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Abstract
The motion of a spinning test particle given by the Mathisson–Papapetrou equations is studied on an exterior vacuum C metric background spacetime describing the accelerated motion of a spherically symmetric gravitational source. We consider circular orbits of the particle around the direction of acceleration of the source. The symmetries of this configuration lead to the reduction of the differential equations of motion to algebraic relations. The spin supplementary conditions as well as the coupling between the spin of the particle and the acceleration of the source are discussed.

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1. Introduction

In a recent paper [1], the absence of spin–acceleration coupling has been examined via wave perturbations of the vacuum C metric [2–6], which represents the static exterior gravitational field of a spherical mass M that is accelerated uniformly with acceleration A such that $MA < 1/(3\sqrt{3})$. This gravitational field may be considered to be a nonlinear superposition of the Rindler and Schwarzschild spacetimes. The aim of the present work is to investigate the motion of a classical spinning test particle in the C metric using the Mathisson–Papapetrou equations [7, 8]. For the sake of simplicity, we confine our analysis to motion along circular orbits about the direction of acceleration. We find that for a given circular orbit the results are unchanged when the particle spin flips and at the same time the sense of motion reverses, a circumstance that is consistent with time-reversal invariance. Our results thus indicate that a spin–acceleration coupling analogous to the spin–rotation coupling [9, 10] does not exist. This particle result thus reinforces and complements the previous wave result [1].

This paper is organized as follows: in section 2, we present a brief discussion of the geometric properties of circular orbits around the direction of acceleration of the C metric. In section 3, we discuss the motion of spinning test particles along the circular orbits for the three standard spin supplementary conditions and determine the allowed orbits and spin directions. Section 4 contains a discussion of our results.

2. Vacuum C metric and circular orbits

In its restricted interpretation, the vacuum C metric describes the static gravitational field associated with a uniformly accelerated mass with $MA < 1/(3\sqrt{3})$ [2, 3]; it is of Petrov-type D and belongs to the Weyl class of solutions of the Einstein equations [5]. In the $\{u, r, \theta, \phi\}$ coordinate system, the C metric takes the form

$$ds^2 = -H \, du^2 - 2du \, dr + 2Ar^2 \sin \theta \, du \, d\theta + \frac{r^2 \sin^2 \theta}{G} \, d\theta^2 + r^2 G \, d\phi^2, \qquad (2.1)$$

where G and H are given by

$$G(\theta) = 1 - \cos^2 \theta - 2MA \cos^3 \theta,$$

$$\begin{aligned} H(r, \theta) = 1 - \frac{2M}{r} &- A^2 r^2 (1 - \cos^2 \theta - 2MA \cos^3 \theta) \\ &- 2Ar \cos \theta (1 + 3MA \cos \theta) + 6MA \cos \theta. \end{aligned} \qquad (2.2)$$

The constants $M > 0$ and $A > 0$ denote the mass and acceleration of the source, respectively. Unless specified otherwise, we choose units such that the gravitational constant and the speed of light in vacuum are unity. Moreover, we assume that the C metric has signature +2; to preserve this signature, we must have $G > 0$. As mentioned above, it turns out that the physical region of interest in this case corresponds to $MA < 1/(3\sqrt{3})$ [4, 11–13] .

The metric (2.1) can be seen to be a nonlinear superposition of two metrics, one associated with a Schwarzschild black hole (case $A = 0$) and the other corresponding to a uniformly accelerating particle (case $M = 0$) [2, 3]. The C metric has event horizons (which are also Killing horizons) given by hypersurfaces of the form $r = r(\theta)$ that are solutions of $H = 0$. These can be determined exactly. To this end, let us introduce the new variable [6]

$$W = \frac{Ar}{1 - Ar \cos \theta} \qquad (2.3)$$

so that $H = 0$ becomes

$$W^3 - W + 2MA = 0. \qquad (2.4)$$

Following [6], let us also introduce an acceleration lengthscale based on $A > 0$: $L_A = \frac{1}{3\sqrt{3}A}$. There are three cases depending on whether M is less than, equal to or greater than L_A. As we are interested only in the case $M < L_A$, the solutions of (2.4) can be written as

$$W_1 = 2\hat{U}, \qquad W_2 = -\hat{U} + \sqrt{3}\hat{V}, \qquad W_3 = -\hat{U} - \sqrt{3}\hat{V}, \qquad (2.5)$$

where

$$\hat{U} + i\hat{V} = \frac{1}{\sqrt{3}} \left(-\frac{M}{L_A} + i\sqrt{1 - \frac{M^2}{L_A^2}} \right)^{1/3}. \qquad (2.6)$$

It is worth noting that, when $M \to 0$, the solution W_1 gives the Rindler horizon $r = [A(1 + \cos \theta)]^{-1}$, while, when $A \to 0$, the solution W_2 gives the Schwarzschild horizon $r = 2M$; W_3 gives instead a negative value for r and should be rejected. Thus,
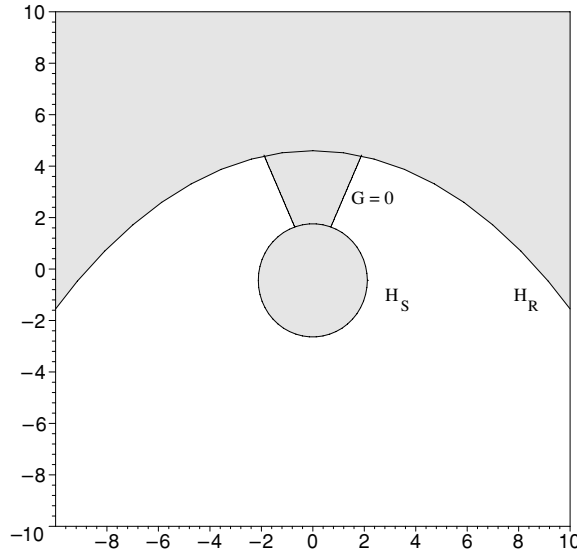
Figure 1. The accessible spacetime region (not shaded) in the (X, Y) plane with $X = \hat{r} \sin\theta$ (abscissa) and $Y = \hat{r} \cos\theta$ (ordinate), $\hat{r} = r/M$, is shown for the value of the parameter $MA = 0.1$. The upper curve represents the Rindler horizon while the lower curve represents the Schwarzschild horizon. The forbidden conical region corresponds to negative values of the metric function G, i.e., to signature changes which are not considered here.

$r = [A(\cos\theta + 1/W_1)]^{-1}$ and $r = [A(\cos\theta + 1/W_2)]^{-1}$ correspond to the modified Rindler and Schwarzschild horizons, respectively.

The C metric has two commuting hypersurface-orthogonal Killing vectors, one timelike (∂_u) and the other spacelike (∂_ϕ).

