Relativistic Effects in the Motion of the Moon

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Abstract

The main general relativistic effects in the motion of the Moon are briefly reviewed. The possibility of detection of the solar *gravitomagnetic* contributions to the mean motions of the lunar node and perigee is discussed.

1 Introduction

In a recent paper, Gutzwiller has provided an admirable review of the oldest three-body problem, namely, the Sun-Earth-Moon system [1]. Some work on the relativistic theory is mentioned in his paper; however, in view of the recent advances in relativistic celestial mechanics this subject deserves a more complete discussion. Here we provide a brief description of the main relativistic effects.

The lunar laser ranging experiment has opened up the possibility of measuring relativistic effects in the motion of the Moon; indeed, the agreement between the standard general relativistic model that contains over a hundred model parameters and the ranging data accumulated over the past three decades is excellent [2,3]. For instance, the post-fit residuals in the Earth-Moon distance are at the centimeter level [2,3]. Simple theoretical estimates lead to the conclusion that the main relativistic effects in the lunar theory are due to the spin-orbit coupling of the Earth-Moon system in the gravitational field of the Sun. The post-Newtonian influence of the solar field on the lunar motion consists of terms that can be classified as either harmonic (i.e. periodic) or secular (i.e. cumulative) in time. It turns out to be very difficult in practice to separate the harmonic terms from the corresponding Newtonian terms with the same periodicities. In effect, the existence of the post-Newtonian harmonic terms leads to small relativistic corrections in the numerical values of certain model parameters that are thereby adjusted by a fit to the ranging data. To give an example of such harmonic effects, we mention our prediction of a 6 cm relativistic tidal variation in the Earth-Moon distance with a period of 1/2 synodic month [4].

The main secular terms turn out to be essentially due to the precessional motion of the Earth-Moon orbital angular momentum in the field of the Sun. The Earth-Moon system can be thought of as an extended gyroscope in orbit about the Sun; we are interested in the description of the motion of the spin axis of this gyroscope with respect to the "fixed" stars (i.e. the sidereal frame). An ideal pointlike test gyroscope carried along a geodesic orbit would exhibit, in the post- Newtonian approximation, geodetic precession due to the orbital motion around the mass of the source as well as gravitomagnetic precession due to the intrinsic rotation of the source; however, the finite size of the gyroscope in this case (i.e. the orbit of the Moon about the Earth) leads to additional tidal effects. The main post-Newtonian gravitoelectric effect, i.e. geodetic precession, results in the advance of the Moon's node and perigee by about 2 arcseconds per century as first predicted by de Sitter already in 1916 [5]. This motion has been measured by Shapiro *et al.* with an accuracy of about one percent [6]. It is a relativistic three-body effect; therefore, we consider in the next section the restricted three-body problem in general relativity and briefly indicate, in particular, the more subtle post-Newtonian gravitomagnetic contributions to the motions of the Moon's node and perigee that are caused by the rotation of the Sun; indeed, solar rotation induces *cumulative* relativistic tidal effects in the Earth-Moon system [4].

2 Restricted Three-Body Problem in General Relativity

In our previous work [4], we developed a new scheme for the approximate treatment of the restricted three-body problem in general relativity. This coordinate-invariant approach is particularly useful for a reliable theoretical description of relativistic (solar) tidal effects in the motion of the Earth-Moon system. We assume that the Moon follows a geodesic in the gravitational field generated by the Earth and the Sun. This field may be calculated as follows: we first imagine that the Earth follows a geodesic in the solar field. Along this geodesic, we set up a geocentric Fermi coordinate system. This system, which involves the tidal field of the Sun, is then enhanced by taking due account of the field of the Earth in the linear approximation. Tidal effects in general relativity involve the projection of the Riemann tensor onto the tetrad frame of the measuring device. Consider the tidal matrix for a test system ("Earth") in free fall in the gravitational field of a rotating mass ("Sun"). In the standard first-order post-Newtonian treatment, the spatial axes of the local tetrad frame along the orbit are obtained by boosting the background Minkowski axes and adjusting scales to maintain orthonormality; the resulting tidal matrix for an approximately circular geodesic orbit turns out to be sinusoidal in time [7]. In this case, the tetrad frame is not parallel-transported, but its motion involves the Lense-Thirring orbital precession as well as the geodetic (i.e. de Sitter-Fokker) precession of the spatial axes. Once the parallel transport of the spatial axes along the orbit is imposed, the gravitomagnetic (i.e. Schiff) precession of the spatial axes would also appear in the first post-Newtonian order. In this order, the tidal matrix for the parallel-transported axes contains a secular term as well that must therefore be a direct consequence of the Schiff precession of the spatial axes [8], in agreement with our previous work [9-11]. The linear growth of this gravitomagnetic contribution to the tidal field poses a problem for the first post-Newtonian approximation: the non-Newtonian "off-diagonal" part of the tidal matrix can diverge in time [9-11]. To avoid this limitation, we have developed a post-Schwarzschild treatment of gravitomagnetic tidal effects; indeed, the concept of relativistic nutation provides a natural resolution of this difficulty by limiting the temporal extent of validity of the post-Newtonian approximation [9-11].

