FINITE POINT CONFIGURATIONS AND PROJECTION THEOREMS IN VECTOR SPACES OVER FINITE FIELDS

A Dissertation

presented to

the Faculty of the Graduate School

University of Missouri

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

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May 2010

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FINITE POINT CONFIGURATIONS AND PROJECTION THEOREMS IN VECTOR SPACES OVER FINITE FIELDS

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ACKNOWLEDGMENTS

First, I would like to express my gratitude to my dissertation advisor, Alex Iosevich, for his support throughout the dissertation process. Without him, this dissertation would not have been possible. I would also like to thank Richard Belshoff, Albert Dixon, and Craig Haile for inspiring me to continue my education. I am also grateful to the members of my graduate committee, William Banks, Peter Casazza, Dix Pettey, and Sergei Kopeikin.

Finally, I would like to thank my family, especially my loving wife and son. Without their support none of this would be possible.

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ABSTRACT

We study a variety of combinatorial distance and dot product related problems in vector spaces over finite fields. First, we focus on the generation of the Special Linear Group whose elements belong to a finite field with q elements. Given $A \subset \mathbb{F}_q$, we use Fourier analytic methods to determine how large A needs to be to ensure that a certain product set contains a positive proportion of all the elements of $SL_2(\mathbb{F}_q)$.

We also study a variety of distance and dot product sets related to the Erdős-Falconer distance problem. In general, the Erdős-Falconer distance problem asks for the number of distances determined by a set of points. The classical Erdős distance problem asks for the minimal number of distinct distances determined by a finite point set in $\mathbb{R}^d$, where $d \geq 2$. The Falconer distance problem, which is the continuous analog of the Erdős distance problem, asks to find $s_0 > 0$ such that if the Hausdorff dimension of E is greater than s_0, then the Lebesgue measure of $\Delta(E)$ is positive.

A generalization of the Erdős-Falconer distance problem in vector spaces over finite fields is to determine the minimal $\alpha > 0$ such that E contains a congruent copy of every k dimensional simplex whenever $|E| \gtrsim q^\alpha$. We improve on known results (for $k > 3$) using Fourier analytic methods, showing that α may be taken to be $\frac{d+k}{2}$. If E is a subset of a sphere, then we get a stronger result which shows that α may be taken to be $\frac{d+k-1}{2}$.

Chapter 1

Introduction

1.1 A Brief Overview

In geometric combinatorics, we often try to answer the following general question: how large does a set need to be in able to ensure that certain geometric properties hold? In recent years, mathematicians have looked to finite fields as models to gain insight to analogous problems in the Euclidean setting. We are hopeful that if a certain property holds in the finite field setting, then an analogous result will hold true in the Euclidean version. However, this is certainly not always the case. It is also a fallacy to believe that the finite field problem is always easier to solve than the Euclidean version. Often, the finite field problem entails complications not present in the Euclidean setting which makes the finite field problems interesting in their own right. In this dissertation, the author studies several geometric combinatorial problems in vector spaces over finite fields related to distance and dot product sets.

To get started, let $\mathbb{F}_q$ denote a finite field with q elements, where in general $q = p^n$ for an odd prime p and a positive integer n. Let $SL_2(\mathbb{F}_q)$ denote the Special Linear Group of two by two matrices with determinant one whose elements belong to a finite field. In the second chapter, we will focus on the generation of $SL_2(\mathbb{F}_q)$.

Given $A \subset \mathbb{F}_q$, define

$$R(A) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(\mathbb{F}_q) : a_{11}, a_{12}, a_{21} \in A \right\}.$$

We determine how large A needs to be to ensure that the product set

$$R(A) \cdot R(A) = \{M \cdot M' : M, M' \in R(A)\}$$

contains a positive proportion of all the elements of $SL_2(\mathbb{F}_q)$. We prove that if $A \subset \mathbb{F}_q \backslash \{0\}$ with $|A| > Cq^{\frac{5}{6}}$, then $|R(A) \cdot R(A)| \geq C'q^3$.

Since the operation of matrix multiplication can be viewed as the dot product of a row vector and a column vector, we were able to make use of a dot product result previously established by D. Hart and A. Iosevich ([11]) which implies that if $|A|$ is much larger than $q^{\frac{3}{4}}$, then

$$|\{(a_{11}, a_{12}, a_{21}, a_{22}) \in A \times A \times A \times A : a_{11}a_{22} + a_{12}a_{21} = t\}| = |A|^4 q^{-1}(1 + o(1)).$$

Our result is partly motivated by the following result due to Harald Helfgott ([14]). See his paper for further background on this problem and related references.

