ON THE ENERGY OF ROTATING GRAVITATIONAL WAVES*

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ABSTRACT

A class of solutions of the gravitational field equations describing vacuum spacetimes outside rotating cylindrical sources is presented. A subclass of these solutions corresponds to the exterior gravitational fields of rotating cylindrical systems that emit gravitational radiation. The properties of these rotating gravitational wave spacetimes are investigated. In particular, we discuss the energy density of these waves using the gravitational stress-energy tensor.

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1. INTRODUCTION

The main purpose of this paper is to present a class of cylindrically symmetric vacuum solutions of the gravitational field equations representing rotating gravitational waves and to study some of the physical properties of such waves. The spacetimes under consideration here are not asymptotically flat in general; therefore, the concepts of energy, momentum, and stress do not make sense in the standard interpretation of general relativity. Nevertheless, it is possible to introduce a local gravitoelectromagnetic stress-energy tensor via a certain averaging procedure [1]. This gravitational stress-energy tensor provides a natural physical interpretation of the Bel-Debever-Robinson tensor that has been used frequently in numerical relativity [2]. In this work, we study the local gravitational stress-energy tensor for free rotating gravitational waves.

Rotating cylindrically-symmetric gravitational waves were first discussed in 1990 [3, 4]. The investigation of these solutions was motivated by Ardavan's discovery of the speed-of-light catastrophe [5] and its implications concerning gravitation [6]. Previous work is generalized in the present paper and an extended class of rotating gravitational wave spacetimes is analyzed. We find that the solution investigated previously [1, 2] is special, since it is the only member of the extended class studied here that represents the propagation of *free* rotating gravitational waves.

In this paper, we consider Ricci-flat spacetimes that are characterized by a metric of the form

$$-ds^2 = e^{2\gamma-2\psi}(-dt^2 + d\rho^2) + \mu^2 e^{-2\psi}(\omega dt + d\phi)^2 + e^{2\psi}dz^2, \tag{1}$$

in cylindrical coordinates $(\rho,\ \phi,\ z)$. Here γ, μ, ψ and ω are functions of t and ρ only; moreover, the speed of light in vacuum is set equal to unity except where indicated otherwise. The spacetime represented by equation (1) admits two commuting spacelike Killing vectors ∂_z and ∂_ϕ. Though ∂_z is hypersurface-orthogonal, ∂_ϕ is not and this fact implies that the isometry group is not orthogonally transitive. Instead of the variables t and ρ, it is convenient to express the gravitational field equations in terms of retarded and advanced times $u = t - \rho$ and $v = t + \rho$, respectively. The field equations then take the form

$$(\mu\psi_v)_u + (\mu\psi_u)_v = 0, \tag{2}$$

$$\mu_{uv} - \frac{l^2}{8}\mu^{-3}e^{2\gamma} = 0, \tag{3}$$

$$\omega_v - \omega_u = l\mu^{-3}e^{2\gamma}, \tag{4}$$

$$\gamma_u = \frac{1}{2\mu_u}(\mu_{uu} + 2\mu\psi_u^2), \tag{5}$$

$$\gamma_v = \frac{1}{2\mu_v}(\mu_{vv} + 2\mu\psi_v^2), \tag{6}$$

where $\psi_u = \partial\psi/\partial u$, etc. Here l is a constant length characteristic of the rotation of the system. Using equation (3), it is possible to eliminate γ from equations (5) and (6); then, we obtain the following equations

$$2\mu^2\psi_u^2 = 3\mu_u^2 - \mu\mu_{uu} + \mu\mu_u\mu_{uuv}(\mu_{uv})^{-1}, \tag{7}$$

$$2\mu^2\psi_v^2 = 3\mu_v^2 - \mu\mu_{vv} + \mu\mu_v\mu_{uvv}(\mu_{uv})^{-1}. \tag{8}$$

The integrability condition for this system — i.e. $\psi_{uv} = \psi_{vu}$ — results in a nonlinear fourth order partial differential equation for μ. Alternatively, one could obtain the same equation for μ by combining equations (2), (7), and (8), which shows the consistency of the field equations (2) – (6). It is important to notice that μ cannot be a function of u or v alone, since this possibility would be inconsistent with equation (3). The partial differential equation of fourth order for μ is, however, identically satisfied if μ is a separable function, i.e. $\mu = \alpha(u)\beta(v)$; this leads, in fact, to the rotating waves discussed earlier [3, 4]. Here we wish to study the *general* solution of the field equations.

In section 2, we discuss the general solution of the field equations. We find that there are two possible classes of solutions: The first class corresponds to the stationary exterior field of a rotating cylindrical source, while the second class appears to represent a mixed situation involving rotating gravitational waves. In fact, a subclass of the latter solutions describes the exterior fields of certain rotating sources that emit gravitational radiation; these solutions approach the special rotating wave solution [3, 4] far from their sources. Indeed, the only solution of the second class corresponding to a pure gravitational wave spacetime is the special solution [3, 4] that is further discussed in section 3. In section 4, some aspects of the energy and momentum of the special rotating gravitational waves are discussed using the gravitoelectromagnetic stress-energy tensor developed in a recent work [1]. The appendices contain some of the detailed calculations.

