

Quantum Theory in Accelerated Frames of Reference

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Abstract

The observational basis of quantum theory in accelerated systems is studied. The extension of Lorentz invariance to accelerated systems via the hypothesis of locality is discussed and the limitations of this hypothesis are pointed out. The nonlocal theory of accelerated observers is briefly described. Moreover, the main observational aspects of Dirac's equation in noninertial frames of reference are presented. The Galilean invariance of nonrelativistic quantum mechanics and the mass superselection rule are examined in the light of the invariance of physical laws under inhomogeneous Lorentz transformations.

1. INTRODUCTION

Soon after Dirac discovered the relativistic wave equation for a spin $\frac{1}{2}$ particle [1], the generally covariant Dirac equation was introduced by Fock and Ivanenko [2] and was studied in great detail by a number of authors [3]. Dirac's equation

$$(i\hbar\gamma^\alpha\partial_\alpha - mc)\psi = 0 \quad (1)$$

transforms under a Lorentz transformation $x'^\alpha = L^\alpha_\beta x^\beta$ as

$$\psi'(x') = S(L)\psi(x), \quad (2)$$

where $S(L)$ is connected with the spin of the particle and is given by

$$S^{-1}\gamma^\alpha S = L^\alpha_\beta\gamma^\beta. \quad (3)$$

The generally covariant Dirac equation can be written as

$$(i\hbar\gamma^\mu\nabla_\mu - mc)\psi = 0, \quad (4)$$

where $\nabla_\mu = \partial_\mu + \Gamma_\mu$ and Γ_μ is the spin connection. Let us consider a class of observers in spacetime with an orthonormal tetrad frame $\lambda^\mu_{(\alpha)}$, i.e.

$$g_{\mu\nu}\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)} = \eta_{(\alpha)(\beta)}, \quad (5)$$

where $\eta_{(\alpha)(\beta)}$ is the Minkowski metric tensor. Then in equation (4), γ^μ is given by $\gamma^\mu = \lambda^\mu_{(\alpha)}\gamma^{(\alpha)}$ and

$$\Gamma_\mu = -\frac{i}{4}\lambda_{\nu(\alpha)}[\lambda^\nu_{(\beta)}]_{;\mu}\sigma^{(\alpha)(\beta)}, \quad (6)$$

where

$$\sigma^{(\alpha)(\beta)} = \frac{i}{2}[\gamma^{(\alpha)}, \gamma^{(\beta)}]. \quad (7)$$

In this way, the generally covariant Dirac equation is minimally coupled to inertia and gravitation.

The standard quantum measurement theory involves ideal inertial observers. However, all actual observers are more or less accelerated. Indeed, the whole observational basis of Lorentz invariance as well as quantum mechanics rests upon measurements performed by accelerated observers. It is therefore necessary to discuss how the measurements of noninertial observers are connected with those of ideal inertial observers. This paper is thus organized into two parts. In the first part, sections 2-4, we consider the basic physical assumptions that underlie the covariant generalization of Dirac's equation. The second part, sections 5-9, are devoted to the physical consequences of this generalization for noninertial frames of reference. In particular, the connection between the relativistic theory and nonrelativistic quantum mechanics in accelerated systems is examined in detail. Section 10 contains a brief discussion.

2. HYPOTHESIS OF LOCALITY

The extension of Lorentz invariance to noninertial systems necessarily involves an assumption regarding what accelerated observers actually measure. What is assumed in the standard theory of relativity is the *hypothesis of locality*, which states that an accelerated observer is pointwise equivalent to an otherwise identical momentarily comoving inertial observer. It appears that Lorentz first introduced such an assumption in his theory of electrons to ensure that an electron—conceived as a small ball of charge—is always Lorentz contracted along its direction of motion [4]. He clearly recognized that this is simply an approximation

based on the assumption that the time in which the electron velocity changes is very long compared to the period of the internal oscillations of the electron (see section 183 on page 216 of [4]).

The hypothesis of locality was later adopted by Einstein in the course of the development of the theory of relativity (see the footnote on page 60 of [5]). In retrospect, the locality assumption fits perfectly together with Einstein's local principle of equivalence to guarantee that every observer in a gravitational field is locally (i.e. pointwise) inertial. That is, Einstein's heuristic principle of equivalence, namely, the presumed local equivalence of an observer in a gravitational field with an accelerated observer in Minkowski spacetime, would lose its operational significance if one did not know what accelerated observers measure. However, combined with the hypothesis of locality, Einstein's principle of equivalence provides a basis for a theory of gravitation that is consistent with (local) Lorentz invariance.

Early in the development of the theory of relativity, the hypothesis of locality was usually stated in terms of the direct acceleration independence of the behavior of rods and clocks. The clock hypothesis, for instance, states that "standard" clocks measure proper time. Thus measuring devices that conform to the hypothesis of locality are usually called "standard". It is clear that inertial effects exist in any accelerated measuring device; however, in a *standard* device these effects are usually expected to integrate to a negligible influence over the duration of each elementary measurement. Thus a standard measuring device is locally inertial [6].

Following the development of the general theory of relativity, the hypothesis of locality was discussed by Weyl [7]. Specifically, Weyl [7] noted that the locality hypothesis was an adiabaticity assumption in analogy with slow processes in thermodynamics.

The hypothesis of locality originates from Newtonian mechanics: the accelerated observer and the otherwise identical momentarily comoving inertial observer have the same position and velocity; therefore, they share the same *state* and are thus pointwise identical in classical mechanics. The evident validity of this assertion for Newtonian point particles means that no new assumption is required in the treatment of accelerated systems of reference in Newtonian mechanics. It should also hold equally well in the classical *relativistic* mechanics of point particles, as originally recognized by Minkowski (see page 80 of [8]). If all physical phenomena could be reduced to *pointlike coincidences* of particles and rays, then the hypothesis of locality would be exactly valid.

The hypothesis of locality is not in general valid, however, in the case of classical wave phenomena. Consider, for instance, the determination of the frequency of an incident electromagnetic wave by a linearly accelerated observer. Clearly, the frequency cannot be determined instantaneously; in fact, the observer needs to measure a few oscillations of the electromagnetic field before a reasonable determination of the frequency becomes operationally possible. Let λ be the characteristic wavelength of the incident radiation and \mathcal{L} be the acceleration length of the observer; then, the hypothesis of locality is approximately valid for $\lambda \ll \mathcal{L}$. Here \mathcal{L} is a length scale that involves the speed of light c and certain scalars formed from the acceleration of the observer such that the acceleration time \mathcal{L}/c characterizes the time in which the velocity of the observer varies appreciably. In an Earth-based laboratory, for instance, the main translational and rotational acceleration lengths would be $c^2/g_{\oplus} \approx 1$ lt-yr and $c/\Omega_{\oplus} \approx 28$ AU, respectively. Thus in most experimental situations λ/\mathcal{L} is negligibly small and any possible deviations from the locality hypothesis are therefore below the current levels of detectability. Indeed, in the ray limit, $\lambda/\mathcal{L} \rightarrow 0$, the hypothesis of locality would be valid; therefore, λ/\mathcal{L} is a measure of possible deviation from the locality postulate.