To illustrate this situation, in figure 1 we have shown the accessible spacetime region in the (r, θ) plane for the value of the metric parameter $MA = 0.1$. In this plot, it is possible to identify both the Rindler horizon (H_R) and the Schwarzschild horizon (H_S) as well as the forbidden conical region corresponding to negative values of the metric function G, i.e., to signature changes.

In the spacetime represented by the metric (2.1), let us consider as 'fiducial observers' the family of static observers with 4-velocity aligned with the Killing direction ∂_u, $e_{\hat{u}} = H^{-1/2}\partial_u$; it is convenient to introduce an orthonormal frame adapted to these observers

$$e_{\hat{u}} = H^{-1/2}\partial_u, \qquad\qquad e_{\hat{r}} = H^{-1/2}[H\partial_r - \partial_u],$$
$$e_{\hat{\theta}} = G^{1/2}\left[Ar\partial_r + \frac{1}{r\sin\theta}\partial_\theta\right], \qquad e_{\hat{\phi}} = \frac{G^{-1/2}}{r}\partial_\phi. \tag{2.7}$$

We are interested in the study of circular motion of test particles along the ϕ direction with constant speed and at a fixed value of the polar angle θ as illustrated in figure 2; so the family of test particles is characterized by the following (timelike) 4-velocity U

$$U = \Gamma_\zeta[\partial_u + \zeta\partial_\phi] = \gamma[e_{\hat{u}} + \nu e_{\hat{\phi}}], \tag{2.8}$$

where ζ and ν are the angular velocity and linear velocity parametrizations, respectively, of the whole family such that

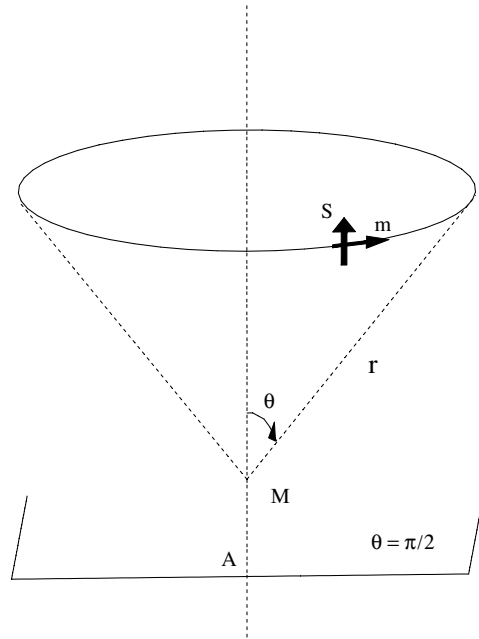$$\nu = r\left(\frac{G}{H}\right)^{1/2}\zeta. \tag{2.9}$$

Figure 2. Schematic diagram representing circular orbits under consideration in this paper. The orbital plane is parallel to the $\theta = \pi/2$ plane. The spherical source M is accelerated with uniform acceleration A in the $\theta = \pi$ direction.

Here, Γ_ζ is defined by the timelike condition $U \cdot U = -1$ as

$$-\Gamma_\zeta^{-2} = g_{uu} + \zeta^2 g_{\phi\phi} = -H + \zeta^2 r^2 G = -\frac{H}{\gamma^2} \tag{2.10}$$

and $\gamma = -U \cdot e_{\hat{u}} = (1 - v^2)^{-1/2} = \Gamma_\zeta H^{-1/2}$ is the Lorentz factor. Note that Γ_ζ, ζ, v and γ are all functions of r and θ only; therefore, they remain constant along the orbit (see figure 2).

We also define the unit direction orthogonal to U in the $(e_{\hat{u}}, e_{\hat{\phi}})$ 2-space

$$E_{\hat{\phi}} = \gamma[v e_{\hat{u}} + e_{\hat{\phi}}], \tag{2.11}$$

which is a vector field defined only along the worldline of U; in fact, its definition requires the relative velocity field v of U with respect to fiducial observers $e_{\hat{u}}$ which only exist along U.

Two quantities are relevant for the discussion below: the acceleration of the line U, $a(U) = \mathrm{D}U/\mathrm{d}\tau_U = \nabla_U U$, where τ_U is the proper time along U, and the directional derivative along U of the local ϕ direction as seen by the observer U, $\mathrm{D}E_{\hat{\phi}}/\mathrm{d}\tau_U = \nabla_U E_{\hat{\phi}}$. A direct evaluation shows that both these quantities belong to the $(e_{\hat{r}}, e_{\hat{\theta}})$ plane and are given by

$$\frac{\mathrm{D}U}{\mathrm{d}\tau_U} = \gamma^2\left[k_{\hat{r}}\left(v^2 - v_{(r)}^2\right)e_{\hat{r}} + k_{\hat{\theta}}\left(v^2 - v_{(\theta)}^2\right)e_{\hat{\theta}}\right], \tag{2.12}$$

$$\frac{\mathrm{D}E_{\hat{\phi}}}{\mathrm{d}\tau_U} = \gamma^2 v\left(\frac{k_{\hat{r}}}{\gamma_{(r)}^2}e_{\hat{r}} + \frac{k_{\hat{\theta}}}{\gamma_{(\theta)}^2}e_{\hat{\theta}}\right), \tag{2.13}$$

where

$$v_{(r)}^2 = \frac{M - Ar^2[GAr + \cos\theta(1 + 3AM\cos\theta)]}{rH}, \qquad \gamma_{(r)}^2 = 1/\left(1 - v_{(r)}^2\right),$$

$$v_{(\theta)}^2 = \frac{GAr}{GAr + \cos\theta(1 + 3AM\cos\theta)}, \qquad \gamma_{(\theta)}^2 = 1/\left(1 - v_{(\theta)}^2\right), \tag{2.14}$$

and the Lie relative curvature [14, 15] of the orbit $k_{(Lie)} \equiv k = -\nabla \ln(\sqrt{g_{\phi\phi}})$ has been introduced with frame components

$$k_{\hat{r}} = -\frac{\sqrt{H}}{r}, \qquad k_{\hat{\theta}} = -\frac{GAr + \cos\theta(1 + 3AM\cos\theta)}{r\sqrt{G}}, \qquad (2.15)$$

and magnitude $\kappa = \left(k_{\hat{r}}^2 + k_{\hat{\theta}}^2\right)^{1/2}$.

