SOME CONJECTURES ABOUT INTEGRAL MEANS OF ∂f AND $\overline{\partial} f$.

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1. **The problems.** In this note we shall discuss some conjectural integral inequalities which are related to quasiconformal mappings, singular integrals, martingales and the calculus of variations. For a function $f : \mathbb{C} \to \mathbb{C}$, denote the formal complex derivatives by

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \quad \overline{\partial} f = \frac{\partial f}{\overline{\partial} z} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

Define a function $L : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ by

$$\begin{aligned} L(z,w) &= |z|^2 - |w|^2, \quad \text{if} \quad |z| + |w| \le 1, \\ &= 2|z| - 1, \quad \text{if} \quad |z| + |w| > 1. \end{aligned}$$

Conjecture 1. *(V. Šverák) Let $f \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. Then*

$$\int_{\mathbb{C}} L(\partial f, \overline{\partial} f) \ge 0. \tag{1.1}$$

Here we denote by $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ denotes the "homogeneous" Sobolev space of complex valued locally integrable functions in the plane whose distributional first derivatives are in L^2 on the plane. Integrals without specified variables are understood to be with respect to Lebesgue measure.

Since $\partial(\overline{f}) = \overline{\overline{\partial} f}$, Conjecture 1 is true if and only if (1.1) always holds when $L(\partial f, \overline{\partial} f)$ in the integral is replaced by $L(\overline{\partial} f, \partial f)$.

The function L, with a plus 1 added to the right hand side, was introduced by Burkholder [Bu4], [Bu5, p.20]. In his setting, the variables z and w are taken from an arbitrary Hilbert space. It

The first author was supported in part by NSF grants DMS-9206319 and DMS-9501293. The second author was supported in part by NSF grant DMS-9424396

appears independently in work of Šverák [Sv1], who considered the question, as yet unresolved, of whether functions belonging to a certain class which contains a function naturally associated to L are "quasiconvex". As will be explained in §5, quasiconvexity of this function implies (1.1).

For $p \in (1,\infty)$, set

$$p^* = \max\,(p,p'), \quad \text{where} \quad \frac{1}{p}+\frac{1}{p'}=1.$$

Again following Burkholder [Bu5, p.16], [Bu3, p.77], [Bu4, p.8], define functions $\Phi_p : \mathbb{C}\times\mathbb{C}\to\mathbb{R}$ by

$$\Phi_p(z,w) = \alpha_p((p^*-1)|z|-|w|)\,(|z|+|w|)^{p-1}, \quad \alpha_p = p(1-\frac{1}{p^*})^{p-1}.$$

For $1<p<2$ and $z,w\in\mathbb{C}$ one calculates

$$\int_0^\infty t^{p-1}L(\frac{z}{t},\frac{w}{t})\,dt = \beta_p\,\Phi_p(z,w), \quad \beta_p = (\frac{1}{2}\,p\,(2-p)\,\alpha_p)^{-1}. \tag{1.2a}$$

Set $M(z,w) = L(z,w)-(|z|^2-|w|^2) = (|w|^2-(|z|-1)^2)1_{(|z|+|w|>1)}$. Then, for $2<p<\infty$,

$$\int_0^\infty t^{p-1}M(\frac{w}{t},\frac{z}{t})\,dt = \gamma_p\,\Phi_p(z,w), \quad \gamma_p = (\frac{1}{2}\,p\,(p-1)\,(p-2)\,\alpha_p)^{-1}. \tag{1.2b}$$

For $f\in \dot W^{1,2}(\mathbb{C})$, one sees by Fourier transforms or otherwise that

$$\int_{\mathbb{C}} |\partial f|^2 - |\overline{\partial} f|^2 = 0.$$

If Conjecture 1 is true, the last three identities imply the truth of

Conjecture 2. *(R. Bañuelos - G.Wang) For $f\in \dot W^{1,p}(\mathbb{C},\mathbb{C})$ holds*

$$\int_{\mathbb{C}} \Phi_p(\partial f,\overline{\partial} f) \ge 0, \quad 1<p<\infty.$$

Bañuelos and Wang arrived at Conjecture 2 in the course of their work [BW1]. The conjecture is stated as Question 1' in [BL, §5], where the reader can find other questions and comments related to the present paper.

From [Bu3, p.77] follows the inequality

$$\Phi_p(z,w) \le (p^*-1)^p|z|^p - |w|^p, \quad z,w\in\mathbb{C}, \quad 1<p<\infty.$$

Thus, if Conjecture 2 is true, then so is the following conjecture, which is due to T.Iwaniec [I1, I3].

Conjecture 3. *(T.Iwaniec) For $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$ holds*

$$\int_{\mathbb{C}} |\overline{\partial} f|^p \leq (p^* - 1)^p \int_{\mathbb{C}} |\partial f|^p, \quad 1 < p < \infty.$$

Like Conjecture 1, each of Conjectures 2 and 3 is true if and only the same inequality always holds when ∂f and $\overline{\partial} f$ are interchanged in the corresponding integral.

All three conjectured inequalities are sharp, if they are true. Let $f(z) = cz$ for $|z| < 1$, $\quad f(z) = c/\overline{z}$ for $|z| > 1$, where c is a nonzero complex constant. Simple calculations show that equality holds for f in Conjecture 1. Equality holds for f in Conjecture 2 when $1 < p \leq 2$ and for $\overline{f}$ when $2 \leq p < \infty$. In section 6, we'll see that equality holds in Conjectures 1 and 2 for a large class of functions in $\dot{W}^{1,2}$ and $\dot{W}^{1,p}$, respectively.

