On a weak type $(1,1)$ inequality for a maximal conjugate function

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1 Introduction

Throughout this paper, N denotes a fixed but arbitrary positive integer, $\mathbf{T}$ denotes the circle group, and $\mathbf{T}^N$ denotes the product of N copies of $\mathbf{T}$. The normalized Lebesgue measure on $\mathbf{T}^N$ will be symbolized by P. For a measurable function f, we let $\|f\|_1^* = \sup_{y>0} y\lambda_f(y)$ where $\lambda_f(y) = P\left(\{x \in \mathbf{T}^N : |f(x)| > y\}\right)$. The integers will be denoted by $\mathbf{Z}$ and the complex numbers by $\mathbf{C}$.

Let $\mathcal{F}_n = \sigma(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ denote the σ-algebra on $\mathbf{T}^N$ generated by the first n coordinate functions. For $f \in L^1(\mathbf{T}^N)$, the conditional expectation of f with respect to $\mathcal{F}_n$ will be denoted $\mathbf{E}(f|\mathcal{F}_n)$. Let

$$d_0(f) = \mathbf{E}(f|\mathcal{F}_0) = \int_{\mathbf{T}^N} f dP,$$

and for $j = 1, \ldots, N$, let $d_j(f) = \mathbf{E}(f|\mathcal{F}_j) - \mathbf{E}(f|\mathcal{F}_{j-1})$. We have the martingale difference decomposition

$$f = \sum_{j=0}^{N} d_j(f). \tag{1}$$

Consider the maximal function corresponding to (1)

$$D(f) = \sup_{1\le n\le N} \left| \sum_{j=0}^{n} d_j(f) \right| = \sup_{1\le n\le N} |\mathbf{E}(f|\mathcal{F}_n)|. \tag{2}$$

A well-known weak type $(1,1)$ maximal inequality due to Doob states that there is a constant a independent of f and N such that

$$\|Df\|_1^* \le a\|f\|_1. \tag{3}$$

Now we recall the conjugate function operator $f \mapsto \widetilde{f}$, defined for all $f \in L^2(\mathbf{T})$ by the multiplier relation

$$\widehat{\widetilde{f}}(n) = -i\mathrm{sgn}(n)\widehat{f}(n), \text{ for all } n \in \mathbf{Z}.$$

By Kolmogorov's Theorem [8, Chap. IV, Theorem (3.16)], the operator $f \mapsto \widetilde{f}$ is of weak type $(1,1)$.

Denote an element of $\mathbf{T}^N$ by $(\theta_1, \theta_2, \ldots, \theta_N)$. Let H_j denote the one-dimensional conjugate function operator defined for functions on $\mathbf{T}^N$ with respect to the θ_j variable. As an operator on $L^2(\mathbf{T}^N)$, H_j is given by the multiplier relation $\widehat{H_j(f)}(z_1, z_2, \ldots, z_N) = -i\text{sgn}(z_j)\widehat{f}(z_1, z_2, \ldots, z_N)$, for all $(z_1, z_2, \ldots, z_N) \in \mathbf{Z}^N$. Plainly, the operators H_j, $j = 1, \ldots, N$, are of weak type $(1, 1)$ on $L^1(\mathbf{T}^N)$ with the same constant as in Kolmogorov's theorem for $L^1(\mathbf{T})$. The conjugate function that we consider is defined for all $f \in L^1(\mathbf{T}^N)$ by

$$H(f) = \sum_{j=1}^{N} H_j(d_j(f)). \tag{4}$$

Since both H_j and d_j are multipliers, they commute. We have

$$H(f) = \sum_{j=1}^{N} d_j(H(f)). \tag{5}$$

The maximal function that we are interested in is defined by

$$M(f) = \sup_{1 \leq n \leq N} \left| \sum_{j=1}^{n} d_j(H_j(f)) \right| = D(H(f)), \tag{6}$$

where D is as in (2). Thus M is the composition of two operators of weak type $(1, 1)$. (The fact that H is of weak type $(1, 1)$ is known, and will not be needed in the proofs. See Remarks 1.2 (a), below. This fact will also follow from our main theorem.) Our goal is to prove the following result.