Analogously, it is easy to derive the directional derivatives along U of $e_{\hat{r}}$ and $e_{\hat{\theta}}$, which instead belong to the $(e_{\hat{u}}, e_{\hat{\phi}})$ plane:

$$\frac{De_{\hat{r}}}{d\tau_U} = \gamma^2 k_{\hat{r}} \left[\left(v^2 - v_{(r)}^2\right)U - \frac{v}{\gamma_{(r)}^2} E_{\hat{\phi}} \right], \qquad (2.16)$$

$$\frac{De_{\hat{\theta}}}{d\tau_U} = \gamma^2 k_{\hat{\theta}} \left[\left(v^2 - v_{(\theta)}^2\right)U - \frac{v}{\gamma_{(\theta)}^2} E_{\hat{\phi}} \right]. \qquad (2.17)$$

The following useful relations hold

$$k_{\hat{r}}^2 v_{(r)}^2 + k_{\hat{\theta}}^2 v_{(\theta)}^2 = \frac{M}{r^3}, \qquad A\sqrt{G} = -k_{\hat{\theta}} v_{(\theta)}^2. \qquad (2.18)$$

A detailed analysis of circular orbits, 'special' for geometrical or kinematical reasons, is considered in [16].

The only non-vanishing components of the 4-acceleration $a(U) = \nabla_U U$ can be conveniently rewritten as

$$a(U)^{\hat{r}} = \gamma^2 k_{\hat{r}}\left(v^2 - v_{(r)}^2\right), \qquad a(U)^{\hat{\theta}} = \gamma^2 k_{\hat{\theta}}\left(v^2 - v_{(\theta)}^2\right). \qquad (2.19)$$

Conditions for circular geodesics, namely, $(a(U)^{\hat{r}}, a(U)^{\hat{\theta}}) = (0, 0)$ (or equivalently, $v = \pm v_{(r)}$ and $v_{(r)}^2 = v_{(\theta)}^2$) can easily be obtained from (2.19); it follows that for a fixed angle θ, only the value $r = r_g$ is allowed with

$$r_g = \frac{1}{2A} \frac{3AMG + (AM)^{1/2}[\cos\theta(4 + AM\cos^3\theta) + 9AM(3 - 2G)]^{1/2}}{G + \cos\theta[\cos\theta + 3AM(2 - AM\cos^3\theta)]}, \qquad (2.20)$$

where

$$v \equiv v_{g\pm} = \pm\sqrt{\frac{Ar_g G}{Ar_g G + \cos\theta(1 + 3MA\cos\theta)}}, \qquad \gamma_g = 1/\sqrt{1 - v_{g\pm}^2}. \qquad (2.21)$$

Note that r_g and $v_{g\pm}$ depend on θ only. In the same diagram of figure 1, the curve of the r and θ values corresponding to circular geodesics is shown in figure 3.

The circular geodesics of the C metric, which represents the gravitational potentials of a Schwarzschild black hole accelerating uniformly with acceleration $A < 1/(3\sqrt{3}M)$ along the $\theta = \pi$ direction, have been obtained with the assumption that $A > 0$; moreover, the geodesics are timelike for $\theta_0 < \theta < \pi/2$ and spacelike for $\pi/2 < \theta < \theta_c$. There is a unique null geodesic for $\theta = \pi/2$ given by $r_g = 3M$ irrespective of the acceleration A so long as $MA < 1/(3\sqrt{3})$. Here θ_0 and θ_c depend upon MA, θ_0 is given by $G(\theta_0) = 0$ and θ_c is such that $r_g(\theta_c)$ lies on the modified Schwarzschild horizon that satisfies $H(r_g(\theta_c), \theta_c) = 0$, see figure 3. In the nonrelativistic limit $MA \ll 1$, the circular geodesics reduce to the orbits that can easily be obtained from the Newtonian theory of gravitation: $(M/r_g^2)\cos\theta = A$ follows from (2.20) and $v_g^2 = (M/r_g)\sin^2\theta$ from (2.20) and (2.21). Furthermore, in the Newtonian limit $\zeta_g^2 = M/r_g^3$.

It is interesting to consider the significance of these orbits in the gravitoelectromagnetic analogy [6]. When one explores the correspondence between classical and quantum
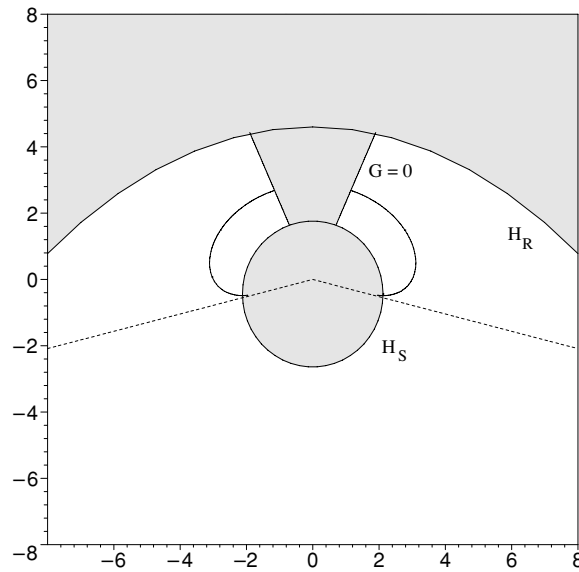
Figure 3. Circular geodesics exist for fixed values of θ only for certain corresponding values of r. The solid line in the allowed region represents the location of the geodesics, which are respectively timelike, null and spacelike above, on and below the equatorial plane $\theta = \pi/2$. The pair of straight lines dropping down from the centre correspond to the limiting value $\theta = \theta_c$. The value of the parameter $MA = 0.1$ is the same as in figure 1.

descriptions of electron motion in the Stark effect, the circular orbits that correspond to those under consideration here describe the classical motion of the electron in a quantum-mechanical phenomenon, namely, the *quadratic* Stark effect in hydrogen [17].

The purpose of this paper is to explore the accelerated motion of classical spinning test particles on circular orbits $r = r(\theta)$ in the C metric. We therefore turn to the investigation of the allowed orbits for particles with spin.

3. Spinning test particles along circular orbits

The motion of a (classical) spinning test particle is described by the Mathisson–Papapetrou equations

$$\frac{\mathrm{D}P^\mu}{\mathrm{d}\tau_U} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} \equiv F_{\mathrm{spin}}{}^\mu, \tag{3.1}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau_U} = P^\mu U^\nu - P^\nu U^\mu, \tag{3.2}$$

where P^μ is the total 4-momentum of the particle and $S^{\mu\nu}$ is the antisymmetric spin tensor. Here, U is the timelike unit tangent vector of the 'centre line' that is used to perform the multipole reduction of the energy–momentum tensor associated with the particle in order to derive equations (3.1) and (3.2).