2. SOLUTION OF THE FIELD EQUATIONS

To solve the field equations (2) – (6), let us introduce the functions U, V, and W by

$$U = \mu_u \gamma_u - \frac{1}{2}\mu_{uu}, \tag{9}$$

$$V = \mu_v \gamma_v - \frac{1}{2}\mu_{vv}, \tag{10}$$

$$W = \mu \gamma_{uv} + \frac{3}{2}\mu_{uv}, \tag{11}$$

and rewrite equations (5) and (6) as

$$\mu\psi_u = \epsilon \ (\mu U)^{1/2}, \qquad \mu\psi_v = \hat{\epsilon} \ (\mu V)^{1/2}, \tag{12}$$

where the symbols ϵ and $\hat{\epsilon}$ represent either $+1$ or -1 (i.e., $\epsilon^2 = \hat{\epsilon}^2 = 1$). Using equation (3), it is straightforward to show that

$$\mu U_v = \mu_u W, \qquad \mu V_u = \mu_v W. \tag{13}$$

Let us now combine relations (12) and (13) in order to satisfy equation (2); the result is

$$\epsilon\, U^{-1/2}(\mu_v U + \mu_u W) + \hat{\epsilon}\; V^{-1/2}(\mu_u V + \mu_v W) = 0. \tag{14}$$

This equation can be written as

$$U(\mu_u V + \mu_v W)^2 = V(\mu_v U + \mu_u W)^2, \tag{15}$$

which holds if either $W^2 = UV$ or $\mu_u^2\; V = \mu_v^2\; U$; that is, equation (15) can be factorized into a fourth order equation and a third order equation, respectively.

Let us first consider the fourth order equation $W^2 = UV$. It follows from this relation and equation (12) that $W = \pm\mu\psi_u\psi_v$. This expression for W along with $U = \mu\psi_u^2$ and $V = \mu\psi_v^2$ can be inserted into equation (13) to obtain relations that when combined with equation (2) in the form

$$2\mu\psi_{uv} = -(\mu_u\psi_v + \mu_v\psi_u) \tag{16}$$

imply

$$\psi_u\psi_v = 0. \tag{17}$$

If ψ is a function of either u or v alone, then equation (2) implies that ψ must be a constant. The spacetime represented by a constant ψ turns out to be a common special case of the solutions discussed in this section and is *flat*.

Let us next consider $\mu_u^2\; V = \mu_v^2\; U$. It follows from equation (3) that

$$\gamma_u = \frac{3}{2}\frac{\mu_u}{\mu} + \frac{1}{2}\frac{\mu_{uvu}}{\mu_{uv}}, \tag{18}$$

$$\gamma_v = \frac{3}{2}\frac{\mu_v}{\mu} + \frac{1}{2}\frac{\mu_{uvv}}{\mu_{uv}}, \tag{19}$$

which together with equations (9) and (10) imply that $\mu_u^2\; V = \mu_v^2\; U$ is essentially equivalent to

$$\frac{\mu_{uvv}}{\mu_v} - \frac{\mu_{vv}\mu_{uv}}{\mu_v^2} = \frac{\mu_{uvu}}{\mu_u} - \frac{\mu_{uu}\mu_{uv}}{\mu_u^2}. \tag{20}$$

Equation (20) can be written as

$$(\ln \mu_v)_{uv} = (\ln \mu_u)_{vu}. \tag{21}$$

Let $f(u)$ and $g(v)$ be arbitrary functions of their arguments and consider the transformation $(u, v) \to (x, y)$ such that

$$x = f(u) + g(v), \tag{22}$$

$$y = f(u) - g(v), \tag{23}$$

$f_u \neq 0$, and $g_v \neq 0$; then, the general solution of equation (21) is that μ should not depend on y, i.e.

$$\mu = \mu(x). \tag{24}$$

Let us now proceed to the calculation of ψ. To this end, let us consider a function $\mathcal{L}(x)$ given by

$$\mathcal{L}^2(x) = \frac{1}{2}\left(3\mu'^2 - \mu\mu'' + \mu\frac{\mu'\mu'''}{\mu''}\right), \tag{25}$$

where a prime indicates differentiation with respect to x. Equations (7) and (8) can now be written as

$$\mu^2\psi_u^2 = f_u^2\ \mathcal{L}^2, \qquad \mu^2\psi_v^2 = g_v^2\ \mathcal{L}^2. \tag{26}$$

Furthermore,

$$\psi_u = \left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\right) f_u, \tag{27}$$

$$\psi_v = \left(\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y}\right) g_v. \tag{28}$$

Combining equations (26) – (28), we find that either ψ is purely a function of x given by $\mu^2(d\psi/dx)^2 = \mathcal{L}^2(x)$ or ψ is purely a function of y given by $\mu^2(d\psi/dy)^2 = \mathcal{L}^2(x)$. These cases will now be discussed in turn.

Case (i): $\psi = \psi(x)$

It follows from equation (2) that in this case

$$\frac{d\psi}{dx} = \frac{C}{\mu(x)}, \tag{29}$$

where C is an integration constant. Thus $\mu(x)$ is determined by $\mathcal{L}^2(x) = C^2$. To solve this differential equation, let $X = \mu'$ and note that

$$\mathcal{L}^2 = \frac{1}{2} X^2 \left(\mu^2 \frac{dX}{d\mu}\right)^{-1} \frac{d}{d\mu}\left(\mu^3 \frac{dX}{d\mu}\right). \tag{30}$$

Moreover, let $S = \mu\; dX/d\mu$; then, $\mathcal{L}^2(x) = C^2$ can be written as

$$\frac{dS}{dX} = 2\left(\frac{C^2}{X^2} - 1\right), \tag{31}$$

which can be integrated to give $S = -2(D + X + C^2/X)$. Here D is an integration constant. It follows from this result that $\mu(x)$ can be found implicitly from $d\mu = X(\mu)dx$, where

$$\mu^2 = \exp\left(-\int \frac{X dX}{X^2 + DX + C^2}\right). \tag{32}$$

We will show that for $C = 0$, ψ is constant and this solution turns out to be flat. If $C \neq 0$, however, we have a new solution.