Theorem 1.1 *There is a constant A independent of N such that for all $f \in L^1(\mathbf{T}^N)$ we have*

$$\|Mf\|_1^* \leq A\|f\|_1, \tag{7}$$

where M is the maximal operator given by (6).

The proof of this theorem is presented in the following section, and is of independent interest. We will show that by changing the time in the Brownian motion that Burkholder, Gundy, and Silverstein used in [3] from a continuous range $[0, \infty)$ to a semi-continuous range $\{1, 2, \ldots\} \times [0, \infty)$, the proofs in [3] can be carried out on $\mathbf{T}^N$, yielding inequalities which are independent of N (e.g., the "good λ" inequality).

We end this section with some remarks concerning the operator H that will not be used in the sequel.

Remarks 1.2 (a) The operator $f \mapsto Hf$ that we defined in (5) is a conjugate function operator of the kind that was introduced and studied by Helson [6]. Helson's definition is in terms of orders on the dual group $\mathbf{Z}^N$. In our case, the operator H can be recast in terms of a lexicographic order on $\mathbf{Z}^N$. As shown in [6], the operator H is bounded from $L^1(\mathbf{T}^N)$ into $L^p(\mathbf{T}^N)$, for any $0 < p < 1$. Indeed it is of weak type $(1, 1)$ (see [1, Theorem 4.3]).

(b) We proved in [1, Theorem 5.4] that the square function $Sf = \left(\sum_{j=1}^{N} |H_j(d_j(f))|^2\right)^{1/2}$ is of weak type $(1, 1)$. It is known that under certain conditions on the martingale, the weak type estimates for the square function and the maximal function are equivalent (see, for example, [2], Assumptions A1-A3). The martingales that we are studying do not satisfy these conditions, and so (7) does not follow from the weak $(1, 1)$ estimates for the square function, by using general facts from probability theory.

2 Proof of Theorem 1.1

For clarity's sake, we start with an outline of the proof, setting in the process our notation, and describing our generalization of the methods in [3].

It is enough to prove (7) with $f \in S(\mathbf{T}^N)$, the space of trigonometric polynomials on $\mathbf{T}^N$. We may also assume that f is real-valued and that $d_0(f) = 0$. Write

$$f(\theta_1, \ldots, \theta_N) = \sum a_{j_1,\ldots,j_N} \theta_1^{j_1} \ldots \theta_N^{j_N},$$

and extend f to a function on $\mathbf{C}^N$ that is harmonic in each variable as follows

$$f(r_1\theta_1, \ldots, r_N\theta_N) = \sum a_{j_1,\ldots,j_N} r_1^{|j_1|} \theta_1^{j_1} \ldots r_N^{|j_N|} \theta_N^{j_N}$$

where r_n is a nonnegative real number, and $|\theta_n| \in \mathbf{T} = \{z : |z| = 1\}$. In this notation, the n-th term in the martingale difference decomposition of f becomes

$$d_n(f) = \sum_{\substack{j_1, j_2, \ldots, j_n \\ j_n \neq 0}} a_{j_1,\ldots,j_n,0,\ldots,0} \theta_1^{j_1} \ldots \theta_n^{j_n}.$$

Since by assumption $d_0(f) = 0$, it follows that

$$d_n(f)(r_1\theta_1, \ldots, r_{n-1}\theta_{n-1}, 0) = 0 \tag{8}$$

for all $n = 0, 1, \ldots, N$.