Imagine, for instance, a set of three orthogonal test gyroscopes falling freely along an inclined circular geodesic orbit with constant radius r ("astronomical unit") about a slowly rotating central body ("Sun") with mass M and proper angular momentum J. The motion of the spin axes of these torque-free gyroscopes, which constitute a local inertial frame (i.e. the geocentric Fermi frame), is essentially governed by the equations of parallel transport along the geodesic orbit. By solving these equations using the post-Schwarzschild approximation scheme that takes M into account to all orders, it can be shown that the average motion of the gyroscope axes with respect to an effective Newtonian (i.e. sidereal) frame consists of a gravitoelectric precessional motion—i.e. geodetic precession that was first completely analyzed by Fokker— together with a complex gravitomagnetic motion that can be loosely described as a combination of precessional movement and a harmonic nodding movement. The latter motion is a new relativistic effect of a rotating mass and has been referred to as *relativistic nutation* [11]. In the post-Newtonian approximation, the nutational terms over a limited time combine with the other gravitomagnetic precessional terms to give the Schiff precession. To see how this comes about, let us denote by τ the proper time of the geodesic orbit and consider a vector normal to the orbital plane (ecliptic) at the beginning of measurement ($\tau = 0$). Relativisticnutation is a periodic variation of the angle between this vector and a gyroscope axis that is Fermi propagated along the orbit. The leading contribution of relativistic nutation to this angle can be written as

$$\Theta_n \approx \xi[\sin(\eta_0 + \omega_F \tau) - \sin \eta_0] \sin \alpha, \tag{1}$$

where η_0 is the azimuthal position of the Earth in the ecliptic at $\tau = 0$ measured from the line of the ascending nodes and $\xi = J/Mr^2\omega$. Here $\omega, \omega^2 = GM/r^3$, approximately describes the orbital frequency in the absence of rotation and α denotes the inclination of the orbit with respect to the equatorial plane of the Sun [12]. The frequency of this nutational oscillation is the Fokker frequency $\omega_F \approx \frac{3}{2}\epsilon\omega$, where $\epsilon = GM/c^2 r$. The nutation amplitude, $\xi \sin \alpha$, does not depend on the speed of light c. This remarkable fact can be traced back to the occurrence of a small divisor [9-11] involving the Fokker frequency. In the post-Newtonian limit of the post-Schwarzschild approximation, Eq. (1) reduces to $\Theta_n \sim \omega_n \tau$, which represents a *precessional* motion with frequency $\omega_n = \xi \omega_F \sin\alpha \cos\eta_0$ about a direction opposite to that of orbital velocity at $\tau = 0$. Thus, relativistic nutation reduces to a *part* of the Schiff precession in the first post-Newtonian approximation. It follows from this analysis that the first post-Newtonian approximation breaks down over timescales of the order of Fokker period $\tau_F = 2\pi/\omega_F$; however, this fact does not diminish the usefulness of the first post-Newtonian approximation for the description of observations in the solar system since in this case the Fokker period is almost immeasurably long (e.g. $\tau_F \simeq 67$ million years for the motion of the Earth about the Sun).

Let us consider the influence of the gravitomagnetic field of the central body ("Sun") on the relative acceleration of two nearby test particles ("Earth" and "Moon") moving along the circular geodesic orbit. The dominant contributions of the gravitomagnetic field of the central body to the tidal matrix, first calculated by the authors [9-11], are proportional to

$$\omega^2 \xi \sin\alpha \sin\left(\frac{1}{2}\omega_F \tau\right), \tag{2}$$

which is directly proportional to the amplitude of relativistic nutation ($\xi \sin\alpha$) and exhibits a maximum (at $\tau = \tau_F/2$) that is independent of the speed of light c. It follows from Eq. (2) that to first order in $\omega_F \tau \ll 1$, the dominant gravitomagnetic amplitude varies linearly with τ. This secular amplitude originates from a coupling of the *nutation part* of Schiff precession with the amplitude ($\sim \omega^2$) of the Newtonian contribution to the gravity gradient [13]. It should be mentioned in passing that the relativistic quadrupole contributions to the tidal matrix have properties quite similar to the gravitomagnetic tidal effect described here [10].