The spacetime metric in this case can be written in a form that depends only on x. That is, it is possible to show — by a transformation of the metric to normal form — that the general solution in Case (i) has an extra timelike or spacelike Killing vector field ∂_y. To this end, let us consider the coordinate transformation $(t, \rho, \phi, z) \to (T, R, \Phi, Z)$, where $f(u) = (T + R)/2$, $g(v) = (-T + R)/2$, $\phi = \Phi + H(T, R)$, and $z = Z$. In this transformation $H(T, R)$ is a solution of the partial differential equation

$$-(F + G)\frac{\partial H}{\partial T} + (F - G)\frac{\partial H}{\partial R} + \frac{2}{l}(F + G)\frac{d\mu}{dR} = 0, \tag{33}$$

where $F = f_u^{-1}$, $G = g_v^{-1}, R = x$, and $T = y$. To prove our assertion, we will show that under this coordinate transformation the spacetime metric in Case (i) takes the form

$$-ds^2 = P(R)(-dT^2 + dR^2) + \mu^2(R)e^{-2\psi(R)}[\Omega(R)dT + d\Phi]^2 + e^{2\psi(R)}dZ^2, \tag{34}$$

which is clearly invariant under a translation in T thus implying the existence of a Killing vector field ∂_T. If $P(R) > 0$, then this metric could represent the *stationary* exterior field of a rotating cylindrical configuration. It follows from a comparison of the metric forms (1) and (34) that

$$P(R) = -\frac{2}{l^2}\mu^3 \ \frac{d^2\mu}{dR^2} \ e^{-2\psi(R)}, \tag{35}$$

$$\frac{\partial H}{\partial T} = -\frac{1}{4}\omega(F - G) + \Omega(R), \tag{36}$$

$$\frac{\partial H}{\partial R} = -\frac{1}{4}\omega(F + G). \tag{37}$$

Equations (36) and (37) can be used in equation (33) to show that $\Omega(R) = 2l^{-1}d\mu/dR$. It remains to show that the integrability condition for equations (36) and (37), i.e. $\partial^2 H/\partial R\partial T = \partial^2 H/\partial T\partial R$, is satisfied. It turns out that this relation is indeed true, since it is equivalent to the field equation (4) for ω.

Finally, let us consider the case $C = 0$ or, equivalently, $\psi =$ constant. It follows from equation (31) that in Case (i)

$$\frac{d^2\mu}{dR^2} = -\frac{2}{\mu}\left[\left(\frac{d\mu}{dR}\right)^2 + D\frac{d\mu}{dR} + C^2\right], \tag{38}$$

and one can show explicitly that the metric form (34) is flat once $C = 0$. For $C \neq 0$, the general solution of Case (i) is not flat; however, its physical properties will not be further discussed in this work, which is devoted to rotating *gravitational waves.*

Case (ii): $\psi = \psi(y)$

It follows from $(d\psi/dy)^2 = \mathcal{L}^2(x)/\mu^2$ that ψ must be a linear function of y, since the left hand side of this equation is purely a function of y while the right hand side is purely a function of x; therefore, each side must be constant. Thus $\psi = ay + b$, where a and b are constants, and $\mathcal{L}^2 = a^2\mu^2$. It turns out that equation (3) is identically satisfied in this case. Using equation (30), the differential equation for μ can be expressed in terms of $X = \mu'$ as

$$X^2 \frac{d}{d\mu}\left(\mu^3 \frac{dX}{d\mu}\right) = 2a^2\mu^4 \frac{dX}{d\mu}. \tag{39}$$

If $a = 0$, ψ is constant and we recover the same flat spacetime solution as in Case (i) with $C = 0$. Therefore, let $a \neq 0$ and consider a new "radial" coordinate r given by $r = (a^2\mu^2)^{-1}$; then,

$$r^2 X^2 \frac{d^2X}{dr^2} + \frac{dX}{dr} = 0. \tag{40}$$

The solution $X =$ constant is unacceptable, since it implies that $\gamma = -\infty$. However, there is another exact solution that is given by

$$X = \pm \left(\frac{3}{2}\, r\right)^{-1/2}, \tag{41}$$

which turns out to correspond to the special rotating gravitational waves [3, 4] that are the subject of the next section. It is possible to transform equation (40) to an autonomous form; to this end, let us define A and B such that $A = 2/(3rX^2)$ and $B = -4X_r/(3X^3)$, where $X_r = dX/dr$. It can then be shown that equation (40) is equivalent to

$$\frac{dB}{dA} = \frac{3}{2}\frac{B(B - A^2)}{A(B - A)}, \tag{42}$$

where $(A, B) = (1, 1)$ represents the special solution (41). This special solution is an isolated singularity of the nonlinear autonomous system (42) and the behavior of characteristics near this point indicates that $(1, 1)$ is a saddle point.

Let us note here certain general features of $X(r)$, which is the solution of equation (40). If $X(r)$ is a solution, then so is $-X(r)$. Moreover, equation (40) can be written as $(X_r^2)_r = -2X_r^2/(r^2X^2)$, which indicates that for $X \neq$ constant the absolute magnitude of the slope of $X(r)$ monotonically decreases as r increases. If $X(r)$ has a zero at $r_0 \neq 0$, then the behavior of $X(r)$ near r_0 is given by

$$X(r) = \pm\sqrt{\frac{2}{r_0}}\left(\frac{r}{r_0}-1\right)^{1/2}\left[1+\frac{1}{2}\left(\frac{r}{r_0}-1\right)-\frac{3}{76}\left(\frac{r}{r_0}-1\right)^2+\dots\right], \tag{43}$$

for $r \geq r_0$. These results are illustrated in figure 1.