By contrast, it seems plausible that when $p \neq 2$ equality never holds in Conjecture 3. To construct sequences which saturate the upper bound in Conjecture 3, take $p \in (1, \infty)$ and $\alpha \in (0, 1/p)$. Define $f_\alpha(z) = z|z|^{-2\alpha}$, if $|z| \leq 1$, $f_\alpha(z) = 1/\overline{z}$, if $|z| \geq 1$. One computes that $\int_{\mathbb{C}} |\partial f_\alpha|^p / \int_{\mathbb{C}} |\overline{\partial} f_\alpha|^p \to (p-1)^p$ as $\alpha \to 1/p$. Thus, $\int_{\mathbb{C}} |\partial(\overline{f_\alpha})|^p / \int_{\mathbb{C}} |\overline{\partial}\,(\overline{f_\alpha})|^p \to (p-1)^{-p}$ as $\alpha \to 1/p$. From these two relations, and the interchangeability of ∂f, $\overline{\partial} f$ in the conjectures, it follows that the constant on the right hand side of Conjecture 3 must be at least $\max^p(p-1, \frac{1}{p-1}) = (p^* - 1)^p$.

Here are three reasons we find these conjectures of interest.

(a) Truth of Conjecture 3 would imply in the limiting case $p \to \infty$ a notable recent theorem of Astala [As1] about area distortion of quasiconformal mappings in the plane.

(b) Let S be the singular integral operator in the plane defined by

$$Sf(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{(z - \zeta)^2} \, |d\zeta|^2. \tag{1.3}$$

Truth of Conjecture 3 would show that the norm of S on $L^p(\mathbb{C}, \mathbb{C})$ is precisely $p^* - 1$.

(c) Falsity of Conjecture 1 would prove, for 2×2 matrix valued functions, a conjecture of Morrey in the calculus of variations which asserts that rank one functions are not necessarily quasiconvex.

In Sections 2-5 we'll elaborate on statements (a), (b), and (c). In Sections 6-8 we'll present some evidence in favor of the conjectures.

We are grateful to Professors Astala, Bañuelos, and Iwaniec for helpful communications, especially to Professor Iwaniec for sharing some of his unpublished notes with us. Thanks go also to N. Arcozzi, D.Burkholder, R. Laugesen and the referee for corrections and comments on the first version of the manuscript. The first author thanks the organizers of the Uppsala conference for their marvelous hospitality and efficiency. Above all, he thanks Matts and Agneta for many years of inspiration and friendship.

2. **Area distortion by quasiconformal mappings.** The integral in (1.3) is a Cauchy principal value. The operator S is sometimes called the Beurling-Ahlfors transform. The general theory of such operators, as developed by Calderón, Zygmund, and others, is presented, for example, in [S]. Among its consequences are the facts that S is bounded on L^p for $1 < p < \infty$, and that S may be defined via Fourier multipliers by

$$(Sf)\hat{}(\xi) = (\overline{\xi}/\xi)\hat{f}(\xi), \quad \xi \in \mathbb{C}. \tag{2.1}$$

Thus, S acts isometrically on $L^2(\mathbb{C}, \mathbb{C})$. It follows also from (2.1) that for appropriate functions f we have

$$S(\overline{\partial} f) = \partial f. \tag{2.2}$$

Because of (2.2), the Ahlfors-Beurling operator plays an important role in the theory of quasiconformal mapping in the plane. See, for example, [LV]. A homeomorphism $F : \mathbb{C} \to \mathbb{C}$ is said to be K quasiconformal, $K \geq 1$, if $F \in W^{1,2}_{\text{loc}}(\mathbb{C}, \mathbb{C})$, and if $|\overline{\partial} F(z)| \leq k|\partial F(z)|$ for a.e $z \in \mathbb{C}$, where $k = (K-1)/(K+1)$. In the 1950's, Bojarski [Bo 1,2] applied the recently proved L^p-boundedness of S to prove that partial derivatives of $K-qc$ maps, which a priori belong to L^2_{loc}, belong in fact to L^p_{loc} for some $p > 2$ which depends only on K. Via Hölder's inequality, this enhanced integrability leads to an inequality for the distortion of area by qc maps. One way to state the area distortion property is as follows: If $F(0) = 0$ and $F(1) = 1$, then for all measurable sets $E \subset (|z| < 1)$,

$$|F(E)| \leq C|E|^{\kappa}, \tag{2.3}$$

where $|\cdot|$ denotes Lebesgue measure, and C and κ depend only on K.

Gehring and Reich [GR] conjectured in 1966 that the best possible, i.e. smallest, κ for which (2.3) is valid should be $\kappa = 1/K$. Prototypical conjectured extremals were the radial stretch maps

$$F_K(z) = z|z|^{\frac{1}{K}-1}.$$

F_K is $K-qc$ for each $K \geq 1$, and satisfies $|F_K(B)| = \pi^{1-\frac{1}{K}}|B|^{1/K}$ for balls B centered at the origin.

The Gehring-Reich conjecture withstood many assaults before it was finally proved in the 1990's by Astala [As1], by means of very innovative considerations involving holomorphic dynamics and thermodynamical formalism. Eremenko and Hamilton [EH] gave a shorter proof of the conjecture using a distillation of Astala's ideas. More background and related results can be found in the survey [As2]. In [N] and [AsM], the distortion results are applied to problems about "homogenization" of composite materials.

To continue our story requires a backup. The weak 1-1 and L^2 boundedness of S imply existence of absolute constants c and α such that for all $E \subset (|z| < 1)$,

$$\int_{|z|<1} |S(1_E)| \leq c|E| \log(\frac{\alpha}{|E|}).$$

Gehring and Reich showed that their area distortion conjecture is more or less equivalent to proving that the smallest c for which some α exists is $c = 1$. Let $||S||_p$ denote the norm of S acting as an operator from $L^p(\mathbb{C}, \mathbb{C})$ into itself. Iwaniec [I1] found that "c=1" is implied by

$$\liminf_{p\to\infty} \frac{1}{p}||S||_p = 1.$$

This implication, together with the examples f_α in Section 1, led Iwaniec to Conjecture 3, which, as noted in (b) at the end of section 1, can be restated as

$$||S||_p = (p^* - 1), \quad 1 < p < \infty. \tag{2.4}$$

Thus, if Conjecture 3 is true, it could be regarded as a significantly stronger form of Astala's area distortion theorem.

