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The Bernoulli Brothers and the Brachistochrone

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The world of Mathematics in the 17th century was rife with rivalries. Isaac Newton and Gottfried Wilhelm von Leibniz both claimed to have developed the methods of Calculus, and their competing claims split the society of mathematicians into factions. Newton's supporters were primarily British mathematicians. Among those in the Leibniz camp was the Swiss mathematician Johann Bernoulli. Another famous rivalry was one between Johann and his older brother Jakob. One of them would develop some mathematical problem and challenge the other to solve it. Sometimes, money was staked on the outcome of the challenge, and the challenges were often rather public.

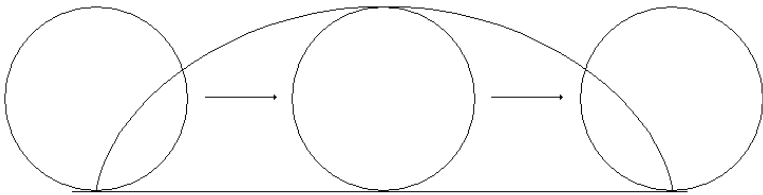
In the June 1696 *Acta Eruditorum*, one of the world's first scientific journals, Johann proposed the following mathematical problem [1, p.645]:

“If two points A and B are given in a vertical plane, to assign to a mobile particle M the path AMB along which, descending under its own weight, it passes from the point A to the point B in the briefest time”.

This problem became known as the Problem of the Brachistochrone (from the Greek words *brachistos* and *chronos*, meaning “shortest” and “time”), and Bernoulli pledged to reveal the name of the solution curve in six months if none were able to find it within that time. Only Leibniz was able to determine the solution within that period, and at his urging,

Bernoulli extended his time limit an additional five months, “in order...that no one might have cause to complain of the shortness of time allotted” [1, p.647]. Bernoulli also restated the problem publicly in January 1697 at the University of Groningen, where he was a Professor of Mathematics, so that “those to whom the...*Acta* is not available” could be given a chance to solve it. Further, he specified that the two given points not be on a single vertical line (since in that special case, the vertical line would be the solution curve), and that the path be frictionless [1, p.647].

This problem was not a new one. In his 1638 publication, *Two New Sciences*, Galileo demonstrated that the Brachistochrone curve was not the straight line between the endpoints, despite being the curve of the shortest distance, and posited that an arc of a circle seemed to be the solution. As it turns out, his supposition was incorrect, but the true answer, the inversion of a curve called the Cycloid, is closely tied to the circle. If a circle is “rolled” along a straight line, the Cycloid is the curve generated by the path of a single point on that rolling circle. The inversion is the same curve, just upside-down.



The Cycloid has many interesting properties. The area under one arch of a cycloid is three times that of its generating circle. The arc length of one arch of a cycloid is eight times the diameter of its generating circle. Christiaan Huygens discovered a particularly remarkable physical property of the inverted Cycloid: that the time it takes for a particle to descend from rest at some point of the curve to the lowest point of the curve is the same, regardless of the starting point (so long as the starting point isn't the lowest point of the curve). Particles starting higher up on the curve accelerate more quickly down

the steeper incline, but must travel a greater distance, while particles starting closer to the lowest point have less distance to travel, but are accelerated less quickly. Because of this property, the inverted Cycloid is often called the “Isochrone” or “Tautochrone” (from the Greek words *isos* and *tauto*, meaning “equal” and “the same”).