Once a solution $X(r)$ of equation (40) is given, one can find a solution of the field equations in Case (ii). It turns out that in general such a solution is not a pure gravitational wave. The only exception is the special solution (41). To demonstrate this, let us first consider a transformation of the spacetime metric (1) to the normal form appropriate for Case (ii). The first step in this transformation will be exactly the same as that given in Case (i) and equation (33), i.e. $(t, \rho, \phi, z) \to (T, R, \Phi, Z)$. Next, we change the radial coordinate to r and let $(T, R, \Phi, Z) \to (\hat{t}, r, \Phi, \hat{Z})$, where $\hat{t} = 2aT$ and $\hat{Z} = la^2 \exp(2b)Z$. In terms of the new system of coordinates, the spacetime metric may be expressed — up to an overall constant factor — as

$$-ds^2 = e^{-\hat{t}}\ \frac{X_r}{X}\ \left(-X^2 d\hat{t}^2 + \frac{dr^2}{r^3}\right) + \frac{e^{-\hat{t}}}{\hat{l}^2 r}(\hat{l}X d\hat{t} + d\Phi)^2 + e^{\hat{t}} d\hat{Z}^2, \tag{44}$$

where $\hat{l} = (la)^{-1}$ is a constant and $X(r)$ is a solution of equation (40). There are four curvature invariants for this Ricci-flat spacetime and it is possible to show that two of these are identically zero. The other two also vanish for the special solution (41); therefore, this special case corresponds to *free* gravitational waves [7]. Specifically, let

$$I_1 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - iR_{\mu\nu\rho\sigma}R^{*\mu\nu\rho\sigma}, \tag{45}$$

$$I_2 = R_{\mu\nu\rho\sigma}R^{\rho\sigma\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} + iR_{\mu\nu\rho\sigma}R^{\rho\sigma\alpha\beta}R^{*}_{\alpha\beta}{}^{\mu\nu}, \tag{46}$$

be the complex invariants of spacetime. We find that for the metric form (44) these are real and are given by

$$I_1 = -e^{2\hat{t}}(rX^4X_r^2)^{-1}(1 - 3rX^2 - 2r^2XX_r - r^3X^3X_r + r^4X^2X_r^2), \tag{47}$$

$$I_2 = -\frac{3}{4} e^{3\hat{t}} (r X^5 X_r^3)^{-1} (1 - 2rX^2 - 2r^2 X X_r + 2r^3 X^3 X_r + r^4 X^2 X_r^2). \qquad (48)$$

It is important to note that I_1 and I_2 vanish for the special solution $X = \pm(3r/2)^{-1/2}$, indicating that it describes the propagation of free gravitational waves. Moreover, all other solutions — which do not describe *free* gravitational waves — are singular for $\hat{t} \to \infty$, since I_1 and I_2 both diverge in the infinite future. Thus all such solutions evolve to states that are ultimately singular. The singular nature of the special solution (41) has been discussed previously [3, 4].

It is clear from equation (40) and figure 1 that as $r \to 0$, $X(r) \to \pm(3r/2)^{-1/2}$ for a subclass of the spacetimes under consideration here. Such solutions correspond to the exterior field of a rotating and radiating cylindrical source such that very far from the symmetry axis the metric nearly describes free rotating gravitational waves given by the unique special solution (41). Indeed, inspection of the metric form (44) reveals that the circumference of a spacelike circle about the symmetry axis and normal to it is proportional to $r^{-1/2}$ at a given time $\hat{t}$; therefore, as $r \to 0$ the metric form (44) describes the asymptotic region very far from the axis. Moreover, it is simple to check that $I_1 \to 0$ and $I_2 \to 0$ as $r \to 0$ for the solution given in figure 1 with $X_r(r = 1) = -1$.

Finally, let us note that the unique special solution corresponding to free gravitational waves (41) can be written in terms of μ and x as $d\mu/dx = \pm(2/3)^{1/2} a\mu$, which can be easily solved to show that μ depends exponentially upon $x = f(u) + g(v)$. This means simply that μ can be written as $\mu = \alpha(u)\beta(v)$, where α and β are arbitrary functions. The next section is devoted to a discussion of the properties of this solution beyond what is already known from previous studies [3, 4].

3. FREE ROTATING GRAVITATIONAL WAVES

Free nonrotating cylindrical gravitational wave solutions of Eintein's equations were first discussed by Beck [8]. These solutions have been the subject of many subsequent investigations [7, 9, 10]. The solutions considered in this section can be interpreted in terms of simple cylindrical waves that rotate [4].

The special solution (41) for free rotating gravitational waves is given by

$$\mu = \alpha(u)\beta(v), \tag{49}$$

$$\psi = \sqrt{\frac{3}{2}}\ln\frac{\alpha}{\beta}, \tag{50}$$

$$\gamma = \frac{1}{2}\ln\left[\frac{8}{l^2}(\alpha\beta)^3\alpha_u\beta_v\right], \tag{51}$$

$$\omega_v - \omega_u = \frac{8}{l}\alpha_u\beta_v. \tag{52}$$

It turns out that in this case equation (2) is equivalent to the scalar wave equation for ψ in the background geometry given by equation (1); therefore, the function ψ — which is a mixture of ingoing and outgoing waves according to equation (50) — has the interpretation of the scalar potential for the free rotating gravitational waves. The solution (49) – (52) cannot be thought of as a collision between outgoing and ingoing gravitational waves, since the field equations do not admit solutions for which $\mu = \mu(u)$ or $\mu = \mu(v)$. That is, there is no purely outgoing solution just as there is no purely ingoing solution.