3. **Norms of singular integral operators and martingale transforms.** The prototypical singular integral operator is the Hilbert transform H, defined for functions on $\mathbb{R}$ by

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

When the Fourier transform $\hat{f}$ is defined as in [S], we have the multiplier equation $(Hf)\hat{}(\xi) = i\frac{\xi}{|\xi|}\hat{f}(x)$. Thus H is an isometry on $L^2(\mathbb{R})$. M.Riesz, in 1927, proved that H is in fact bounded on $L^p(\mathbb{R})$, for $1 < p < \infty$. The sharp L^p bounds for real valued functions were found by Pichorides [Pi] in 1972. Let $||H||_p$ denote the norm of H acting on $L^p(\mathbb{R}, \mathbb{R})$. Recall that $p^* = \max(p, p')$.

Pichorides's Theorem. $||H||_p = \cot \frac{\pi}{2p^*}, \quad 1 < p < \infty.$

Let F be the harmonic extension of $f + iHf$ to the upper half plane. Then F is holomorphic. If ϕ is subharmonic on the range of F then $\phi \circ F$ is subharmonic. Pichorides proved his theorem by making a good choice for ϕ. According to [G], the best constant for Riesz's theorem was found independently by B.Cole, whose work established a generalized version of the theorem in the context of "Jensen measures" on uniform algebras.

Verbitsky [V], and a little later Essén [E], gave a shorter proof of Pichorides's theorem by finding an even better choice for the subharmonic function ϕ. Grafakos [Gr] found a still shorter proof. Verbitsky and Essén also proved sharp bounds for the analytic projection operator $I + iH$, where I denotes the identity, acting as an operator from $L^p(\mathbb{R}, \mathbb{R})$ to $L^p(\mathbb{R}, \mathbb{C})$:

$$||I + iH||_p = \csc \frac{\pi}{2p^*}. \tag{3.1}$$

As explained in [Pe], the norm of H acting on $L^p(\mathbb{R}, \mathbb{C})$ is still $\cot \frac{\pi}{2p^*}$. But for $I + iH$ acting on $L^p(\mathbb{R}, \mathbb{C})$ the norm is apparently not known. See [Pe], [KV]. The sharp weak 1-1 constant for H acting on $L^1(\mathbb{R}, \mathbb{R})$ was found by B.Davis in 1974, but seems to be not known for H acting on $L^1(\mathbb{R}, \mathbb{C})$. See [Pe], [Bu4, p.6] for discussion.

The Beurling-Ahlfors operator S furnishes one analogue of the Hilbert transform in dimension 2. But the most basic generalizations to higher dimensions of the Hilbert transform are the *Riesz transforms* $R_j,\ j = 1, ..., n$. In terms of Fourier multipliers, they are defined by

$$(R_j f)\hat{}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

and in terms of integrals by convolution with the kernel $C_n x_j/|x|^{n+1}$, where
$C_n = \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}$.

Let $||R_j||_p$ denote the norm of R_j acting on $L^p(\mathbb{R}^n, \mathbb{R})$ or $L^p(\mathbb{R}, \mathbb{C})$. T.Iwaniec and G.Martin [IM 3], proved that R_j has the same norm as H.

Iwaniec-Martin Theorem. $||R_j||_p = \cot \frac{\pi}{2p^*}, \quad 1 < p < \infty.$

Iwaniec and Martin use the method of rotations to show that $||R_j||_p$ is bounded above by Pichorides's constant. The lower estimate for $||R_j||_p$ is proved by a simple but clever "transference" argument.

Bañuelos and Wang [BW1] obtained another proof of $||R_j||_p \leq \cot\,\frac{\pi}{2p^*}$ as a consequence of a " $\cot\,\frac{\pi}{2p^*}$ theorem" they proved for transformations of certain stochastic integrals. In addition, they proved that (3.1) holds when H is replaced by one of the R_j. Apparently, the L^p norms of operators such as $I \bigoplus R_1 \bigoplus R_2 : L^p(\mathbb{R}^n, \mathbb{R}) \to L^p(\mathbb{R}^n, \mathbb{R}^3)$ remain unknown when $n \geq 2$ and $p \neq 2$.

Arcozzi [Ar1], see also [Ar2], [ArL], carried the martingale methods over to compact manifolds, Lie groups, and Gauss space. Among other things, he proves that for suitable definitions of Riesz transforms R on n-spheres and on certain compact Lie groups again hold $||R||_p = \cot\,\frac{\pi}{2p^*}$ and $||I + iR||_p = \csc\,\frac{\pi}{2p^*}$. For general compact Lie groups, Arcozzi proves that one has the upper bounds $||R||_p \leq \cot\,\frac{\pi}{2p^*}$ and $||I + iR||_p \leq \csc\,\frac{\pi}{2p^*}$.

Recall that Conjecture 3 can be stated as $||S||_p = p^* - 1$, where S is the Ahlfors-Beurling operator (1.3). This conjecture differs from the established result $||R_j||_p = \cot\,\frac{\pi}{2p^*}$ in two respects: (i) The kernel z^{-2} for S is even, so the method of rotations is not applicable. (ii) The kernel for S is complex-valued.

At least two sharp inequalities for S, not involving L^p, do exist. See [EH] and [I2]. Additional evidence for Conjecture 3 is provided in [AIS], where it is shown that the operator $I - S\mu$ is invertible in L^p for all functions $\mu \in L^\infty(\mathbb{C}, \mathbb{C})$ with $||\mu||_{L^\infty} \leq k$ if and only if $k < 1/p^*$.

In [IM 1,2,3], Iwaniec and Martin introduce operators S_n which operate on functions $f : \mathbb{R}^n \to \Lambda$, where Λ is the usual Grassmann algebra of $\mathbb{R}^n$. The operator S_2 can be identified with S. Iwaniec and Martin conjecture that in all dimensions, one still has $||S||_p = p^* - 1$. They point out that such a result would have strong consequences for the regularity theory of quasiregular maps in $\mathbb{R}^n$. For subsequent work on S_n, see [BL]. The survey [I4] discusses sundry related subjects in n dimensions.

