Hardy martingales and Jensen's Inequality

Nakhlé H. Asmar and Stephen J. Montgomery–Smith
Department of Mathematics
University of Missouri–Columbia
Columbia, Missouri 65211 U. S. A.

Abstract

Hardy martingales were introduced by Garling and used to study analytic functions on the N-dimensional torus $\mathbf{T}^N$, where analyticity is defined using a lexicographic order on the dual group $\mathbf{Z}^N$. We show how, by using basic properties of orders on $\mathbf{Z}^N$, we can apply Garling's method in the study of analytic functions on an arbitrary compact abelian group with an arbitrary order on its dual group. We illustrate our approach by giving a new and simple proof of a famous generalized Jensen's Inequality due to Helson and Lowdenslager [5].

1 Introduction

Suppose that G is a nonzero compact connected abelian group with an infinite (torsion-free) dual group Γ, and normalized Haar measure λ. For $1 \leq p < \infty$, the Banach space of measurable functions f such that $|f|^p$ is integrable will be denoted by $L^p(G)$, and the Banach space of essentially bounded measurable functions on G will be denoted by $L^\infty(G)$. We use the symbols $\mathbf{N}, \mathbf{Z}, \mathbf{R}$, and $\mathbf{C}$ to denote the natural numbers, the integers, the real numbers, and the complex numbers respectively. The circle group will be denoted by $\mathbf{T}$ and will be parametrized as $\{e^{it} : \ 0 \leq t < 2\pi\}$.

A subset P of Γ is called an *order* if it satisfies the following three axioms:

$$P \cap (-P) = \{0\}, P \cup (-P) = \Gamma, \text{ and } P + P = P.$$

Given an order $P \subset \Gamma$, we define a signum function with respect to P, sgn_P, by: $\mathrm{sgn}_P(\chi) = -1, 0$, or 1, according as $\chi \in (-P) \setminus \{0\}, \chi = 0$, or $\chi \in P \setminus \{0\}$. A function $f \in L^1(G)$ is called *analytic* if its Fourier transform $\widehat{f}$ vanishes off P. This notion of analyticity was introduced by Helson and Lowdenslager [5] and [6] in connection with prediction theory, and since then it has been extensively studied because of its independent interest. For $1 \leq p \leq \infty$, following [6], we let $H^p(G)$ denote the space of analytic functions in $L^p(G)$. To be specific about the order, we will also use the notation $H^p_P(G)$.

Recently, Garling [3] introduced Hardy martingales and used them in [4] to prove various properties of analytic functions on $\mathbf{T}^N$, the N-dimensional torus. (See §3 below for a review of these notions.) Analyticity in [4] was defined with respect to the following lexicographic order on the dual group $\mathbf{Z}^N$:

$$
\begin{aligned}
P^* &= \{0\}\bigcup\{(m_1, m_2, \ldots, m_N) \in \mathbf{Z}^N : \ m_1 > 0, m_2 = \ldots = m_N = 0\} \\
&\quad \bigcup\{(m_1, m_2, \ldots, m_N) \in \mathbf{Z}^N : \ m_2 > 0, m_3 = \ldots = m_N = 0\} \\
&\quad \bigcup \ldots \bigcup\{(m_1, m_2, \ldots, m_N) \in \mathbf{Z}^N : \ m_N > 0\}.
\end{aligned}
$$

Our goal in this paper is to show how Garling's approach in [4] can be applied in the setting of an arbitrary order on the dual group, by using basic properties of orders on $\mathbf{Z}^N$. We will derive these properties in Section 2. In Section 3, we illustrate our approach by giving a simple proof of a generalized Jensen's Inequality for functions in $H^1(G)$, due to Helson and Lowdenslager [5]. The inequality states that for any $f \in H^1_P(G)$ we have

$$
\left|\int_G f(x)d\lambda(x)\right| \leq \exp\left(\int_G \log|f(x)|d\lambda(x)\right). \tag{1}
$$

This inequality plays an important role in the study of analytic measures on groups, factorization of $H^1(G)$ functions, and the invariant subspace theory in $H^2(G)$ (see [5] and [6]). The only available proof of this inequality is the original one which uses the full strength of the methods in [5]. The proof that we offer is straightforward and follows directly from the corresponding inequality for functions in $H^1(\mathbf{T})$.