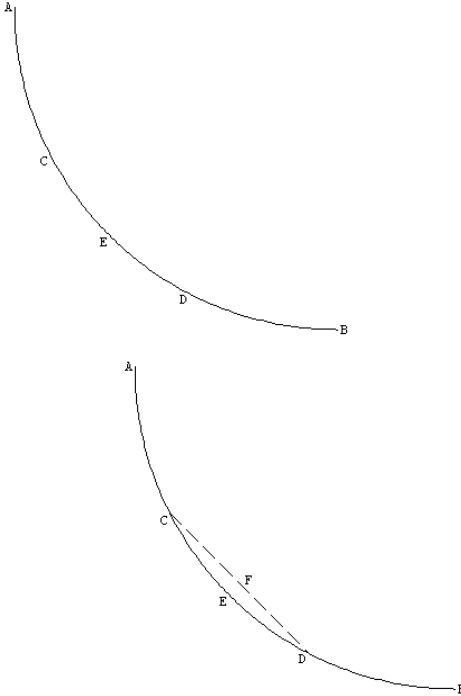
In the proclamation in which Bernoulli restated the Brachistochrone problem, he remarked on the fact that so few had theretofore solved it, and while he didn’t explicitly state any names, a challenge to Isaac Newton can be readily inferred from the phrasing [1, p.648]. Shortly thereafter, the problem came to Newton’s attention. Within twelve hours of receiving it, he solved it, and the result was published in the January 1697 issue of *Philosophical Transactions*, a publication of the Royal Society. In the *Acta Eruditorum* of May 1697, Bernoulli submits his own proof of the solution, along with those of his brother Jakob and the Marquis de L’Hospital, as well as the excerpt from *Philosophical Transactions* containing Newton’s solution. He also acknowledges the solutions of Leibniz and Ehrenfried Walther von Tschirnhaus. Of the six mathematicians whose solutions are mentioned or included, only L’Hospital’s is incorrect.

Johann’s proof is rather innovative. Operating under Fermat’s principle—that light always travels from one point to another along the path that takes the least time—he demonstrates that a beam of light traveling through (and being refracted by) differentially thin layers of varying transparent materials will travel along a Cycloid path. While this does effectively validate Bernoulli’s claim that the problem was a useful one to science, and not merely hypothetical speculation, his method cannot be generalized to apply to other fields or circumstances. His brother Jakob’s method, on the other hand, is generalizable, and was later developed into what is known as the Calculus of Variations.

Jakob precedes his proof with the following Lemma—a sort of preliminary proof, the result of which he uses to help complete his primary proof. My following analysis appears in square brackets.

Lemma:

Let $ACEDB$ be the desired curve along which a heavy point falls from A to B in the shortest time, and let C and D be two points on it as close together as we like.



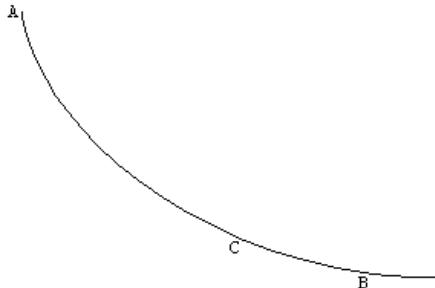
Then the segment of arc CED is among all segments of arc with C and D as end points the segment that a heavy point falling from A traverses in the shortest time. [In other words, there are no other curve segments with endpoints C and D through which a heavy point falling from A would pass in less time than it would through CED .

Demonstrating this fact is the purpose of the Lemma, as it allows him (in his subsequent proof) to focus on any segment of the curve, rather than the curve in its entirety.]

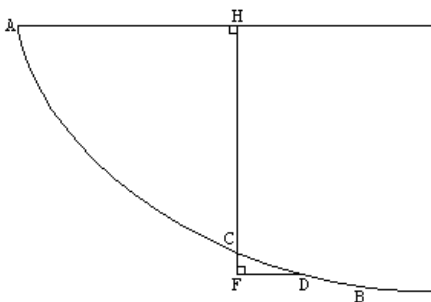
[Proof of Lemma:] Indeed, if another segment of arc CFD were traversed in a shorter time, then the point would move along $ACFDB$ in a shorter time than along $ACEDB$, which is contrary to our supposition.

[End of Proof of Lemma.]

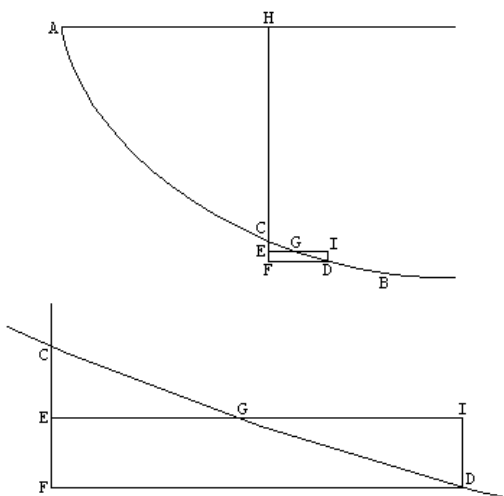
[Proof of the Brachistochrone Problem:] Hence in a plane arbitrarily inclined to the horizon (the plane need not be [vertical]), take ACB as the required curve, on which a heavy point from A reaches B in a shorter time than on any other curve in this plane.