4. **Differential subordination.** The Bañuelos-Wang work continues a line of sharp constant investigation initiated by Burkholder in the late 1970's. The survey [Bu5] contains a good bibliography for this rich and varied body of work. Here we'll confine discussion to the parts most pertinent to Conjectures 1 and 2.

Following [Bu5, p.16], let $\{f_n\}$ and $\{g_n\}$, $n \geq 0$, be Hilbert space valued martingales with respect to the same filtration on some probability space $(\Omega, \mathcal{F}, P)$. Denote the corresponding difference sequences by $\{d_n\}$ and $\{e_n\}$, so that

$$f_n = \sum_{k=0}^{n} d_k, \quad g_n = \sum_{k=0}^{n} e_k.$$

For $p \geq 1$, write $||f_n||_p$ for the L^p norm of f_n with respect to P. Then $||f||_p \equiv \lim_{n\to\infty} ||f_n||_p$ exists.

Burkholder's Theorem. *Suppose that, for all $k \geq 0$ and $P - a.e$ $\omega \in \Omega$,*

$$|e_k(\omega)| \leq |d_k(\omega)|. \tag{4.1}$$

Then

$$||g||_p \leq (p^* - 1)||f||_p, \tag{4.2}$$

and the constant $p^* - 1$ is best possible. If $0 < ||f||_p < \infty$, then equality occurs in (4.2) if and only if $p = 2$ and equality holds a.e in (4.1) for all $k \geq 0$.

Burkholder's proof of (4.2) is similar in spirit to the proofs of Pichorides's theorem: He shows that, with Φ_p the function we defined in section 1, the sequence of expectations $E\Phi_p(f_n, g_n)$ is nondecreasing for $n \geq 0$, with $E\Phi_p(f_0, g_0) \geq 0$. It follows that the analogue of Conjecture 2 holds in Burkholder's setting, and hence so does the analogue (4.2) of Conjecture 3.

By 1984, Burkholder [Bu1] had proved that (4.1) implies (4.2) whenever the martingales are real valued. Among other features of the proof in [Bu1] is a reduction to the case when $e_k = \epsilon_k d_k$, where the ϵ_k are constants, each of which is 1 or -1. Then f and g are somewhat like conjugate harmonic functions, and the transform $f \to g$ can be viewed as an analogue of the Hilbert transform. Extension of (4.1) $\implies$ (4.2) to the Hilbert space valued case followed in 1988.

When martingales f and g as above satisfy (4.1), Burkholder says that g is differentially subordinate to f. He introduced also, in [Bu4], the notion of differentially subordinate harmonic functions. If u and v are Hilbert space valued harmonic functions on a domain $D \subset \mathbb{R}^n$, then v is said to be differentially subordinate to u if $|\nabla v(x)| \leq |\nabla u(x)|$ at each $x \in D$. For example, when $n = 2$ each member of a pair of conjugate harmonic functions is differentially subordinate to the other. It turns out that $\Phi_p(u, v)$ is subharmonic in D when v is differentially subordinate to u. Let $x_0 \in D$, and let μ denote the harmonic measure of D at x_0. If we assume also that $|v(x_0)| \leq |u(x_0)|$, then the subharmonicity leads to the inequality

$$||v||_p \leq (p^* - 1)||u||_p, \quad 1 < p < \infty, \tag{4.3}$$

where the L^p norm is taken with respect to μ.

It is not known if p^*-1 is best possible in (4.3). Pichorides's theorem implies that the best constant c_p must satisfy $c_p \geq \cot \frac{\pi}{2p^*}$. Related papers about differential subordination include [Bu6], [C1], and [C2].

The function L apparently first appears in [Bu4]. A related function appears in [Bu2, (8)]. In [Bu4], Burkholder proves integral inequalities analogous to our Conjecture 1 for differentially subordinate martingales f, g and differentially subordinate harmonic functions u, v with $|v(x_0)| \leq |u(x_0)|$. These inequalities, valid in the Hilbert space valued case, imply, for $n \geq 0$, the weak-type inequalities

$$P(|f_n| + |g_n| \geq 1) \leq 2||f||_1, \quad \mu(|u| + |v| \geq 1) \leq 2||u||_1. \tag{4.4}$$

The constant 2 in (4.4) is best possible even when $|f_n| + |g_n|$ is replaced by $|g_n|$ and $|u| + |v|$ by $|v|$. See [Bu4, p.11] and [Bu6, Remark 13.1].

Burkholder proved versions of his discrete parameter martingale results for some continuous parameter martingales. Bañuelos and Wang [BW1,2] and Wang [W] extended the theory to cover a wider class of continuous parameter martingales. Theorem 1 of [BW1], about real-valued differentially subordinate martingales, when combined with a probabilistic representation of the Riesz transforms due to Gundy and Varopoulos [GV], leads to the new proof of the upper bound in the Iwaniec-Martin theorem and the proof of (3.1) with H replaced by R_j mentioned at the end of the section 3. Theorem 2 of [BW1] leads to the following L^p estimates for the Beurling- Ahlfors transform S.

Bañuelos-Wang Theorem.

$$||S||_p \leq 4(p^* - 1), \quad 1 < p < \infty. \tag{4.5}$$

The constant 4 in (4.4) is the smallest known at present which works for all p. Recall that $||S||_2 = 1$, and that Conjecture 3 may be stated as $||S||_p = p^* - 1$.