Consider a classical particle of mass m and charge q under the influence of an external force \mathbf{f}_{ext} . The accelerated charge radiates electromagnetic radiation with a typical wavelength $\lambda \sim \mathcal{L}$, where \mathcal{L} is the acceleration length of the particle. We would expect that a significant breakdown of the locality hypothesis occurs in this case, since $\lambda/\mathcal{L} \sim 1$ in the interaction of the particle with the electromagnetic field. The violation of the hypothesis of locality implies that the state of the particle cannot be characterized by its position and velocity. This is indeed the case, since the equation of motion of the radiating particle in the nonrelativistic approximation is given by the Abraham-Lorentz equation

$$m \frac{d\mathbf{v}}{dt} - \frac{2q^2}{3c^3} \frac{d^2\mathbf{v}}{dt^2} + \dots = \mathbf{f}_{\text{ext}} \quad (8)$$

which implies that position and velocity are not sufficient to specify the state of the radiating charged particle [9].

To discuss quantum mechanics in an accelerated system of reference, it is therefore useful to investigate the status of the hypothesis of locality vis-a-vis the basic principles of quantum theory. The physical interpretation of wave functions is based on the notion of wave-particle duality. On the other hand, the locality hypothesis is valid for classical particles and is in

general violated for classical waves. This circumstance provides the motivation to develop a nonlocal theory of accelerated systems that would go beyond the hypothesis of locality and would be consistent with wave-particle duality. Such a theory has been developed [10] and can be employed, in principle, to describe a nonlocal Dirac equation in accelerated systems of reference. Some of the main aspects of the nonlocal theory are described in section 4.

3. ACCELERATION TENSOR

It follows from the hypothesis of locality that an accelerated observer in Minkowski space-time carries an orthonormal tetrad $\lambda^\mu_{(\alpha)}$, where $\lambda^\mu_{(0)} = dx^\mu/d\tau$ is its four-velocity vector that is tangent to its worldline and acts as its local temporal axis. Here τ is the proper time along the worldline of the accelerated observer. To avoid unphysical situations, we assume throughout that the observer is accelerated only for a finite period of time. The local spatial frame of the observer is defined by the unit spacelike axes $\lambda^\mu_{(i)}$, $i = 1, 2, 3$. The tetrad frame is transported along the worldline in accordance with

$$\frac{d\lambda^\mu_{(\alpha)}}{d\tau} = \Phi_\alpha{}^\beta \lambda^\mu_{(\beta)}, \quad (9)$$

where

$$\Phi_{\alpha\beta} = -\Phi_{\beta\alpha} \quad (10)$$

is the antisymmetric *acceleration tensor*. In close analogy with the Faraday tensor, the acceleration tensor consists of “electric” and “magnetic” components. The “electric” part is characterized by the translational acceleration of the observer such that $\Phi_{0i} = a_i(\tau)$, where $a_i = A_\mu \lambda^\mu_{(i)}$ and $A^\mu = d\lambda^\mu_{(0)}/d\tau$ is the four-acceleration vector of the observer. The “magnetic” part is characterized by the rotation of the local spatial frame with respect to a locally nonrotating (i.e. Fermi-Walker transported) frame such that $\Phi_{ij} = \epsilon_{ijk} \Omega^k$, where $\Omega(\tau)$ is the rotation frequency. The elements of the acceleration tensor, and hence the spacetime scalars $\mathbf{a}(\tau)$ and $\Omega(\tau)$, completely determine the local rate of variation of the state of the observer. It proves useful to define the acceleration lengths $\mathcal{L} = c^2/a$ and c/Ω , as well as the corresponding acceleration times $\mathcal{L}/c = c/a$ and $1/\Omega$, to indicate respectively the spatial and temporal scales of variation of the state of the observer. Let λ be the intrinsic length scale of the phenomenon under observation; then, we expect that the deviation from the hypothesis of locality should be proportional to λ/\mathcal{L} .

It follows from a detailed analysis that if D is the spatial dimension of a standard measuring device, then $D \ll \mathcal{L}$ [6]. Such devices are necessary for the determination of the local frame of the accelerated observer. In fact, this circumstance is analogous to the correspondence principle: while we are interested in the deviations from the hypothesis of locality, such nonlocal effects are expected to be measured with standard measuring devices.

4. NONLOCALITY

Imagine an accelerated observer in a background global Minkowski spacetime and let $\psi(x)$ be a basic incident radiation field. The observer along its worldline passes through a continuous infinity of hypothetical momentarily comoving inertial observers; therefore, let $\hat{\psi}(\tau)$ be the field measured by the hypothetical inertial observer at the event characterized by the proper time τ . The local spacetime of the hypothetical inertial observer is related to the background via a proper Poincaré transformation $x' = Lx + s$; hence, $\psi'(x') = \Lambda(L)\psi(x)$, so that $\Lambda = 1$ for a scalar field. We therefore assume that along the worldline $\hat{\psi}(\tau) = \Lambda(\tau)\psi(\tau)$, where Λ belongs to a matrix representation of the Lorentz group.

Suppose that $\hat{\Psi}(\tau)$ is the field that is actually measured by the accelerated observer. What is the connection between $\hat{\Psi}(\tau)$ and $\hat{\psi}(\tau)$? The hypothesis of locality postulates the pointwise equivalence of $\hat{\Psi}(\tau)$ and $\hat{\psi}(\tau)$, i.e. it requires that $\hat{\Psi}(\tau) = \hat{\psi}(\tau)$. On the other hand, the most general linear relation between $\hat{\Psi}(\tau)$ and $\hat{\psi}(\tau)$ consistent with causality is

$$\hat{\Psi}(\tau) = \hat{\psi}(\tau) + \int_{\tau_0}^{\tau} K(\tau, \tau') \hat{\psi}(\tau') d\tau', \quad (11)$$

where τ_0 is the initial instant of the observer's acceleration. Equation (11) is manifestly Lorentz invariant, since it involves spacetime scalars. The kernel $K(\tau, \tau')$ must be directly proportional to the observer's acceleration, since $\hat{\Psi} = \hat{\psi}$ for an inertial observer. The ansatz (11) differs from the hypothesis of locality by an integral over the past worldline of the observer. In fact, this nonlocal part is expected to vanish for $\lambda/\mathcal{L} \rightarrow 0$. The determination of a radiation field by an accelerated observer involves a certain spacetime average according to equation (11) and this circumstance is consistent with the viewpoint developed by Bohr and Rosenfeld [11].

