GRAVITATIONAL SUPERENERGY TENSOR

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Abstract

We provide a physical basis for the local gravitational superenergy tensor. Furthermore, our gravitoelectromagnetic deduction of the Bel-Debever-Robinson superenergy tensor permits the identification of the gravitational stress-energy tensor. This *local* gravitational analog of the Maxwell stress-energy tensor is illustrated for a plane gravitational wave.

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The concept of energy and the law of conservation of energy play crucial roles in all physical theories. The notion of particle energy is already present in Newtonian mechanics. In more general physical theories, the notion of energy becomes more sophisticated, while still retaining its fundamental significance. In general relativity, the energy associated with matter is represented by a stress-energy tensor $T_M^{\alpha\beta}$ that allows us to define, in a consistent way, the *local* energy density of matter as measured by a given observer. The condition $T_{M\ \ ;\beta}^{\alpha\beta} = 0$ represents the law of *local* conservation of energy of matter, but it does not guarantee, in general, that the total energy is conserved. This is because $T_M^{\alpha\beta}$ contains only the contribution of matter as well as all nongravitational fields, while the gravitational energy, which is also expected to contribute to the total energy, has not been taken into account. However, in general relativity there is no physically meaningful notion of energy of the gravitational field. In fact, one would expect that in general relativity the gravitational energy should be given by an expression quadratic in the first derivatives of the metric, such as the Landau-Lifshitz pseudotensor. However, it follows from Einstein's principle of equivalence that it is not possible to construct *locally* a covariant expression for energy only in terms of the spacetime metric and its first derivatives. In this paper, we present a physical derivation of an expression that is, in a certain average sense, the *local* stress-energy tensor of the gravitational field, an expression which is analogous to the stress-energy tensor of the electromagnetic field. In fact, the validity of this novel gravitational stress-energy tensor rests upon the approximate correspondence – in a quasi-inertial neighborhood surrounding the worldline of a geodesic observer – between gravitation and electromagnetism. Our gravitoelectromagnetic stress-energy tensor turns out to be directly proportional to the Bel-Debever-Robinson tensor, which is thereby furnished with a proper physical interpretation.

The electromagnetic field is endowed with a symmetric traceless stress-energy tensor $\Theta^{\alpha\beta}$ that is given in terms of the Faraday tensor $F^{\alpha\beta}$ by

$$\Theta^{\alpha\beta} = \frac{1}{8\pi}(F^{\alpha\gamma}F^{\beta}_{\ \gamma} + {}^*F^{\alpha\gamma}\ {}^*F^{\beta}_{\ \gamma}), \tag{1}$$

where ${}^*F_{\alpha\beta}$ is the dual of $F_{\alpha\beta}$. In 1958, Bel suggested the notion of a gravitational superenergy tensor by constructing an analogous fourth-rank tensor from the Riemann tensor $R_{\mu\nu\rho\sigma}$, namely,

$$T^{\alpha\beta\gamma\delta} = \frac{1}{2}\left(R^{\alpha\mu\gamma\nu} R^{\beta}{}_{\mu}{}^{\delta}{}_{\nu} + {}^{*}R^{\alpha\mu\gamma\nu}\, {}^{*}R^{\beta}{}_{\mu}{}^{\delta}{}_{\nu} \right), \tag{2}$$

which is unambiguously defined only for spacetimes with $R_{\mu\nu} = \Lambda g_{\mu\nu}$ [1]. In this paper, we assume that the cosmological constant vanishes ($\Lambda = 0$), and the signature of the Ricci flat spacetime is $+2$; moreover, units are chosen such that Newton's constant of gravitation and the speed of light in vacuum are set equal to unity.

The gravitational superenergy tensor (2) has been used to define the local energy density and Poynting vector for source-free gravitational fields [1]. Contributions to this approach have also been made by Debever [2] and Robinson [3]; a detailed treatment of this topic as well as further references to the original literature is contained in Zakharov's monograph [4].

The Bel-Debever-Robinson (BDR) superenergy tensor $T^{\alpha\beta\gamma\delta}$ is totally symmetric and traceless. Moreover, it satisfies a conservation law $T^{\alpha\beta\gamma\delta}{}_{;\delta} = 0$ in analogy with $\Theta^{\alpha\beta}{}_{;\beta} = 0$ for source-free electromagnetic fields. The BDR tensor has been particularly useful in numerical relativity and has therefore been the subject of studies by a number of investigators [5]. However, a basic derivation as well as physical interpretation of the superenergy tensor has been lacking thus far. It is the purpose of the present work to ameliorate this situation.