Here are some ideas from the proof of (4.5). Take a rapidly decreasing smooth function $f : \mathbb{C} \to \mathbb{C}$. Extend f to a harmonic function in the half space $\mathbb{R}^3_+$, denoted also by f. Then, using Itô's formula,

$$f(B_0) = \int_{-\infty}^{0} \nabla f(B_s) \cdot dB_s,$$

where B_s is the $\mathbb{R}^3_+$ - valued Gundy-Varopoulos "background radiation process", and $\nabla f \cdot dB_s$ is the complex number obtained by splitting ∇f into real and imaginary parts, then taking dot products in $\mathbb{R}^3$. For suitable functions A whose values are complex $3 \times 3-$ matrices, define random variables $A * f$ by

$$A * f = \int_{-\infty}^{0} (A(B_s)\nabla f(B_s)) \cdot dB_s.$$

When the limit 0 in the integrals is replaced by $t \in (-\infty, 0)$, one obtains complex valued martingales to which the extended Burkholder theory is applicable. Let $||A|| = \sup\{|A(z)v| : z \in \mathbb{R}^3_+,\ v \in \mathbb{C}^3,\ |v| \le 1\}$, where $|.|$ denotes the Euclidean norm in $\mathbb{C}^3$. From Theorem 2 of [BW1] follows

$$||A * f||_p \le ||A||(p^* - 1)||f(B_0)||_p. \tag{4.6}$$

Now S can be expressed in terms of Riesz transforms: $S = {R_2}^2 - {R_1}^2 + 2iR_1R_2$. If A is taken to be the constant matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2i \\ 0 & 2i & -2 \end{pmatrix},$$

it turns out that

$$Sf(z) = E(A * f \mid B_0 = z), \quad z \in \mathbb{C}. \tag{4.7}$$

The conditional expectation operator in (4.7) is a contraction on L^p when $p \ge 1$, and the distribution of B_0 on $\mathbb{C}$ is Lebesgue measure. These facts together with (4.6) yield

$$||Sf||_p \le ||A * f||_p \le ||A||\ (p^* - 1)||f(B_0)||_p = (p^* - 1)||A||\ ||f||_p. \tag{4.8}$$

Calculation gives $||A|| = 4$. So (4.5) follows from (4.8).

How is the pair ∂f, $\overline{\partial} f$ like a pair of differentially subordinate martingales or harmonic functions? That, it seems, is what we really need to know to get the full conjectured result $||Sf||_p \le (p^* - 1)||f||_p$ by the route of this section.

5. **Quasiconvex and rank one convex functions.** Let $\mathbb{R}^{nm}$ denote the set of all $m \times n$ matrices with real coefficients. A function $\Psi : \mathbb{R}^{nm} \to \mathbb{R}$ is said to be rank one convex on $\mathbb{R}^{nm}$ if for each $A,\ B \in \mathbb{R}^{nm}$ with rank $B = 1$ the function

$$h(t) \equiv \Psi(A + tB), \quad t \in \mathbb{R},$$

is convex. Ψ is said to be quasiconvex on $\mathbb{R}^{nm}$ if it is locally integrable and for each $A \in \mathbb{R}^{nm}$, each bounded domain $D \subset \mathbb{R}^n$ and each compactly supported Lipschitz function $f : D \to \mathbb{R}^m$ holds

$$\frac{1}{|D|} \int_D \Psi(A + \nabla f) \geq \Psi(A).$$

If $n = 1$ or $m = 1$ then Ψ is quasiconvex or rank one convex if and only if it is convex. If $m \geq 2$ and $n \geq 2$, then convexity $\implies$ quasiconvexity $\implies$ rank one convexity. See [D1], where one finds also a discussion of polyconvexity, a property which lies in between convexity and quasiconvexity. Additional relevant works include [DDGR], [AD], [D2], [Sv1], [Sv2], and [Sv3].

Morrey [M, p.26] conjectured in 1952 that rank one convexity does not imply quasiconvexity when m and n are both ≥ 2. Šverák [Sv2], in 1992, proved that Morrey's conjecture is correct if $m \geq 3$ and $n \geq 2$. The cases $m = 2$, $n \geq 2$ remain open.

Define $\alpha : \mathbb{R}^{2,2} \to \mathbb{C}^2$ by $\alpha(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = (z_1, z_2)$, where

$$z_1 = \frac{1}{2}((a + d) + i(c - b)), \quad z_2 = \frac{1}{2}((a - d) + i(c + b)).$$

For $f : \mathbb{C} \to \mathbb{C}$, represent ∇f as a real 2×2 matrix in the usual way: $\nabla f = \left[\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right]$, where u and v are the real and imaginary parts of f. Then $\alpha(\nabla f) = (\partial f, \overline{\partial} f)$.

Recall that the Burkholder-Šverák function $L : \mathbb{C}^2 \to \mathbb{R}$ is defined by

$$\begin{aligned} L(z_1, z_2) &= |z_1|^2 - |z_2|^2, \quad \text{if} \quad |z_1| + |z_2| \leq 1, \\ &= 2|z_1| - 1, \quad \text{if} \quad |z_1| + |z_2| \geq 1. \end{aligned}$$

Define $L_1 = L \circ \alpha$. Then, for $D \subset \mathbb{C}$,

$$\int_D L(\partial f, \overline{\partial} f) = \int_D L_1(\nabla f).$$

Thus, Conjecture 1 may be restated as: L_1 is quasiconvex at 0.

In [Sv1], Šverák introduced a class of functions containing L_1 whose members he proved to be rank one convex, and noted that he was unable to determine if these functions are quasiconvex. We supply below a simple proof that L_1 is rank one convex.

For $A, B \in \mathbb{R}^{2,2}$, write $\alpha(A) = (z_1, z_2)$, $\alpha(B) = (w_1, w_2)$. If $\operatorname{rank} B \le 1$, then $|w_1| = |w_2|$. Let $a = |z_1|^2 - |z_2|^2$, $b = 2\mathrm{Re}\,(z_1\overline{w_1} - z_2\overline{w_2})$, and $I = \{t \in \mathbb{R} : |z_1 + tw_1| + |z_2 + tw_2| < 1\}$. Then, for $g(t) = L_1(A + tB)$, we have, when $\operatorname{rank} B \le 1$,

$$\begin{aligned} g(t) &= a + bt, \quad t \in I, \\ &= 2|z_1 + tw_1| - 1, \quad t \in \mathbb{R} \setminus I. \end{aligned}$$

Now g is continuous, I is either empty or a bounded interval, and $t \to |z_1 + tw_1|$ is convex. It follows that g is convex on $\mathbb{R}$. Hence, L_1 is rank one convex. Thus, if Conjecture 1 is false, then Morrey's conjecture for $m = n = 2$ will be confirmed.

It can also be shown that L_1 is not polyconvex. One way to do this is to show that L_1 does not satisfy condition (6) on [D1, p.107] when $A = 0$.