Equation (11) has the form of a Volterra integral equation. According to Volterra's theorem [12], the relationship between $\hat{\Psi}$ and $\hat{\psi}$ (and hence ψ) is unique in the space of

continuous functions. Volterra's theorem has been extended to the Hilbert space of square-integrable functions by Tricomi [13].

To determine the kernel K , we postulate that a basic radiation field can never stand completely still with respect to an accelerated observer. This physical requirement is a generalization of a well-known consequence of Lorentz invariance to all observers. That is, the invariance of Maxwell's equations under the Lorentz transformations implies that electromagnetic radiation propagates with speed c with respect to all inertial observers. That this is the case for any basic radiation field is reflected in the Doppler formula, $\omega' = \gamma(\omega - \mathbf{v} \cdot \mathbf{k})$, where $\omega = c|\mathbf{k}|$. An inertial observer moving uniformly with speed v that approaches c measures a frequency ω' that approaches zero, but the wave will never stand completely still ($\omega' \neq 0$) since $v < c$; hence, $\omega' = 0$ implies that $\omega = 0$. Generalizing this situation to arbitrary accelerated observers, we demand that if $\hat{\Psi}$ turns out to be a constant, then ψ must have been constant in the first place. The Volterra-Tricomi uniqueness result then implies that for any true radiation field ψ in the inertial frame, the field $\hat{\Psi}$ measured by the accelerated observer will vary in time. Writing equation (11) as

$$\hat{\Psi}(\tau) = \Lambda(\tau)\psi(\tau) + \int_{\tau_0}^{\tau} K(\tau, \tau')\Lambda(\tau')\psi(\tau')d\tau', \quad (12)$$

we note that our basic postulate that a constant $\hat{\Psi}$ be associated with a constant ψ implies

$$\Lambda(\tau_0) = \Lambda(\tau) + \int_{\tau_0}^{\tau} K(\tau, \tau')\Lambda(\tau')d\tau', \quad (13)$$

where we have used the fact that $\hat{\Psi}(\tau_0) = \Lambda(\tau_0)\psi(\tau_0)$. Given $\Lambda(\tau)$, equation (13) can be used to determine $K(\tau, \tau')$; however, it turns out that $K(\tau, \tau')$ cannot be uniquely specified in this way. To go forward, it originally appeared most natural from the standpoint of phenomenological nonlocal theories to postulate that $K(\tau, \tau')$ is only a function of $\tau - \tau'$ [10]; however, detailed investigations later revealed that such a convolution kernel can lead to divergences in the case of nonuniform acceleration [14]. It turns out that the only physically acceptable solution of equation (13) is of the form [15, 16]

$$K(\tau, \tau') = k(\tau') = -\frac{d\Lambda(\tau')}{d\tau'}\Lambda^{-1}(\tau'). \quad (14)$$

In the case of uniform acceleration, equation (14) and the convolution kernel both lead to the same constant kernel. The kernel (14) is directly proportional to the acceleration of the observer and is a simple solution of equation (13), as can be verified by direct substitution.

Moreover, if the acceleration of the observer is turned off at τ_f , then the unique kernel (14) vanishes for $\tau > \tau_f$. Thus for $\tau > \tau_f$, the nonlocal contribution to the field in equation (11) is simply a constant memory of the past acceleration of the observer that is in principle measurable. This constant memory is simply canceled in a measuring device whenever the device is reset.

For a scalar field $\Lambda = 1$ and hence the kernel (14) vanishes. As will be demonstrated in section 8, it follows from the locality of such a field that for scalar radiation of frequency ω , an observer rotating uniformly with frequency Ω will measure $\omega' = \gamma(\omega - M\Omega)$, where $M = 0, \pm 1, \pm 2, \dots$. Thus $\omega' = 0$ for $\omega = M\Omega$ and our basic physical postulate is violated: the scalar radiation stands completely still for all observers rotating uniformly about the same axis with frequency Ω . It therefore follows from the nonlocal theory of accelerated observers that a pure scalar (or pseudoscalar) radiation field does not exist. Such fields can only be composites formed from other basic fields. This consequence of the nonlocal theory is consistent with present observational data, as they show no trace of a fundamental scalar (or pseudoscalar) field.

4.1. Nonlocal Field Equations

It follows from the Volterra equation (11) with kernel (14) that

$$\hat{\psi} = \hat{\Psi} + \int_{\tau_0}^{\tau} r(\tau, \tau') \hat{\Psi}(\tau') d\tau', \quad (15)$$

where $r(\tau, \tau')$ is the resolvent kernel. Imagine that a nonlocal field Ψ exists in the background Minkowski spacetime such that an accelerated observer with a tetrad frame $\lambda^\mu_{(\alpha)}$ measures

$$\hat{\Psi} = \Lambda \Psi. \quad (16)$$

The relationship between Ψ and ψ can then be simply worked out using (15), namely,

$$\psi = \Psi + \int_{\tau_0}^{\tau} \tilde{r}(\tau, \tau') \Psi(\tau') d\tau', \quad (17)$$

where \tilde{r} is related to the resolvent kernel by

$$\tilde{r}(\tau, \tau') = \Lambda^{-1}(\tau) r(\tau, \tau') \Lambda(\tau'). \quad (18)$$

It is possible to extend equation (17) to a class of accelerated observers such that $\psi(x)$ within a finite region of spacetime is related to a nonlocal field $\Psi(x)$ by a suitable extension

of equation (17). The local field $\psi(x)$ satisfies certain partial differential equations; therefore, it follows from (17) that Ψ would satisfy certain Lorentz-invariant nonlocal field equations. In this way, the nonlocal Maxwell equations have been derived explicitly for certain linearly accelerated systems [17]. It turns out that in general the field equations remain nonlocal even after the cessation of accelerated motion.