Theorem 1. *(Helfgott) Let p be a prime. Let E be a subset of $SL_2(\mathbb{Z}/p\mathbb{Z})$ not contained in any proper subgroup.*

- *Assume that $|E| < p^{3-\delta}$ for some fixed $\delta > 0$. Then*

  $$|E \cdot E \cdot E| > c|E|^{1+\epsilon},$$

  where $c > 0$ and $\epsilon > 0$ depend only on δ.

- *Assume that $|E| > p^{\delta}$ for some fixed $\delta > 0$. Then there is an integer $k > 0$, depending only on δ, such that every element of $SL_2(\mathbb{Z}/p\mathbb{Z})$ can be expressed as a product of at most k elements of $E \cup E^{-1}$.*

In the third chapter we will turn our attention to problems related to the Erdős-Falconer distance problem. The classical Erdős distance problem asks for the minimal number of distinct distances determined by a finite point set in $\mathbb{R}^d$ where $d \geq 2$. More precisely, the problem is to find the smallest possible size of $\Delta(E)$ in terms of the size of E where $\Delta(E) = \{||x-y|| : x, y \in E\}$ and $E \subset \mathbb{R}^d$ is finite. The Erdős conjecture is that $|\Delta(E)| \gtrapprox |E|^{2/d}$, and taking E to be a subset of the integer lattice shows the exponent $2/d$ is the best possible . Erdős showed in [7] that $|\Delta(E)| \gtrsim |E|^{1/d}$. Here, and throughout, $X \lessapprox Y$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon N^\epsilon Y$. Similarly, $X \lesssim Y$ means that there exists $C > 0$ independent of q such that $X \leq CY$.

The continuous analog of this problem, called the Falconer distance problem, asks for the optimal threshold such that the set of distances determined by a subset of $\mathbb{R}^d$, $d \geq 2$, of larger dimension has positive Lebesgue measure. That is, if $E \subset \mathbb{R}^d$, then the problem is to find $s_0 > 0$ such that if the Hausdorff dimension of E is greater than s_0, then the Lebesgue measure of $\Delta(E)$ is positive. It is conjectured that s_0 may be taken to be $\frac{d}{2}$.

Neither problem is close to being completely solved. See [17], [16], and the references contained therein for the latest developments on the Erdős distance problem. See [6] and the references contained therein for the best known exponents for the Falconer distance problem.

In vector spaces over finite fields, one may use the same definition of $\Delta(E)$, $E \subset \mathbb{F}_q^d$, by defining $\|x\| = x_1^2 + \cdots + x_d^2$. While $||\cdot||$ does not satisfy the metric space definition of distance, it is still a rigid invariant in the sense that if $||x-y|| = ||x'-y'||$, then there exists $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$, the group of special orthogonal matrices, such that $x' = Ox + \tau$ and $y' = Oy + \tau$.

One may again ask for the smallest possible size of $\Delta(E)$ in terms of the size of E. There are several issues to contend with here. First, if $E = \mathbb{F}_q^d$, the whole vector space, then $\Delta(E) = \mathbb{F}_q$ which implies $|\Delta(E)| = |E|^{1/d}$. Also observe that if q is a prime congruent to 1 $(mod\ 4)$, then there exists $i \in \mathbb{F}_q$ such that $i^2 = -1$. This allows us to construct a set Z in $\mathbb{F}_q^2$, $Z = \{(t, it) : t \in \mathbb{F}_q\}$, such that $\Delta(Z) = \{0\}$.

The first non-trivial result on the Erdős-Falconer distance problem in vector spaces over finite fields was obtained by Bourgain, Katz and Tao in [1]. They consider the case $d = 2$ and get around E being the whole vector space by assuming that $|E| \lesssim q^{2-\epsilon}$ for some $\epsilon > 0$. They avoid the existence of i by assuming that q is a prime $\equiv 3$ $(mod\ 4)$. As a result they prove that $|\Delta(E)| \gtrsim |E|^{\frac{1}{2}+\delta}$, where δ is a function of ϵ.

In [15] Iosevich and Rudnev solve an analog of the Falconer distance problem for general fields. They prove that if $|E| \geq 2q^{\frac{d+1}{2}}$, then $\Delta(E) = \mathbb{F}_q$ directly in line with Falconer's result ([8]) in the Euclidean setting which says that if the Hausdorff dimension of a set is greater than $\frac{d+1}{2}$ then the Lebesgue measure of the distance set is positive. Hart, Iosevich, Koh, and Rudnev discovered in [12] that the exponent $\frac{d+1}{2}$ is sharp in odd dimensions. In even dimensions, it is still possible that the correct exponent is $\frac{d}{2}$ in analogy with Falconer's conjecture.

