Ultrarelativistic motion: inertial and tidal effects in Fermi coordinates

2005 Class. Quantum Grav. 22 195

(http://iopscience.iop.org/0264-9381/22/1/013)

INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **22** (2005) 195–205 doi:10.1088/0264-9381/22/1/013

Ultrarelativistic motion: inertial and tidal effects in Fermi coordinates

C Chicone[1] **and B Mashhoon**[2]

[1] Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA
[2] Department of Physics and Astronomy, University of Missouri-Columbia, Columbia, MO 65211, USA

E-mail: MashhoonB@missouri.edu

Received 7 October 2004, in final form 8 November 2004
Published 9 December 2004
Online at stacks.iop.org/CQG/22/195

Abstract

Fermi coordinates are the natural generalization of inertial Cartesian coordinates to accelerated systems and gravitational fields. We study the motion of ultrarelativistic particles and light rays in Fermi coordinates and investigate inertial and tidal effects beyond the critical speed $c/\sqrt{2}$. In particular, we discuss the black-hole tidal acceleration mechanism for ultrarelativistic particles in connection with a possible origin for high-energy cosmic rays.

PACS number: 04.20.Cv

1. Introduction

A physically meaningful interpretation of the measurement of relative motion from the point of view of an accelerated observer in a gravitational field requires the introduction of a special coordinate system (i.e. Fermi coordinates) along the worldline of the observer. In this coordinate system, the equations of relative motion reveal tidal and inertial effects for ultrarelativistic motion (with speed exceeding the critical value $c/\sqrt{2}$) that are contrary to Newtonian expectations. While there are general treatments of inertial and tidal effects in Fermi coordinates [1–4] and the special case of ultrarelativistic motion of particles has been discussed in our recent papers [5–7], the purpose of this work is to present a more systematic and complete description of motion beyond the critical speed $c/\sqrt{2}$. To this end, the equations of motion in Fermi coordinates are discussed in section 2 and inertial effects are considered in sections 3, 4 and the appendix. Section 5 is devoted to tidal effects. A summary and brief discussion of our results is presented in section 6.

2. Equations of motion in Fermi coordinates

Imagine an accelerated observer in a general relativistic spacetime following a worldline $\bar{x}^\mu(\tau)$, where τ is the proper time along its trajectory. The local axes of the observer are given by an orthonormal tetrad frame $\lambda^\mu{}_{(\alpha)}$ that is carried along its path according to

$$\frac{D\lambda^\mu{}_{(\alpha)}}{d\tau} = \Phi_{(\alpha)}{}^{(\beta)}\lambda^\mu{}_{(\beta)}, \tag{1}$$

where $\Phi_{(\alpha)(\beta)}$ is the antisymmetric acceleration tensor, $\lambda^\mu{}_{(0)} = d\bar{x}^\mu/d\tau$ is the local temporal direction and $\lambda^\mu{}_{(i)}$, $i = 1, 2, 3$, form the local spatial triad, and where (here and throughout this paper) Greek indices run from 0 to 3, Latin indices run from 1 to 3, the signature of the metric is +2 and $c = 1$, unless specified otherwise.

In analogy with the electromagnetic Faraday tensor, we decompose the acceleration tensor into its 'electric' and 'magnetic' components $\Phi_{(\alpha)(\beta)} \to (-\mathbf{a}, \boldsymbol{\omega})$; these are given by the translational acceleration $\mathbf{a}(\tau)$ and the rotational frequency $\boldsymbol{\omega}(\tau)$, respectively. That is, the acceleration of the reference trajectory is given by $a^i\lambda^\mu{}_{(i)}$ and $\boldsymbol{\omega}$ is the frequency of rotation of the spatial triad with respect to a local nonrotating (i.e. Fermi–Walker transported) triad.

Let us next establish a Fermi coordinate system $(T, \mathbf{X})$ in a neighbourhood of the reference worldline [1]. It turns out that Fermi coordinates can be assigned uniquely only to spacetime events that are within a cylindrical region of finite radius along the observer's worldline. At each event $\bar{x}^\mu(\tau)$ on the observer's worldline, consider all spacelike geodesic curves that are normal to this reference worldline at this event. Each event x^μ on the resulting hypersurface and within the cylindrical region under consideration is connected to $\bar{x}^\mu(\tau)$ by a unique spacelike geodesic curve that starts at $\bar{x}^\mu(\tau)$ and whose tangent vector ξ^μ at this event is normal to the reference worldline (that is, $\xi_\mu\lambda^\mu{}_{(0)} = 0$). The event x^μ is assigned the Fermi coordinates $X^\mu = (T, \mathbf{X})$, where $T = \tau$ and

$$X^i = \sigma\xi^\mu\lambda_\mu{}^{(i)}. \tag{2}$$

Here σ is the proper length of the normal spacelike geodesic segment connecting $\bar{x}^\mu(\tau)$ to x^μ. The Fermi coordinate system is admissible in a cylindrical spacetime region around $\bar{x}^\mu(\tau)$ with radius $|\mathbf{X}| \sim \mathcal{R}$, where $\mathcal{R}$ is the minimum radius of curvature of spacetime along the reference worldline [2].