The approach that we take is to consider a martingale on a time structure that is part continuous and part discrete. Our notion of time is $\mathcal{T} = \{1, 2, \ldots, N\} \times [0, \infty[$ with the order $(m, s) < (n, t)$ if and only if $m < n$ or $m = n$ and $s < t$. Construct N independent complex Brownian motions $c_{n,t} = a_{n,t} + i b_{n,t}$ $(1 \leq n \leq N,\ t \geq 0)$ each one starting at 0. Define stopping times $\tau_n = \inf\{\, t : |c_{n,t}| \geq 1\}$.

Define an increasing family of sigma fields $(\mathcal{A}_{(n,t)} : (n, t) \in \mathcal{T})$, where $\mathcal{A}_{(n,t)}$ is the sigma field generated by the functions $c_{m,s}$ for $(m, s) \leq (n, t)$. Then we define a process over our new time structure by:

$$\begin{aligned} F_{n,t} &= f(c_{1,\tau_1}, \ldots, c_{n-1,\tau_{n-1}}, c_{n,\tau_n \wedge t}, 0, \ldots, 0) \\ &= \sum_{k=0}^{n-1} d_k(f)(c_{1,\tau_1}, \ldots, c_{k,\tau_k}) + d_n(f)(c_{1,\tau_1}, \ldots, c_{n,t\wedge\tau_n}). \end{aligned} \tag{9}$$

Since $\tau_n < \infty$ a.s., it follows that a.s., for sufficiently large (n, t), we have $F_{n,t} = F_\infty$, where

$$F_\infty = \sum_{k=0}^{N} d_k(c_{1,\tau_1}, \ldots, c_{k,\tau_k}) = f(c_{1,\tau_1}, \ldots, c_{N,\tau_N}).$$

We will show that the family of functions $(F_{n,t})$ is a martingale relative to $\mathcal{A}_{(n,t)}$. To be able to use results from the classical theory of martingales, it is convenient to label the family $(F_{n,t})$ by a continuous time parameter. This can be done by forming an order preserving bijection between $\mathcal{T} \cup \{\infty\}$ and $[0, N]$ as follows:

$$\phi(n, t) = n - 1 + t/(t + 1), \text{ and } \phi(\infty) = N.$$

Because $c_{n,t}$ is a.s. continuous in t, and also $\tau_n < \infty$ a.s., it follows that $F_{\phi^{-1}(t)}$ is a continuous time martingale on $[0, N]$. Let $\tilde{F}_{n,t}$ be constructed from Hf as in (9). Define the Brownian maximal function

$$F^* = \sup_{0 \leq t \leq N} |F_{\phi^{-1}(t)}|,$$

and let $\tilde{F}^*$ be defined similarly by using $\tilde{F}_{n,t}$. The proof of the desired inequality (7) will proceed in four steps.
Step 1: $\|F_\infty\|_1 = \|f\|_1$;
Step 2: $\|F^*\|^*_{1,\infty} \leq \|F_\infty\|_1$;
Step 3: $\left\|\tilde{F}^*\right\|^*_{1,\infty} \leq c \, \|F^*\|^*_{1,\infty}$;
Step 4: $\|Mf\|^*_{1,\infty} \leq \left\|\tilde{F}^*\right\|^*_{1,\infty}$.

We now proceed with the proofs. Suppose that $c_t = a_t + ib_t$ is a complex Brownian motion starting at 0. Let A_t be the sigma field generated by c_s for $s \leq t$. Let $\tau = \inf\{t : |c_t| \geq 1\}$.
Suppose that v is a real-valued trigonometric polynomial on $\mathbf{T} = \{|z| = 1\}$, and extend v to be harmonic on $\mathbf{C}$. It follows from [5, Theorem 4.1] that $v(c_t)$ is a martingale, and $v(c_t)$ is A_t-measurable. The following lemma, is a simple consequence of this fact and Doob's Optional Stopping Theorem.

Lemma 2.1 *With the above notation, if μ is a stopping time such that $\mu \leq \tau$, then*

$$\mathbf{E}(v(c_\mu)|A_t) = v(c_{t\wedge\mu}).$$

[**?**, Theorem (3.2), p.65]. We have Using Lemma (2.1), we can establish a basic property of the functions $(F_{n,t})$.

