e_k$ such that $\mathbf{f}(x) = \min\{v_i(x)/e_i\}$. Furthermore, since R is analytically unramified, there exists a constant L such that for all $x \in R$, $f(x) \le \lfloor \mathbf{f}(x) \rfloor \le f(x) + L$ ([Re, Theorem 5.32 and 4.16]).

Now, by Theorem 2.3, for each v_i there exists a positive real number α_i such that if $c \in (I_t^{[q]} : u_t^q)$ then $v_i(c) \ge \alpha_i q$. Combined with the valuation theorem we see that $\mathbf{f}(c) \ge \min\{q\alpha_i/e_i\}$. Let $\alpha = \min\{\alpha_i/e_i\}$. Then $f(c) \ge \alpha q - L - 1$. Let $s = \mu(\mathbf{m})$. Choose $q_1 > 1/\alpha$, $q_2 \ge L+1$, and $q_3 \ge s$ (all powers of p). Set $q_0 = q_1q_2q_3$. Then $f(c) \ge \alpha q - (L+1) \ge q/q_1 - (L+1) \ge q/q_1q_2 - 1 \ge (q/q_0)s - 1$. A simple combinatorial argument shows that $\mathbf{m}^{(q/q_0)s-1} \in \mathbf{m}^{[q/q_0]}$. Hence $c \in \mathbf{m}^{[q/q_0]}$. □

§3. Tight closure in flat extension maps

We show in this section that extending a weakly (respectively, strongly) F-regular ring by a flat map with sufficiently nice Gorenstein closed fiber yields another weakly (resp., strongly) F-regular ring. These results are Theorems 3.4 and 3.6 (see also Corollary 3.5 for the F-regular case).

By saying that $\phi : (R, \mathbf{m}) \to (S, \mathbf{n})$ is flat we mean that ϕ is flat and that $\phi(\mathbf{m}) \subseteq \mathbf{n}$. Since the map is flat we then know that given ideals $A, B \subseteq R$ we have $AS :_S BS = (A :_R B)S$ (B finitely generated). The next lemma merely asserts that modding out by elements which are regular in the closed fiber preserves flatness.

Lemma 3.1. *Let $\phi : (R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat map. Let $z_1, \dots, z_d \in S$ be elements whose images in $S/\mathbf{m}S$ are a regular sequence. Then for any ideal I generated by monomials in the z's, the ring S/IS is flat over R.*

Proof. See, for example [HH4, Theorem 7.10a,b]. □

The next proposition shows that tight closure behaves well for irreducible $\mathbf{m}$-primary ideals when extending to S. Given a sequence of elements $\mathbf{z} = z_1, \dots, z_d$ we will use $\mathbf{z}^{[t]}$ to denote $z_1^t, \dots, z_d^t$.

Proposition 3.2. *Let $\phi : (R, \mathbf{m}, K) \to (S, \mathbf{n}, L)$ be a flat map with Gorenstein closed fiber. Let $\mathbf{z} = z_1, \dots, z_d \in S$ be elements whose images form a s.o.p. in $S/\mathbf{m}S$. Let $I \subseteq R$ be such that $l(R/I) < \infty$ and $\dim_K(0 :_{R/I} m) = 1$. Suppose that either*

(1) *R and S have a common test element and $S/\mathbf{m}S$ is F-injective, or*
(2) *$c \in S - \mathbf{m}S$ is a test element for S, and $S/\mathbf{m}S$ is F-rational, or*
(3) *R is excellent, $\widehat{R}$ is a domain, and $S/\mathbf{m}S$ is F-rational.*

Then I is tightly closed in R $\iff$ for all $t > 0$, $IS + (\mathbf{z})^{[t]}S$ is tightly closed in S $\iff$ there exists $t > 0$ such that $IS + (\mathbf{z})^{[t]}S$ is tightly closed in S.

Proof. Let $b \in S$ have as its image the socle element in $S/\mathbf{m}S + (\mathbf{z})S$. Let $u \in R$ be the socle element mod I. Then the socle element of $S/(IS + (\mathbf{z})S)$ is ub since the map $R/I \to R/I \otimes S = S/IS$ is flat with Gorenstein fibers (there is only one fiber).