The observer, at each instant $\tau = T$ of its proper time 'sees' a locally Euclidean three-dimensional space and 'instantaneously' determines distances within it using the (spacelike) geodesic lengths beginning from its position at the spatial origin of the new Fermi coordinates. A nearby particle worldline that punctures the sequence of three-dimensional spaces at $\mathbf{X} = (X(T), Y(T), Z(T))$ is thus a graph over the reference worldline and each point on the graph is connected to the observer by a 'locally straight' line (i.e. a geodesic) that is normal to the observer's worldline. From the viewpoint of the observer, the particle has relative coordinate velocity $\mathbf{V} = d\mathbf{X}/dT$ and relative coordinate acceleration $d\mathbf{V}/dT$.

The general equation of motion for a test particle of mass m in the Fermi frame is

$$\frac{d^2X^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{ds}\frac{dX^\beta}{ds} = A^\mu, \tag{3}$$

where s is the proper time along its worldline, $-ds^2 = g_{\mu\nu}(T, \mathbf{X})\, dX^\mu\, dX^\nu$ and $mA^\mu = F^\mu$ is the external force acting on the particle.

Equation (3) can be written as the system

$$\frac{d^2T}{ds^2} + \Gamma^0_{\alpha\beta}\frac{dX^\alpha}{ds}\frac{dX^\beta}{ds} = A^0, \tag{4}$$

$$\frac{d^2X^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dX^\alpha}{ds}\frac{dX^\beta}{ds} = A^i. \tag{5}$$

Using the identity

$$\frac{d^2X^i}{ds^2} = \frac{d^2T}{ds^2}V^i + \Gamma^2\frac{d^2X^i}{dT^2}, \tag{6}$$

and the Lorentz factor of the particle

$$\Gamma(T, \mathbf{X}) := \frac{dT}{ds}, \tag{7}$$

equation (5) can be expressed as

$$\frac{d^2X^i}{dT^2} + \left(\Gamma^i_{\alpha\beta} - \Gamma^0_{\alpha\beta}V^i\right)\frac{dX^\alpha}{dT}\frac{dX^\beta}{dT} = \frac{1}{\Gamma^2}(A^i - A^0V^i). \tag{8}$$

Let $U^\mu := dX^\mu/ds$. We note that $U^\mu = \Gamma(1, \mathbf{V})$ and the physical content of the equation of motion (3) is contained in equation (8) together with $U_\mu U^\mu = -1$ and $U_\mu A^\mu = 0$. These can be written respectively as

$$\Gamma = \frac{1}{\sqrt{-g_{00} - 2g_{0i}V^i - g_{ij}V^iV^j}} \tag{9}$$

and

$$A^0 = -\frac{g_{0j} + g_{ij}V^i}{g_{00} + g_{0i}V^i}A^j. \tag{10}$$

It is crucial to recognize that in Fermi coordinates the velocity $\mathbf{V}$ satisfies the condition $|\mathbf{V}| \leqslant 1$ only at $\mathbf{X} = 0$; indeed, away from the reference trajectory $|\mathbf{V}|$ could in principle exceed unity in accordance with the equation of motion (8).

3. Inertial effects

It is natural to begin our discussion with an accelerated observer in Minkowski spacetime. In this case, spacelike geodesic segments are straight lines; therefore, the construction of Fermi coordinates is simple. In fact, equation (2) reduces to $x^\mu - \bar{x}^\mu(\tau) = X^i\lambda^\mu{}_{(i)}(\tau)$. Differentiating this relation we find that

$$dx^\mu = \left[P\lambda^\mu{}_{(0)} + Q^j\lambda^\mu{}_{(j)}\right]dX^0 + \lambda^\mu{}_{(i)}\,dX^i, \tag{11}$$

where P and Q are given by

$$P(T, \mathbf{X}) = 1 + \mathbf{a}(T)\cdot\mathbf{X}, \qquad \mathbf{Q}(T, \mathbf{X}) = \boldsymbol{\omega}(T)\times\mathbf{X}. \tag{12}$$

The components $g_{\mu\nu}$ of the Minkowski metric tensor in Fermi coordinates are given by

$$g_{00} = -P^2 + Q^2, \qquad g_{0i} = Q_i, \qquad g_{ij} = \delta_{ij}. \tag{13}$$

Moreover, $\det(g_{\mu\nu}) = -P^2$ and the inverse metric is given by

$$g^{00} = -\frac{1}{P^2}, \qquad g^{0i} = \frac{Q_i}{P^2}, \qquad g^{ij} = \delta_{ij} - \frac{Q_iQ_j}{P^2}. \tag{14}$$