Take on it two points C and D infinitesimally close together [practically the same point, in other words. Bear in mind that, even though (for the sake of clarity) C and D don't look like they are infinitesimally close in the following diagrams, it is something he is taking for granted] and draw the horizontal line AH , the vertical CH , and DF [perpendicular] to it.

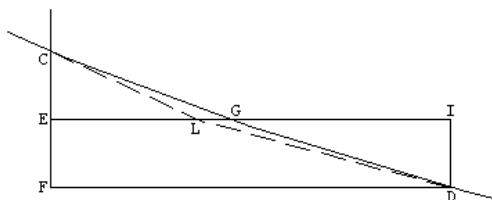


Take E halfway between C and F and complete [rectangle $EIDF$] by means of the line EI [and DI].



On EI we now must determine point G such that the time of fall through $CG +$ the time of fall through GD [which is denoted by $t_{CG} + t_{GD}$, being sure to keep in mind that the fall begins at point A] is a minimum. [In other words, given points C and D , infinitesimally close, on ACB , we want to find a formula that describes how to place point G between them.]

If we now take on the line EI another point L such that GL is incomparably small as compared to EG [so he is assuming that G and L are nearly the same point. He has already assumed that CD is infinitesimally small, so he is taking GL to be infinitesimally small compared to that], and if we draw CL and DL ,



then, [since he is assuming that CGD and CLD are effectively the same path:]

$$t_{CL} + t_{DL} = t_{CG} + t_{GD}$$

[here, “=” is technically only a very close approximation, and should be

taken to mean that the difference between the two things being “equated” is negligibly small] and hence [by subtracting t_{CL} and t_{GD} from both sides]

$$t_{DL} - t_{GD} = t_{CG} - t_{CL}.$$

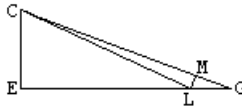
I now reason as follows. According to the nature of the fall of heavy bodies [see Appendix 1 for explanation],

$$\frac{CG}{CE} = \frac{t_{CG}}{t_{CE}}, \quad \text{and} \quad \frac{CL}{CE} = \frac{t_{CL}}{t_{CE}},$$

hence [subtracting the right from the left, we have]

$$\frac{CG - CL}{CE} = \frac{t_{CG} - t_{CL}}{t_{CE}}.$$

If we take a point M on CG such that $CG - CL = GM$,



then we have, because of the similarity of the [“infinitesimal”] triangles LMG and CEG , [that $\frac{GL}{CG} = \frac{GM}{EG}$. Then, multiplying by CG and dividing by CE , we see that $\frac{GL}{CE} = \frac{CG * GM}{CE * EG}$. Since $GM = CG - CL$, we have that

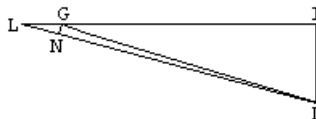
$$\frac{GL}{CE} = \frac{CG * (CG - CL)}{CE * EG} = \frac{CG}{EG} * \frac{(CG - CL)}{CE}. \quad \text{Thus, since } \frac{CG - CL}{CE} = \frac{t_{CG} - t_{CL}}{t_{CE}}, \text{ we have]}$$

$$\frac{GL}{CE} = \frac{CG * (t_{CG} - t_{CL})}{EG * t_{CE}}. \quad (\#)$$

In the same way, we find, according to the nature of the fall of heavy bodies,

$$\frac{GD}{EF} = \frac{t_{GD}}{t_{EF}}, \quad \text{and} \quad \frac{DL}{EF} = \frac{t_{DL}}{t_{EF}}, \quad \text{hence} \quad \frac{DL - GD}{EF} = \frac{t_{DL} - t_{GD}}{t_{EF}}.$$