We assume here the centre line to be a circular orbit, i.e., U is given by (2.8). From (3.2), contracting both sides of the second equation with U_ν, one has

$$P^\mu = -(U \cdot P)U^\mu - U_\nu \frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau_U} \equiv mU^\mu + P_s^\mu, \tag{3.3}$$

where m is the component of $-P$ along U and it is usually referred to as the mass of the

particle under the Pirani supplementary conditions (see below). Equation (3.2) implies that

$$S_{\hat{u}\hat{\phi}} = 0, \qquad S_{\hat{r}\hat{\theta}} = 0, \qquad k_{\hat{r}}\left(v_{(r)}^2 S_{\hat{\theta}\hat{u}} + v S_{\hat{\theta}\hat{\phi}}\right) - k_{\hat{\theta}}\left(v_{(\theta)}^2 S_{\hat{r}\hat{u}} + v S_{\hat{r}\hat{\phi}}\right) = 0. \tag{3.4}$$

The spin tensor then takes the form

$$S = \omega^{\hat{r}} \wedge [S_{\hat{r}\hat{u}}\omega^{\hat{u}} + S_{\hat{r}\hat{\phi}}\omega^{\hat{\phi}}] + \omega^{\hat{\theta}} \wedge [S_{\hat{\theta}\hat{u}}\omega^{\hat{u}} + S_{\hat{\theta}\hat{\phi}}\omega^{\hat{\phi}}], \tag{3.5}$$

where $\{\omega^{\hat{u}}, \omega^{\hat{r}}, \omega^{\hat{\theta}}, \omega^{\hat{\phi}}\}$ is the dual frame of (2.7). It is clear from (3.3) that P_s is orthogonal to U; moreover, it turns out to be also aligned with $E_{\hat{\phi}}$

$$P_s = m_s E_{\hat{\phi}}, \tag{3.6}$$

where

$$m_s \equiv \|P_s\| = \gamma \left[k_{\hat{r}}\left(v S_{\hat{r}\hat{u}} + v_{(r)}^2 S_{\hat{r}\hat{\phi}}\right) + k_{\hat{\theta}}\left(v S_{\hat{\theta}\hat{u}} + v_{(\theta)}^2 S_{\hat{\theta}\hat{\phi}}\right)\right] \tag{3.7}$$

is the component of P along $E_{\hat{\phi}}$. It follows from (3.3) and (3.6) that the total 4-momentum P can be written in the form $P = \mu U_p$, where μ represents the magnitude of P and the unit timelike direction U_p is given by

$$U_p = \gamma_p[e_{\hat{u}} + v_p e_{\hat{\phi}}], \qquad v_p = \frac{v + m_s/m}{1 + v m_s/m}, \qquad \mu = \frac{\gamma}{\gamma_p}(m + v m_s). \tag{3.8}$$

Here $\gamma_p = \left(1 - v_p^2\right)^{-1/2}$, as expected.

Let us now consider the equation of motion (3.1). The spin force is given by

$$F^{(\mathrm{spin})} = \gamma \frac{M}{r^3}[(2S_{\hat{u}\hat{r}} + v S_{\hat{r}\hat{\phi}})e_{\hat{r}} - (S_{\hat{u}\hat{\theta}} + 2v S_{\hat{\theta}\hat{\phi}})e_{\hat{\theta}}], \tag{3.9}$$

while the term on the left-hand side of equation (3.1) can be written, from (3.3) and (3.6), as

$$\frac{DP}{d\tau_U} = ma(U) + m_s \frac{DE_{\hat{\phi}}}{d\tau_U}, \tag{3.10}$$

where $a(U)$ and $DE_{\hat{\phi}}/d\tau_U$ are given by (2.12) and (2.13), respectively, and the quantities μ, m and m_s remain constant along the worldline of U. Hence, (3.1) can be written as

$$\gamma\left[m\left(v^2 - v_{(r)}^2\right) + m_s \frac{v}{\gamma_{(r)}^2}\right]k_{\hat{r}} - \frac{M}{r^3}(2S_{\hat{u}\hat{r}} + v S_{\hat{r}\hat{\phi}}) = 0,$$

$$\gamma\left[m\left(v^2 - v_{(\theta)}^2\right) + m_s \frac{v}{\gamma_{(\theta)}^2}\right]k_{\hat{\theta}} + \frac{M}{r^3}(S_{\hat{u}\hat{\theta}} + 2v S_{\hat{\theta}\hat{\phi}}) = 0. \tag{3.11}$$

Finally, it proves useful to introduce the quadratic invariant

$$s^2 = \tfrac{1}{2}S_{\mu\nu}S^{\mu\nu} = -S_{\hat{r}\hat{u}}^2 - S_{\hat{\theta}\hat{u}}^2 + S_{\hat{r}\hat{\phi}}^2 + S_{\hat{\theta}\hat{\phi}}^2, \tag{3.12}$$

which is not in general constant along the trajectory of the spinning particle. It is constant, however, in the situation under consideration in this paper.

Summarizing, the equations to be considered are: (a) the constraint equations (3.4) that result from the evolution of the spin tensor whose nontrivial components are given in (3.5); (b) the two equations of motion (3.11); (c) the definition of m_s given in (3.7) and (d) the quadratic invariant s introduced in (3.12). To solve these equations and, hence, to discuss the features of the motion we need three further supplementary conditions. We shall follow the standard approaches existing in the literature:

1. Corinaldesi–Papapetrou [18] conditions (CP): $S^{uv} = 0$,
2. Pirani [19] conditions (P): $S^{\mu\nu}U_\nu = 0$,
3. Tulczyjew [20] conditions (T): $S^{\mu\nu}P_\nu = 0$.

It is helpful to introduce the spin vector

$$S_{(SC)\beta} = \tfrac{1}{2}\eta^\alpha{}_{\beta\gamma\delta}(U_{(SC)})_\alpha S^{\gamma\delta}, \tag{3.13}$$

where $U_{(SC)} = (e_{\hat{u}}, U, U_p)$ for the CP, P, T conditions, respectively, and $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$ is the unit volume 4-form constructed with the Levi-Civita alternating symbol $\epsilon_{\alpha\beta\gamma\delta}$, such that $\epsilon_{\hat{u}\hat{r}\hat{\theta}\hat{\phi}} = 1$. It follows that in general $s^2 = S_\mu S^\mu$.