Suppose that I is tightly closed. There is no loss of generality in taking $t = 1$. If $IS + (\mathbf{z})S$ is not tightly closed in S then we have $c(ub)^q \in (I^{[q]} + (\mathbf{z})^{[q]})S$ for all q. In case (1) we may take $c \in R^\circ$, so that

$$b^q \in (I^{[q]} + (\mathbf{z})^{[q]})S :_S cu^q = (I^{[q]} :_R cu^q)S + (\mathbf{z})^{[q]}S \subseteq \mathbf{m}S + (\mathbf{z})^{[q]}S$$

for all $q \gg 0$. The first equality is a consequence of flatness, while the inclusion follows since $u \notin I^*$. By our assumption that $S/\mathbf{m}S$ is F-injective we reach the contradictory conclusion that $b \in ((\mathbf{z}) + \mathbf{m})S$. In case (2) we have

$$cb^q \in (I^{[q]} + (\mathbf{z})^{[q]})S :_S u^q = (I^{[q]} :_R u^q)S + (\mathbf{z})^{[q]}S \subseteq \mathbf{m}S + (\mathbf{z})^{[q]}S$$

for all $q \gg 0$. As $S/\mathbf{m}S$ is F-rational, it is a domain, so $c \neq 0$ in $S/\mathbf{m}S$. This contradicts our hypothesis that $S/\mathbf{m}S$ is F-rational (in fact it is enough to assume that I is Frobenius closed to reach this conclusion). In case (3) we can choose q_0 as in Proposition 2.4, and then

$$c(b^{q_0})^{q/q_0} \in (I^{[q]} + (\mathbf{z})^{[q]})S :_S u^q = (I^{[q]} :_R u^q)S + (\mathbf{z})^{[q]}S \subseteq \mathbf{m}^{[q/q_0]}S + ((\mathbf{z})^{[q_0]})^{[q/q_0]}$$

for all q/q_0. But then $b^{q_0} \in (\mathbf{m}S + (\mathbf{z})^{[q_0]})^*$. By persistence, the image of b^{q_0} is in $((\mathbf{z})^{[q_0]}S/\mathbf{m}S)^*$, which contradicts the F-rationality of $S/\mathbf{m}S$.

Suppose now that $IS + (\mathbf{z})^{[t]}S$ is tightly closed in S for all t, but I is not tightly closed in R. Then $u \in (IR)^* \subseteq (I + (\mathbf{z})^{[t]})^*$ (since $R^\circ \subseteq S^0$). But then $u \in \cap_t (IS + (\mathbf{z})^{[t]}S)^* \cap R \subseteq \cap_t (IS + (\mathbf{z})^{[t]}S) \cap R \subseteq IS \cap R = IR$.

Finally, suppose that $(IS + (\mathbf{z})^{[t_0]})S$ is tightly closed for some t_0. Given any t, the socle element of $(IS + (\mathbf{z})^{[t]})S$ is $(z_1 \cdots z_d)^{t-1}ub$. If $c((z_1 \cdots z_d)^{t-1}ub)^q \in (IS + (\mathbf{z})^{[t]})^{[q]}$ then by flatness, $c((z_1 \cdots z_d)^{t_0-1}ub)^q \in (IS + (\mathbf{z})^{[t_0]})^{[q]}$. Therefore, one such ideal tightly closed shows that all such ideals are tightly closed. □

To deal with strong F-regularity we need to give a similar proposition with R/I replaced by the injective hull $E_R(R/\mathbf{m})$. Suppose that we can write $E = E_R(R/\mathbf{m}) = \varinjlim_t R/J_t$, the set $\{u_t\} \subseteq R$ is a collection of elements such that $u_t \mapsto u_{t+1}$ in the map $R/J_t \to R/J_{t+1}$ and the image of each u_t in E is the socle element of E. It suffices that R be approximately Gorenstein [Ho2] (e.g., excellent and normal, or even reduced) to obtain E in this manner. In particular an F-finite ring is excellent [Ku], so a reduced F-finite ring is approximately Gorenstein.