For $A \in \mathbb{R}^{2,2}$, let $|A|^2 = a^2 + b^2 + c^2 + d^2$. Let

$$E = \{A \in \mathbb{R}^{2,2} : (|A|^2 + 2\det \mathrm{A})^{1/2} + (|A|^2 - 2\det \mathrm{A})^{1/2} \le 2\}.$$

Then

$$\begin{aligned} L_1(A) &= \det \mathrm{A}, \quad A \in E, \\ &= (|A|^2 + 2\det \mathrm{A})^{1/2} - 1, \quad A \in \mathbb{R}^{2,2} \setminus E. \end{aligned}$$

Some rank one convex functions which look something like L_1 are studied in [DDGR] and [Sv3].

The connection between Morrey's conjecture and the Beurling-Ahlfors transform is discussed also in [As2] and [BL].

6. **Stretch Functions.** There is a large class of functions for which equality holds in Conjectures 1 and 2. Write $z = re^{i\theta}$. Functions $f : \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = g(r)e^{i\theta},$$

where g is a nonnegative locally Lipschitz function on $(0, \infty)$ with

$$g(0) \equiv g(0+) = 0, \quad \text{and} \quad \lim_{r \to \infty} g(r) = 0,$$

will be called stretch functions. Let $\mathcal{S}$ denote the set of all stretch functions. For $f \in \mathcal{S}$, we have

$$\partial f = \frac{1}{2}(g' + r^{-1}g), \quad \overline{\partial} f = \frac{1}{2}e^{2i\theta}(g' - r^{-1}g), \quad |\partial f| + |\overline{\partial} f| = \max(r^{-1}g, |g'|). \tag{6.1}$$

Let $\mathcal{S}_1$ denote the subclass of $f \in \mathcal{S}$ such that, for a.e. $r \in [0, \infty)$, holds

$$|g'(r)| \leq r^{-1}g(r). \tag{6.2}$$

For example, for each $\alpha \in (0, 1]$, $\beta \in (0, 1]$, and positive constant c, the functions

$$\begin{aligned} f(z) &= cr^{\alpha}e^{i\theta}, \quad |z| \leq 1, \\ &= cr^{-\beta}e^{i\theta}, \quad |z| \geq 1, \end{aligned} \tag{6.3}$$

belong to $\mathcal{S}_1$.

Theorem 1. *If $f \in \mathcal{S}_1 \cap \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$, then*

$$\int_{\mathbb{C}} L(\partial f, \overline{\partial} f) = 0. \tag{6.4}$$

If $f \in \mathcal{S}_1 \cap \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, then

$$\int_{\mathbb{C}} \Phi_p(\partial f, \overline{\partial} f) = 0, \quad 1 < p \leq 2, \quad \int_{\mathbb{C}} \Phi_p(\overline{\partial} f, \partial f) = 0, \quad 2 \leq p < \infty. \tag{6.5}$$

Proof. From (1.1a), (1.1b) and an approximation argument, it follows that we need only prove (6.4). Let $h(r) = g(r)/r$. Then h is continuous and nonincreasing on $(0, \infty)$, with $\lim_{r \to \infty} h(r) = 0$. From the last equation in (6.1) follows $|\partial f(z)| + |\overline{\partial} f(z)| = h(r)$.

Let $E = \{r \in (0, \infty) : h(r) > 1\}$. Then

$$\begin{aligned} L(\partial f(z), \overline{\partial} f(z)) &= r^{-1}g(r) + g'(r) - 1, \quad r \in E, \\ &= r^{-1}g(r)g'(r), \quad r \notin E. \end{aligned} \tag{6.6}$$

Thus,

$$\begin{aligned} rL(\partial f(z), \overline{\partial} f(z)) &= \frac{d}{dr}(rg - \frac{1}{2}r^2), \quad r \in E, \\ &= \frac{1}{2}\frac{d}{dr}g^2, \quad r \notin E. \end{aligned} \tag{6.7}$$

Now E is either empty, or is a single interval $(0, R]$, with $0 < R < \infty$. Moreover, $g(0) = 0$ and $\lim_{r\to\infty} g(r) = 0$. If E is nonempty, then from (6.7) follows

$$\frac{1}{2\pi}\int_{\mathbb{C}} L(\partial f, \overline{\partial} f) = (Rg(R) - \frac{1}{2}R^2) - \frac{1}{2}g^2(R) = -\frac{1}{2}(g(R) - R)^2. \tag{6.8}$$

The definition of R implies that $g(R) = R$. Hence (6.4) is true when E is nonempty. If E is empty, then it follows again from (6.7) that the integral on the left hand side of (6.8) equals 0. Theorem 1 is proved.

Theorem 2. *If* $f \in \mathcal{S} \cap \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$, *then*

$$\int_{\mathbb{C}} L(\partial f, \overline{\partial} f) \geq 0. \tag{6.9}$$

If $f \in \mathcal{S} \cap \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$, $1 < p < \infty$, *then*

$$\int_{\mathbb{C}} \Phi_p(\partial f, \overline{\partial} f) \geq 0. \tag{6.10}$$

Thus, Conjectures 1 and 2 are true for stretch functions. According to Theorem 1, the equality sign holds in (6.9) for the stretch functions which also satisfy (6.2). When $1 < p \leq 2$, the equality sign holds in (6.10) for stretch functions which satisfy (6.2), while for $2 \leq p < \infty$ equality holds for their complex conjugates.

As was pointed out to us by Iwaniec, by no means do all extremals for Conjectures 1 and 2 belong to $\mathcal{S}_1$. For example, start with the unit disk B in the plane. For $j = 1, 2, ...$, let $B_j = \{z : |z - a_j| < r_j\}$ be disjoint sub-disks of B, and let $\{f_j\}$ be a sequence in $\mathcal{S}_1$, with $f_j(z) = z$ on $|z| = 1$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} f(z) &= a_j + r_j f_j(\frac{z - a_j}{r_j}), \quad \text{if} \quad z \in B_j, \\ &= z, \quad \text{if} \quad z \in B \setminus \cup_{j=1}^{\infty} B_j, \\ &= 1/\overline{z}, \quad \text{if} \quad z \in \mathbb{C} \setminus B. \end{aligned}$$

Then, for each choice of $\{f_j\}$, equality holds for f in Conjecture 1, and for f in Conjecture 2 when $1 < p \le 2$. Equality holds for $\overline{f}$ in Conjecture 2 when $2 \le p < \infty$.