2 Orders on $\mathbf{Z}^N$

Our goal in this section is to prove a useful property of orders which states that, given an arbitrary order P on $\mathbf{Z}^N$ and a finite subset E of P, there is an isomorphism of $\mathbf{Z}^N$ onto itself mapping E into the lexicographic order P^*. We start by recalling from [7, Appendix A] definitions and properties of some special subsets of discrete groups. Throughout this section, Γ denotes an infinite torsion-free abelian group.

Definition 2.1 A finite subset $S = \{b_1, b_2, \ldots, b_k\} \subset \Gamma$ is *independent* if $0 \notin S$ and if $n_1b_1 + n_2b_2 + \ldots + n_kb_k = 0$ implies that $n_1 = n_2 = \ldots = n_k = 0$ (each n_j is an integer). An infinite subset is called *independent* if every one of its finite subsets is independent. If an independent set S generates Γ then S is called a *basis*. As a convention, if $B = \{b_1, b_2, \ldots\}$ is a basis in Γ and $x \in \Gamma$, we will write $b_j(x)$ for the jth coordinate of x in that basis. Hence $x = \sum_{j=1}^k b_j(x)b_j$, for some positive integer k.

Definition 2.2 A subgroup H of a torsion-free group Γ is called *pure* in Γ if whenever $x \in \Gamma, n \neq 0, \ nx \in H$, then $x \in H$.

Pure subgroups will arise in our proofs as kernels of homomorphisms. The concept of pure subgroups is very important in the study of the structure of abelian groups. (See [7, Appendix A].) In what follows, we list some basic properties related to this concept, which will be needed in the sequel.

Remark 2.3 (a) Suppose that H and K are subgroups of Γ, and that $H \subset K \subset \Gamma$. It is easy to see that if K is pure in Γ and H is pure in K, then H is pure in Γ.
The following is a very important property of pure subgroups of $\mathbf{Z}^N$.
(b) If H is a proper pure subgroup of $\mathbf{Z}^N$, then H is isomorphic to $\mathbf{Z}^\nu$ for some $\nu < N$. In this case, there is a basis of $\mathbf{Z}^N$ of the form $\{b_1, b_2, \ldots, b_\nu, \ldots, b_N\}$ such that $\{b_1, b_2, \ldots, b_\nu\}$ is a basis of H (see [7, Theorem A. 26]). The number ν is called the *rank*, or the dimension, of H, and will be denoted $r(H)$.
Using (a), we can generalize (b) as follows.
(c) If $\{0\} \subset H_1 \subset H_2 \subset \ldots \subset H_k = \mathbf{Z}^N$ is a sequence of subgroups of $\mathbf{Z}^N$ such that H_j is a proper pure subgroup of H_{j+1}, then there is a basis of $\mathbf{Z}^N$, $B = \{b_1, \ldots, b_{r(H_1)}, \ldots, b_{r(H_j)}, \ldots, b_{r(H_k)}\}$, such that, for $j = 1, \ldots, k$, $\{b_1, \ldots, b_{r(H_j)}\}$ is a basis of H_j.

In addition to these notions, we need the following result, [1, Theorem (2.5)], that describes orders in $\mathbf{Z}^N$ in terms of a decreasing sequence of subgroups and corresponding separating real-valued homomorphisms.

Theorem 2.4 *Let P be an arbitrary order on $\mathbf{Z}^N$. There are a strictly increasing sequence of subgroups*

$$\{0\} = C_0 \subset C_1 \subset \ldots \subset C_{k-1} \subset C_k = \mathbf{Z}^N, \tag{2}$$

and a sequence $\{L_j\}_{j=1}^k$ of real-valued homomorphisms of $\mathbf{Z}^N$ such that, for $j = 1, 2, \ldots, k$, we have
(i) $L_j(C_{j-1}) = \{0\}$;
(ii) $\operatorname{sgn}_P(\chi) = \operatorname{sgn}(L_j(\chi))$, for all $\chi \in C_j \setminus C_{j-1}$.