A consistent physical interpretation of the Mathisson–Papapetrou equations requires that $\hat{s} := s/(mr) \ll 1$. For $s = 0$, the orbit of the test particle coincides with the geodesic orbit $r_g = r_g(\theta_g)$; therefore, we expect that the circular orbit $r = r(\theta)$ of the spinning particle should be close to such a geodesic. Hence, we assume that

$$r = r_g + \delta_r, \qquad \theta = \theta_g + \delta_\theta, \tag{3.14}$$

where δ_r and δ_θ are of order $\hat{s}$ and will only be considered to linear order in what follows. Any given function of the particle orbit, $F(r, \theta)$, can then be written to first order in δ_r and δ_θ as

$$F(r, \theta) = F(r_g, \theta_g) + \left(\frac{\partial F}{\partial r}\right)_{r=r_g, \theta=\theta_g} \delta_r + \left(\frac{\partial F}{\partial \theta}\right)_{r=r_g, \theta=\theta_g} \delta_\theta. \tag{3.15}$$

In the following, we shall obtain for each SC explicit expressions in terms of r and θ for both the linear velocity ν and the spin parameter $\hat{s}$ associated with the orbit of the spinning particle. Close to a geodesic, we shall show that all the SC cases can be summarized as

$$\nu \simeq \nu_{g\pm} + \sigma \Delta\nu^{(SC)}, \qquad \hat{s} \simeq \sigma \Delta\hat{s}^{(SC)}, \tag{3.16}$$

where the notation $\sigma = \text{sign}(\nu_{g\pm})$ is conveniently introduced, so that $\sigma = +1$ for a counterclockwise orbit and $\sigma = -1$ for a clockwise orbit (cf figure 2). The corresponding angular velocity ζ and its reciprocal are

$$\zeta_\pm \simeq \zeta_{g\pm} + \sigma \Delta\zeta^{(SC)}, \qquad \frac{1}{\zeta_\pm} \simeq \frac{1}{\zeta_{g\pm}} - \frac{\sigma}{\zeta_{g\pm}^2}\Delta\zeta^{(SC)}, \tag{3.17}$$

respectively, with $\zeta_{g\pm} = (\nu_{g\pm}\sqrt{H_g})/(r_g\sqrt{G_g})$. As a result, the orbital frequency turns out to depend on the spin parameter. Explicit expressions for all of the quantities introduced above can be found in the appendix.

3.1. The Corinaldesi–Papapetrou (CP) supplementary conditions

These are *conceptually* the weakest conditions that have been imposed on the spin tensor; in general, they do not even ensure that the quadratic invariant $S_{\mu\nu}S^{\mu\nu}$ is conserved along the path. In the simple configuration under consideration here, the CP supplementary conditions require $S_{\hat{u}\hat{r}} = S_{\hat{u}\hat{\theta}} = 0$. Equations (3.4) as well as (3.12) are identically satisfied with

$$(S_{\hat{r}\hat{\phi}}, S_{\hat{\theta}\hat{\phi}}) = \frac{s}{\kappa}(k_{\hat{r}}, k_{\hat{\theta}}), \tag{3.18}$$

where s can be of any sign. The spin vector,

$$S_{(CP)\beta} = \tfrac{1}{2}\eta^\alpha{}_{\beta\gamma\delta}(e_{\hat{u}})_\alpha S^{\gamma\delta}, \tag{3.19}$$

spatial with respect to $e_{\hat{u}}$ in this case is given by

$$S_{(CP)} = -S_{\hat{r}\hat{\phi}}e_{\hat{\theta}} + S_{\hat{\theta}\hat{\phi}}e_{\hat{r}} = \frac{s}{\kappa}[-k_{\hat{r}}e_{\hat{\theta}} + k_{\hat{\theta}}e_{\hat{r}}]. \tag{3.20}$$

It follows from (3.7) that

$$m_s = \frac{M}{r^3}\frac{s\gamma}{\kappa}. \tag{3.21}$$

Finally, the spin force (see (3.9)) is given by

$$F^{(\text{spin})} = \frac{s\gamma\nu}{\kappa}\frac{M}{r^3}[k_{\hat{r}}e_{\hat{r}} - 2k_{\hat{\theta}}e_{\hat{\theta}}], \qquad (3.22)$$

so that equations (3.11) reduce to

$$\left(\nu^2 - \nu_{(r)}^2\right)\left(s + \frac{m}{\gamma\nu}\kappa\frac{r^3}{M}\right) = 0, \qquad s + \left(\frac{m}{\gamma\nu}\kappa\frac{r^3}{M}\right)\frac{\nu^2 - \nu_{(\theta)}^2}{3 - 2\nu^2 - \nu_{(\theta)}^2} = 0. \qquad (3.23)$$

Now, by eliminating s, we get the solution for $\nu = \nu_{\pm}^s$. If $\nu^2 \neq \nu_{(r)}^2$, then plugging the first of equations (3.23) into the second, one gets $\nu_{\pm}^s = \pm 1$ and so $s = 0$, which must be rejected. The only possibility is then $\nu = \nu_{(r)}$ (see (2.14)), with s given by

$$s = -\frac{m}{\gamma_{(r)}\nu_{(r)}}\kappa\frac{r^3}{M}\frac{\nu_{(r)}^2 - \nu_{(\theta)}^2}{3 - 2\nu_{(r)}^2 - \nu_{(\theta)}^2}. \qquad (3.24)$$

Close to a geodesic, from (3.23) we obtain

$$\Delta\nu^{(\text{CP})} = \Delta_{\nu_{(r)}}^r \delta_r + \Delta_{\nu_{(r)}}^\theta \delta_\theta,$$

$$\Delta\hat{s}^{(\text{CP})} = -\frac{2}{3}\gamma_g\kappa_g\frac{r_g^2}{M}\left[\delta_r\left(\Delta_{\nu_{(r)}}^r - \Delta_{\nu_{(\theta)}}^r\right) + \delta_\theta\left(\Delta_{\nu_{(r)}}^\theta - \Delta_{\nu_{(\theta)}}^\theta\right)\right]; \qquad (3.25)$$

the linear velocity ν and the corresponding angular velocity ζ and its reciprocal are thus given by (3.16) and (3.17), respectively.

3.2. The Pirani (P) supplementary conditions

It was first demonstrated by Pirani [19] that the Mathisson–Papapetrou equations with the supplementary conditions $S^{\mu\nu}U_\nu = 0$ imply that the spin vector is Fermi–Walker transported along the trajectory. Subsequently, Taub [21] demonstrated that these equations correspond to the motion of a classical point particle with 'intrinsic' spin. The Mathisson–Papapetrou equations were investigated in [22, 23] and it was found that of the various supplementary conditions the only one that produces a consistent result in the massless limit is $S^{\mu\nu}U_\nu = 0$ [23]. In fact, the trajectory in this case is a null geodesic with the spin vector either parallel or antiparallel to the direction of motion; furthermore, the helicity of the particle is conserved in an orientable spacetime [23].