Internal evidence, together with Burkholder's martingale results, suggests that when $p \ne 2$ there are no nontrivial functions for which equality is achieved in Conjecture 3.

Proof of Theorem 2. As with Theorem 1, it suffices to prove (6.9). In our proof of (6.9) we shall assume that g is continuously differentiable on $[0, \infty)$. The case when g is locally Lipschitz then follows by an approximation argument.

Let $E = \{r \in (0, \infty) : |\partial f(z)| + |\overline{\partial} f(z)| > 1\}$. Then, from (6.1),

$$\begin{aligned} L(\partial f(z), \overline{\partial} f(z)) &= |r^{-1} g(r) + g'(r)| - 1, \quad r \in E, \\ &= r^{-1} g(r) g'(r), \quad r \notin E. \end{aligned} \tag{6.11}$$

For $r \in [0, \infty)$, define $F(r) = r^{-1} g + g' - 1$. If $r \in E$, then

$$F(r) \le L(\partial f(z), \overline{\partial} f(z)), \tag{6.12}$$

by (6.11). If $r \notin E$, then the third equation in (6.1) implies that $g(r) \le r$ and $|g'(r)| \le 1$. Hence, for $r \notin E$,

$$F(r) - L(\partial f(z), \overline{\partial} f(z)) = r^{-1} g + g' - 1 - r^{-1} g g' = (1 - r^{-1} g)(g' - 1) \le 0.$$

Thus, (6.12) holds for all $r \in [0, \infty)$.

If the set $\{r \in [0, \infty) : g(r) \ge r\}$, is nonempty, let R denote its supremum. If the set is empty, define $R = 0$. Then $0 \le R < \infty$, since $g = o(1)$ at ∞, and $g(R) = R$. Since (6.12) holds for $r \in [0, \infty)$, we have

$$\frac{1}{2\pi} \int_{|z|<R} L(\partial f, \overline{\partial} f) \ge \int_0^R r F(r)\, dr = (R g(R) - \frac{1}{2} R^2) = \frac{1}{2} g^2(R). \tag{6.13}$$

Let $G(r) = L(\partial f(z), \overline{\partial} f(z))$. Then $rG = gg'$ on $[0, \infty) \setminus E$, by (6.11), and

$$\frac{1}{2\pi} \int_{|z|>R} L(\partial f, \overline{\partial} f) = \int_R^\infty g g'\, dr + \int_{E \cap (R,\infty)} (rG - gg')\, dr. \tag{6.14}$$

From (6.13) and (6.14), it follows that

$$\frac{1}{2\pi}\int_{\mathbb{C}} L(\partial f, \overline{\partial} f) \geq \int_{E\cap(R,\infty)} (rG - gg')\,dr. \tag{6.15}$$

Since $g(r) < r$ for $r > R$, it follows from (6.1) that $E \cap (R,\infty) = \{r \in (R,\infty) : |g'(r)| > 1\}$. If $E \cap (R,\infty)$ is empty, then (6.9) follows from (6.15). Assume $E \cap (R,\infty)$ is nonempty. Then it is a finite or countable union of open intervals $(r_1, r_2) \subset (R,\infty)$, on each of which either g' is everywhere > 1 or g' is everywhere < -1. The hypothesis $f \in \dot{W}^{1,2}$ insures that all endpoints r_2 are finite. To prove (6.9), it suffices, in view of (6.15), to prove that, for each such (r_1, r_2),

$$\int_{r_1}^{r_2} (rG - gg')\,dr \geq 0. \tag{6.16}$$

Suppose that $g' > 1$ on (r_1, r_2). Then, on (r_1, r_2),

$$rG - gg' = g + rg' - r - gg' = -\frac{1}{2}\frac{d}{dr}(r-g)^2.$$

Hence,

$$\int_{r_1}^{r_2} (rG - gg')\,dr = \frac{1}{2}[(r_1 - g(r_1))^2 - (r_2 - g(r_2))^2]. \tag{6.17}$$

But

$$g' > 1 \implies g(r_2) - g(r_1) > r_2 - r_1 \implies r_1 - g(r_1) > r_2 - g(r_2) > 0.$$

Thus, the integral in (6.17) is > 0.

Suppose that $g' < -1$ on (r_1, r_2). Then, on (r_1, r_2),

$$rG - gg' = -g - rg' - r - gg' = -\frac{1}{2}\frac{d}{dr}(r+g)^2.$$

Hence,

$$\int_{r_1}^{r_2} (rG - gg')\,dr = \frac{1}{2}[(r_1 + g(r_1))^2 - (r_2 + g(r_2))^2]. \tag{6.18}$$

But

$$g' < -1 \implies g(r_2) - g(r_1) < -r_2 + r_1 \implies r_1 + g(r_1) > r_2 + g(r_2) > 0.$$

Thus, the integral in (6.18) is > 0. The proof of (6.9) is complete.

7. **Some other partial results.** There are a few other classes of functions, in addition to the stretch functions, for which we can confirm Conjectures 1 and or 2.

Theorem 3. *For $a, b \in \mathbb{C},\ k = 1, 2, 3, \ldots,$ Conjecture 1 is true for*

$$\begin{aligned} f(z) &= az^k + b\overline{z}^k, \quad |z| \le 1, \\ &= a\overline{z}^{-k} + bz^{-k}, \quad |z| \ge 1. \end{aligned}$$

Theorem 4. *For $1 < p < \infty$, Conjecture 2 is true for $f \in \dot{W}^{1,p}(\mathbb{C}, \mathbb{C})$ provided*

$$f \quad \text{is harmonic in} \quad \mathbb{C} \cup \{\infty\} \setminus \{|z| = 1\}, \tag{7.1}$$

or

$$f = F \circ f_1 \quad \text{or} \quad f = \overline{F} \circ f_1, \tag{7.2}$$

where $f_1 \in \mathcal{S}$ and F is holomorphic on $f_1(\mathbb{C})$,

Recall that $\mathcal{S}$ denotes the class of stretch functions defined in Section 6. The function Φ_p is homogeneous but the function L is not; that is the main reason we can verify Conjecture 2 for more functions than we can for Conjecture 1.