For the lexicographic order P^* (see Section 1), Theorem 2.4 is obvious. In this case, we have $k = N$; $C_0 = \{0\}, C_1 = \{x \in \mathbf{Z}^N : x_2 = \ldots = x_N = 0\}, \ldots, C_{N-1} = \{x \in \mathbf{Z}^N : x_N = 0\}$, $C_N = \mathbf{Z}^N$; and $L_j(x_1, x_2, \ldots, x_N) = x_j$ for $j = 1, \ldots, N$.

Since C_{k-1} is the kernel of a homomorphism of $\mathbf{Z}^N$, it follows immediately that C_{k-1} is a pure subgroup of $\mathbf{Z}^N$. Similarly, C_{k-2} is a pure subgroup of C_{k-1}. Now by Remark 2.3 (a), it follows that C_{k-2} is a pure subgroup of $\mathbf{Z}^N$. Continuing in this fashion, we obtain the following simple proposition.

Proposition 2.5 *In the notation of Theorem 2.4, we have that every C_j, $j = 1, \ldots, k$, is a pure subgroup of $\mathbf{Z}^N$.*

Let H denote a nonzero pure subgroup of $\mathbf{Z}^N$. Write an element $x \in H$ as $x = \sum_{l=1}^{r(H)} b_l(x) b_l$ where $\{b_l\}_{l=1}^{r(H)}$ is a basis of H. A nonzero homomorphism L of H into $\mathbf{R}$ is said to *have integer coefficients* if

$$L(x) = \sum_{l=1}^{r(H)} \alpha_l b_l(x)$$

where $\{\alpha_l\}_{l=1}^{r(H)} \subset \mathbf{Z}$. Note that if L has integer coefficients with respect to one basis, then it has integer coefficients with respect to all bases.

Lemma 2.6 *Suppose that $H \neq \{0\}$ is a pure subgroup of $\mathbf{Z}^N$, and L is a nonzero homomorphism of H with integer coefficients. Let $K = \ker L = \{x \in H : \ L(x) = 0\}$. Then there is an element $h \neq 0 \in H$ such that*

$$H = K \oplus < h >,$$

where $< h >$ denotes the subgroup of $\mathbf{Z}^N$ generated by h. In other words, $\ker L$ has codimension 1 in H.

Proof. Since H is a pure subgroup, we may without loss of generality assume that $H \equiv \mathbf{Z}^\nu$. The homomorphism L may be considered as a homomorphism from the linear vector space $\mathbf{Q}^\nu$ (over the field $\mathbf{Q}$) into $\mathbf{Q}$. Since the rank of this mapping is clearly 1, its kernel has dimension $\nu - 1$, and so contains $\nu - 1$ independent vectors over $\mathbf{Q}$. By multiplying these vectors by a large enough integer, we get $\nu - 1$ independent vectors of $\mathbf{Z}^\nu$ belonging to $\ker L$. These vectors necessarily form a basis for K. Now, since $\ker L$ is a pure subgroup of $\mathbf{Z}^\nu$, the lemma follows from Remark 2.3 (c).

Some more items of notation are needed before we state the main theorem of this section. To be specific about an order P on a group Γ, we will sometimes write (Γ, P). Given a nonvoid finite subset S of $(\mathbf{Z}^N, P)$, we will write S_j to denote the set

$$S_j = S \cap (C_j \setminus C_{j-1}), \quad j = 1, 2, \ldots, \ k,$$

where C_j is the subgroup of $\mathbf{Z}^N$ given by Theorem 2.4. According to Theorem 2.4, there are at most $k \leq N$ such sets S_j, uniquely determined by P. The case of a lexicographic order is particularly interesting to us. In that case, we will use the notation S_j^* for the sets S_j. Hence, for $j = 1, \ldots, N$, we have

$$S_j^* = \{x \in S : \ x_j \neq 0, x_{j+1} = \ldots = x_N = 0\}.$$

Theorem 2.7 *Let S be a finite nonvoid subset of $\mathbf{Z}^N$, let P be an arbitrary order on $\mathbf{Z}^N$, and let $\{C_j\}_{j=0}^k$ be as in Theorem 2.4. There is an isomorphism*

$$\psi \ : (\mathbf{Z}^N, P) \longrightarrow (\mathbf{Z}^N, P^*)$$

such that $\psi(S \cap P) \subset P^*$, and $\psi(S \cap (-P)) \subset (-P^*)$. Moreover, for $j = 1, \ldots, k$, we have