4.2. Nonlocal Electrodynamics

To confront the nonlocal theory with observation, it is useful to derive the physical consequences of nonlocal electrodynamics in systems that undergo translational and rotational accelerations and compare the predictions of the theory with observational data. It turns out that for accelerated systems the experimental data available at present do not have sufficient sensitivity to distinguish between the standard theory (based on the locality hypothesis) and the nonlocal theory. In the case of linearly accelerated systems, it may be possible to reach the desired level of sensitivity with the acceleration of grains using high-intensity femtosecond lasers [18, 19]. For a uniformly rotating observer in circular motion, one can compare the predictions of nonlocal electrodynamics with the nonrelativistic quantum mechanics of electrons in circular atomic orbits or about uniform magnetic fields in the correspondence limit. If the nonlocal theory corresponds to reality, its predictions should be closer to quantum mechanical results in the correspondence regime than those of the standard local theory of accelerated systems. This turns out to be the case for the simple cases that have been worked out in detail [20]. Let us now return to the standard physical consequences of Dirac's equation in noninertial systems of reference. In the following sections, emphasis will be placed on the main inertial effects and their observational aspects in matter-wave interferometry.

5. INERTIAL PROPERTIES OF A DIRAC PARTICLE

The physical consequences that follow from the Dirac equation in systems of reference that undergo translational and rotational accelerations have been considered by a number of authors [21]-[24]. In particular, the work of Hehl and Ni [25] has elucidated the general inertial properties of a Dirac particle. In their approach, standard Foldy-Wouthuysen [26]

transformations are employed to decouple the positive and negative energy states such that the Hamiltonian for the Dirac particle may be written as

$$\mathcal{H} = \beta \left(mc^2 + \frac{p^2}{2m} \right) + \beta m \mathbf{a} \cdot \mathbf{x} - \boldsymbol{\Omega} \cdot (\mathbf{L} + \mathbf{S}) \quad (19)$$

plus higher-order terms. Here $\beta m \mathbf{a} \cdot \mathbf{x}$ is an inertial term due to the translational acceleration of the reference frame, while the inertial effects due to the rotation of the reference frame are reflected in $-\boldsymbol{\Omega} \cdot (\mathbf{L} + \mathbf{S})$.

Before proceeding to a detailed discussion of these inertial terms in sections 6-9, it is important to observe that Obukhov [27] has recently introduced certain exact ‘‘Foldy-Wouthuysen’’ (FW) transformations to decouple the positive and negative energy states of the Dirac particle. Such a FW transformation is defined up to a unitary transformation, which introduces a certain level of ambiguity in the physical interpretation. That is, it is not clear from [27] what one could predict to be the observable consequences of Dirac’s theory in noninertial systems and gravitational fields. For instance, in Obukhov’s exact FW transformation, an inertial term of the form $-\frac{1}{2} \mathbf{S} \cdot \mathbf{a}$ appears in the Hamiltonian [27]; on the other hand, it is possible to remove this term by a unitary transformation [27]. The analog of this term in a gravitational context would be $\frac{1}{2} \mathbf{S} \cdot \mathbf{g}$. Thus the energy difference between the states of a Dirac particle with spin polarized up and down in a laboratory on the Earth would be $\frac{1}{2} \hbar g_{\oplus} \approx 10^{-23} \text{eV}$, which is a factor of five larger than what can be detected at present [28]. A detailed examination of spin-acceleration coupling together with theoretical arguments for its absence is contained in [29].

The general question raised in [27] has been treated in [30]. It appears that with a proper choice of the unitary transformation such that physical quantities would correspond to simple operators, the standard FW transformations of Hehl and Ni [25] can be recovered [30]. Nevertheless, a certain phase ambiguity can still exist in the wave function corresponding to the fact that the unitary transformation may not be unique. This phase problem exists even in the nonrelativistic treatment of quantum mechanics in translationally accelerated systems as discussed in detail in section 9.

6. ROTATION

It is possible to provide a simple justification for the rotational inertial term in the Hamiltonian (19). Let us start with the classical nonrelativistic Lagrangian of a particle $L = \frac{1}{2}mv^2 - W$, where W is a potential energy. Under a transformation to a rotating frame of reference, $\mathbf{v} = \mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r}$, the Lagrangian takes the form

$$L' = \frac{1}{2}m(\mathbf{v}' + \boldsymbol{\Omega} \times \mathbf{r})^2 - W, \quad (20)$$

where W is assumed to be invariant under the transformation to the rotating frame. The canonical momentum of the particle $\mathbf{p}' = \partial L' / \partial \mathbf{v}' = \mathbf{p}$ is an invariant and we find that $H' = H - \boldsymbol{\Omega} \cdot \mathbf{L}$, where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the invariant angular momentum of the particle. Let us note that this result of Newtonian mechanics [31] has a simple relativistic generalization: the rotating observer measures the energy of the particle to be $E' = \gamma(E - \mathbf{v} \cdot \mathbf{p})$, where $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$; therefore, $E' = \gamma(E - \boldsymbol{\Omega} \cdot \mathbf{L})$.

This local approach may be simply extended to nonrelativistic quantum mechanics, where the hypothesis of locality would imply that [32]

$$\psi'(\mathbf{x}', t) = \psi(\mathbf{x}, t), \quad (21)$$

since the rotating measuring devices are assumed to be locally inertial. Thus $\psi'(\mathbf{x}', t) = R\psi(\mathbf{x}, t)$, where

$$R = \hat{T} e^{\frac{i}{\hbar} \int_0^t \boldsymbol{\Omega}(t') \cdot \mathbf{J} dt'}. \quad (22)$$

Here \hat{T} is the time-ordering operator and we have replaced \mathbf{L} by $\mathbf{J} = \mathbf{L} + \mathbf{S}$, since the *total* angular momentum is the generator of rotations [32]. It follows that from the standpoint of rotating observers, $H\psi = i\hbar\partial\psi/\partial t$ takes the form $H'\psi' = i\hbar\partial\psi'/\partial t$, where

$$H' = RHR^{-1} - \boldsymbol{\Omega} \cdot \mathbf{J}. \quad (23)$$

For the case of the single particle viewed by uniformly rotating observers, H' can be written as

$$H' = \frac{1}{2m}(\mathbf{p}' - m\boldsymbol{\Omega} \times \mathbf{r})^2 - \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2 - \boldsymbol{\Omega} \cdot \mathbf{S} + W, \quad (24)$$

where $-\frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2$ is the standard centrifugal potential and $-\boldsymbol{\Omega} \cdot \mathbf{S}$ is the spin-rotation coupling term [32]. The Hamiltonian (24) is analogous to that of a charged particle in a uniform magnetic field; this situation is a reflection of the Larmor theorem. The corresponding

analog of the Aharonov-Bohm effect is the Sagnac effect for matter waves [33]. This effect is discussed in the next section.