The spacetime given by equations (49) – (52) is singular. In fact, the analysis of the corresponding curvature indicates that moving singular cylinders appear whenever α, β, α_u, or β_v vanishes. It is interesting to consider the nature of the symmetry axis for rotating waves, since in these solutions the axis does not in general satisfy the condition of elementary flatness. If for an infinitesimal spacelike circle around the axis of symmetry the ratio of circumference to radius goes to 2π as the radius goes to zero, the condition of elementary flatness is satisfied for the axis under consideration. In our case, this means that $\mu^2/(e^{2\gamma}\rho^2) \to 1$ as $\rho \to 0$. For simple cylindrical waves (i.e., Beck's solution) we have $\mu = \rho$ and hence the condition of elementary flatness is that $\gamma \to 0$ as $\rho \to 0$. In general, Beck's fields can be divided into two classes: the Einstein-Rosen waves [9] and the Bonnor-Weber-Wheeler waves [10]. In the former class, the axis does not satisfy the condition of elementary flatness; in fact, the axis is not regular either. It is a singularity of spacetime and is therefore interpreted as the source of the cylindrical waves which are otherwise free of singularities. In the latter case, the axis does satisfy the condition of elementary flatness and is, moreover, regular. The waves presumably originate at infinity: Incoming waves implode on the axis and then move out to infinity with no singularities in the finite regions of spacetime. It is therefore clear that *no* caustic cylinders appear regardless of the nature of the axis. They do appear, however, when the waves *rotate*. We are therefore led to regard the appearance of moving singular cylinders in our solution as being due to the rotation of the waves. It is important to emphasize the absence of a direct causal connection between the violation of elementary flatness at the axis and the presence of singular cylinders: the axis is static while the singularity is in motion, and the condition of elementary flatness involves the gravitational potentials ($g_{\mu\nu}$) while the singularity of the field has to do with the spacetime curvature. In fact, it is possible to find instances of exact solutions for which the axis is elementary flat but singular [11].

In spacetime regions between the extrema of α and β, the solution (49) – (52) can be reduced to a normal form in which α and β are linear functions of their arguments [3]. With further elementary coordinate transformations, the normal form can be reduced to two special solutions which are of interest: (*a*) $\alpha = \sigma u, \beta = \sigma v$ and (*b*) $\alpha = \sigma u, \beta = -\sigma v$. Here $\sigma = \pm(l_0)^{-1/2}$, where l_0 is a constant length whose introduction is necessary on dimensional grounds. The metrics for these cases are, respectively,

$$-ds^2 = \frac{8}{l_0^4 l^2}(uv)^3 \left(\frac{u}{v}\right)^{-\sqrt{6}} (-dt^2 + d\rho^2) + \frac{1}{l_0^2}(uv)^2 \left(\frac{u}{v}\right)^{-\sqrt{6}} \left(\frac{8}{l_0 l}\rho dt + d\phi\right)^2 + \left(\frac{u}{v}\right)^{\sqrt{6}} dz^2 \tag{53}$$

and

$$-ds^2 = -\frac{8}{l_0^4 l^2}(-uv)^3 \left(-\frac{u}{v}\right)^{-\sqrt{6}} (-dt^2 + d\rho^2) + \frac{1}{l_0^2}(uv)^2 \left(-\frac{u}{v}\right)^{-\sqrt{6}} \left(-\frac{8}{l_0 l}\rho dt + d\phi\right)^2 + \left(-\frac{u}{v}\right)^{\sqrt{6}} dz^2. \tag{54}$$

To ensure that the spacetime metric is real, case (a) must be limited to $(u>0, v>0)$ or $(u<0, v<0)$. Similarly, case (b) must be limited to $(u>0, v<0)$ or $(u<0, v>0)$. In either case the hypersurfaces $u=0$ and $v=0$ are curvature singularities. Since the laws of physics break down very close to these surfaces, it appears that the consideration of boundary conditions across such surfaces would be without physical significance. In case (b), ρ is a temporal coordinate and t is a spatial coordinate. The transformation $(t \rightarrow \rho,\ \rho \rightarrow t)$, or equivalently $(u \rightarrow -u,\ v \rightarrow v)$, brings the metric to the form

$$-ds^2 = \frac{8}{l_0^4 l^2}(uv)^3 \left(\frac{u}{v}\right)^{-\sqrt{6}} (-dt^2 + d\rho^2) + \frac{1}{l_0^2}(uv)^2 \left(\frac{u}{v}\right)^{-\sqrt{6}} \left(-\frac{8}{l_0 l} t d\rho + d\phi\right)^2 + \left(\frac{u}{v}\right)^{\sqrt{6}} dz^2, \tag{55}$$

which reduces to case (a) with a further coordinate transformation $\phi \rightarrow \phi + 8t\rho/(l_0 l)$. Therefore, only the normal form in case (a) will be considered in the rest of this paper.

The geodesic equation for free rotating gravitational wave spacetimes is discussed in appendix A.