Proposition 3.3. *Let $(R, \mathbf{m}, K) \to (S, \mathbf{n}, L)$ be a flat map of F-finite reduced rings with Gorenstein closed fiber.*

1. *If $Rc^{1/q} \subseteq R^{1/q}$ splits for some q (over R) and $S/\mathbf{m}S$ is F-injective then $Sc^{1/q} \subseteq S^{1/q}$ splits for some q (over S).*
2. *If 0 is Frobenius closed in $E_R(K)$, $S/\mathbf{m}S$ is F-rational and $c \in S - \mathbf{m}S$ then there exists q such that $Sc^{1/q} \subseteq S^{1/q}$ splits (over S).*

Proof. Choose $\mathbf{z} = z_1, \dots, z_d \in S$ elements which generate a s.o.p. in $S/\mathbf{m}S$. By [HH4, Lemma 7.10] we have $E_S(L) = \varinjlim_v S/(\mathbf{z}^{[v]}) \otimes_R E_R(K) = \varinjlim_{v,t} S/(\mathbf{z}^{[v]}) \otimes_R R/J_t = \varinjlim_t S/(\mathbf{z}^{[t]}, J_t)S$. If $b \in S$ generates the socle element in $S/(\mathbf{m} + (\mathbf{z}))S$ then the image of $(z_1 \cdots z_d)^{t-1} b u_t$ in $S/((\mathbf{z}^{[t]}) + J_t)S$ maps to the socle element of E_S (where u_t is as given above).

In case (1), if for all q the inclusion $Sc^{1/q} \to S^{1/q}$ fails to split, by [Ho1, Theorem 1 and Remark 2] for all q there exists t_q such that

$$c(z_1 \cdots z_d)^{(t_q-1)q} b^q u_{t_q}^q \in ((\mathbf{z}^{[t_q]}), J_{t_q})^{[q]} S.$$

Hence $(z_1 \cdots z_d)^{(t_q-1)q} b^q \in ((\mathbf{z}), J_{t_q})^{[q]} :_S cu_{t_q}^q \subseteq (J_{t_q}^{[q]} :_R cu_{t_q}^q)S + (\mathbf{z}^{[t_q]})^{[q]}S \subseteq \mathbf{m}S + (\mathbf{z}^{[t_q]})^{[q]}S$ for $q \gg 0$ (we are using here that if $Rc^{1/q} \subseteq R^{1/q}$ splits for some q then $Rc^{1/q'} \subseteq R^{1/q'}$ splits for all $q' \geq q$). Thus $b^q \in \mathbf{m}S + (\mathbf{z})^{[q]}$ since $S/\mathbf{m}S$ is CM. This contradicts the F-injectivity of $S/\mathbf{m}S$.

To see (2), if there is no splitting we obtain

$$c(z_1 \cdots z_d)^{(t_q-1)q} b^q \in (\mathbf{z}^{[t_q]}, J_{t_q})^{[q]} :_S u_{t_q}^q \subseteq (J_{t_q}^{[q]} :_R u_{t_q}^q)S + ((\mathbf{z}^{[t_q]})^{[q]}S \subseteq \mathbf{m}S + ((\mathbf{z}^{[t_q]})^{[q]}S$$

and hence $cb^q \in \mathbf{m}S + (\mathbf{z})^{[q]}$. This contradicts the F-rationality of $S/\mathbf{m}S$. □

We can now give our main theorems on the extension of weakly and strongly F-regular rings by flat maps with Gorenstein closed fiber.

Theorem 3.4. *Let $\phi : (R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat map. Assume that $S/\mathbf{m}S$ is Gorenstein and R is weakly F-regular and CM. Suppose that either*

1. *$c \in R^\circ$ is a common test element for R and S, and $S/\mathbf{m}S$ is F-injective, or*
2. *$c \in S - \mathbf{m}S$ is a test element for S and $S/\mathbf{m}S$ is F-rational, or*
3. *R is excellent and $S/\mathbf{m}S$ is F-rational.*

Then S is weakly F-regular.