$$\psi\left(S_j\right) = \left(\psi(S)\right)^*_{r(C_j)}.$$

Proof. We apply Theorem 2.4 and use its notation. We will construct a basis B of $\mathbf{Z}^N$ of the form

$$B = \{c_1, \ldots, c_{r(C_1)}, \ldots, c_{r(C_j)}, \ldots, c_{r(C_k)}\},$$

so that, in that basis, every element of S_j has a nonzero $r(C_j)$-th component, and $x \in S_j \cap P$ if and only if $x \in S_j$ and $c_{r(C_j)}(x) > 0$. Then the theorem will follow by setting

$$\psi\left(\sum_{l=1}^N c_l(x) c_l\right) = \sum_{l=1}^N c_l(x) e_l$$

where $\{e_1, \ldots, e_N\}$ denotes the standard basis in $\mathbf{Z}^N$. We now proceed to show how to construct B. Without loss of generality, we may assume that S_j is not empty for all $j = 1, \ldots, k$. Let

$$B_1 = \{b_1, \ldots, b_{r(C_1)}, \ldots, b_{r(C_j)}, \ldots, b_{r(C_k)}\},$$

denote a basis of $\mathbf{Z}^N$ with the property that $\{b_1, \ldots, b_{r(C_1)}, \ldots, b_{r(C_j)}\}$ is a basis for C_j (Remark 2.3 (c)). For $x \in \mathbf{Z}^N$, we write $x = \sum_{l=1}^N b_l(x) b_l$. Expressing the homomorphism L_j in the basis B_1, we have for $x \in C_j$,

$$L_j(x) = \sum_{l=r(C_{j-1})+1}^{r(C_j)} \beta_l b_l(x), \tag{3}$$

where $\beta_l \in \mathbf{R}$, because $L_j(C_{j-1}) = \{0\}$. Since, by Theorem 2.4 (ii), $L_j(x) > 0$ for all $x \in S_j \cap P$, and $L_j(x) < 0$ for all $x \in S_j \cap (-P)$, and since S is finite, we can replace the coefficients β_l in (3) by integers α_l so that

$$\sum_{l=r(C_{j-1})+1}^{r(C_j)} \alpha_l b_l(x) > 0$$

if $x \in S_j \cap P$, and

$$\sum_{l=r(C_{j-1})+1}^{r(C_j)} \alpha_l b_l(x) < 0$$

if $x \in S_j \cap (-P)$. (First replace the real numbers α_l in (3) by rational numbers, then multiply by a sufficiently large positive integer.) Define a homomorphism L_j^* for all $x \in \mathbf{Z}^N$ by

$$L_j^*(x) = \sum_{l=r(C_{j-1})+1}^{r(C_j)} \alpha_l b_l(x).$$

Plainly, $L_j^*(x) \neq 0$, for all $x \in S_j$. Let $D_j = \{x \in C_j : \ L_j^*(x) = 0\}$. Then $D_j \supset C_{j-1}$, and, by Lemma 2.6, D_j has codimension 1 in C_j. Let $h_j \in C_j$ be such that

$$C_j = D_j \oplus < h_j > .$$

We can and do choose h_j so that $L_j^*(h_j) > 0$. Now consider the basis

$$B = \{c_1, \ldots, c_{r(D_1)}, h_1 = c_{r(C_1)}, \ldots, c_{r(D_j)}, h_j = c_{r(C_j)}, \ldots, c_{r(D_k)}, h_k = c_{r(C_k)}\},$$

where $\{c_1, \ldots, c_{r(D_1)}, h_1, \ldots, c_{r(D_j)}\}$ is a basis for D_j, and hence

$$\{c_1, \ldots, c_{r(D_1)}, h_1, \ldots, c_{r(D_j)}, h_j\}$$

is a basis for C_j. For $x \in S_j$ we have $L_j^*(x) \neq 0$, and so $h_j(x) \neq 0$. Also, $x \in S_j \cap P$ if and only if $x \in S_j$ and $L_j^*(x) > 0$, if and only if $x \in S_j$ and $h_j(x) > 0$. This shows that B has the desired properties and completes the proof of the theorem.