4. THE GRAVITATIONAL STRESS-ENERGY TENSOR

In a recent paper [1], a local gravitoelectromagnetic stress-energy tensor $T_{\mu v}$ has been defined for Ricci-flat spacetimes via a certain averaging procedure in a Fermi frame along the path of a geodesic observer. That is,

$$\begin{aligned} T_{\mu\nu} &= \frac{L^2}{12\pi} T_{\mu\nu\rho\sigma} \lambda^{\rho}_{(0)} \lambda^{\sigma}_{(0)} \\ &= \frac{L^2}{12\pi} T_{\mu\nu(0)(0)}\ , \end{aligned} \tag{56}$$

where L is a length-scale characteristic of the field under consideration, $T_{\mu\nu\rho\sigma}$ is the Bel-Debever-Robinson tensor [7, 12] defined by

$$T_{\mu\nu\rho\sigma} = \frac{1}{2}(R_{\mu\xi\rho\zeta} R_{\nu\ \sigma}^{\ \xi\ \zeta} + R_{\mu\xi\sigma\zeta} R_{\nu\ \rho}^{\ \xi\ \zeta}) - \frac{1}{16} g_{\mu\nu} g_{\rho\sigma} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, \tag{57}$$

and $\lambda^{\mu}_{(0)} = dx^{\mu}/d\tau$ is the vector tangent to the timelike path of the observer. The gravitational stress-energy tensor is expected to provide *approximate* measures of the average gravitoelectromagnetic energy, momentum, and stress in the neighborhood of the observer. There does not seem to be any direct connection between $T_{\mu\nu}$ and the Landau-Lifshitz pseudotensor; this point is further discussed in appendix B.

It would be interesting to study the energy of rotating gravitational waves. However, the spacetime metric for the rotating waves (44) is given explicitly only for the case of the exact solution (41) of the differential equation (40). This unique solution is approached asymptotically by the rotating wave solutions very far from the symmetry axis. For the sake of simplicity, we therefore restrict attention to the special solution (41) that, according to the results presented in section 3, corresponds to the *free* rotating waves.

To determine the energy density of this radiation field measured by a geodesic observer, it is first necessary to consider a solution of the geodesic equation discussed in appendix A. Let us therefore choose a "radial" geodesic of the free rotating waves given by

$$t = \frac{l_0}{2}(\zeta^{1/p} + \zeta^{1/q}), \tag{58}$$

$$\rho = -\frac{l_0}{2}(\zeta^{1/p} - \zeta^{1/q}), \tag{59}$$

$$\phi = \frac{\sqrt{5}}{2}\left(\zeta^{2/p} - \zeta^{2/q} - \sqrt{\frac{3}{2}}\,\zeta^{4/5}\right) + \phi_0, \tag{60}$$

$$z = z_0, \tag{61}$$

where we have used equations $(A11)-(A14)$ of appendix A together with the metric in normal form (53), i.e. $\alpha(u) = u/l_0^{1/2}$ and $\beta(v) = v/l_0^{1/2}$, and we have fixed the constants in these equations such that $\tau_1 = \tau_2 = l_0$ and

$$2c_1 = l_0^{1-p/2}, \qquad 2c_2 = l_0^{1-q/2}\ .$$

Note that the scaling parameter l_0 is therefore related to the intrinsic rotation parameter l via $2l_0 = \sqrt{5}l$. Moreover, ϕ_0 and z_0 are constants and ζ is defined by

$$\zeta = 1 - \tau/l_0. \tag{62}$$

The observer under consideration here starts at the symmetry axis $\rho = 0$, which is regular at $\tau = 0$, and returns to it at $\tau = l_0$ when it is singular. The trajectory of this observer $\rho(\phi)$ is depicted in figure 2.

Imagine now an orthonormal tetrad frame $\lambda^\mu_{(\alpha)}$ that is parallel propagated along this timelike worldline. It is simple to work out explicitly two axes of the tetrad: the time axis and the spatial axis parallel to the symmetry axis. These are given by $\lambda^\mu_{(0)} = dx^\mu/d\tau$ using equations (58) – (61) and

$$\lambda^\mu_{(3)} = \zeta^{-3/5}\ \delta^\mu_3, \tag{63}$$

respectively. It follows from the projection of $T_{\mu\nu\rho\sigma}$ given by equation (57) on these axes that along the geodesic the gravitational radiation energy density is given by

$$T_{(0)(0)} = \frac{36}{625\pi}\frac{L^2}{(l_0 - \tau)^4}, \tag{64}$$

while the energy flux along the symmetry axis vanishes, $T_{(0)(3)} = 0$, as expected. There is, however, radiation pressure along the z-axis and is given by

$$T_{(3)(3)} = \frac{3}{125\pi}\frac{L^2}{(l_0 - \tau)^4}. \tag{65}$$

The energy density and the pressure measured by the geodesic observer both diverge at the singularity $\tau = l_0$. Note that $T_{(3)(3)}/T_{(0)(0)} = 5/12$, which is less than unity as would be expected for the ratio of pressure to density. The characteristic length-scale associated with rotating gravitational waves is l; therefore, the constant length L in expressions (64) and (65) could be chosen to be simply proportional to $l = 2l_0/\sqrt{5}$.