Lemma 2.2 *In the above notation, we have that $\mathbf{E}(F_\infty|\mathcal{A}_{n,t}) = F_{n,t}$, and hence that $(F_{n,t})$ is a martingale. Consequently, $(F_{\phi^{-1}(t)})$ is a continuous time martingale for $t \in [0, N]$.*

Proof. First, it is clear that if $k < n$, then

$$\mathbf{E}(d_k(c_{1,\tau_1}, \ldots, c_{k,\tau_k})|\mathcal{A}_{n,t}) = d_k(c_{1,\tau_1}, \ldots, c_{k,\tau_k}),$$

because $d_k(c_{1,\tau_1}, \ldots, c_{k,\tau_k})$ is $\mathcal{A}_{n,t}$ measurable. Also, if $k > n$, then

$$\mathbf{E}(d_k(c_{1,\tau_1}, \ldots, c_{k,\tau_k})|\mathcal{A}_{n,t}) = \mathbf{E}(\mathbf{E}(d_k(c_{1,\tau_1}, \ldots, c_{k,\tau_k})|\mathcal{A}_{k,0})|\mathcal{A}_{n,t}) = 0,$$

by Lemma (2.1) and (8). Similarly, by the same lemma, it also follows that if $k = n$, then

$$\mathbf{E}(d_k(c_{1,\tau_1}, \ldots, c_{k,\tau_k})|\mathcal{A}_{n,t}) = d_k(c_{1,\tau_1}, \ldots, c_{k,t\wedge\tau_k})$$

and hence $\mathbf{E}(F_\infty|\mathcal{A}_{n,t}) = F_{n,t}$. This proves that $(F_{n,t})$ is a martingale. The rest of the lemma is obvious.
Proof of Steps 1, 2, 4 Because of Lemma (2.2), Step 2 follows from Doob's Maximal

Inequality for continuous time martingales (see [4, Chapter VII, Section 11]). Step 1 also follows from the uniform distribution of Brownian motion over $\mathbf{T}$ (see [7, Corollary 3.6.2]). Step 4 is also a consequence of the same property of Brownian motion. We give details. We have

$$\begin{aligned} \tilde{F}^* &= \sup_{(n,t)} |\tilde{F}_{n,t}| \geq \sup_n |\tilde{F}_{n,\tau_n}| \\ &= \sup_n \left| \sum_{m=0}^{n} H_m(d_m(f))(c_{1,\tau_1}, \ldots, c_{m,\tau_m}) \right|. \end{aligned}$$

But since $(c_{1,\tau_1}, \ldots, c_{m,\tau_m})$ is equidistributed with $(\theta_1, \ldots, \theta_m)$, the right side of the displayed inequalities is equidistributed with $\sup_n |\sum_{m=0}^n H_m(d_m(f))(\theta_1, \ldots, \theta_m)|$, and Step 4 follows.

Proof of Step 3. The proof may be done as in [3, Theorem 4]. We provide the details to show the role of analyticity on $\mathbf{T}^N$. Here we call a function $\phi \in L^1(\mathbf{T}^N)$ analytic if its Fourier transform is supported in the half-space

$$\mathcal{O} = \{0\} \bigcup_{j=1}^{N} \{(m_1, m_2, \ldots, m_N) \in \mathbf{Z}^N : \ m_j > 0, m_{j+1} = \ldots, m_N = 0\}.$$

The following basic properties of analytic functions on $\mathbf{T}^N$ are easy to prove.

- A function $\phi \in L^1(\mathbf{T}^N)$ is analytic if and only if each term in its martingale difference decomposition, $d_j(\phi)$ $(j = 1, \ldots, N)$, is analytic in the j-th variable θ_j and has zero mean, i.e., $d_j(\phi) \in H_0^1(\mathbf{T})$.
- If ϕ is analytic then ϕ^2 is also analytic. (This follows from $\mathcal{O} + \mathcal{O} = \mathcal{O}$.)
- If ϕ is a trigonometric polynomial on $\mathbf{T}^N$, then $\phi + iH(\phi)$ is analytic.