7. SAGNAC EFFECT

The term $-\mathbf{\Omega} \cdot \mathbf{L}$ in the Hamiltonian (19) signifies the coupling of the orbital angular momentum of the particle with the rotation of the reference frame and is responsible for the Sagnac effect exhibited by the Dirac particle. The corresponding Sagnac phase shift is given by

$$\Delta\Phi_{\text{Sagnac}} = \frac{2m}{\hbar} \int \mathbf{\Omega} \cdot d\mathbf{A}, \quad (25)$$

where \mathbf{A} is the area of the interferometer. Equation (25) can be expressed as

$$\Delta\Phi_{\text{Sagnac}} = \frac{2\omega}{c^2} \int \mathbf{\Omega} \cdot d\mathbf{A}, \quad (26)$$

where $mc^2 \approx \hbar\omega$ and ω is the de Broglie frequency of the particle. Equation (26) is equally valid for electromagnetic radiation of frequency ω .

For matter waves, the Sagnac effect was first experimentally measured for Cooper pairs in a rotating superconducting Josephson-junction interferometer [34]. Using slow neutrons, Werner et al. [35] measured the Sagnac effect with $\mathbf{\Omega}$ as the rotation frequency of the Earth. The result was subsequently confirmed with a rotating neutron interferometer in the laboratory [36]. Significant advances in atom interferometry have led to the measurement of the Sagnac effect for neutral atoms as well. This was first achieved by Riehle et al. [37] and has been subsequently developed with a view towards achieving high sensitivity for atom interferometers as inertial sensors [38]. In connection with charged particle interferometry, the Sagnac effect has been observed for electrons by Hasselbach and Nicklaus [39].

The Sagnac effect has significant and wide-ranging applications and has been reviewed in [40].

8. SPIN-ROTATION COUPLING

The transformation of the wave function to a uniformly rotating system of coordinates involves $(t, r, \theta, \phi) \rightarrow (t, r, \theta, \phi + \Omega t)$ in spherical coordinates, where Ω is the frequency of rotation about the z axis. If the dependence of the wave function on ϕ and t is of the

form $\exp(iM\phi - iEt/\hbar)$, then in the rotating system the temporal dependence of the wave function is given by $\exp[-i(E - \hbar M\Omega)t/\hbar]$. The energy of the particle measured by an observer at rest in the rotating frame is

$$E' = \gamma(E - \hbar M\Omega), \quad (27)$$

where $\gamma = t/\tau$ is the Lorentz factor due to time dilation. Here $\hbar M$ is the *total* angular momentum of the particle along the axis of rotation; in fact, $M = 0, \pm 1, \pm 2, \dots$, for a scalar or a vector particle, while $M \mp \frac{1}{2} = 0, \pm 1, \pm 2, \dots$, for a Dirac particle.

In the JWKB approximation, equation (27) may be expressed as $E' = \gamma(E - \boldsymbol{\Omega} \cdot \mathbf{J})$ and hence

$$E' = \gamma(E - \boldsymbol{\Omega} \cdot \mathbf{L}) - \gamma\boldsymbol{\Omega} \cdot \mathbf{S}. \quad (28)$$

It follows that the energy measured by the observer is the result of an instantaneous Lorentz transformation together with an additional term

$$\delta H = -\gamma\boldsymbol{\Omega} \cdot \mathbf{S}, \quad (29)$$

which is due to the coupling of the intrinsic spin of the particle with the frequency of rotation of the observer [32]. The dynamical origin of this term can be simply understood on the basis of the following consideration: The intrinsic spin of a free particle remains fixed with respect to the underlying global inertial frame; therefore, from the standpoint of observers at rest in the rotating system, the spin precesses in the opposite sense as the rotation of the observers. The Hamiltonian responsible for this inertial motion is given by equation (29). The relativistic nature of spin-rotation coupling has been demonstrated by Ryder [41]. Let us illustrate these ideas by a thought experiment involving the reception of electromagnetic radiation of frequency ω by an observer that rotates uniformly with frequency Ω . We assume for the sake of simplicity that the plane circularly polarized radiation is normally incident on the path of the observer, i.e. the wave propagates along the axis of rotation. We are interested in the frequency of the wave ω' as measured by the rotating observer. A simple application of the hypothesis of locality leads to the conclusion that the measured frequency is related to ω by the transverse Doppler effect, $\omega'_D = \gamma\omega$, since the instantaneous rest frame of the observer is related to the background global inertial frame by a Lorentz transformation. On the other hand, a different answer emerges when we focus attention on

the measured electromagnetic field rather than the propagation vector of the radiation,

$$F_{(\alpha)(\beta)}(\tau) = F_{\mu\nu}\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)}, \quad (30)$$

where $F_{\mu\nu}$ is the Faraday tensor of the incident radiation and $\lambda^\mu_{(\alpha)}$ is the orthonormal tetrad of the rotating observer. The nonlocal process of Fourier analysis of $F_{(\alpha)(\beta)}$ results in [42]

$$\omega' = \gamma(\omega \mp \Omega), \quad (31)$$

where the upper (lower) sign refers to positive (negative) helicity radiation. We note that in the eikonal limit $\Omega/\omega \rightarrow 0$ and the instantaneous Doppler result is recovered. The general problem of electromagnetic waves in a (uniformly) rotating frame of reference has been treated in [43].

It is possible to understand equation (31) in terms of the relative motion of the observer with respect to the field. In a positive (negative) helicity wave, the electric and magnetic fields rotate with the wave frequency ω ($-\omega$) about its direction of propagation. Thus the rotating observer perceives that the electric and magnetic fields rotate with frequency $\omega - \Omega$ ($-\omega - \Omega$) about the direction of wave propagation. Taking due account of time dilation, the observed frequency of the wave is thus $\gamma(\omega - \Omega)$ in the positive helicity case and $\gamma(\omega + \Omega)$ in the negative helicity case. These results illustrate the phenomenon of helicity-rotation coupling for the photon, since (31) can be written as $E' = \gamma(E - \mathbf{S} \cdot \boldsymbol{\Omega})$, where $E = \hbar\omega$, $\mathbf{S} = \hbar\hat{\mathbf{H}}$ and $\hat{\mathbf{H}} = \pm c\mathbf{k}/\omega$ is the unit helicity vector.