To provide a physical basis for the analogy with electrodynamics, essential use will be made here of the notion of a *Fermi system*. Consider an observer freely falling in a gravitational field; the observer carries a tetrad frame $\lambda^{\mu}_{(\alpha)}$, where $\lambda^{\mu}_{(0)} = dx^{\mu}/d\tau$ is the observer's velocity and τ is the proper time along its path. The Fermi system – which is the simplest generalization of a local inertial frame along the path of the observer – is a geodesic coordinate system based on a nonrotating frame along the observer's worldline [6]. Let $X^{\alpha} = (\tau,\mathbf{X})$ be the Fermi coordinates along the observer's worldline; the spacetime metric in these coordinates is given by ${}^{F}g_{\alpha\beta} = \eta_{\alpha\beta} + {}^{F}h_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the Minkowski metric and ${}^{F}h_{\alpha\beta}(\tau,\mathbf{X})$ may be expressed as a power series in $\mathbf{X}$ away from the observer with coefficients that depend on the Riemannian curvature of spacetime. Only the lowest order terms away from the path will be taken into account throughout this paper. It is useful to define the gravitoelectric and gravitomagnetic potentials Φ_g and $\mathbf{A}_g$, respectively, via ${}^F h_{00} = 2\Phi_g$ and ${}^F h_{0i} = -2(\mathbf{A}_g)_i$ in analogy with electrodynamics.

It follows from the construction of the Fermi system that

$$\Phi_g = -\frac{1}{2}\ {}^F R_{0i0j}(\tau) X^i X^j, \tag{3}$$

$$(\mathbf{A}_g)_i = \frac{1}{3}\ {}^F R_{0jik}(\tau) X^j X^k, \tag{4}$$

where ${}^F R_{\alpha\beta\gamma\delta}(\tau) = R_{(\alpha)(\beta)(\gamma)(\delta)} = R_{\mu\nu\rho\sigma}\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)}\lambda^\rho_{(\gamma)}\lambda^\sigma_{(\delta)}$ is the curvature as measured by the fiducial observer. A discussion of the general properties of this curvature is beyond the scope of this paper [7,8]. We now define the gravitoelectric ($\mathcal{E}$) and gravitomagnetic ($\mathcal{B}$) fields in complete analogy with electrodynamics; hence,

$$\mathcal{E}_i(\tau, \mathbf{X}) = \ {}^F R_{0i0j}(\tau) X^j, \tag{5}$$

$$\mathcal{B}_i(\tau, \mathbf{X}) = -\frac{1}{2}\,\epsilon_{ijk}\ {}^F R_{jk0l}(\tau) X^l, \tag{6}$$

which agree with the identification of "electric" and "magnetic" components of the Riemann tensor. The curvature tensor generally consists of "electric", "magnetic," and "spatial" components; in a Ricci flat region, however, the "spatial" components are given by the "electric" components. Therefore, Eqs. (5) and (6) contain the full information regarding the gravitational field along the path. Moreover, it is possible to express the gravitational field equations in the Fermi system in a form that is analogous to the Maxwell equations using Eqs. (5) and (6). This partial agreement between the field equations of gravitation and electrodynamics is well known and will not be further elaborated here [9].

The Lorentz force law is needed in addition to Maxwell's equations to produce a complete picture of classical electrodynamics. In the Fermi system, this equation to first order in velocity is

$$m\frac{d^2\mathbf{X}}{d\tau^2} = q_{_E}\mathcal{E} + q_{_B}\frac{d\mathbf{X}}{d\tau}\times\mathcal{B}, \tag{7}$$