The Christoffel symbols can be evaluated using (13) and (14); the nonzero components are

$$\Gamma^0_{00} = \frac{\mathbf{S}\cdot\mathbf{X}}{P}, \qquad \Gamma^0_{0i} = \frac{a_i}{P}, \qquad \Gamma^i_{0j} = -\left(\epsilon_{ijk}\omega^k + \frac{Q_ia_j}{P}\right), \tag{15}$$

$$\Gamma^i_{00} = Pa_i - \frac{\mathbf{S}\cdot\mathbf{X}}{P}Q_i + [\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{X}) + \dot{\boldsymbol{\omega}}\times\mathbf{X}]_i, \tag{16}$$

where an overdot denotes differentiation with respect to T and $\mathbf{S}$ is defined by

$$\mathbf{S}(T) := \dot{\mathbf{a}} + \mathbf{a} \times \boldsymbol{\omega}. \tag{17}$$

In Minkowski spacetime, the Fermi coordinates are admissible for $P^2 > Q^2$. We note that the structure of the boundary region $P^2 = Q^2$ has been discussed in detail [8]. In general, the boundary of the admissible region at a given time T is a real quadric cone. If $\boldsymbol{\omega} = 0$, this surface degenerates into coincident planes. Moreover, if $\boldsymbol{\omega} \neq 0$, but $\mathbf{a} \cdot \boldsymbol{\omega} = 0$, the boundary surface is a hyperbolic cylinder for $|\boldsymbol{\omega}|^2 < |\mathbf{a}|^2$, a parabolic cylinder for $|\boldsymbol{\omega}|^2 = |\mathbf{a}|^2$ and an elliptic cylinder for $|\boldsymbol{\omega}|^2 > |\mathbf{a}|^2$. As is well known, for $\mathbf{a} = 0$, the boundary surface is a circular cylinder of radius $|\boldsymbol{\omega}|^{-1}$.

We now consider the equations of motion in Fermi coordinates. Equations (9) and (10) are given by

$$\frac{1}{\Gamma^2} = P^2 - (\mathbf{Q} + \mathbf{V})^2 \geqslant 0, \tag{18}$$

$$A^0 = \frac{(\mathbf{Q} + \mathbf{V}) \cdot \mathbf{A}}{P^2 - Q^2 - \mathbf{Q} \cdot \mathbf{V}}. \tag{19}$$

Hence, equation (8) has the form

$$\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} + 2\boldsymbol{\omega} \times \mathbf{V} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{X}) + \dot{\boldsymbol{\omega}} \times \mathbf{X} + P\mathbf{a} - \frac{1}{c^2 P}(\mathbf{Q} + \mathbf{V})(\mathbf{S} \cdot \mathbf{X} + 2\mathbf{a} \cdot \mathbf{V}) = \mathcal{F}, \tag{20}$$

where $\mathcal{F}$ represents the external force per unit mass

$$\mathcal{F} = \frac{1}{\Gamma^2}\left[\mathbf{A} - \frac{(\mathbf{Q} + \mathbf{V}) \cdot \mathbf{A}}{c^2(P^2 - Q^2 - \mathbf{Q} \cdot \mathbf{V})}\mathbf{V}\right]. \tag{21}$$

It is interesting to note that in equation (20) the purely rotational inertial accelerations are essentially the same as in the nonrelativistic theory [7, 9].

Inertial accelerations have been discussed by a number of authors [3, 4, 7, 10–15]. In particular, it has been shown [7] that for $\boldsymbol{\omega} = 0$ and $\dot{\mathbf{a}} = 0$, the inertial acceleration experienced by the particle parallel to its motion is given to the lowest order in $\mathbf{a}$ by $-\mathbf{a} \cdot \hat{\mathbf{V}}(1 - 2V^2/c^2)$, where $\hat{\mathbf{V}} = \mathbf{V}/V$ is the unit vector tangent to the spatial path of the particle; therefore, there is a sign reversal for $V > V_c = c/\sqrt{2}$ with consequences that are contrary to Newtonian expectations.

Note that the acceleration $\mathbf{A}$ and the inertial acceleration $\mathbf{a}$ occur in rather different ways in equation (20). This difference is a consequence of the absolute character of acceleration in the theory of relativity; that is, the fact that an observer is accelerated is independent of the choice of coordinates. In particular, while the critical speed associated with $\mathbf{A}$ is c (on the basis of equation (21)), the critical speed associated with $\mathbf{a}$ in equation (20) is $c/\sqrt{2}$.

To obtain the limiting case of lightlike motion, we let $\mathrm{d}s = m\,\mathrm{d}\lambda$, where λ is an affine parameter. In the limit as $m \to 0$, we have that $P^2 = (\mathbf{Q} + \mathbf{V})^2$; hence, equation (20) is valid with $\mathcal{F} = 0$ (see the appendix).