If we take on DL the point N such that $DL - GD = LN$,



then we have, because of the similarity of [“infinitesimal”] triangles

LNG and GID , [that $\frac{GL}{LN} = \frac{DG}{GI}$. Then, multiplying by LN and dividing by EF , we see that $\frac{GL}{EF} = \frac{DG * LN}{EF * GI} = \frac{DG * (DL - GD)}{EF * GI} = \frac{DG}{GI} * \frac{DL - GD}{EF}$, recalling that $LN = DL - GD$. Thus, since $\frac{DL - GD}{EF} = \frac{t_{DL} - t_{GD}}{t_{EF}}$, and recalling that $EF = CE$ by construction, we have]

$$\frac{GL}{CE} = \frac{DG * (t_{DL} - t_{GD})}{GI * t_{EF}}. \quad (##)$$

By comparison [of (#) and (##)] we obtain

$$\frac{CG * (t_{CG} - t_{CL})}{EG * t_{CE}} = \frac{DG * (t_{DL} - t_{GD})}{GI * t_{EF}},$$

and [multiplying both sides of the equation by the right-hand denominator, dividing by the left-hand numerator, and recalling that $t_{DL} - t_{GD} = t_{CG} - t_{CL}$, we see that]

$$\frac{GI * t_{EF}}{EG * t_{CE}} = \frac{DG * (t_{DL} - t_{GD})}{CG * (t_{CG} - t_{CL})} = \frac{DG}{CG}.$$

But [since C and E are infinitesimally close together, the acceleration due to gravity over CE is negligible, so we can treat the speed of the falling object at point C as the average speed, \bar{v} . Then, because $v_C = \bar{v} = \sqrt{2g * y_C}$ (see Appendix 1 for this extrapolation), and since $\bar{v} = \frac{CE}{t_{CE}}$,

we can see that $t_{CE} = \frac{CE}{\sqrt{2g * y_C}} = \frac{CE}{\sqrt{2g * CH}}$. In the same way, since E and F

are infinitesimally close together and $CE = EF$ by construction, we have

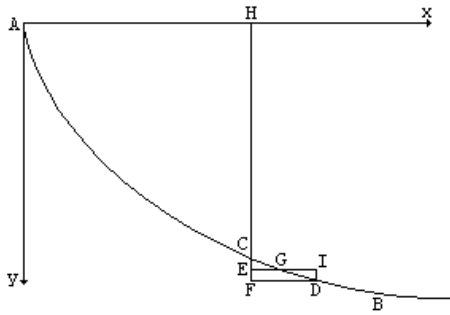
$t_{EF} = \frac{CE}{\sqrt{2g * EH}}$. Then we see that $\frac{t_{EF}}{t_{CE}} = \frac{CE * \sqrt{2g * CH}}{CE * \sqrt{2g * EH}} = \frac{\sqrt{CH}}{\sqrt{EH}}$, so] according to

the law of gravity we have

$$\frac{GI * t_{EF}}{EG * t_{CE}} = \frac{GI * \sqrt{CH}}{EG * \sqrt{EH}},$$

and therefore finally:

$$\frac{GI * \sqrt{CH}}{EG * \sqrt{EH}} = \frac{DG}{CG}.$$



Now EG and GI are elements of the abscissa AH , CG and DG are elements of the curve $[ACB]$, CH and EH their ordinates, and CE and EF elements of the ordinate. [In the notation of modern differential Calculus, we would refer to $EG + GI$ as dx , where AH is some nonnegative value of x , as on the axes above; we would refer to $CG + DG$ as ds ; CH and EH are some nonnegative values of y , as on the axes above; and we would refer to $CE + EF$ as dy . Note: by these labels, we are assuming that the force of gravity is pulling downward in the positive y direction. Further, since GL is “incomparably small as compared to EG ,” GL would be referred to as d^2x , the second differential of x .] The problem can therefore be reduced to the purely geometric one of determining the curve of which the [curve] elements are directly proportional to the elements of the abscissa and indirectly proportional to the square roots of the ordinates. [In other words:

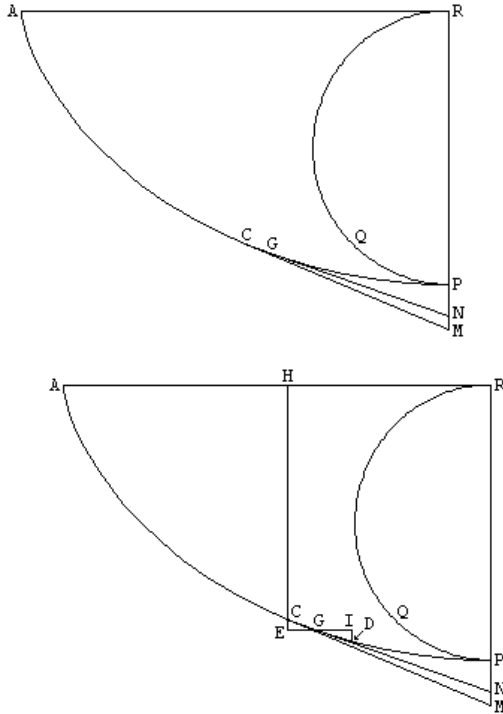
$ds * \sqrt{y} = k * dx$, where k is some nonnegative constant. Then, since $ds^2 = dx^2 + dy^2$, we can square both sides to get $(dx^2 + dy^2) * y = k^2 * dx^2$. Then, $y * dy^2 = (k^2 - y) * dx^2$. Dividing by the quantity $(k^2 - y)$ and

taking the square root of both sides, we have $dy * \sqrt{\frac{y}{k^2 - y}} = dx$, which is of identical form to the differential equation for the cycloid. Rather than stop at that, however, Jakob goes on to prove geometrically that the cycloid fits the equation he derived in the previous proof.]

I find that this property belongs to the Isochrone of Huygens, which therefore is also the Oligochrone [Jakob’s name for the Brachistochrone, from the Greek *oligo*, meaning “scant”], namely the

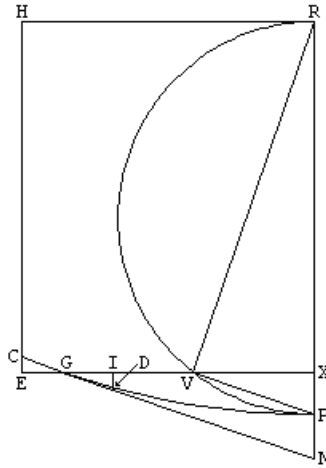
cycloid [2, pp.396-398].

[Proof that the cycloid has the properties determined for the curve of least descent:] Let ACP be a semicycloid; let CM and GN be tangents to the curve in C and G ; let PQR be the [left] semicircle of the generating circle of ACP [extend PR to M . Drop a perpendicular from H on AR through C to E and draw lines EI and DI as before].



[See the figure below. Extend EI to PR , intersecting the semicircle in V and its diameter in X . Draw PV and RV . Recalling that G and D are assumed to be infinitesimally close together, we can treat D as though it is a point on tangent GN . A property of cycloids that Jakob uses here is that Differential Triangle DGI is similar to Triangles PVX and NGX . A property of semicircles he uses is that Triangles PVX , VRX , and PRV are

similar.]



Then [by the similarity of the aforementioned triangles] we have

$$\frac{GI}{DG} = \frac{GX}{GN} = \frac{VX}{PV} = \frac{RX}{RV}$$

[Then, from the equation of the semicircle

$$VX = \sqrt{\left(\frac{1}{2}PR\right)^2 - \left(RX - \frac{1}{2}PR\right)^2} = \sqrt{RX * PR - RX^2}$$

, we can use the Pythagorean

Theorem on Triangle VRX to find that

$$RV = \sqrt{RX^2 + VX^2} = \sqrt{RX^2 + RX * PR - RX^2} = \sqrt{RX * PR}.$$

Thus, recalling that $RX = EH$, we have]

$$\frac{GI}{DG} = \frac{RX}{RV} = \frac{\left(\sqrt{RX}\right)^2}{\sqrt{RX * PR}} = \frac{\sqrt{RX}}{\sqrt{PR}} = \frac{\sqrt{EH}}{\sqrt{PR}}. \quad (\&)$$

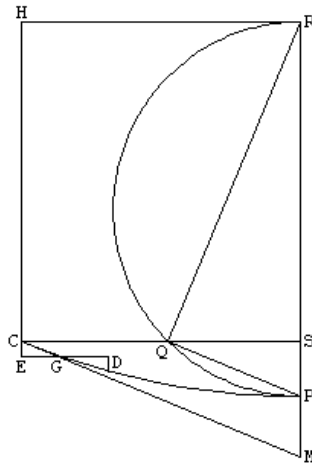
[Now we draw a perpendicular from C to PR , intersecting the semicircle and its diameter at Q and S , respectively. Because of the same properties of semicircles and cycloids from before, we have that