Getting back to the proof of Step 3, let

$$g(r_1\theta_1, \ldots, r_N\theta_N) = f(r_1\theta_1, \ldots, r_N\theta_N) + iH(f)(r_1\theta_1, \ldots, r_N\theta_N),$$

and let

$$h = g^2.$$

Both g and h are analytic on $\mathbf{T}^N$. Hence the functions $d_m(g)(\theta_1, \ldots, r_m\theta_m)$ and $d_m(h)(\theta_1, \ldots, r_m\theta_m)$ are analytic in the m-th variable. Form the functions $G_{n,t}$ and $H_{n,t}$ as in (9). By Lemma (2.2), $G_{n,t}$ and $H_{n,t}$ are martingales relative to $\mathcal{A}_{n,t}$. We claim that, because of analyticity, we have

$$H_{n,t} = G_{n,t}^2. \tag{10}$$

To see this, write

$$g(\theta_1, \ldots, \theta_N) = \sum_{k=1}^{N} d_k(g)(\theta_1, \ldots, \theta_k)$$

and

$$h(\theta_1, \ldots, \theta_N) = \sum_{k=1}^{N} d_k(h)(\theta_1, \ldots, \theta_k).$$

Then, since all the exponents of θ_n are positive, we get

$$\left(\sum_{k=1}^{n-1} d_k(g)(\theta_1,\ldots,\theta_k) + d_n(g)(\theta_1,\ldots,r_n\theta_n)\right)^2 = \sum_{k=1}^{n-1} d_k(h)(\theta_1,\ldots,\theta_k)+d_n(h)(\theta_1,\ldots,r_n\theta_n)$$

and (10) easily follows. Consequently, since the functions $H_{n,t}$ form a martingale relative to the σ-algebra $\mathcal{A}_{n,t}$, we have that $G^2_{n,t}$ is a martingale relative to this σ-algebra. With this fact in hands, we can now proceed with the proof of Step 3 in exactly the same way as in [3, pp. 148-149]. We need a lemma.

Lemma 2.3 *Suppose that μ and ν are stopping times with $\mu \le \nu$ a. e. Let f be a real-valued trigonometric polynomial on $\mathbf{T}^N$ with $\int f dP = 0$. Then,*

$$\|\tilde{F}_\nu - \tilde{F}_\mu\|_2 = \|F_\nu - F_\mu\|_2.$$

Proof. Using the fact that $G^2_{n,t}$ is a martingale, we get

$$0 = \mathbf{E}(G^2_0) = \mathbf{E}(G^2_\mu).$$

Similarly, $\mathbf{E}(G^2_\nu) = 0$. Hence, $\mathbf{E}F^2_\mu = \mathbf{E}\tilde{F}^2_\mu$ and $\mathbf{E}F^2_\nu = \mathbf{E}\tilde{F}^2_\nu$. Next, we show that $\mathbf{E}(F_\mu F_\nu) = \mathbf{E}(F^2_\mu)$, and $\mathbf{E}(\tilde{F}_\mu\tilde{F}_\nu) = \mathbf{E}(\tilde{F}^2_\mu)$. We start with the first equality. Using Doob's Optional Sampling Theorem and basic properties of the conditional expectation, we see that

$$\mathbf{E}(F_\nu|F_\mu) = F_\mu,$$
$$F_\mu\mathbf{E}(F_\nu|F_\mu) = F^2_\mu,$$

and so

$$\mathbf{E}(F_\mu F_\nu|F_\mu) = F^2_\mu.$$

Integrating both sides of the last equality, we get $\mathbf{E}(F_\mu F_\nu) = \mathbf{E}(F^2_\mu)$. The second equality can be proved similarly. Thus