In the case under consideration here, the P supplementary conditions ($S^{\mu\nu}U_\nu = 0$) require

$$S_{\hat{u}\hat{r}} = S_{\hat{r}\hat{\phi}}\nu, \qquad S_{\hat{u}\hat{\theta}} = S_{\hat{\theta}\hat{\phi}}\nu. \qquad (3.26)$$

Using these relations one finds that equations (3.4) as well (3.12) are identically satisfied with

$$(S_{\hat{r}\hat{\phi}}, S_{\hat{\theta}\hat{\phi}}) = \frac{\gamma s}{\Sigma}\left(\frac{k_{\hat{r}}}{\gamma_{(r)}^2}, \frac{k_{\hat{\theta}}}{\gamma_{(\theta)}^2}\right), \qquad (3.27)$$

where

$$\Sigma = \left(\frac{k_{\hat{r}}^2}{\gamma_{(r)}^4} + \frac{k_{\hat{\theta}}^2}{\gamma_{(\theta)}^4}\right)^{1/2} \qquad (3.28)$$

and, as before, s can be of any sign. The spin vector

$$S_{(P)\beta} = \tfrac{1}{2}\eta^\alpha{}_{\beta\gamma\delta}U_\alpha S^{\gamma\delta} \qquad (3.29)$$

spatial with respect to U in this case is given by

$$S_{(P)} = \frac{1}{\gamma}[-S_{\hat{r}\hat{\phi}}e_{\hat{\theta}} + S_{\hat{\theta}\hat{\phi}}e_{\hat{r}}] = \frac{s}{\Sigma}\left[-\frac{k_{\hat{r}}}{\gamma_{(r)}^2}e_{\hat{\theta}} + \frac{k_{\hat{\theta}}}{\gamma_{(\theta)}^2}e_{\hat{r}}\right]. \qquad (3.30)$$

Then, from (3.7) we have that

$$m_s = -s\gamma^2 \frac{\gamma_{(\theta)}^2 k_{\hat{r}}^2 (v^2 - v_{(r)}^2) + \gamma_{(r)}^2 k_{\hat{\theta}}^2 (v^2 - v_{(\theta)}^2)}{\gamma_{(r)}^2 \gamma_{(\theta)}^2 \Sigma}. \tag{3.31}$$

Finally, the spin force (see (3.9)) is given by

$$F^{(\text{spin})} = \frac{3s\gamma^2 v}{\Sigma} \frac{M}{r^3} \left(\frac{k_{\hat{r}}}{\gamma_{(r)}^2} e_{\hat{r}} - \frac{k_{\hat{\theta}}}{\gamma_{(\theta)}^2} e_{\hat{\theta}} \right), \tag{3.32}$$

so that equations (3.11) reduce to

$$s = -\frac{m\gamma_{(r)}^2 \Sigma}{v} \frac{v^2 - v_{(r)}^2}{(\kappa^2 - \frac{4M}{r^3}) - \gamma^2 \Sigma^2}, \qquad s = -\frac{m\gamma_{(\theta)}^2 \Sigma}{v} \frac{v^2 - v_{(\theta)}^2}{(\kappa^2 + \frac{2M}{r^3}) - \gamma^2 \Sigma^2}, \tag{3.33}$$

by solving both equations with respect to s. By eliminating s in (3.33), we get the solution $v = v_{\pm}^s$:

$$v_{\pm}^s = \pm \left[1 - \frac{\frac{6M}{r^3} - \Sigma^2 (\gamma_{(\theta)}^2 - \gamma_{(r)}^2)}{(\kappa^2 + \frac{2M}{r^3})\gamma_{(r)}^2 - (\kappa^2 - \frac{4M}{r^3})\gamma_{(\theta)}^2} \right]^{1/2}. \tag{3.34}$$

Close to a geodesic, we obtain from (3.33) and (3.34) that

$$\Delta v^{(P)} = \frac{1}{3} \left\{ \frac{\kappa_g^2 r_g^3}{2M} v_{g\pm}^2 \left[(\Delta_{v_{(r)}}^r - \Delta_{v_{(\theta)}}^r)\delta_r + (\Delta_{v_{(r)}}^\theta - \Delta_{v_{(\theta)}}^\theta)\delta_\theta \right] \right.$$

$$\left. + (\Delta_{v_{(r)}}^r + 2\Delta_{v_{(\theta)}}^r)\delta_r + (\Delta_{v_{(r)}}^\theta + 2\Delta_{v_{(\theta)}}^\theta)\delta_\theta \right\}, \tag{3.35}$$

$$\Delta\hat{s}^{(P)} = \frac{\kappa_g}{3} \frac{r_g^2}{M} \left[\delta_r (\Delta_{v_{(r)}}^r - \Delta_{v_{(\theta)}}^r) + \delta_\theta (\Delta_{v_{(r)}}^\theta - \Delta_{v_{(\theta)}}^\theta) \right];$$

the linear velocity v and the corresponding angular velocity ζ and its reciprocal are given by (3.16) and (3.17), respectively.

3.3. The Tulczyjew (T) supplementary conditions

The general theory of the motion of an extended particle in a gravitational field has been developed by Dixon (see [24] and references therein) on the basis of the dynamical equations $\nabla_\nu T^{\mu\nu} = 0$. In Dixon's approach, when attention is limited to a pole–dipole particle and higher multipole moments of the extended body are neglected, one recovers the Mathisson–Papapetrou equations with the supplementary conditions $S^{\mu\nu} P_\nu = 0$. These require, in the case under consideration here, that

$$S_{\hat{u}\hat{r}} = S_{\hat{r}\hat{\phi}} v_p, \qquad S_{\hat{u}\hat{\theta}} = S_{\hat{\theta}\hat{\phi}} v_p. \tag{3.36}$$

Using these relations one finds that equations (3.4) as well as (3.12) are identically satisfied with

$$(S_{\hat{r}\hat{\phi}}, S_{\hat{\theta}\hat{\phi}}) = \frac{\gamma_p s}{\Lambda} \left(k_{\hat{r}} (v - v_p v_{(r)}^2), k_{\hat{\theta}} (v - v_p v_{(\theta)}^2) \right), \tag{3.37}$$

where

$$\Lambda = \left[k_{\hat{r}}^2 (v - v_p v_{(r)}^2)^2 + k_{\hat{\theta}}^2 (v - v_p v_{(\theta)}^2)^2 \right]^{1/2}. \tag{3.38}$$

The spin vector

$$S_{(T)\beta} = \tfrac{1}{2} \eta^\alpha{}_{\beta\gamma\delta} (U_p)_\alpha S^{\gamma\delta} \tag{3.39}$$

spatial with respect to U_p in this case is given by

$$S_{(T)} = \frac{1}{\gamma_p}[-S_{\hat{r}\hat{\phi}}e_{\hat{\theta}} + S_{\hat{\theta}\hat{\phi}}e_{\hat{r}}] = \frac{s}{\Lambda}[-k_{\hat{r}}(v - v_p v_{(r)}^2)e_{\hat{\theta}} + k_{\hat{\theta}}(v - v_p v_{(\theta)}^2)e_{\hat{r}}]. \tag{3.40}$$