3 Hardy martingales and Jensen's Inequality

We start this section by reviewing the concept of Hardy martingales from [4]. Let N be a fixed positive integer, and let $e^{i\theta_n}$ denote the n-th coordinate evaluation function on $\mathbf{T}^N$. Let $\mathcal{F}_n = \sigma(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ denote the σ-algebra generated by the first n coordinate functions. For $f \in L^1(\mathbf{T}^N)$, the conditional expectation of f with respect to $\mathcal{F}_n$ will be denoted $\mathbf{E}(f|\mathcal{F}_n)$. For $n = 0, 1, 2, \ldots, N$, the function $\mathbf{E}(f|\mathcal{F}_n)$ is constructed from f by projecting the Fourier transform of f on $\mathbf{Z}^n$, where here $\mathbf{Z}^0 = \{0\}$, and

$$\mathbf{Z}^n = \{(k_1, k_2, \ldots, k_n, 0, \ldots, 0) : \ k_j \in \mathbf{Z}\}.$$

The finite sequence $(\mathbf{E}(f|\mathcal{F}_n))_{n=1}^N$ forms a martingale relative to $(\mathcal{F}_n)_{n=1}^N$. (A detailed analysis of martingales on groups with a decreasing sequence of subgroups can be found in [2, Chapter 5].) A complex-valued martingale (f_n) on $\mathbf{T}^N$ is called a *Hardy martingale* if $\mathbf{E}(f_{n+1}e^{ik\theta_{n+1}}|\mathcal{F}_n) = 0$ for $k > 0$ and all $n = 0, \ldots, N-1$.

Hardy martingales arise naturally when studying analytic functions in $H^1(\mathbf{T}^N)$. Indeed, as observed in [4], for $1 \le p \le \infty$, we have

$$H^p_{P^*}(\mathbf{T}^N) = \left\{ f \in L^p(\mathbf{T}^N) : \ (\mathbf{E}(f|\mathcal{F}_n))_{n=1}^N \text{ is a Hardy martingale} \right\}, \tag{4}$$

where here, as in §1, P^* denotes the lexicographic order on $\mathbf{Z}^N$.

It is instructive to justify this fact, and show, in the process, how Hardy martingales are related to the usual Hardy spaces on the circle group. For this purpose, we recall the notion of martingale differences series. Let

$$d_0(f) = \int_{\mathbf{T}^N} f dx, \tag{5}$$

and for $j = 1, \ldots, N$, let

$$d_j(f) = \mathbf{E}(f|\mathcal{F}_j) - \mathbf{E}(f|\mathcal{F}_{j-1}).$$

Thus $d_j(f)$ $(j = 1, \ldots, N)$ is constructed from f by projecting the Fourier transform of f on the set difference $\mathbf{Z}^j \setminus \mathbf{Z}^{j-1}$. So $d_j(f)$ may be formally represented as the Fourier series:

$$d_j(f) = \sum_{k=-\infty, k\neq 0}^{\infty} f_{j,k}(\theta_1, \ldots, \theta_{j-1}) e^{ik\theta_j}, \tag{6}$$

where $f_{j,k}(\theta_1, \ldots, \theta_{j-1})$ is a function of $\theta_1, \ldots, \theta_{j-1}$ only. We have the *martingale difference series* decomposition

$$f = \sum_{j=0}^{N} d_j(f). \tag{7}$$

As observed in [4], and is easy to check, f is in $H^1_{P^*}(\mathbf{T}^N)$ if and only if, for $j = 1, \ldots, N$,

$$d_j(f) = \sum_{k=1}^{\infty} f_{j,k}(\theta_1, \ldots, \theta_{j-1}) e^{ik\theta_j}. \tag{8}$$

Hence, as a function of θ_j, the function $d_j(f)$ belongs to $H^1(\mathbf{T})$ for $j = 1, 2, \ldots, N$. From these observations, (4) follows easily.

We now return to inequality (1), and prove a special case of it.