APPENDIX A: GEODESICS IN ROTATING WAVE SPACETIMES

Starting with the metric form given by equation (1), let

$$\mathcal{L}_g = -\frac{1}{2}\left(\frac{ds}{d\lambda}\right)^2 = \frac{1}{2}[e^{2\gamma-2\psi}(-\dot{t}^2+\dot{\rho}^2)+\mu^2 e^{-2\psi}(\omega\dot{t}+\dot{\phi})^2+e^{2\psi}\dot{z}^2], \qquad (A1)$$

where $\dot{t} = dt/d\lambda$, etc. Since the Lagrangian $(A1)$ does not depend upon ϕ and z, there are two constants of the motion p_ϕ and p_z given by

$$p_\phi = \frac{\partial \mathcal{L}_g}{\partial \dot{\phi}} = \mu^2 e^{-2\psi}(\omega\dot{t}+\dot{\phi}) \ , \qquad (A2)$$

$$p_z = \frac{\partial \mathcal{L}_g}{\partial \dot{z}} = e^{2\psi}\dot{z}. \qquad (A3)$$

Moreover, we have

$$\frac{d}{d\lambda}(\omega p_\phi - e^{2\gamma-2\psi}\dot{t}) = \frac{\partial \mathcal{L}_g}{\partial t}, \qquad (A4)$$

and

$$\frac{d}{d\lambda}(e^{2\gamma-2\psi}\dot{\rho}) = \frac{\partial \mathcal{L}_g}{\partial \rho}. \qquad (A5)$$

Let us now focus attention on “radial” geodesics such that the momenta associated with azimuthal and vertical motions vanish. It is then convenient to write the equations of motion in terms of radiation coordinates u and v. We find that equations $(A2) - (A5)$ reduce to

$$\frac{d}{d\lambda}\left(e^{2\gamma-2\psi}\frac{du}{d\lambda}\right) = (e^{2\gamma-2\psi})_v \ \frac{du}{d\lambda}\frac{dv}{d\lambda} \qquad (A6)$$

and

$$\frac{d}{d\lambda}\left(e^{2\gamma-2\psi}\frac{dv}{d\lambda}\right) = (e^{2\gamma-2\psi})_u \ \frac{du}{d\lambda}\frac{dv}{d\lambda}, \qquad (A7)$$

which imply that

$$e^{2\gamma-2\psi} \ \frac{du}{d\lambda}\frac{dv}{d\lambda}$$

is a constant along the path. This constant vanishes for a null geodesic, so that “radial” null geodesics correspond to $u = t - \rho =$ constant or $v = t + \rho =$ constant, where λ is the affine parameter along the path.

We are mainly interested in timelike "radial" geodesics, hence we set $\lambda = \tau$, where τ is the proper time along the path. Then, with $\tilde{U} = du/d\tau$ and $\tilde{V} = dv/d\tau$, we have $\tilde{U}\tilde{V} = e^{2\psi - 2\gamma}$. It follows from ($A6$) and ($A7$) that $\tilde{U}_v \tilde{V}^2 = \tilde{V}_u \tilde{U}^2$. The general solution is given by

$$\frac{du}{d\tau} = \frac{1}{2}\left(\frac{\partial \mathcal{S}}{\partial u}\right)^{-1}, \tag{A8}$$

$$\frac{dv}{d\tau} = \frac{1}{2}\left(\frac{\partial \mathcal{S}}{\partial v}\right)^{-1}, \tag{A9}$$

where $\mathcal{S}(u, v)$ is any solution of the differential equation

$$4\,\frac{\partial \mathcal{S}}{\partial u}\frac{\partial \mathcal{S}}{\partial v} = e^{2\gamma - 2\psi}. \tag{A10}$$

This is, in fact, the Hamilton-Jacobi equation for "radial" timelike geodesics, and $\mathcal{S}$ along the path differs from the proper time by a constant.

It is simple to illustrate a class of solutions of the eikonal equation ($A10$) via separation of variables, i.e.

$$\mathcal{S}(u, v) = -c_1 \alpha^p(u) - c_2 \beta^q(v), \tag{A11}$$

where $p = 4 - \sqrt{6}$, $q = 4 + \sqrt{6}$, and c_1 and c_2 are constants such that

$$c_1 c_2 = \frac{1}{5l^2}. \tag{A12}$$

Equations ($A8$) and ($A9$) now have solutions

$$\alpha(u) = \left(\frac{\tau_1 - \tau}{2c_1}\right)^{1/p}, \tag{A13}$$

$$\beta(v) = \left(\frac{\tau_2 - \tau}{2c_2}\right)^{1/q}, \tag{A14}$$

where τ_1 and τ_2 are constants of integration. More explicitly, let us consider the normal form of the metric with $\alpha(u) = l_0^{-1/2} u$ and $\beta(v) = l_0^{-1/2} v$ as employed in section 4. The geodesic would then hit the null singular hypersurface $u = 0$ (or $v = 0$) at $\tau = \tau_1$ (or

$\tau = \tau_2$). The singular nature of these moving cylinders can be seen from the fact that the geodesic can not be continued past them since then u or v would become complex.

APPENDIX B: THE LANDAU-LIFSHITZ PSEUDOTENSOR IN RIEMANN NORMAL COORDINATES

The energy of a gravitational field — if it can be defined at all — is nonlocal according to general relativity. On the other hand, the Bel-Debever-Robinson tensor is locally defined. A connection could perhaps be established between these concepts if the energy-momentum pseudotensor of the gravitational field is expressed in Riemann normal coordinates about a typical event in spacetime.