4. Rindler observer

For a Rindler observer [16] (that is, an observer in hyperbolic motion in Minkowski spacetime), consider an inertial frame in which the Rindler observer is at rest at $(0, 0, 0, z_0)$ and has uniform translational acceleration g along the z-axis. The observer's worldline is given by

$$\bar{t} = \frac{1}{g}\sinh g\tau, \qquad \bar{x} = \bar{y} = 0, \qquad \bar{z} = z_0 + \frac{1}{g}(-1 + \cosh g\tau). \tag{22}$$

The nonrotating orthonormal tetrad frame along this worldline has the form

$$\lambda^{\mu}{}_{(0)} = \gamma(1, 0, 0, \beta), \qquad \lambda^{\mu}{}_{(1)} = (0, 1, 0, 0), \tag{23}$$

$$\lambda^{\mu}{}_{(2)} = (0, 0, 1, 0), \qquad \lambda^{\mu}{}_{(3)} = \gamma(\beta, 0, 0, 1), \tag{24}$$

where $\beta = \tanh g\tau$ and $\gamma = \cosh g\tau$, and the transformation from Fermi to inertial coordinates is

$$t = \left(Z + \frac{1}{g}\right)\sinh gT, \qquad x = X, \qquad y = Y, \tag{25}$$

$$z = z_0 - \frac{1}{g} + \left(Z + \frac{1}{g}\right)\cosh gT. \tag{26}$$

The Rindler coordinates are admissible for $T, X, Y \in (-\infty, \infty)$ and $Z \in (-1/g, \infty)$. We note that the manifold where $Z = -1/g$ is the Rindler horizon. It is related to the null cone in the inertial frame, since equations (25) and (26) imply that $(Z+1/g)^2 = (z - z_0 + 1/g)^2 - t^2$.

Imagine a free particle moving along the z-axis with speed v_0, $-1 < v_0 < 1$, according to

$$z = v_0 t + z_0. \tag{27}$$

From the viewpoint of the Rindler observer, this motion is given by

$$Z = \frac{1}{g}\left(-1 + \frac{1}{\cosh gT - v_0 \sinh gT}\right). \tag{28}$$

For $v_0 < 0$, the particle descends monotonically towards the horizon. For $v_0 > 0$, its path crosses the path of the Rindler observer ($Z = 0$) at two events corresponding to $T = t = 0$ and $T = T_1$, where

$$T_1 = \frac{1}{g}\ln\left(\frac{1+v_0}{1-v_0}\right). \tag{29}$$

This value corresponds to the inertial time $t_1 = 2\gamma_0^2 v_0/g$, where γ_0 is the Lorentz factor associated with v_0. At either event

$$\left.\frac{\mathrm{d}^2 Z}{\mathrm{d}T^2}\right|_{Z=0} = -g\big(1 - 2v_0^2\big). \tag{30}$$

On the other hand, $\dot{Z}\,(T = 0) = v_0$ and $\dot{Z}\,(T = T_1) = -v_0$. That is, the free particle ascends from $Z = 0$ to $Z_0 = (\gamma_0 - 1)/g$ at $T_0 = T_1/2$ such that $\tanh gT_0 = v_0$, then descends back to $Z = 0$ at T_1 and finally approaches the horizon $Z = -1/g$ as $T \to \infty$. For $v_0 = 1/\sqrt{2}$, the particle has no acceleration at the crossing events.

We note that for ultrarelativistic motion beyond the critical speed ($v_0 > 1/\sqrt{2}$), the inertial acceleration of the free particle has the opposite sign at crossing events compared to intuitive expectations based on Newtonian mechanics; but other aspects of the motion are not affected. For instance, the critical speed does not appear to play a role in the proper temporal intervals between the crossing events. The relevant proper time of the Rindler observer T_1 given by equation (29) is always less than the proper time of the free particle $t_1/\gamma_0 = 2\gamma_0 v_0/g$, since for $0 < v_0 < 1$,

$$\ln\left(\frac{1+v_0}{1-v_0}\right) < \frac{2v_0}{\sqrt{1-v_0^2}}. \tag{31}$$

That is, the length of the geodesic segment is maximum. This circumstance, usually called 'the twin paradox', is a reflection of the absolute character of acceleration in the theory of relativity.

For the limiting case of a null ray ($v_0 = \pm 1$), $P = 1 + gZ = \pm\dot{Z} = \exp(\pm gT)$ and $\ddot{Z} = g\exp(\pm gT)$. The ray thus ascends monotonically towards infinity or descends monotonically towards the horizon always with positive acceleration. Moreover, the Fermi coordinate speed of the ray can range from zero to infinity.

5. Tidal effects

We now turn our attention to motion in a gravitational field [1–4]. The spacetime metric in Fermi coordinates then takes the form

$$g_{00} = -P^2 + Q^2 - {}^F R_{0i0j} X^i X^j + O(|\mathbf{X}|^3), \tag{32}$$

$$g_{0i} = Q_i - \tfrac{2}{3}{}^F R_{0jik} X^j X^k + O(|\mathbf{X}|^3), \tag{33}$$

$$g_{ij} = \delta_{ij} - \tfrac{1}{3}{}^F R_{ikjl} X^k X^l + O(|\mathbf{X}|^3), \tag{34}$$

where

$${}^F R_{\alpha\beta\gamma\delta}(T) = R_{\mu\nu\rho\sigma}\lambda^{\mu}{}_{(\alpha)}\lambda^{\nu}{}_{(\beta)}\lambda^{\rho}{}_{(\gamma)}\lambda^{\sigma}{}_{(\delta)} \tag{35}$$

is the projection of the Riemann tensor on the tetrad frame of the observer.