Proposition 3.1 *Suppose that $f \in H^1_{P^*}(\mathbf{T}^N)$. Then,*

$$\exp\left(\int_{\mathbf{T}^N} \log|\sum_{j=0}^{n} d_j(f)|dx\right) \leq \exp\left(\int_{\mathbf{T}^N} \log|\sum_{j=0}^{n+1} d_j(f)|dx\right)$$

and

$$\left|\int_{\mathbf{T}^N} f dx\right| \leq \exp\left(\int_{\mathbf{T}^N} \log|f|dx\right).$$

Proof. The second inequality follows by applying the first inequality repeatedly (for $n = 0, 1, \ldots, N-1$) and using (5) and (7). Let us prove the first inequality. As we observed above, the function $\theta_{n+1} \mapsto d_{n+1}(f)$ is in $H^1(\mathbf{T})$. Using the one-dimensional version of Jensen's Inequality (1) for functions in $H^1(\mathbf{T})$ (see [10, Inequality (3.2), and Theorem 3.11]), we obtain

$$\log\left|\frac{1}{2\pi}\int_{\theta_{n+1}}\left(\sum_{j=0}^{n+1} d_j(f)\right)d\theta_{n+1}\right| \leq \frac{1}{2\pi}\int_{\theta_{n+1}} \log\left|\sum_{j=0}^{n+1} d_j(f)\right| d\theta_{n+1}.$$

But

$$\int_{\theta_{n+1}} d_{n+1}(f)d\theta_{n+1} = 0,$$

and $d_j(f)$ is constant in θ_{n+1} for $j = 0, 1, \ldots, n$. Thus,

$$\log\left|\sum_{j=0}^{n} d_j(f)\right| \leq \frac{1}{2\pi}\int_{\theta_{n+1}} \log\left|\sum_{j=0}^{n+1} d_j(f)\right| d\theta_{n+1}.$$

Integrating with respect to the remaining variables, we get

$$\int_{\mathbf{T}^N} \log\left|\sum_{j=0}^{n} d_j(f)\right| dx \leq \int_{\mathbf{T}^N} \log\left|\sum_{j=0}^{n+1} d_j(f)\right| dx,$$

which completes the proof of the proposition.

For the remainder of the proof, we need the following density result.

Lemma 3.2 *Let G be a compact abelian group with dual ordered by P, and let Y be a dense subspace of $H^1_P(G)$. If (1) is true for all f in Y, then (1) is true for all f in $H^1_P(G)$.*

Proof. We first note that for a given function $f \in L^1(G)$ that (1) holds if and only if for all $0 < p < 1$ we have

$$\left|\int_G f d\lambda\right|^p \leq \int_G |f|^p d\lambda. \tag{9}$$

Indeed (1) follows from (9) by letting p tend to zero (see [9, (13.32)(ii)]). Now suppose that (1) holds. Then, for any $0 < p < 1$, we have

$$\left|\int_G f d\lambda\right|^p \leq \exp\left(\int_G \log\left(|f|^p\right) d\lambda\right) \leq \int_G |f|^p d\lambda,$$

where the last inequality follows from [9, (13.32) (i)].

Now fix $0 < p < 1$ and $f \in H^1_P(G)$. Let (f_n) be a sequence in Y such that $f_n \to f$ in $L^1(G)$. We have that $f_n \to f$ in L^p, and from the inequality $\int_G ||f_n|^p - |f|^p| \, d\lambda \leq \int_G |f_n - f|^p \, d\lambda$, it follows that $\int_G |f_n|^p \, d\lambda \to \int_G |f|^p \, d\lambda$ (see [9, Theorem (13.17) and (13.25) (a)]). Since (9) holds for every f_n, it follows immediately that it also holds for f.

The next step is to establish (1) for arbitrary orders on $\mathbf{Z}^N$. At this point we will appeal to Theorem 2.7.

Proposition 3.3 *Suppose that $f \in H^1_P(\mathbf{T}^N)$. Then,*

$$\left|\int_{\mathbf{T}^N} f dx\right| \leq \exp\left(\int_{\mathbf{T}^N} \log|f| dx\right).$$