Let us now turn to the potentially observable effects of the solar gravitomagnetic field on the lunar motion. The lunar path is determined by the Newton-Jacobi equation

$$\frac{d^2x^i}{d\tau^2} + \frac{Gm}{R^3}x^i = -K_{ij}(\tau)x^j, \tag{3}$$

where $x^i, i = 1, 2, 3$, represent the geocentric Fermi coordinates of the Moon, m is the total mass of the Earth-Moon system and $R(\tau)$ denotes the Earth-Moon distance depending on the proper time τ measured along the geocentric path around the Sun. Here K is the tidal matrix. Equation (3) describes the motion of the Moon with respect to a geocentric local inertial frame [14]. Using the equation of relative motion (3), we have calculated—among other things—the influence of the tidal field of the Sun on the orbital angular momentum of the Moon with respect to the Earth. To express the result with respect to the sidereal frame, we choose as our sidereal reference framethe geocentric Fermi frame at $\tau = 0$. This frame is related to the Fermi frame at time τ by a rotation matrix that incorporates the relativistic precession and nutation of the Fermi frame with respect to the sidereal frame. In the first post-Newtonian approximation, this motion reduces to a (Fokker plus Schiff) precession. Let D denote this rotation matrix, then $\mathcal{L}_i = D_{ij}L_j$, where the sidereal components $(\mathcal{L}_i)$ of the orbital angular momentum are obtained from a transformation of the geocentric components (L_j) with

$$D_{ij} = \delta_{ij} - \epsilon_{ijk}\Phi_k \quad , \quad \mathbf{\Phi} = \int_0^\tau \omega_{FS}(\tau')d\tau'. \tag{4}$$

Here the analysis is limited to the first post-Newtonian approximation and ω_{FS} represents the frequency of (Fokker plus Schiff) precession. The direction of Schiff precession is not fixed along the Earth's orbit; therefore, $\mathbf{\Phi}$ contains (cumulative) secular terms (which represent simple precession) together with (harmonic) nutational terms of frequency 2ω and amplitude of order $\alpha\epsilon\xi$. Averaging over the latter terms, the dominant secular terms in $\mathbf{\Phi}$ are given by

$$\Phi_1 \sim \frac{1}{2}\alpha\epsilon\xi\omega\tau\sin\eta_0 \quad , \quad \Phi_2 \sim \left(-\frac{3}{2}+\xi\right)\epsilon\omega\tau \quad , \quad \Phi_3 \sim \frac{1}{2}\alpha\epsilon\xi\omega\tau\cos\eta_0 \tag{5}$$

with respect to the geocentric Fermi frame at $\tau = 0$, which has its first axis essentially along the radial position of the Earth, the third axis approximately along the direction of motion of the Earth and the second axis normal to the ecliptic (in a direction opposite to the Earth's orbital angular momentum about the Sun). We note that for $\xi = 0$, Eq. (5) expresses the de Sitter-Fokker effect that has been observed by Shapiro *et al.* [6]. To illustrate our approach, let us use Eq. (3) to determine the value of $\mathbf{L}$, which is the angular momentum of the Moon in a circular orbit about the Earth with respect to the Fermi frame, averaged over orbital motions of the Earth about the Sun (with frequency ω) and the Moon about the Earth (with frequency Ω). Then

$$\frac{d\langle\mathbf{L}\rangle}{d\tau} = \tilde{\omega}\times\mathbf{L_0} \quad , \tag{6}$$

where $\mathbf{L_0}$ is the unperturbed orbital angular momentum with respect to the geocentric Fermi frame and $\tilde{\omega}$ is given by

$$\tilde{\omega}^1 \approx -\tilde{\omega}_0\alpha\epsilon\xi\left(2\sin\eta_0+\frac{3}{2}\omega\tau\cos\eta_0\right), \tag{7}$$

$$\tilde{\omega}^2 \approx \tilde{\omega}_0(1-6\epsilon\xi), \tag{8}$$

$$\tilde{\omega}^3 \approx -\tilde{\omega}_0\alpha\epsilon\xi\left(2\cos\eta_0-\frac{3}{2}\omega\tau\sin\eta_0\right) \tag{9}$$

to first order in the tidal perturbation characterized by the Newtonian regression frequency $\tilde{\omega}_0 = 3\omega^2/4\Omega$, which corresponds to a period of nearly 18 years [15,16]. It is clear from Eqs. (4)-(9) that the motion of $\langle\mathcal{L}\rangle$ can be expressed as a *Newtonian* regression modulated by long-term (secular) *relativistic* perturbations characterized by the de Sitter-Fokker, Schiff and gravitomagnetic tidal effects. To illustrate this point, let us assume for the sake of simplicity that in the absence of relativistic effects the lunar orbital angular momentum undergoes a steady regression of frequency $\tilde{\omega}_0$ and that once relativistic effects are included the average motion in the Fermi frame is one of precession with the frequency given by Eqs. (7)-(9). It then follows that the expression for $\langle \mathcal{L}_2 \rangle$, i.e. the average of the second sidereal component of the lunar orbital angular momentum, contains a dominant gravitomagnetic contribution of the form