Differential Triangle CEG is similar to Triangles PSQ , MSC , QSR , and PQR . Additionally, just as before, from the equation of the semicircle

$$QS = \sqrt{\left(\frac{1}{2}PR\right)^2 - \left(RS - \frac{1}{2}PR\right)^2} = \sqrt{RS * PR - RS^2}$$

, we can use the Pythagorean

Theorem on Triangle QSR to determine that $QR = \sqrt{RS * PR}$.]



Then we have [recalling that $RS = CH$]

$$\frac{CG}{EG} = \frac{CM}{CS} = \frac{PQ}{QS} = \frac{QR}{RS} = \frac{\sqrt{RS} * \sqrt{PR}}{(\sqrt{RS})^2} = \frac{\sqrt{PR}}{\sqrt{RS}} = \frac{\sqrt{PR}}{\sqrt{CH}} \quad (\&\&)$$

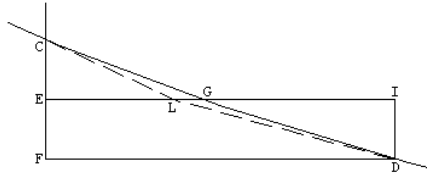
Therefore [multiplying (&) by (&&) and dividing $\frac{GI}{EG}$ by the result yields]

$$\frac{DG}{CG} = \frac{GI * \sqrt{PR} * \sqrt{CH}}{EG * \sqrt{PR} * \sqrt{EH}} = \frac{GI * \sqrt{CH}}{EG * \sqrt{EH}},$$

as was desired [3, p.213]. [End of Proof.]

This problem attracted some of the most famous European mathematicians of the period to attempt to uncover its solution. It was yet another outlet for the fierce competition so typical of that era, and contributed to many scientific advances, including the development of a new field of Calculus. Like Bernoulli, many were fascinated by the fact that the inverted Cycloid is the solution to both the Brachistochrone and Isochrone problems, and others are fascinated by it even today.

Appendix 1



The average speed of a moving body is the distance of travel divided by the time of travel. The statements (regarding “the nature of the fall of heavy bodies”) made by Jakob are equivalent to saying

$$\frac{CG}{t_{CG}} = \frac{CE}{t_{CE}} = \frac{CL}{t_{CL}},$$

which means that the average speed is the same over CG , CE , and CL .

Since C and G are infinitesimally close, we can treat the slope of the curve—and, consequently, the acceleration due to gravity on a body moving along the curve—as constant from C to G . Thus, by the Mean Speed Rule, the average speed, \bar{v} , of a body moving from C to G is

$$\bar{v} = \frac{1}{2}(v_C + v_G).$$

The mean speed rule also holds true for a body moving from C to E or from C to L .

The law of conservation of energy states that the sum of kinetic energy (energy due to movement) and potential energy (energy due to position) remains constant. Assuming that gravity is pulling in the direction of positive y , this can be expressed as

$$\frac{1}{2}m*(v_1)^2 + -m*g*y_1 = \frac{1}{2}m*(v_2)^2 + -m*g*y_2.$$

If we divide everything by m , the mass, and take the initial velocity and y value to be zero, then

$$\frac{1}{2}(v_1)^2 - g*y_1 = 0 = \frac{1}{2}(v_2)^2 - g*y_2.$$

Rearranging and taking the square root of both sides shows

$$v_2 = \sqrt{2g*y_2},$$

so an object’s speed at any point along the curve is directly proportional to the square root of its y value. Thus, since the y value is the same for

points E , L , and G , the object's speed would also be the same at those points. Then the mean speeds over CE , CL , and CG are equal, as was stated.

Works Cited

1. *A Source Book in Mathematics*. Ed. David Eugene Smith. Dover. 1959.
2. *A Source Book in Mathematics, 1200-1800*. Ed. Dirk J. Struik. Harvard University Press. 1969.
3. *Acta Eruditorum*. May, 1697. Ed. Otto Mencke.