It follows from (31) that for a slowly moving detector $\gamma \approx 1$ and

$$\omega' \approx \omega \mp \Omega, \quad (32)$$

which corresponds to the phenomenon of *phase wrap-up* in the Global Positioning System (GPS) [44]. In fact, equation (32) has been verified for $\omega/(2\pi) \approx 1$ GHz and $\Omega/(2\pi) \approx 8$ Hz by means of the GPS [44]. For $\omega \gg \Omega$, the modified Doppler and aberration formulas due to the helicity-rotation coupling are [45]

$$\omega' = \gamma[(\omega - \hat{\mathbf{H}} \cdot \boldsymbol{\Omega}) - \mathbf{v} \cdot \mathbf{k}], \quad (33)$$

$$\mathbf{k}' = \mathbf{k} + \frac{1}{v^2}(\gamma - 1)(\mathbf{v} \cdot \mathbf{k})\mathbf{v} - \frac{1}{c^2}\gamma(\omega - \hat{\mathbf{H}} \cdot \boldsymbol{\Omega})\mathbf{v}, \quad (34)$$

and similar formulas can be derived for any spinning particle. Circularly polarized radiation is routinely employed for radio communication with artificial satellites as well as Doppler

tracking of spacecraft. In general, the rotation of the emitter as well as the receiver should be taken into account. It follows from (33) that ignoring helicity-rotation coupling would lead to a systematic Doppler bias of magnitude $c\Omega/\omega$. In the case of the Pioneer spacecraft, the anomalous acceleration resulting from the helicity-rotation coupling has been shown to be negligibly small [46].

A half-wave plate flips the helicity of a photon that passes through it. Imagine a half-wave plate that rotates uniformly with frequency Ω and an incident positive helicity plane wave of frequency ω_{in} that propagates along the axis of rotation. It follows from (32) that $\omega' \approx \omega_{\text{in}} - \Omega$. The spacetime of a uniformly rotating system is stationary; therefore, ω' remains fixed inside the plate. The radiation that emerges from the plate has frequency ω_{out} and negative helicity; hence, equation (32) implies that $\omega' \approx \omega_{\text{out}} + \Omega$. Thus the rotating half-wave plate is a frequency shifter: $\omega_{\text{out}} - \omega_{\text{in}} \approx -2\Omega$. In general, any rotating spin flipper can cause an up/down energy shift given by $-2\mathbf{S} \cdot \boldsymbol{\Omega}$ as a consequence of the spin-rotation coupling. The frequency-shift phenomenon was first discovered in microwave experiments [47] and has subsequently been used in many optical experiments (see [45] for a list of references).

Regarding the spin-rotation coupling for fermions, let us note that for experiments in a laboratory fixed on the Earth, we must add to every Hamiltonian the spin-rotation-gravity term

$$\delta H \approx -\mathbf{S} \cdot \boldsymbol{\Omega} + \mathbf{S} \cdot \boldsymbol{\Omega}_P, \quad (35)$$

where the second term is due to the gravitomagnetic field of the Earth. That is, the rotation of the Earth causes a dipolar gravitomagnetic field (due to mass current), which is locally equivalent to a rotation by the gravitational Larmor theorem. In fact, $\boldsymbol{\Omega}_P$ is the frequency of precession of an ideal fixed test gyro and is given by

$$\boldsymbol{\Omega}_P \approx \frac{G}{c^2 r^5} [3(\mathbf{J} \cdot \mathbf{r})\mathbf{r} - \mathbf{J}r^2], \quad (36)$$

where J is the proper angular momentum of the central source. It follows from (35) that for a spin $\frac{1}{2}$ particle, the difference between the energy of the particle with spin up and down in the laboratory is characterized by $\hbar\Omega_{\oplus} \sim 10^{-19}\text{eV}$ and $\hbar\Omega_P \sim 10^{-29}\text{eV}$, while the present experimental capabilities are in the 10^{-24}eV range [28]. In fact, indirect observational evidence for the spin-rotation coupling has been obtained [48] from the analysis of experiments

that have searched for anomalous spin-gravity interactions [49]. Further evidence for spin-rotation coupling exists based on the analysis of muon $g - 2$ experiment [51].

An experiment to measure directly the spin-rotation coupling for a spin $\frac{1}{2}$ particle was originally proposed in [32]. This involved a large-scale neutron interferometry experiment with polarized neutrons on a rotating platform [52]. A more recent proposal [53] employs a rotating neutron spin flipper and hence is much more manageable as it avoids a large-scale interferometer. The slow neutrons from a source are longitudinally polarized and the beam is coherently split into two paths that contain neutron spin flippers, one of which rotates with frequency Ω about the direction of motion of the neutrons. In this leg of the interferometer, an energy shift $\delta H = -2\mathbf{S} \cdot \boldsymbol{\Omega}$ is thus introduced. The two beams are brought back together and the interference beat frequency Ω is then measured. It is interesting to note that a beat frequency in neutron interferometry has already been measured in another context [54]; therefore, similar techniques can be used in the proposed experiment [53].

Some general remarks on the calculation of the phase shift are in order here. One starts from the relation $\hbar d\Phi = -E dt + \mathbf{p} \cdot d\mathbf{x}$ for the phase $\Phi(\mathbf{x}, t)$ of the neutron wave in the JWKB approximation. Integrating from the source (\mathbf{x}_S, t_S) to the detector (\mathbf{x}_D, t_D) , we find

$$\hbar\Phi(\mathbf{x}_D, t_D) = \hbar\Phi(\mathbf{x}_S, t_S) - \int_{t_S}^{t_D} E dt + \int_{\mathbf{x}_S}^{\mathbf{x}_D} \mathbf{p} \cdot d\mathbf{x}. \quad (37)$$

Assuming equal amplitudes, the detector output is proportional to

$$|e^{i\Phi_1} + e^{i\Phi_2}|^2 = 2(1 + \cos \Delta\Phi), \quad (38)$$

where $\Phi_1(\Phi_2)$ refers to the phase accumulated along the first (second) beam and $\Delta\Phi = \Phi_1 - \Phi_2$. It is usually assumed that the two beams are coherently split at the source; therefore,

$$\Phi_1(\mathbf{x}_S, t_S) = \Phi_2(\mathbf{x}_S, t_S). \quad (39)$$

We thus find

$$\hbar \Delta\Phi = - \int_{t_S}^{t_D} \Delta E dt + \oint \mathbf{p} \cdot d\mathbf{x}. \quad (40)$$

In stationary situations, it is possible to assume that $E_1 = E_2 = p_0^2/(2m)$, where (for $i = 1, 2$)

$$E_i = \frac{p_i^2}{2m} + \delta H_i. \quad (41)$$

Thus $\Delta E = 0$ and the calculation of the phase shift (40) can be simply performed if the perturbations δH_1 and δH_2 are small. It then follows from (41) that if $\delta \mathbf{p}$ is the perturbation in neutron momentum due to δH such that $\mathbf{p} - \delta \mathbf{p}$ is the “unperturbed” momentum with magnitude p_0 , then

$$\mathbf{v} \cdot \delta \mathbf{p} = -\delta H, \quad (42)$$

where \mathbf{v} is the neutron velocity. Hence, the *extra* phase shift due to the perturbation is given by

$$\Delta \Phi = \frac{1}{\hbar} \oint \delta \mathbf{p} \cdot d\mathbf{x} = \frac{1}{\hbar} \int_S^D (-\delta H_1 + \delta H_2) dt. \quad (43)$$

Consider, as an example, the Sagnac effect in the rotating frame, where $E = p^2/(2m) + \delta H$ with $\delta H = -\boldsymbol{\Omega} \cdot \mathbf{L}$. Thus equation (43) can be written as $\hbar \Delta \Phi = \oint \boldsymbol{\Omega} \cdot (m\mathbf{r} \times d\mathbf{r})$, since $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. In this way, one immediately recovers equation (25). The approach described here was originally employed for the calculation of the phase shift due to the spin-rotation coupling in a uniformly rotating system in [32].