A classical result due to Furstenberg, Katznelson and Weiss ([9]) states that if $E \subset \mathbb{R}^2$ has positive upper Lebesgue density, then for any $\delta > 0$, the δ-neighborhood of E contains a congruent copy of a sufficiently large dilate of every three-point configuration. Bourgain ([2]) showed that for arbitrary three-point configurations it is not possible to replace the thickened set E_δ by E. He did this by giving an example of a degenerate triangle where all three vertices are on the same line whose large dilates could not be placed in E. In [2] Bourgain applied Fourier analytic techniques to prove that a set E of positive upper Lebesgue density will always contain a sufficiently large dilate of every non-degenerate k-point configuration where $k < d$. If $k \geq d$, it is not currently known whether the δ-neighborhood assumption is necessary.

In combinatorics and geometric measure theory the study of k-simplices, that is $k+1$ points spanning a k-dimensional subspace, up to congruence may be rephrased in terms of distances. Asking whether a particular translated and rotated copy of a k-simplex occurs in a set E is equivalent to asking whether the set of $\binom{k+1}{2}$ distances determined by that k-simplex is obtained by some $k+1$ point subset of E. Notice that if we set $k = 1$ then this is equivalent to the already discussed Erdős and Falconer distance problems.

One may then phrase the following generalization of the Erdős-Falconer distance problem in vector spaces over finite fields. How large does E need to be to ensure that E contains a congruent copy of every or at least a positive proportion of all k-simplices? Observe that the lack of order in a finite field makes the notion of a sufficiently large dilation meaningless, which is why dilations are not used.

The first investigation into this was done by Hart and Iosevich in [10] (see also [13]). It was shown that if a subset E of $\mathbb{F}_q^d, d > \binom{k+1}{2}$ is such that $|E| \gtrsim q^{\frac{k}{k+1}d+\frac{k}{2}}$ then E contains a congruent copy of every k dimensional simplex. This was improved using graph theoretic methods by Vinh ([18]) who obtained the same conclusion for

E such that $|E| \gtrsim q^{\frac{d-1}{2}+k}, d \geq 2k$. In the case of triangles in $\mathbb{F}_q^2$, Covert , Hart, Iosevich, and Uriarte-Tuero ([5]) showed that if E has density greater than ρ for some $Cq^{-1/2} \leq \rho \leq 1$ with a sufficiently large constant $C > 0$, then the set of triangles determined by E, up to congruence, has density greater than $c\rho$. Vinh ([19]) has shown that for $|E| \gtrsim q^{\frac{d+2}{2}}$ then the set of triangles, up to congruence, has density greater than c.

In this dissertation, we improve on known k-simplices results for $k > 3$ using Fourier analytic methods. We show that if $|E| \gtrsim q^{\frac{d+k}{2}}$, $d \geq k$, then E contains a congruent copy of every k dimensional simplex.

If E is subset of a sphere S where $S = \{x \in \mathbb{F}_q^d : \|x\| = 1\}$, then one has for $x, y \in E$ that $\|x - y\| = 2 - 2x \cdot y$. Therefore, determining distances is equivalent to determining dot products. Under this assumption on E we obtain a stronger result. We show that if $|E| \gtrsim q^{\frac{d+k-1}{2}}$, then E contains a congruent copy of every k dimensional simplex.

The only sharpness example we have at this point is the Cartesian product of sub-spaces. If $q = p^2$, then there exists a subset of $\mathbb{F}_q^d$ of size exactly $q^{\frac{d}{2}}$ such that all the distances among the vertices of a k-simplex are elements of $\mathbb{F}_p$ and thus a positive proportion of k-simplices cannot possibly be realized. We conjecture that in odd dimensions, the exponent $\frac{d+k}{2}$ is sharp. In even dimensions, we believe the exponent $\frac{d+k-1}{2}$ to be the best possible.

1.2 Basic Formulas

We shall make use of the following basic formulas of Fourier analysis on $\mathbb{F}_q^d$. Let $f : \mathbb{F}_q^d \to \mathbb{C}$ and let χ denote a non-trivial additive character on $\mathbb{F}_q$. Define the Fourier Transform by the relation

$$\widehat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x).$$

Also recall that the Fourier inversion theorem is given by

$$f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m) \widehat{f}(m)$$

and the Plancherel theorem is given by

$$\sum_{m \in \mathbb{F}_q^d} \left|\widehat{f}(m)\right|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

We shall also frequently use the following orthogonality property which is given by

$$q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(x \cdot m) = \begin{cases} 1 & : m = 0 \\ 0 & : otherwise \end{cases}$$

Chapter 2

Rapid Generation of $SL_2(\mathbb{F}_q)$

2.1 Statement of Results

Recall the following definition.