To simplify matters, we concentrate on tidal effects only and assume that the observer follows a geodesic along which a Fermi coordinate system is constructed based on a parallel-propagated tetrad frame. Moreover, we neglect external forces on the test particle. In this case, the equations of motion of the free particle in the Fermi system, i.e. equations (8)–(10), reduce to

$$\frac{\mathrm{d}^2 X^i}{\mathrm{d}T^2} + {}^F R_{0i0j} X^j + 2\,{}^F R_{ikj0} V^k X^j + \left(2\,{}^F R_{0kj0} V^i V^k + \frac{2}{3}{}^F R_{ikj\ell} V^k V^\ell + \frac{2}{3}{}^F R_{0kj\ell} V^i V^k V^\ell\right) X^j + O(|\mathbf{X}|^2) = 0, \tag{36}$$

and

$$\frac{1}{\Gamma^2} = 1 - V^2 + {}^F R_{0i0j} X^i X^j + \frac{4}{3}{}^F R_{0jik} X^j V^i X^k + \frac{1}{3}{}^F R_{ikj\ell} V^i X^k V^j X^\ell + O(|\mathbf{X}|^3) \geqslant 0. \tag{37}$$

Equality holds for null rays ($\Gamma = \infty$). In this case the right-hand side of equation (37) is a first integral of the differential equation (36) and higher-order tidal terms cannot be neglected. The Fermi coordinate system is admissible within a cylindrical region along the observer's worldline with $|\mathbf{X}| < \mathcal{R}$, where $\mathcal{R}^{-2}(T)$ is the supremum of $|{}^F R_{\alpha\beta\gamma\delta}(T)|$. Equation (36) is the *geodesic deviation equation* in Fermi coordinates, since it represents the relative motion of the free particle with respect to the fiducial observer that has been assumed here to follow a geodesic.

To illustrate the tidal effects for test particles and null rays, we consider the gravitational field of a Kerr black hole. The Kerr metric, in Boyer–Lindquist coordinates (t, r, ϑ, ϕ), is given by

$$-\mathrm{d}s^2 = -\mathrm{d}t^2 + \Sigma\left(\frac{1}{\Delta}\,\mathrm{d}r^2 + \mathrm{d}\vartheta^2\right) + (r^2 + a^2)\sin^2\vartheta\,\mathrm{d}\phi^2 + 2GM\frac{r}{\Sigma}(\mathrm{d}t - a\sin^2\vartheta\,\mathrm{d}\phi)^2, \tag{38}$$

where $\Sigma = r^2 + a^2\cos^2\vartheta$ and $\Delta = r^2 - 2GMr + a^2$. The Kerr source has mass M and angular momentum $J = Ma$. For a black hole, the specific angular momentum a is such that $0 \leqslant a \leqslant GM$. The observer is taken to be on an escape trajectory along the axis of rotation: it starts at $\tau = 0$ with $r_0 > \sqrt{3}a$ on this axis and moves radially outward reaching infinity with zero kinetic energy. Its geodesic worldline is given by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{r^2 + a^2}{r^2 - 2GMr + a^2}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{\frac{2GMr}{r^2 + a^2}}. \tag{39}$$

To determine Fermi coordinates, we choose a nonrotating orthonormal tetrad frame along the observer's worldline such that $\lambda^\mu{}_{(3)}$ is parallel to the z-axis. In (t, r, ϑ, ϕ) coordinates, we have

$$\lambda^\mu{}_{(0)} = (\dot{t}, \dot{r}, 0, 0), \qquad \lambda^\mu{}_{(3)} = (\dot{t}\dot{r}, 1, 0, 0), \tag{40}$$

where $\dot{t} = \mathrm{d}t/\mathrm{d}\tau$ and $\dot{r} = \mathrm{d}r/\mathrm{d}\tau$ are given in display (39). Using the axial symmetry of the spacetime, $\lambda^\mu{}_{(1)}$ and $\lambda^\mu{}_{(2)}$ can be chosen uniquely up to a rotation about the z-axis. Once a pair is chosen at $\tau = 0$, $\lambda^\mu{}_{(1)}$ and $\lambda^\mu{}_{(2)}$ are parallel propagated along the reference worldline. We do not require the explicit transformation from Boyer–Lindquist to Fermi coordinates to obtain the curvature components that appear in the approximate equations of motion (36)–(37). Indeed, the symmetries of the Riemann tensor make it possible to represent these quantities as elements of a symmetric 6×6 matrix with indices that range over the set $\{01, 02, 03, 23, 31, 12\}$. The result is