where m is the inertial mass of a test particle, and $q_{_E}$ and $q_{_B}$ are the gravitoelectric and gravitomagnetic charges of the particle, respectively. It follows from the gravitational Larmor theorem [10] that $q_{_E} = -m$ and $q_{_B} = -2m$; the negative signs of the gravitoelectromagnetic charges reflect the attractive nature of gravitation, while $q_{_B}/q_{_E} = 2$ since gravitation is a spin-2 field. Equation (7) turns out to be the generalized Jacobi equation [11], which is the equation of motion of a free test particle *relative* to the fiducial observer, valid to first order in $\mathbf{X}$ and $d\mathbf{X}/d\tau$. This agreement between the deviation equation and the Lorentz force law is a remarkable result, since it makes it possible, in principle, to introduce the stresss-energy tensor for gravitoelectromagnetism in complete analogy with the standard deduction of the electromagnetic stress-energy tensor. However, the electromagnetic stress-energy tensor is defined globally while the gravitoelectromagnetic stress-energy tensor would have to be defined in a narrow tube along the timelike path of the observer in spacetime. The invariance under translations in time and space is ultimately responsible for the existence of conservation laws of energy and momentum, respectively, as well as the stress-energy tensor in Minkowski spacetime. The existence of the gravitational pseudotensor can be similarly justified in an asymptotically Minkowskian spacetime manifold. On the other hand, the gravitoelectromagnetic stress-energy tensor would owe its existence to an inertial Fermi region (called the "tidal frame" in [12]) in the form of a thin Minkowskian cylinder in spacetime along the timelike path of the observer.

It is possible to define a gravitational Faraday tensor in the Fermi system by

$$\mathcal{F}_{\alpha\beta} = -\,{}^{F}R_{\alpha\beta 0l}X^l, \tag{8}$$

which contains Eqs. (5) and (6). Let us define the stress-energy tensor associated with $\mathcal{F}_{\alpha\beta}$ in the standard manner by

$$\mathcal{T}^{\alpha\beta} = \frac{1}{4\pi}(\mathcal{F}^{\alpha}_{\ \gamma}\mathcal{F}^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta}\mathcal{F}_{\gamma\delta}\mathcal{F}^{\gamma\delta}), \tag{9}$$

which is symmetric and traceless by construction. The substitution of Eq. (8) in Eq. (9) reveals that $\mathcal{T}^{\alpha\beta}$ is zero along the path of the fiducial observer ($\mathbf{X}= 0$); this is ultimately a consequence of Einstein's principle of equivalence. On the other hand, $\mathcal{T}^{\alpha\beta}$ is nonzero in a neighborhood of this path as the gravitoelectromagnetic field initially varies linearly with distance away from the fiducial observer. Imagine a second observer with a worldline only infinitesimally distant from the fiducial observer; if a second Fermi system is established along this path and our construction of the gravitoelectromagnetic stress-energy tensor is repeated, then this tensor would vanish identically along the second path but would give a finite result along the original reference path. It follows from these considerations that the proper stress-energy tensor along the path of an observer must be defined through an averaging procedure. At any given proper time τ, let us average the tensor given by Eq. (9) over a limiting tube along the worldline. More precisely, consider a sphere of proper radius $\epsilon L,\ 0 < \epsilon \ll 1$, centered around $\mathbf{X}= 0$ at a given τ such that

$$T^{(\alpha)(\beta)}(\tau) = \lim_{\epsilon\to 0} \epsilon^{-2} < \mathcal{T}^{\alpha\beta} >, \tag{10}$$

where the angular brackets denote the operation of averaging over the sphere. Here L is a constant invariant length scale that is otherwise unspecified; in practice, it could be a characteristic length scale for the problem under consideration (e.g., in the exterior field of a black hole, L could be the mass of the black hole), or, in the absence of a natural scale in spacetime as in a pure gravitational radiation field, one could set L equal to the Planck length. Using the fact that $< X^i X^j >= (\epsilon^2 L^2/3)\delta_{ij}$, we can express Eq. (10) in arbitrary coordinates as

$$T_{\mu\nu} = \frac{L^2}{12\pi} T_{\mu\nu\rho\sigma} \lambda^{\rho}_{(0)} \lambda^{\sigma}_{(0)}, \tag{11}$$

where $T_{\mu\nu\rho\sigma}$ is symmetric and traceless in the first pair of indices and symmetric in the second pair of indices by construction, and is given by

$$T_{\mu\nu\rho\sigma} = \frac{1}{2}(R_{\mu\xi\rho\varsigma} R_{\nu}{}^{\xi}{}_{\sigma}{}^{\varsigma} + R_{\mu\xi\sigma\varsigma} R_{\nu}{}^{\xi}{}_{\rho}{}^{\varsigma}) - \frac{1}{4} g_{\mu\nu} R_{\alpha\beta\rho\gamma} R^{\alpha\beta}{}_{\sigma}{}^{\gamma}. \tag{12}$$