Moreover, from (3.7) we have that

$$m_s = -\frac{s\gamma\gamma_p}{\Lambda}[k_{\hat{r}}^2(vv_p - v_{(r)}^2)(v - v_p v_{(r)}^2) + k_{\hat{\theta}}^2(vv_p - v_{(\theta)}^2)(v - v_p v_{(\theta)}^2)], \tag{3.41}$$

and the spin force (see (3.9)) is given by

$$F^{(spin)} = \frac{s\gamma\gamma_p}{\Lambda}\frac{M}{r^3}[(v + 2v_p)(v - v_p v_{(r)}^2)k_{\hat{r}}e_{\hat{r}} - (2v + v_p)(v - v_p v_{(\theta)}^2)k_{\hat{\theta}}e_{\hat{\theta}}]. \tag{3.42}$$

Now, by solving both equations (3.11) with respect to s, we get

$$s = -m\frac{\gamma}{\gamma_p}\frac{v^2 - v_{(r)}^2}{\frac{\gamma^2}{\gamma_{(r)}^2}v\tilde{m}_s + \frac{M}{\Lambda r^3}(2v_p + v)(v_p v_{(r)}^2 - v)},$$

$$s = -\frac{m}{\gamma\gamma_p\gamma_{(\theta)}^2}\frac{\gamma^2 - \gamma_{(\theta)}^2}{\frac{\gamma^2}{\gamma_{(\theta)}^2}v\tilde{m}_s - \frac{M}{\Lambda r^3}(v_p + 2v)(v_p v_{(\theta)}^2 - v)}, \tag{3.43}$$

where $\tilde{m}_s = m_s/(s\gamma\gamma_p)$; by eliminating s, we have that v_p must satisfy the following equation:

$$\mathcal{A}v_p^2 + \mathcal{B}v_p + \mathcal{C} = 0, \tag{3.44}$$

with

$$\mathcal{A} = \frac{3M}{r^3}v_{(r)}^2(v^2 - v_{(\theta)}^2),$$

$$-\frac{\mathcal{B}}{v} = (v_{(r)}^2 - v_{(\theta)}^2)[(v^2 + v_{(r)}^4)k_{\hat{r}}^2 + (v^2 + v_{(\theta)}^4)k_{\hat{\theta}}^2]$$

$$\qquad - \frac{M}{r^3}[(v_{(r)}^2 + 2v_{(\theta)}^2)(1 + v^2) - 3(v^2 + v_{(r)}^2 v_{(\theta)}^2)],$$

$$\mathcal{C} = -3\frac{M}{r^3}v^2(v^2 - v_{(r)}^2). \tag{3.45}$$

Let $v_p^{(\pm)} = (-\mathcal{B} \pm \sqrt{\Delta})/(2\mathcal{A})$ be the solutions of (3.44). The reality condition for these solutions requires that v take values outside the regions implicitly defined by the equation $\Delta \equiv \mathcal{B}^2 - 4\mathcal{A}\mathcal{C} = 0$. By substituting $v_p = v_p^{(\pm)}$ into either of equations (3.43), we obtain a relation between v and s, which must be considered together with the following further equation directly resulting from the definition (3.8) of v_p:

$$s = -m\frac{v - v_p}{\gamma\gamma_p\tilde{m}_s(1 - vv_p)}\bigg|_{v_p = v_p^{(\pm)}}. \tag{3.46}$$

As a result, solutions for both quantities $v_{\pm}^s$ and s can be derived explicitly from (3.43) and (3.46).

Close to a geodesic we obtain the same results as in the case of P supplementary conditions with

$$v_p^{(\pm)} = v + O(\hat{s}^2); \tag{3.47}$$

so the linear velocity v and the corresponding angular velocity ζ and its reciprocal are given by (3.16) and (3.17), respectively, with $\Delta v^{(T)} \equiv \Delta v^{(P)}$.

4. Discussion

The test-particle approximation in the case of the Mathisson–Papapetrou equations for a spinning particle implies that the acceleration of the worldline due to the spin force defined by (3.1) should be relatively small. This means, in the case under consideration, that qualitatively

$$\text{spin force/Newtonian force} \sim \frac{[(GM/r^3)s/c]}{[GMm/r^2]} = \frac{s}{(mcr)} \ll 1, \qquad (4.1)$$

since in the multipole expansion around U, the dipole term should be smaller than the monopole term; we neglect quadrupole and higher order terms in (3.1) and (3.2). The condition (4.1) can be expressed as $r_M \ll r$, where r_M is the Møller radius of the test mass [25].

It follows from this condition that spinning test particles can be maintained on circular orbits in the vacuum C metric only for allowed values of r and θ that are close to those of geodesic circular orbits that have been discussed in section 2 for spinless test particles. Thus, we have considered a one-parameter family of solutions in each of the CP, P and T cases. That is, the spinning particle follows a circular orbit such that $r - r_g$ and $\theta - \theta_g$ are linear in $\hat{s}$ and higher order terms in $\hat{s}$ have been neglected. We find that the orbital frequency is in general spin dependent, but there is no clock effect in contrast to the Schwarzschild case [26], since for a given orbit, the orbital frequency is unchanged under the transformation $\nu \to -\nu$ and $\hat{s} \to -\hat{s}$. In fact, it follows from (3.17) that the difference in the arrival times after one complete revolution with respect to a static observer vanishes

$$\Delta t_{(+,+;-,-)} = 2\pi \left(\frac{1}{\zeta_{(SC,+,+)}} + \frac{1}{\zeta_{(SC,-,-)}} \right) = 0, \qquad (4.2)$$

where $\zeta_{(SC,\pm,\pm)}$ denotes the angular velocity of U derived under a particular choice of supplementary conditions and corresponding to a $\sigma = \pm$ orbit with spin-up/down alignment. This is a consequence of the fact that the linear velocity ν and the spin parameter $\hat{s}$ must both have the same sign as $\nu_{g\pm}$, see (3.16). In the Schwarzschild case, instead, when the motion is confined to the equatorial plane the Mathisson–Papapetrou equations give rise to a unique relation between ν and $\hat{s}$, which becomes linear in the limit of small values of the spin parameter. All combinations of the relative signs between rotation and the two spin-up and spin-down orientations along the z-axis are allowed, and so a non-zero clock effect can appear.

One may consider this circumstance intuitively as follows: in the case of the Schwarzschild circular equatorial orbits of a spinning test particle with its spin normal to the equatorial plane, one may imagine a transformation of coordinates to the frame comoving with the test particle. In this frame, the Schwarzschild source follows circular *geodesic* orbits about the spinning test particle; therefore, one would qualitatively expect the existence of a gravitomagnetic clock effect proportional to the specific angular momentum of the test particle. This effect would then be expected to occur in the original coordinate system as well. It is interesting to observe that a similar argument for the case of the C metric would fail, since the orbits of the source in the frame comoving with the test particle would not be geodesics by assumption.