$$\begin{bmatrix} E & B \\ B & -E \end{bmatrix}, \tag{41}$$

where E and B are 3×3 matrices that are symmetric and traceless and correspond respectively to the electric and magnetic parts of the Riemann tensor. Moreover, $E = \mathrm{diag}(-k/2, -k/2, k)$ and $B = \mathrm{diag}(-q/2, -q/2, q)$, where

$$k = -2GM\frac{r(r^2 - 3a^2)}{(r^2 + a^2)^3}, \tag{42}$$

$$q = 2GMa\frac{3r^2 - a^2}{(r^2 + a^2)^3}. \tag{43}$$

The curvature components are computed along the worldline of the observer, i.e. at $(T, \mathbf{0})$ in Fermi coordinates. The electric and magnetic curvatures in (42) and (43), respectively, only depend upon r. Therefore, it suffices to integrate the equation for $\dot{r}$ in (39) to obtain $r(\tau)$, which when substituted in (42) and (43) with $\tau \to T$ results in $k(T) < 0$ and $q(T) > 0$, respectively. The equation of motion (36) can thus be written as the system

$$\ddot{X} - \tfrac{1}{2}kX\left(1 - 2\dot{X}^2 + \tfrac{4}{3}\dot{Y}^2 - \tfrac{2}{3}\dot{Z}^2\right) + \tfrac{1}{3}k\dot{X}(5Y\dot{Y} - 7Z\dot{Z}) \\ + q[\dot{X}\dot{Y}\dot{Z}X - \dot{Z}Y(1 + \dot{X}^2) - 2\dot{Y}Z] = 0, \tag{44}$$

$$\ddot{Y} - \tfrac{1}{2}kY\left(1 - 2\dot{Y}^2 + \tfrac{4}{3}\dot{X}^2 - \tfrac{2}{3}\dot{Z}^2\right) + \tfrac{1}{3}k\dot{Y}(5X\dot{X} - 7Z\dot{Z}) \\ - q[\dot{X}\dot{Y}\dot{Z}Y - \dot{Z}X(1 + \dot{Y}^2) - 2\dot{X}Z] = 0, \tag{45}$$

$$\ddot{Z} + kZ\left[1 - 2\dot{Z}^2 + \tfrac{1}{3}(\dot{X}^2 + \dot{Y}^2)\right] + \tfrac{2}{3}k\dot{Z}(X\dot{X} + Y\dot{Y}) - q(X\dot{Y} - \dot{X}Y)(1 - \dot{Z}^2) = 0, \tag{46}$$

where we have neglected higher-order tidal terms for the sake of simplicity. These generalized Jacobi equations reduce to the standard Jacobi equations when the relative motion is so slow that the velocity terms can be neglected.

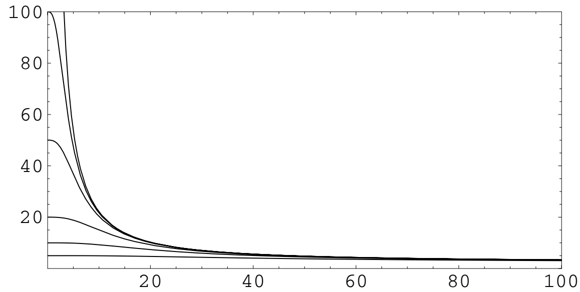

Figure 1. Graphs of $\Gamma_{\|}$ defined in (48) versus $T/(GM)$ approximated by numerical integration of equation (47) with initial $\Gamma_{\|}$ equal to 5, 10, 20, 50, 100 and 1000. We assume that at $T = 0, r_0 = 20GM$ and $Z(0) = 0$. In these graphs $a = GM$; however, the corresponding graphs with $a = 0$ are indistinguishable from those given here.

Consider, for instance, the one-dimensional motion of the test particle along the z-direction. Equations (44)–(46) with $X(T) = Y(T) = 0$ for $T \geqslant 0$ reduce to

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}T^2} + k(1 - 2\dot{Z}^2)Z = 0. \tag{47}$$

The behaviour of the solutions of this equation, which contains the critical speed $V_c = 1/\sqrt{2}$, has been studied in detail in our previous work [5]. Suppose that $Z(T = 0) = 0$ and $\dot{Z}(T = 0) > 1/\sqrt{2}$; then, the particle starts from the observer's location and decelerates along the z-axis towards the critical speed $1/\sqrt{2}$. In this case Γ, given by equation (37), reduces to

$$\Gamma_{\|} = \frac{1}{\sqrt{1 - \dot{Z}^2 + kZ^2}}, \tag{48}$$

whose behaviour is depicted in figure 1. This extends our previous results [5] and is consistent with the rapid decrease of curvature k away from the black hole.