Let x^μ be the Riemann normal coordinates in the neighborhood of some point ("origin") in spacetime; then,

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, x^\alpha x^\beta + \cdots, \qquad (B1)$$

$$\Gamma^\mu_{\nu\rho} = -\frac{1}{3}(R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\nu\sigma})x^\sigma + \cdots. \qquad (B2)$$

The Landau-Lifshitz pseudotensor is quadratic in the connection coefficients by construction; therefore, $t^{L-L}_{\mu\nu}$ is — at the lowest order — quadratic in Riemann normal coordinates. Hence,

$$t^{L-L}_{\mu\nu,\alpha\beta} = \frac{c^4}{144\pi G}\Theta_{\mu\nu\alpha\beta} + \cdots, \qquad (B3)$$

where $\Theta_{\mu\nu\alpha\beta}$ is symmetric in its first and second pairs of indices by construction and is given by

$$\Theta_{\mu\nu\alpha\beta} = \frac{1}{2}(R^{\rho\sigma}{}_{\mu\alpha} R_{\nu\sigma\rho\beta} + R^{\rho\sigma}{}_{\mu\beta} R_{\nu\sigma\rho\alpha}) + \frac{7}{2}(R_{\mu\rho\sigma\alpha} R_\nu{}^{\rho\sigma}{}_\beta + R_{\mu\rho\sigma\beta} R_\nu{}^{\rho\sigma}{}_\alpha)$$
$$-\frac{3}{8}\eta_{\mu\nu}\eta_{\alpha\beta} R_{\rho\sigma\kappa\delta} R^{\rho\sigma\kappa\delta}. \qquad (B4)$$

This expression should be compared and contrasted with equation (57) that expresses the Bel-Debever-Robinson tensor in a similar form. There is no simple relationship between $\Theta_{\mu\nu\alpha\beta}$ and $T_{\mu\nu\alpha\beta}$; however, one can show that

$$\Theta_{\mu\nu\alpha\beta} - 7\, T_{\mu\nu\alpha\beta} = \frac{1}{16}\eta_{\alpha\beta}\eta_{\mu\nu}K + \frac{1}{4}(R^{\rho\sigma}{}_{\mu\alpha}R_{\rho\sigma\nu\beta} + R^{\rho\sigma}{}_{\mu\beta}R_{\rho\sigma\nu\alpha}) \qquad (B5)$$

in Riemann normal coordinates. Here K is the Kretschmann scalar, i.e. $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$.

REFERENCES

[1] Mashhoon B, McClune J C and Quevedo H "Gravitational Superenergy Tensor" preprint 1996

[2] Wheeler J A 1977 *Phys. Rev. D* **16** 3384; Ibáñez J and Verdaguer E 1985 *Phys. Rev. D* **31** 251; Isenberg J, Jackson M and Moncrief V 1990 *J. Math. Phys.* **31** 517; Bréton N, Feinstein A and Ibáñez J 1993 *Gen. Rel. Grav.* **25** 267

[3] Mashhoon B and Quevedo H 1990 *Phys. Lett. A* **151** 464

[4] Quevedo H and Mashhoon B 1991 in *Relativity and Gravitation: Classical and Quantum*, Proceedings of the Conference SILARG VII, edited by J. C. D'Olivo *et al.* (World Scientific, Singapore, 1991) pp. 287–293. There is an error on p. 289 of this paper in the discussion of the exterior field of a rotating cylinder. Contrary to the statement in the paper, the exterior vacuum field is not always static. It is, in fact, stationary for high angular momentum. This has been shown by Bonnor W B 1980 *J. Phys. A* **13** 2121. B. M. is grateful to Professor Bonnor for clarifying remarks. Moreover, a typographical error occurs in equation (9) : $\mu_{,uv}$ should be replaced by $\mu_{,vv}$.

[5] Ardavan H 1984 *Phys. Rev. D* **29** 207; 1989 *Proc. Roy. Soc. London A* **424** 113

[6] Ardavan H 1984 in *Classical General Relativity*, edited by Bonnor W B, Islam J N and MacCallum M A H (Cambridge University Press, Cambridge, 1984) pp.5–14

[7] Zakharov V D *Gravitational Waves in Einstein's Theory* (Halsted, New York, 1973), translated by R N Sen from the 1972 Russian edition

[8] Beck G 1925 *Z. Physik* **33** 713

[9] Einstein A and Rosen N 1937 *J. Franklin Inst.* **223** 43; Rosen N 1954 *Bull. Res. Council Israel* **3** 328

[10] Bonnor W B 1957 *J. Math. Mech.* **6** 203; Weber J and Wheeler J A 1957 *Rev. Mod. Phys.* **29** 509

[11] Van den Bergh N and Wils P 1986 *Class. Quantum Grav.* **2** 229

[12] Bel L 1958 *C. R. Acad. Sci. Paris* **247** 1094; 1959 *ibid.* **248** 1297; 1962 *Cah. Phys.* **16** 59; Debever R 1958 *Bull. Soc. Math. Belg.* **10** 112; Bel L 1962 in *Colloques Internationaux du CNRS*, Paris, pp.119–126

FIGURE CAPTIONS

Figure 1. The plot of $X(r)$ versus r for three solutions of equation (40) with the boundary conditions that at $r = 1$, $X = 2$ and $X_r = -1, 0, 1$. The function $-X(r)$ is plotted in this figure as well. Note that the solution for $X_r(r = 1) = -1$ approaches the special solution (41), also depicted here, for $r \to 0$. The behavior of the solution for $X_r(r = 1) = 1$ near $r_0 \simeq 0.22$ is in accordance with equation (43).

Figure 2. The functions $\rho(\tau)/l_0$ and $\phi(\tau)$ given by equations (59) and (60), respectively, with $\phi_0 = \pi/2$ are plotted here as polar coordinates (i.e., abscissa $= \rho \cos\phi$ and ordinate $= \rho \sin\phi$). This represents the trajectory of a particle following the "radial" geodesic given by equations (58) – (61). The particle starts at $\tau = 0$ from $\rho = 0$ and moves counterclockwise until it returns to a singular axis ($\rho = 0$) at $\tau = l_0$. The return trip takes only about ten percent of the total proper time l_0.