$$\langle \mathcal{L}_2 \rangle_{\text{secular}} \approx 2\alpha\beta\epsilon\xi(\mu R_0^2 \Omega)\omega\tau \sin(\eta_0 + \zeta_0 - \tilde{\omega}_0 \tau), \tag{10}$$

where β, μ, R_0 and ζ_0 denote, respectively, the inclination of the lunar orbit with respect to the ecliptic ($\approx 5^\circ$), the mass of the Moon, the mean Earth-Moon separation and the longitude of the ascending node of the orbit of the Moon measured from the first axis of the sidereal frame. These simple considerations that are based on an initial circular orbit only indicate the nature of the secular terms involved; clearly, extensive calculations are necessary for a complete treatment.

3 Discussion

The results of our theoretical work are of particular interest for the description of dominant relativistic gravitational effects in the motion of the Moon, especially the gravitomagnetic tidal component of the motion of the orbital angular momentum of the Moon. It is important to point out that the eccentricities of the orbit of the Earth around the Sun and the orbit of the Moon about the Earth should be taken into account; we have ignored them in our preliminary analysis [16]. As lunar laser ranging data further accumulate, it may become possible in the future to deduce the angular momentum of the Sun from the measurement of the solar gravitomagnetic contributions to the mean motions of the lunar node and perigee.

It is interesting to compare our secular gravitomagnetic tidal terms with hypothetical terms that might indicate a temporal variation of the gravitational "constant" G. Our results have thus far been based on a secular term proportional to τ in the tidal matrix K in Eq. (3); however, as can be seen from the middle term in Eq. (3), similar effects could be produced if such a term appears in G instead. We have shown that our predictions are similar to a variation of G in Eq. (3) at the level of 10^{-16} yr^{-1}; moreover, there are significant differences between the two effects that can be used to separate them [4,9]. The present upper limit on $|\dot{G}/G|$ is at the level of 10^{-12} yr^{-1}; therefore, it may be a long while before the gravitomagnetic effects in the motion of the Moon become detectable.

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[12] For the motion of the Earth about the Sun, $\alpha \approx 7^\circ$, $\epsilon \approx 10^{-8}$ and $\xi \approx 2 \times 10^{-5}$ based on the standard value of solar angular momentum [cf. C.W. Allen, *Astrophysical Quantities*, 3rd ed. (Athlone, London, 1973)]. The Fermi frame adopted in this paper is such that the nutation vanishes at $\tau = 0$. This frame can be obtained from the results given in Ref. [11]; see, especially, p. 506.

[13] In this connection, it is also important to note that ω_n vanishes for an equatorial orbit ($\alpha = 0$) in contrast to the frequency of the *full* Schiff precession.

[14] The precise definition of the notion of a local geocentric frame was formulated in [11] and further developed in S.M. Kopeikin, Celestial Mech. **44**, 87 (1988) and V.A. Brumberg and S.M. Kopeikin, Nuovo Cimento B **103**, 63 (1989).

[15] It follows from a more complete treatment of the Newtonian problem that the mean motion of the lunar node can be characterized by a backward movement of frequency $\tilde{\omega}_0 N$, which corresponds to a period of about 18.61 years. Similarly, the mean motion of the perigee can be characterized by a forward movement of frequency $\tilde{\omega}_0 P$, which corresponds to a period of about 8.85 years. The theoretical expressions for N and P are rather complicated and depend on ω/Ω as well as the orbital eccentricities, etc. The first two terms of N and P in terms of $\nu = \omega/\Omega$ are given by $N = 1 - 3\nu/8 - \cdots$ and $P = 1 + 75\nu/8 + \cdots$. A detailed discussion of this subtle problem is

given by D. Brouwer and G.M. Clemence, *Celestial Mechanics* (Academic Press, New York, 1961), Ch. 12, especially pp. 320-323.

[16] If our analysis is extended by including a slight eccentricity for the lunar orbit, then the motion of the Runge-Lenz vector indicates an average forward precession of perigee with frequency $\tilde{\omega}_0(1 - 6\epsilon\xi)$. The relativistic gravitoelectric correction to this result is of order ϵ^2 in the post-Newtonian scheme; however, if the standard Schwarzschild coordinates are used in the analysis (cf. Ref. [4]), $\tilde{\omega}_0 \to \tilde{\omega}_0(1 + 3\epsilon)$, so that a slight increase in the forward (backward) precession rate of the perigee (node) would occur to first order in the tidal perturbation. This supersedes (and partly corrects) an assertion in Ref. (21) of our Nuovo Cimento paper (cf. Ref. [4]) regarding the comparison of our results with Robertson's "solar effect." However, a more complete analysis of the relativistic three-body problem is necessary for a reliable calculation of the motion of the lunar perigee.