A more interesting situation arises for motion normal to the direction of rotation of the black hole. Equations (44)–(46) with $Y(T) = Z(T) = 0$ for $T \geqslant 0$ reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}T^2} - \frac{1}{2}k(1 - 2\dot{X}^2)X = 0, \tag{49}$$

which turns out to be the general equation for one-dimensional radial motion perpendicular to the rotation axis of the black hole as a consequence of the axial symmetry of the configuration under consideration here. The corresponding Lorentz factor is given by (37) as

$$\Gamma_{\perp} = \frac{1}{\sqrt{1 - \dot{X}^2 - \frac{1}{2}kX^2}}. \tag{50}$$

We note that for ultrarelativistic motion, the particles gain tidal energy and reach the speed of light as in figure 2 for an extreme Kerr black hole ($a = GM$) and in figure 3 for a Schwarzschild black hole ($a = 0$). It follows from a comparison of figures 2 and 3 that the tidal acceleration mechanism does not depend sensitively upon the specific angular momentum of the black hole; in fact, it appears that the time it takes for $\Gamma_{\perp}$ to diverge is in general slightly longer for $a = GM$ as compared to the $a = 0$ case. Moreover, the divergence of $\Gamma_{\perp}$ indicates that our test-particle approach breaks down.

In two recent papers [6], we approximated $\Gamma_{\perp}$ by $(1 - \dot{X}^2)^{-1/2}$ and plotted its approach towards infinity as a consequence of the ultrarelativistic tidal acceleration mechanism. The present calculation of $\Gamma_{\perp}$ is based on an improved approximation scheme and goes beyond

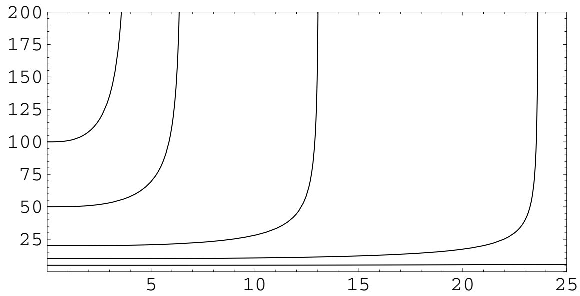

Figure 2. Graphs of $\Gamma_\perp$ defined in (50) versus $T/(GM)$ approximated by numerical integration of equation (49) with $a = GM$ and initial $\Gamma_\perp$ equal to 5, 10, 20, 50 and 100. We assume that at $T = 0$, $r_0 = 20GM$ and $X(0) = 0$. Note that $\Gamma_\perp$ diverges at $T/(GM) \approx 23.7$, 13.1, 6.5 and 4.0 for the initial Lorentz factors $\Gamma_\perp(0) = 10$, 20, 50 and 100, respectively.

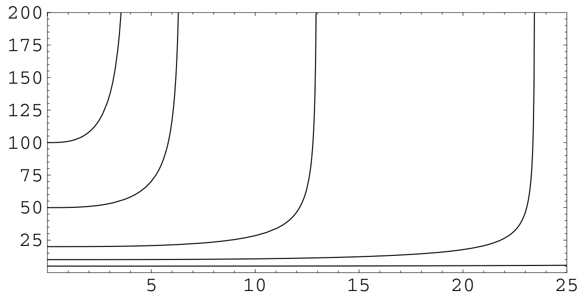

Figure 3. Graphs of $\Gamma_\perp$ defined in (50) versus $T/(GM)$ approximated by numerical integration of equation (49) with $a = 0$ and initial $\Gamma_\perp$ equal to 5, 10, 20, 50 and 100. We assume that at $T = 0$, $r_0 = 20GM$ and $X(0) = 0$. Note that $\Gamma_\perp$ diverges at $T/(GM) \approx 23.5$, 13.0, 6.4 and 3.9 for the initial Lorentz factors $\Gamma_\perp(0) = 10$, 20, 50 and 100, respectively.

our previous work. To reach the speed of light—or, equivalently, for the timelike particle worldline to become null—the Fermi speed $\dot{X}$ should reach a value that is greater than unity as k in equation (50) is negative. We expect that this value of $\dot{X}$ is reached at a later time T as demonstrated by a comparison of figures 2 and 3 with our previous results [6]. Even the present calculations are approximate as they are based on dropping the higher-order tidal terms. It is clear that at a sufficiently long time $T \sim \mathcal{R}$, the Fermi coordinate system breaks down. The kinematic breakdown of Fermi coordinates, i.e. their inadmissibility, is logically independent of the divergence of $\Gamma_\perp$, which indicates the failure of the dynamical theory presented here. Our test-particle approximation ignores any back reaction; furthermore, for charged particles the electromagnetic field configuration near the black hole could enhance or hinder the tidal acceleration mechanism. In any case, this breakdown of our dynamics nevertheless implies that sufficiently energetic particles emerge from the vicinity of the black hole after having experienced tidal acceleration by the gravitational field of the collapsed source. In this way, highly energetic particles may be created by microquasars in our galaxy. It is important to emphasize that the tidal acceleration mechanism is independent of the horizon structure of the black hole. These results are interesting in connection with the origin of the highest energy cosmic rays since cosmic rays with energies above the Greisen–Zatsepin–Kuzmin limit ($\sim 10^{20}$ eV) are not expected to reach the Earth from distant galaxies [17].