Theorem 3 can be proved by direct computation. The proof of Theorem 4 requires computation plus the fact that $p'th$ means of subharmonic functions on circles increase as the radius increases. We'll confine ourselves to sketching the proof of (7.1) when $p > 2$.

Suppose that $f \in \dot{W}^{1,p}(\mathbb{C})$ is harmonic in $|z| < 1$ and in $1 < |z| \le \infty$. Then there exist holomorphic functions g and h in $|z| < 1$ such that

$$\begin{aligned} f(z) &= g(z) + \overline{h}(z), \quad |z| < 1, \\ &= f(1/\overline{z}), \quad |z| > 1. \end{aligned}$$

Let $p > 2$. Then computation gives

$$\begin{aligned} \frac{1}{\alpha_p} \int_{\mathbb{C}} \Phi_p(\partial f, \overline{\partial} f) &= \int_{|z|<1} ((p-1)|g'(z)| - |h'(z)|)(|g'(z)| + |h'(z)|)^{p-1}\, dx\, dy \\ &+ \int_{|z|>1} ((p-1)|h'(1/\overline{z})| - |g'(1/\overline{z})|)(|h'(1/\overline{z})| + |g'(1/\overline{z})|)^{p-1} |z|^{-2p}\, dx\, dy \\ &= 2\pi \int_0^1 ((p-1) - r^{2p-4}) I_1(r)\, r\, dr \ + \ 2\pi \int_0^1 ((p-1) r^{2p-4} - 1) I_2(r)\, r\, dr, \end{aligned}$$

where $I_1(r)$, $I_2(r)$ are the respective mean values on the circle $|z| = r$ of the functions $|g'|(|g'| + |h'|)^{p-1}$ and $|h'|(|g'| + |h'|)^{p-1}$. The logarithms of these functions are subharmonic, hence so are the functions themselves. Thus, I_1 and I_2 are nondecreasing functions of r on $[0,1]$. From $I_2 \nearrow$, one easily shows that the integral containing I_2 is nonnegative. The integral containing I_1 is clearly nonnegative, because its integrand is. Hence, $\int_{\mathbb{C}} \Phi_p(\partial f, \overline{\partial} f) \geq 0$.

8. **Numerical Evidence.** In this section, we present numerical evidence in favor of Conjecture 1. Let $\mathbb{T}$ be the space $[0,1]$ with 0 and 1 identified. Then $W^{1,2}(\mathbb{T}^2, \mathbb{C})$ will denote the Sobolev space of complex valued functions $f : [0,1]^2 \to \mathbb{C}$ such that $f(0,y) \equiv f(1,y)$, $f(x,0) \equiv f(x,1)$, and both f and its distributional derivatives are in L_2. We will work with the following conjecture, which is equivalent to Conjecture 1.

Conjecture 4. *Let $f \in W^{1,2}(\mathbb{T}^2, \mathbb{C})$. Then*

$$\int_{\mathbb{T}^2} L(\partial f, \overline{\partial} f) \geq 0.$$

The approach is to consider piecewise linear functions described as follows. Let N be a natural number. Let p_n be the fractional part of n/N (so that $p_{N+n} = p_n$). Split $\mathbb{T}^2$ into triangles $\Delta^+_{m,n}$ with corners (p_m, p_n), (p_{m+1}, p_n), (p_m, p_{n+1}), and triangles $\Delta^-_{m,n}$ with corners (p_m, p_n), (p_{m-1}, p_n), (p_m, p_{n-1}).

We will say that $u : \mathbb{T}^2 \to \mathbb{C}$ is an element of $\mathcal{P}_N$ if u is continuous, and linear on each of the triangles $\Delta^+_{m,n}$ and $\Delta^-_{m,n}$. In this way, once one knows that u is an element of $\mathcal{P}_N$, then u is totally determined by its values at $(p_m, p_n)_{0 \leq m,n \leq N-1}$. Thus $\mathcal{P}_N$ is a $2N^2$ real dimensional space. Let $\iota : \mathbb{R}^{2N^2} \to \mathcal{P}_N$ denote an isomorphism. Our goal is to check whether the function $F_N : \mathbb{R}^{2N^2} \to \mathbb{R}$ always takes positive values, where

$$F_N(x) = \int_{\mathbb{T}^2} L(\partial(\iota x), \overline{\partial}(\iota x)).$$

In fact, by an approximation argument, Conjecture 4 is equivalent to showing that $F_N(x) \geq 0$ for all $x \in \mathbb{R}^{2N^2}$ and all $N \geq 1$.

We obtained much numerical evidence to support this conjecture. The algorithm was to choose a vector $x \in \mathbb{R}^{2N^2}$ at random, then minimize F_N, with x as starting point, using the conjugate gradient method described in Chapter 10.6 in [PTVF]. This was done for various values of N, ranging from 6 to 100. In every case, it was found, up to machine precision, that F_N always takes non-negative values. The results were verified independently using Maple.

To implement this algorithm, it was necessary to compute the gradient ∇F_N. Because of the special nature of this function, the computations needed to do this were not much more arduous than the computations required for F_N. The formulae required to find ∇F_N were determined using Maple.

Other interesting facts emerged. For a given $x \in \mathbb{R}^{2N^2}$, we may consider the function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(t) = F_N(tx).$$

It was found that this function is always increasing for $t \geq 0$, and always decreasing for $t \leq 0$. However, it was also found that the function h is *not* necessarily convex.

This last fact is interesting, because if in Conjecture 4 the function L were to be replaced by a convex function, then h would be convex.

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