In nonstationary situations, such as the proposed experiment using a rotating spin flipper, $\Delta E \neq 0$ and hence there is a beat phenomenon in addition to a phase shift. In fact, it follows from the analysis of that experiment [53] that $\Delta E = -\hbar\Omega$ for $t > t_{\text{out}}$, when the neutron exits the spin flippers. Hence $\Delta \Phi$ contains $\Omega(t_D - t_{\text{out}})$ in addition to a phase shift.

It is important to mention briefly the modification of spin-rotation coupling by the nonlocal theory of accelerated observers (section 4). Equation (27) implies that E' can be positive, zero or negative. When $E' = 0$, the wave stands completely still with respect to the static observers in the rotating system. This is contrary to the basic postulate of the nonlocal theory; therefore, the only modification in equation (27) occurs for the $E' = 0$ case. This circumstance is discussed in detail in [20].

9. TRANSLATIONAL ACCELERATION

Before treating quantum mechanics in translationally accelerated systems, it proves useful to digress here and discuss the transition from Lorentz invariance to Galilean invariance in quantum mechanics. What is the transformation rule for a Schrödinger wave function under a Galilean boost ($t = t'$, $\mathbf{x} = \mathbf{x}' + \mathbf{V}t$)? It follows from Lorentz invariance that for a spinless particle

$$\phi(x) = \phi'(x'), \quad (44)$$

where ϕ is a scalar wave function that satisfies the Klein-Gordon equation

$$\left(\square + \frac{m^2 c^2}{\hbar^2}\right) \phi(x) = 0. \quad (45)$$

To obtain the Schrödinger equation from (45) in the nonrelativistic limit, we set

$$\phi(x) = \varphi(\mathbf{x}, t) e^{-i \frac{m c^2}{\hbar} t}. \quad (46)$$

Then, (45) reduces to

$$-\frac{\hbar^2}{2m} \nabla^2 \varphi = i \hbar \frac{\partial \varphi}{\partial t} - \frac{\hbar^2}{2m c^2} \frac{\partial^2 \varphi}{\partial t^2}. \quad (47)$$

Neglecting the term proportional to the second temporal derivative of φ in the nonrelativistic limit ($c \rightarrow \infty$), we recover the Schrödinger equation for the wave function φ .

Under a Lorentz boost, (44) and (46) imply that

$$\varphi(\mathbf{x}, t) e^{-i \frac{m c^2}{\hbar} t} = \varphi'(\mathbf{x}', t') e^{-i \frac{m c^2}{\hbar} t'}, \quad (48)$$

where

$$t = \gamma \left(t' + \frac{1}{c^2} \mathbf{V} \cdot \mathbf{x}' \right). \quad (49)$$

It follows from

$$t - t' = \frac{1}{c^2} \left(\mathbf{V} \cdot \mathbf{x}' + \frac{1}{2} V^2 t' \right) + O\left(\frac{1}{c^4}\right) \quad (50)$$

that in the nonrelativistic limit ($c \rightarrow \infty$),

$$\varphi(\mathbf{x}, t) = e^{i \frac{m}{\hbar} (\mathbf{V} \cdot \mathbf{x}' + \frac{1}{2} V^2 t)} \varphi'(\mathbf{x}', t). \quad (51)$$

This is the standard transformation formula for the Schrödinger wave function under a Galilean boost.

On the other hand, we expect from equations (2) and (44) that in the absence of spin, the wave function should turn out to be an invariant. Writing equation (48) in the form

$$\varphi(\mathbf{x}, t) e^{-i \frac{m c^2}{\hbar} t} = [\varphi'(\mathbf{x}', t') e^{i \frac{m c^2}{\hbar} (t-t')}] e^{-i \frac{m c^2}{\hbar} t}, \quad (52)$$

we note that the nonrelativistic wave function may be assumed to be an invariant under a Galilean transformation

$$\psi(\mathbf{x}, t) = \psi'(\mathbf{x}', t), \quad (53)$$

where

$$\psi(\mathbf{x}, t) = \varphi(\mathbf{x}, t), \quad \psi'(\mathbf{x}', t) = e^{i \frac{m}{\hbar} (\mathbf{V} \cdot \mathbf{x}' + \frac{1}{2} V^2 t)} \varphi'(\mathbf{x}', t). \quad (54)$$

That is, in this approach the phase factor in (51) that is due to the relativity of simultaneity belongs to the wave function itself.

The form invariance of the Schrödinger equation under Galilean transformations was used by Bargmann [55] to show that under the Galilei group, the wave function transforms as in (51). Bargmann used this result in a thought experiment involving the behavior of a wave function under the following four operations: a translation (\mathbf{s}) and then a boost (\mathbf{V}) followed by a translation ($-\mathbf{s}$) and finally a boost ($-\mathbf{V}$) to return to the original inertial system. It is straightforward to see from equation (51) that the original wave function $\varphi(\mathbf{x}, t)$ is related to the final one $\varphi'(\mathbf{x}, t)$ by

$$\varphi(\mathbf{x}, t) = e^{-i\frac{m}{\hbar}\mathbf{s}\cdot\mathbf{V}} \varphi'(\mathbf{x}, t). \quad (55)$$

The phase factor in (55) leads to the *mass superselection rule*, namely, one cannot coherently superpose states of particles of different inertial masses [55, 56]. This rule guarantees strict conservation of mass in nonrelativistic quantum mechanics. The physical significance of this superselection rule has been critically discussed by Giulini [57] and more recently by Greenberger [58]. The main point here is that only Lorentz invariance is fundamental, since the nonrelativistic limit ($c \rightarrow \infty$) is never actually realized.

It should be clear from the preceding discussion that no mass superselection rule is encountered in the second approach based on the invariance of the wave function (53). It follows from the hypothesis of locality that the two distinct methods under discussion here carry over to the quantum mechanics of accelerated systems [59].