6. Discussion

Fermi coordinates constitute a geodesic coordinate system that is a natural extension of the standard Cartesian system and is indispensable for the theory of measurement in relativistic physics. We have discussed the general equation of motion of a pointlike test particle, as well as the limiting case of a null ray, in Fermi coordinates. Inertial and tidal effects of ultrarelativistic particles with speeds above the critical speed $c/\sqrt{2} \simeq 0.7c$ have been emphasized. This work goes beyond our previous work in this direction and strengthens the basis for our results and conclusions [5–7].

Acknowledgments

BM is grateful to Donato Bini and Robert Jantzen for helpful correspondence.

Appendix. Null rays

Let us compute the effective external force per unit mass given by equation (21) for the electromagnetic case, namely,

$$A^\mu = \frac{\hat{q}}{m} F^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \tag{A.1}$$

in accordance with the Lorentz force law for the motion of a particle of charge $\hat{q}$ in an external field $F_{\mu\nu}$. Neglecting radiation reaction, we find that

$$\mathcal{F}^i = \hat{q}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}T}\right)\left(F^i{}_0 + F^i{}_j V^j - F^0{}_j V^i V^j\right), \tag{A.2}$$

where λ is the affine parameter defined by $\mathrm{d}s = m\,\mathrm{d}\lambda$.

A massless charged particle does not exist; therefore, we assume that $\hat{q} \to 0$ as $m \to 0$. With this assumption, it turns out that the massless limit of the trajectory of a charged particle is a null geodesic.

There exist nongeodesic null curves in Minkowski spacetime. These may be physically interpreted as follows: imagine the path of a null electromagnetic ray that is reflected from a collection of mirrors fixed in different places in space. The path consists of straight (i.e. geodesic) null segments separated by mirrors. Next imagine a limiting situation involving an infinite number of such idealized pointlike mirrors. The resulting smooth curve is a nongeodesic null curve. An example of such a curve is

$$x^\mu(\lambda) = (\alpha\lambda, \alpha\sin\theta\sin\lambda, \alpha\sin\theta\cos\lambda, \alpha\lambda\cos\theta), \tag{A.3}$$

where α and θ are constants. The generalization to curved spacetime is straightforward.

References

[1] Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland)
[2] Marzlin K-P 1994 *Gen. Rel. Grav.* **26** 619
Marzlin K-P 1994 *Phys. Rev.* D **50** 888
Marzlin K-P 1996 *Phys. Lett.* A **215** 1
[3] Mashhoon B 1975 *Astrophys. J.* **197** 705
Mashhoon B 1977 *Astrophys. J.* **216** 591
[4] Ni W-T and Zimmermann M 1978 *Phys. Rev.* D **17** 1473
Li W-Q and Ni W-T 1979 *J. Math. Phys.* **20** 1473
Li W-Q and Ni W-T 1979 *J. Math. Phys.* **20** 1925

[5] Chicone C and Mashhoon B 2002 *Class. Quantum Grav.* **19** 4231
Chicone C and Mashhoon B 2004 *Int. J. Mod. Phys.* D **13** 945
[6] Chicone C and Mashhoon B 2004 *Preprint* astro-ph/0404170
Chicone C and Mashhoon B 2004 *Preprint* astro-ph/0406005
[7] Chicone C and Mashhoon B 2004 *Class. Quantum Grav.* **21** L139–44
[8] Mashhoon B 2003 *Advances in General Relativity and Cosmology* ed G Ferrarese (Bologna: Pitagora) pp 323–34
[9] de Felice F 1995 *Phys. Rev.* A **52** 3452
[10] Estabrook F B and Wahlquist H D 1964 *J. Math. Phys.* **5** 1629
[11] DeFacio B, Dennis P W and Retzloff D G 1978 *Phys. Rev.* D **18** 2813
DeFacio B, Dennis P W and Retzloff D G 1979 *Phys. Rev.* D **20** 570
[12] Jaffe J and Shapiro I I 1972 *Phys. Rev.* D **6** 405
[13] Jantzen R T, Carini P and Bini D 1992 *Ann. Phys., NY* **215** 1
Bini D, Carini P and Jantzen R T 1995 *Class. Quantum Grav.* **12** 2549
[14] Moliner I, Portilla M and Vives O 1995 *Phys. Rev.* D **52** 1302
[15] Bunchaft F and Carneiro S 1998 *Class. Quantum Grav.* **15** 1557
[16] Rindler W 1969 *Am. J. Phys.* **34** 1174
Rindler W 2001 *Relativity: Special, General and Cosmological* (Oxford: Oxford University Press)
[17] Anchordoqui L A, Dermer C D and Ringwald A 2004 *Preprint* hep-ph/0403001