Proof. Lemma 3.2 shows that it is enough to consider the case of a trigonometric polynomial $f \in H^1_P(\mathbf{T}^N)$. Write

$$f = \sum_{\chi \in S} a_\chi \chi$$

where $a_\chi \in \mathbf{C}$, and S is a finite subset of P. Apply Theorem 2.7 and obtain an isomorphism ψ of $\mathbf{Z}^N$ such that $\psi(S) \subset P^*$. Let ϕ be the adjoint homomorphism of ψ. Thus ϕ is an automorphism of $\mathbf{T}^N$ onto itself such that

$$\psi(\chi)(x) = \chi \circ \phi(x)$$

for all $\chi \in \mathbf{Z}^N$ and all $x \in \mathbf{T}^N$. Moreover, for all $\chi \in S$, the character $\chi \circ \phi = \psi(\chi)$ is in P^*. Since ϕ is an automorphism of $\mathbf{T}^N$, it maps the normalized Haar measure to itself. Using this last observation and Proposition 3.1, we obtain

$$\begin{aligned} \log\left|\int_{\mathbf{T}^N} f \, dx\right| &= \log\left|\int_{\mathbf{T}^N} f \circ \phi \, dx\right| = \log\left|\int_{\mathbf{T}^N} \sum_{\chi \in S} a_\chi \psi(\chi) \, dx\right| \\ &\leq \int_{\mathbf{T}^N} \log\left|\sum_{\chi \in S} a_\chi \psi(\chi)\right| dx = \int_{\mathbf{T}^N} \log|f| \, dx, \end{aligned}$$

which completes the proof of the proposition.

We are only a step away from completing the proof of (1). The standard reduction that remains to be done is based on the Weil formula. We present the details for the sake of completeness.

Proof of (1). Throughout this proof, P will denote an arbitrary order on Γ. By Lemma 3.2, it is enough to consider trigonometric polynomials in $H^1_P(G)$. Let

$$f = \sum_{\chi \in S} a_\chi \chi,$$

where S is a nonvoid finite subset of P, and $a_\chi \in \mathbf{C}$. Let $\langle S \rangle$ denote the subgroup of Γ generated by S. Since Γ is torsion-free, $\langle S \rangle$ is isomorphic to $\mathbf{Z}^N$ for some positive integer N. Let G_0 denote the annihilator in G of $\mathbf{Z}^N$. Then G/G_0 is topologically isomorphic to $\mathbf{T}^N$ and its dual group is $\langle S \rangle \equiv \mathbf{Z}^N$. We order $\mathbf{Z}^N$ by intersecting it with P. Let Π denote the natural homomorphism of G onto G/G_0. Since f is constant on cosets of G_0, there is a trigonometric polynomial on $\mathbf{T}^N$, $f^\dagger$, such that

$$f = f^\dagger \circ \Pi.$$

Clearly $f^\dagger \in H^1_{\mathbf{Z}^N \cap P}(\mathbf{T}^N)$. Now, using the Weil formula [8, Theorem (28.54) (iii)] and Proposition 3.3, we find that

$$\begin{aligned} \log \left| \int_G f d\lambda \right| &= \log \left| \int_{\mathbf{T}^N} f^\dagger dx \right| \\ &\leq \int_{\mathbf{T}^N} \log \left| f^\dagger \right| dx = \int_G \log |f| \, d\lambda, \end{aligned}$$

which yields the desired inequality and completes the proof of (1).

Acknowledgements The work of the authors was supported by separate grants from the National Science Foundation (U.S.A.).

References

[1] N. Asmar, *The conjugate function on the finite dimensional torus*, Can. Math. Bull., **32**, No.2 (1989), 140–148.

[2] R. E. Edwards and G. I. Gaudry, "Littlewood-Paley and Multiplier Theory", Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 90, Springer-Verlag, Berlin, 1977.

[3] D. J. H. Garling, *On martingales with values in a complex Banach space*, Math. Proc. Camb. Phil. Soc. **104** (1988), 399–406.

[4] D. J. H. Garling, *Hardy martingales and the unconditional convergence of martingales*, Bull. London Math. Soc. **23** (1991), 190–192.

[5] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **99** (1958), 165–202.

[6] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables II*, Acta Math. **106** (1961), 175–212.

[7] E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis I," 2nd Edition, Grundlehren der Math. Wissenschaften, Band 115, Springer–Verlag, Berlin 1979.

[8] E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis II," Grundlehren der Math. Wissenschaften in Einzeldarstellungen, Band 152, Springer–Verlag, New York, 1970.

[9] E. Hewitt and K. Stromberg, "Real and Abstract Analysis," Graduate Texts in Mathematics, Vol. 25, 2nd printing, Springer–Verlag, New York, 1969.

[10] Y. Katznelson, "An Introduction to Harmonic Analysis", John Wiley, New York, 1968.