$$\begin{aligned}\mathbf{E}(F_\mu - F_\nu)^2 &= \mathbf{E}F^2_\mu + \mathbf{E}F^2_\nu - 2\mathbf{E}(F_\mu F_\nu)\\ &= \mathbf{E}F^2_\mu + \mathbf{E}F^2_\nu - 2\mathbf{E}(F^2_\mu)\\ &= \mathbf{E}F^2_\nu - \mathbf{E}(F^2_\mu)\\ &= \mathbf{E}(\tilde{F}_\mu - \tilde{F}_\nu)^2,\end{aligned}$$

which completes the proof.

The above lemma enables us to establish a fundamental inequality. This is our version of the 'good λ' inequality for conjugate functions on $\mathbf{T}^N$.

Lemma 2.4 *With the notation of the previous lemma, let $\alpha \ge 1$ and $\beta > 1$. Then there is a constant c, depending only on α and β, such that whenever $\lambda > 0$ satisfies*

$$P(G^* > \lambda) \le \alpha P(G^* > \beta\lambda),$$

then

$$P(G^* > \lambda) \le c\,P(c\,F^* > \lambda).$$

Proof. Define stopping times

$$\mu = \inf\{\, (n,t) \in \mathcal{T} : |G_{n,t}| > \lambda\}, \nu = \inf\{\, (n,t) \in \mathcal{T} : |G_{n,t}| > \beta\lambda\}.$$

If the set $\{\, (n,t) : |G_{n,t}| > \lambda\}$ is empty, then we set $\mu = \infty$. Otherwise μ is such that $|G_{n,t}| \le \lambda$ whenever $(n,t) < \mu$, and $|G_\mu| = \lambda$. We define ν similarly. Also, we have that $\mu \le \nu$, that $|G_\mu| = \lambda$ on the set $\{\mu \ne \infty\} = \{G^*_\infty > \lambda\}$, and that $|G_\nu| = \beta\lambda$ on the set $\{\nu \ne \infty\} = \{G^* > \beta\lambda\}$. Thus if λ satisfies the hypothesis of the lemma, then

$$\begin{aligned} \mathbf{E}(\chi_{G^*>\lambda}(F_\nu - F_\mu)^2) &= \|F_\nu - F_\mu\|_2^2 \\ &= \frac{1}{2}\|G_\nu - G_\mu\|_2^2 \\ &\ge \frac{1}{2}(\beta\lambda - \lambda)^2 P(G^* > \beta\lambda) \\ &\ge c\lambda^2 P(G^*_\infty > \lambda). \end{aligned}$$

Also

$$\mathbf{E}(\chi_{G^*>\lambda}(F_\nu - F_\mu)^4) \le \|G_\nu - G_\mu\|_4^4 \le c\lambda^4 P(G^*_\infty > \lambda).$$

Thus, by a lemma of Paley and Zygmund [8, Chapter V, (8,26)],

$$P(G^* > \lambda) \le cP(c|F_\nu - F_\mu| > \lambda).$$

Since $|F_\nu - F_\mu| \le 2F^*$, the lemma follows.

Now let us finish by proving Step 3. It is sufficient to show $\|G^*\|^*_{1,\infty} \le c\ \|F^*\|^*_{1,\infty}$. Suppose that

$$\|G^*\|^*_{1,\infty} = \sup_{\lambda>0} \lambda P(G^* > \lambda) = A.$$

Pick λ_0 such that $2\lambda_0 P(G^* > 2\lambda_0) \ge A/2$. Then $\lambda_0 P(G^* > \lambda_0) \le A$, and thus λ_0 satisfies the hypothesis of the lemma with $\alpha = 4$ and $\beta = 2$. Then it follows that

$$\|F^*\|^*_{1,\infty} \ge \lambda_0 P(cF^* > \lambda_0) \ge cA/4,$$

as desired.

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