Appendix. Expansion of kinematical quantities around a geodesic

The expansion of the metric functions G and H, the linear velocities $\nu_{(r)}$ and $\nu_{(\theta)}$ and the components of the Lie curvature close to a selected geodesic can be expressed as

$$G \simeq G_g + 2A \sin\theta_g \frac{G_g r_g}{\nu_{g\pm}^2 \gamma_{g\pm}^2} \delta_\theta \equiv G_g + \Delta_G^\theta \delta_\theta,$$

$$H \simeq H_g + 2 \frac{\nu_{g\pm}^2 H_g}{r_g} \delta_r + 2A \sin\theta_g (1 - Ar_g \cos\theta_g)[r_g - 3M(1 - Ar_g \cos\theta_g)]\delta_\theta$$

$$\equiv H_g + \Delta_H^r \delta_r + \Delta_H^\theta \delta_\theta,$$

$$k_{\hat{r}} \simeq -\frac{\sqrt{H_g}}{r_g} + \frac{\sqrt{H_g}}{\gamma_{g\pm}^2 r_g^2} \delta_r - \frac{\Delta_H^\theta}{2 r_g \sqrt{H_g}} \delta_\theta \equiv k_{\hat{r}}^g + \Delta_{k_{\hat{r}}}^r \delta_r + \Delta_{k_{\hat{r}}}^\theta \delta_\theta,$$

$$k_{\hat{\theta}} \simeq -\frac{A\sqrt{G_g}}{\nu_{g\pm}^2} + \frac{A\sqrt{G_g}}{\nu_{g\pm}^2 \gamma_{g\pm}^2 r_g} \delta_r + \frac{1}{2 r_g \sqrt{G_g}} \left[2 \sin\theta_g (1 + 6AM \cos\theta_g) + Ar_g \Delta_G^\theta \left(\frac{1}{\nu_{g\pm}^2} - 2 \right) \right] \delta_\theta$$

$$\equiv k_{\hat{\theta}}^g + \Delta_{k_{\hat{\theta}}}^r \delta_r + \Delta_{k_{\hat{\theta}}}^\theta \delta_\theta,$$

$$\kappa \simeq \left(k_{\hat{r}}^{g2} + k_{\hat{\theta}}^{g2} \right)^{1/2} + \frac{k_{\hat{r}}^g \Delta_{k_{\hat{r}}}^r + k_{\hat{\theta}}^g \Delta_{k_{\hat{\theta}}}^r}{\left(k_{\hat{r}}^{g2} + k_{\hat{\theta}}^{g2} \right)^{1/2}} \delta_r + \frac{k_{\hat{r}}^g \Delta_{k_{\hat{r}}}^\theta + k_{\hat{\theta}}^g \Delta_{k_{\hat{\theta}}}^\theta}{\left(k_{\hat{r}}^{g2} + k_{\hat{\theta}}^{g2} \right)^{1/2}} \delta_\theta \equiv \kappa_g + \Delta_\kappa^r \delta_r + \Delta_\kappa^\theta \delta_\theta,$$

$$\nu_{(r)} \simeq \nu_{g\pm} - \left[\frac{A^2 G_g r_g}{H_g \nu_{g\pm}} \left(\frac{1}{\nu_{g\pm}^2} + \frac{1}{2} \right) + \frac{\nu_{g\pm}}{2 r_g} \left(1 + 2\nu_{g\pm}^2 \right) \right] \delta_r$$

$$+ \nu_{g\pm} \left\{ \frac{A}{\Delta_H^r} \left[-Ar_g \Delta_G^\theta + \sin\theta_g (1 + 6AM \cos\theta_g) \right] - \frac{\Delta_H^\theta}{2 H_g} \right\} \delta_\theta$$

$$\equiv \nu_{g\pm} + \Delta_{\nu_{(r)}}^r \delta_r + \Delta_{\nu_{(r)}}^\theta \delta_\theta,$$

$$\nu_{(\theta)} \simeq \nu_{g\pm} + \frac{\nu_{g\pm}}{2 \gamma_{g\pm}^2 r_g} \delta_r + \frac{\nu_{g\pm}}{2 \gamma_{g\pm}^2} \left[\frac{\Delta_G^\theta}{G_g} + \frac{2 \sin^2\theta_g}{\Delta_G^\theta} (1 + 6AM \cos\theta_g) \right] \delta_\theta$$

$$\equiv \nu_{g\pm} + \Delta_{\nu_{(\theta)}}^r \delta_r + \Delta_{\nu_{(\theta)}}^\theta \delta_\theta, \tag{A.1}$$

where $G_g = G(\theta_g)$ and $H_g = H(r_g, \theta_g)$. As a result, we obtain for each choice of SC the following approximate expressions for the linear velocity ν and the spin parameter $\hat{s}$:

$$\nu \simeq \nu_{g\pm} + \left[\Delta_\nu^{r\,(\text{SC})} \delta_r + \Delta_\nu^{\theta\,(\text{SC})} \delta_\theta \right] \equiv \nu_{g\pm} + \sigma \Delta\nu^{(\text{SC})}$$

$$\hat{s} \simeq \left[\Delta_{\hat{s}}^{r\,(\text{SC})} \delta_r + \Delta_{\hat{s}}^{\theta\,(\text{SC})} \delta_\theta \right] \equiv \sigma \Delta\hat{s}^{(\text{SC})}. \tag{A.2}$$

The corresponding angular velocity ζ and its reciprocal are then given by

$$\zeta_\pm \simeq \zeta_{g\pm} + \sigma \frac{|\zeta_{g\pm}|}{|\nu_{g\pm}|} \left\{ \Delta\nu^{(\text{SC})} - \frac{|\nu_{g\pm}|}{2} \left[\frac{\delta_r}{r_g} \left(2 - r_g \frac{\Delta_H^r}{H_g} \right) + \delta_\theta \left(\frac{\Delta_G^\theta}{G_g} - \frac{\Delta_H^\theta}{H_g} \right) \right] \right\}$$

$$\equiv \zeta_{g\pm} + \sigma \Delta\zeta^{(\text{SC})} \tag{A.3}$$

and

$$\frac{1}{\zeta_\pm} \simeq \frac{1}{\zeta_{g\pm}} - \frac{\sigma}{|\zeta_{g\pm}| \|\nu_{g\pm}\|} \left\{ \Delta\nu^{(\text{SC})} - \frac{|\nu_{g\pm}|}{2} \left[\frac{\delta_r}{r_g} \left(2 - r_g \frac{\Delta_H^r}{H_g} \right) + \delta_\theta \left(\frac{\Delta_G^\theta}{G_g} - \frac{\Delta_H^\theta}{H_g} \right) \right] \right\}$$

$$\equiv \frac{1}{\zeta_{g\pm}} - \frac{\sigma}{\zeta_{g\pm}^2} \Delta\zeta^{(\text{SC})}. \tag{A.4}$$

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