This tensor is identical with that first defined by Bel [1] in 1958 for $R_{\mu\nu} = \Lambda g_{\mu\nu}$ and coincides with Eq. (2). It is important to remark, however, that in our gravitoelectromagnetic deduction, i.e. Eqs. (3) - (12), no restriction has been placed on the Ricci tensor; therefore, Eqs. (11) and (12) define the gravitoelectromagnetic part of the local stress-energy tensor of a general gravitational field. Furthermore, the linear treatment of the fields in Eqs. (5) and (6) is sufficient to obtain the general results in Eqs. (11) and (12) since terms of higher order would simply drop out of Eq. (10) in the limit $\epsilon \to 0$.

To define the *local* gravitational stress-energy tensor at an event in spacetime, a geodesic observer with ideal gyroscope axes is needed at the event; then, $T_{(\alpha)(\beta)} = L^2 T_{(\alpha)(\beta)(0)(0)}/12\pi$ for the observer, and the stress-energy tensor for any other observer at that event can be obtained from $T_{(\alpha)(\beta)}$ by an appropriate Lorentz transformation. Moreover, the local physical quantities defined in this way (i.e. energy density, energy flux, momentum density, and the gravitational stresses) have now their proper dimensionality as a consequence of the introduction of the constant length L. The freedom in the choice of L implies, however, that these quantities are fixed up to a constant scale factor.

To illustrate these results, let us consider a plane gravitational wave given by

$$-ds^2 = -dt^2 + dx^2 + U^2(e^{2h}dy^2 + e^{-2h}dz^2), \tag{13}$$

which propagates along the x–axis and is linearly polarized. Here U and h are functions of $u = t - x$ and are related by $U'' + h'^2\, U = 0$, where a prime indicates differentiation with respect to u. This gravitational field is of Petrov type N and is a member of a class of plane wave spacetimes that has been extensively studied [13].

In the coordinate system under consideration here, observers located at fixed spatial coordinates follow geodesics. Furthermore, the natural tetrad system $\lambda^{\mu}_{(\alpha)}$ associated with these observers is nonrotating. The nonzero elements of $\lambda^{\mu}_{(\alpha)}$ are given by $\lambda^0_{(0)} = \lambda^1_{(1)} = 1$, $\lambda^2_{(2)} = U^{-1}\exp(-h)$, and $\lambda^3_{(3)} = U^{-1}\exp(h)$. The local gravitational stress-energy tensor according to these observers is given by

$$(T^{(\alpha)(\beta)}) = \frac{L^2}{6\pi}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} K^2(u), \tag{14}$$

$$K(u) = h'' + 2\frac{U'}{U}h'. \tag{15}$$

The spacetime is flat if $K(u) = 0$. All the nonzero components of $T_{(\alpha)(\beta)(\gamma)(\delta)}$ can be obtained from Eqs. (14) and (15). As expected, the only nonzero component of the gravitational stresses is the local pressure of the radiation along the x–axis.

Let ρ_g be the local energy density of the gravitational radiation field according to the fiducial static observers; then, $\rho_g = L^2K^2(u)/6\pi$. What is the energy density $\hat{\rho}_g$ measured by observers boosted along the x–axis with speed β? The boost can be expressed in terms of coordinates as $u = \Delta\hat{u}$, $\Delta v = \hat{v}$, $y = \hat{y}$, and $z = \hat{z}$, where $v = t + x$ and $\Delta = (1-\beta)^{1/2}/(1+\beta)^{1/2}$ is the Doppler factor. By transforming the tetrads at a given event, one can show explicitly that $\hat{\rho}_g = \Delta^4\rho_g$. Alternatively, this result may be obtained by noting that under the boost the metric (13) and hence the magnitudes of U and h remain invariant; then, the form of Eq. (15) leads directly to the formula for $\hat{\rho}_g$.

It follows that the local gravitational radiation energy density transforms in this case like the *square* of the energy density in standard field theory in Minkowski spacetime. This could possibly have observable consequences for the local inertia of gravitational radiation; however, such radiation can not be physically confined as a consequence of the universality of gravitational interaction.

The application of these concepts to rotating gravitational waves [14] will be published elsewhere.

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