Proof. To see that S is weakly F-regular it suffices to show that there exists a sequence of irreducible tightly closed ideals of S cofinite with the powers of $\mathbf{n}$. As R is weakly F-regular (so normal) and CM it is approximately Gorenstein. Say that $\{J_t\}$ is a sequence of irreducible ideals cofinite with the powers of $\mathbf{m}$. Let $\mathbf{z} = z_1, \dots, z_d \in S$ be elements which form a s.o.p. in $S/\mathbf{m}S$. Then $(J_t + \mathbf{z}^{[t]})S$ is a sequence of irreducible ideals in S cofinal with the powers of $\mathbf{n}$. By Proposition 3.2, in cases (1), (2), and (3), the ideals $(J_t + \mathbf{z}^{[t]})S$ are tightly closed in S (in case (3), $\widehat{R}$ is still weakly F-regular, so is a domain).

Therefore S is weakly F-regular. We note that in case (2) we may weaken the assumption that R is weakly F-regular to the assumption that R is F-pure (see the comment in the proof of Proposition 3.2, part (2)). $\square$

The next corollary should be compared with [HH4, Theorem 7.25(c)].

Corollary 3.5. *Let $(R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat map of excellent rings with Gorenstein fibers. Suppose that the generic fiber is F-rational and all other fibers are F-injective. If R is F-regular then S is F-regular.*

Proof. By hypothesis the generic fiber is Gorenstein and F-rational, therefore there is a $c \in R^\circ$ which is a common completely stable test element. F-regularity is local on the prime ideals of S and the fiber of such a localization is the localization of a fiber, hence Gorenstein and F-injective (the property of F-injectivity is easily seen to localize). Therefore Theorem 3.4(1) always applies. $\square$

Theorem 3.6. *Let $(R, \mathbf{m}, K) \to (S, \mathbf{n}, L)$ be a flat map of F-finite reduced rings with Gorenstein closed fiber. Assume that R is strongly F-regular. If $S/\mathbf{m}S$ is F-rational then S is strongly F-regular.*

Proof. We must show that there exists an element $c \in S^0$ such that S_c is strongly F-regular and $Sc^{1/q} \subseteq S^{1/q}$ splits for some q.

If there exists $c \in R^\circ$ such that S_c is strongly F-regular (i.e., a power of c is a common test element for R and S) then we are done by Proposition 3.3(1). Even if R and S have no (apparent) common test element, however, we claim that there exists $c \in S - \mathbf{m}S$ such that S_c is strongly F-regular. Once we have shown this, the theorem follows by Proposition 3.3(2).

Since the non-strongly F-regular locus is closed [HH1, Theorem 3.3] it suffices to show that $S_{\mathbf{m}S}$ is strongly F-regular, for then there exists an element $c \in S - \mathbf{m}S$ such that S_c is strongly F-regular. Let $B = S_{\mathbf{m}S}$. Then $R \to B$ is flat and the closed fiber is a field. In particular $E_B(B/\mathbf{m}B) = E_R(K) \otimes_R B$. As R is strongly F-regular (so normal) it is approximately Gorenstein. Say $E_R = \varinjlim_t R/J_t$ with socle element mapped to by u_t (as before). Then $u_t \in B/J_tB$ will still map to the socle element u in E_B. Suppose that $u \in 0^*_{E_B}$. This means there exists $b \in B_0$ such that for all q there exists t_q such that $bu^q_{t_q} \in J^{[q]}_{t_q}B$. Hence $b \in J^{[q]}_{t_q} :_B u^q_{t_q} = (J^{[q]}_{t_q} :_R u^q_{t_q})B$. Note that R is an excellent normal domain, so its completion remains a domain. Thus by Proposition 2.4 we see that as $q \to \infty$, $(J^{[q]}_{t_q} :_R u^q_{t_q})$ gets into larger and larger powers of the maximal ideal, since 0 is tightly closed in E_R. Thus $b \in \cap_N \mathbf{m}^N B = 0$, a contradiction. $\square$

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