Let us therefore consider the transformation to an accelerated system

$$\mathbf{x} = \mathbf{x}' + \int_0^t \mathbf{V}(t') dt', \quad (56)$$

where $\mathbf{a} = d\mathbf{V}/dt$ is the translational acceleration vector. Starting from the Schrödinger equation $H\psi = i\hbar\partial\psi/\partial t$ and assuming the invariance of the wave function, $\psi(\mathbf{x}, t) = \psi'(\mathbf{x}', t)$, as in the second approach, we find that $\psi'(\mathbf{x}', t) = U\psi(\mathbf{x}', t)$, where

$$U = e^{\frac{i}{\hbar} \int_0^t \mathbf{V}(t') \cdot \mathbf{p} dt'}. \quad (57)$$

It follows that ψ' satisfies the Schrödinger equation $H'\psi' = i\hbar\partial\psi'/\partial t$ with the Hamiltonian

$$H' = UHU^{-1} - \mathbf{V}(t) \cdot \mathbf{p}, \quad (58)$$

where \mathbf{p} is the invariant canonical momentum. Writing $H = p^2/(2m) + W$, where W is the invariant potential energy, we find

$$\left[\frac{1}{2m}(\mathbf{p} - m\mathbf{V})^2 - \frac{1}{2}mV^2 + W \right] \psi' = i\hbar \frac{\partial \psi'}{\partial t}. \quad (59)$$

Let

$$\psi'(\mathbf{x}', t) = e^{i\frac{m}{\hbar}[\mathbf{V}\cdot\mathbf{x}' + \frac{1}{2}\int_0^t V^2(t')dt']} \varphi'(\mathbf{x}', t), \quad (60)$$

then $\varphi'(\mathbf{x}', t)$ satisfies the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\nabla'^2 + m\mathbf{a}\cdot\mathbf{x}' + W \right) \varphi' = i\hbar \frac{\partial \varphi'}{\partial t}, \quad (61)$$

where $\nabla' = \nabla$ follows from (56). It is important to recognize that $\varphi'(\mathbf{x}', t)$ is the wave function from the standpoint of the accelerated system according to the first (Bargmann) approach. Here the acceleration potential $m\mathbf{a}\cdot\mathbf{x}'$, where $-\nabla'(m\mathbf{a}\cdot\mathbf{x}') = -m\mathbf{a}$ is the inertial force acting on the particle, corresponds to the inertial term that appears in (19). The existence of this inertial potential has been verified experimentally by Bonse and Wroblewski [60] using neutron interferometry. In connection with the problem of the wave function in the accelerated system—i.e. whether it is φ' or ψ' —a detailed examination of the experimental arrangement in [60] reveals that this experiment cannot distinguish between the two methods that differ by the phase factor given in equation (60). Specifically, the interferometer in [60] oscillated in the horizontal plane and the intensity of the outgoing beam was measured at the inversion points of the oscillation at which the magnitude of acceleration was maximum but $\mathbf{V} = 0$; therefore, the phase factor in question was essentially unity. To conclude our discussion, it is interesting to elucidate further the physical origin of this phase factor using classical mechanics [32].

Under the transformation (56), $\mathbf{v} = \mathbf{v}' + \mathbf{V}(t)$ and the Lagrangian of a classical particle $L = \frac{1}{2}mv^2 - W$, with $L(\mathbf{x}, \mathbf{v}) = L'(\mathbf{x}', \mathbf{v}')$, becomes $L' = \frac{1}{2}m(\mathbf{v}' + \mathbf{V})^2 - W$ in the accelerated system. In classical mechanics, there are two natural and equivalent ways to deal with this Lagrangian. The first method consists of writing [31]

$$L' = \frac{1}{2}mv'^2 - m\mathbf{a}\cdot\mathbf{x}' - W + \frac{dF}{dt}, \quad (62)$$

where F is given, up to a constant, by

$$F = m\mathbf{V}(t)\cdot\mathbf{x}' + \frac{1}{2}m\int_0^t V^2(t')dt'. \quad (63)$$

The total temporal derivative in (62) does not affect the classical dynamics in accordance with the action principle and hence we confine our attention to $L'_1 = \frac{1}{2}mv'^2 - m\mathbf{a} \cdot \mathbf{x}' - W$. The momentum in this case is $\mathbf{p}' = m\mathbf{v}'$ and the Hamiltonian is thus given by

$$H'_1 = \frac{p'^2}{2m} + m\mathbf{a} \cdot \mathbf{x}' + W, \quad (64)$$

which corresponds to the Hamiltonian in the Schrödinger equation (61). The second method deals with L' without subtracting out dF/dt . In this case, the momentum is the invariant canonical momentum $\mathbf{p} = m(\mathbf{v}' + \mathbf{V})$ and the Hamiltonian is

$$H' = \frac{p^2}{2m} - \mathbf{p} \cdot \mathbf{V} + W, \quad (65)$$

which corresponds to equation (58) and the Hamiltonian in the Schrödinger equation (59).

In classical mechanics, the two methods represent the same dynamics. Quantum mechanically, however, there is a phase difference, which can be easily seen from the path integral approach. That is,

$$\psi'(\mathbf{x}', t) = \Sigma e^{\frac{i}{\hbar}S'}, \quad (66)$$

where S' is the classical action,

$$S' = \int L'(\mathbf{x}', \mathbf{v}') dt. \quad (67)$$

It follows from (62) that

$$S' = S'_1 + F, \quad (68)$$

where S'_1 is the action corresponding to L'_1 . Using (68) and the fact that

$$\varphi'(\mathbf{x}', t) = \Sigma e^{\frac{i}{\hbar}S'_1}, \quad (69)$$

we find

$$\psi'(\mathbf{x}', t) = e^{\frac{i}{\hbar}F} \varphi'(\mathbf{x}', t), \quad (70)$$

in agreement with equation (60).

It would be interesting to devise an experiment of the Bonse-Wroblewski [60] type that could distinguish between the two methods and hence remove the phase ambiguity in the treatment of translationally accelerated systems.

10. DISCUSSION

The main observational consequences of Dirac's equation in noninertial frames of reference are related to the Sagnac effect, the spin-rotation coupling and the Bonse-Wroblewski effect. These inertial effects can be further elucidated by interferometry experiments involving matter waves. In particular, a neutron interferometry experiment has been proposed for the direct measurement of inertial effect of intrinsic spin. Moreover, neutron interferometry experiments involving translationally accelerated interferometers may help resolve the phase ambiguity in the description of the wave function from the standpoint of a translationally accelerated system.

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