Definition 2. *Given $A \subset \mathbb{F}_q$, let*

$$R(A) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(\mathbb{F}_q) : a_{11}, a_{12}, a_{21} \in A \right\}.$$

Notice that the size of $R(A)$ is exactly $|A|^3$. Our main result in [4] is the following.

Theorem 3. *Let $A \subset \mathbb{F}_q \backslash \{0\}$ with $|A| \geq Cq^{\frac{5}{6}}$. Then there exists $C' > 0$ such that*

$$|R(A) \cdot R(A)| \geq C'|SL_2(\mathbb{F}_q)| \geq C''q^3. \tag{2.1.1}$$

Remark 1. *Observe that if $q = p^2$, then $\mathbb{F}_q$ contains $\mathbb{F}_p$ as a sub-field. Since $R(\mathbb{F}_p)$ is a sub-group of $SL_2(\mathbb{F}_q)$ we see that the threshold assumption on the size of A in Theorem 3 cannot be improved beyond $|A| \geq q^{\frac{1}{2}}$.*

Since the operation of matrix multiplication can be viewed as the dot product of a row vector and a column vector, we were able to make use of the following dot product result due to Hart and Iosevich ([11]).

Theorem 4. *Let* $E \subset \mathbb{F}_q^d$, $d \geq 2$, *and define*

$$\nu(t) = |\{(x,y) \in E \times E : x \cdot y \equiv x_1y_1 + \cdots + x_dy_d = t\}|.$$

Then

$$\nu(t) = |E|^2 q^{-1} + \mathcal{D}(t),$$

where for every $t > 0$,

$$|\mathcal{D}(t)| < |E| q^{\frac{d-1}{2}}.$$

In particular, if $|E| > q^{\frac{d+1}{2}}$, *then* $\nu(t) > 0$ *and as* E *grows beyond this threshold,*

$$\nu(t) = |E|^2 q^{-1}(1 + o(1)).$$

Observe that Theorem 4 implies that if $E = A \times A \subset \mathbb{F}_q^2$ and $|A|$ is much larger than $q^{\frac{3}{4}}$, then

$$|\{(a_{11}, a_{12}, a_{21}, a_{22}) \in A \times A \times A \times A : a_{11}a_{22} + a_{12}a_{21} = t\}| = |A|^4 q^{-1}(1 + o(1)). \quad (2.1.2)$$

This is what we actually use in the proof of Theorem 3.

The basic idea behind the argument below is the following. Let $T \in SL_2(\mathbb{F}_q)$ and define

$$\nu(T) = |\{(S, S') \in R(A) \times R(A) : S \cdot S' = T\}|.$$

We prove below that

$$\sqrt{var(\nu)} \leq C|A|^3 q^{-\frac{1}{2}},$$

where variance is defined, in the usual way as

$$\mathbb{E}\left((\nu - \mathbb{E}(\nu))^2\right),$$

with the expectation defined, also in the usual way, as

$$\mathbb{E}(\nu) = |SL_2(\mathbb{F}_q)|^{-1} \sum_{T \in SL_2(\mathbb{F}_q)} \nu(T) = |A|^6 |SL_2(\mathbb{F}_q)|^{-1} = |A|^6 q^{-3}(1+o(1)).$$

One can then check by a direct computation that $\sqrt{var(\nu)}$ is much smaller than $\mathbb{E}(\nu)$ if $|A| \geq Cq^{\frac{5}{6}}$, with C sufficiently large, and we conclude that in this regime, $\nu(T)$ is concentrated around its expected value $\mathbb{E}(\nu) = |A|^6 q^{-3}(1+o(1))$.

2.2 Proof of Theorem 3

We are looking to solve the equation

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \frac{1+a_{12}a_{21}}{a_{11}} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & \frac{1+b_{12}b_{21}}{b_{11}} \end{pmatrix} = \begin{pmatrix} t & \alpha \\ \beta & \frac{1+\alpha\beta}{t} \end{pmatrix},$$

which leads to the equations

$$a_{11}b_{11} + a_{12}b_{12} = t, \tag{2.2.1}$$

$$\frac{b_{21}}{b_{11}} t + \frac{a_{12}}{b_{11}} = \alpha,$$

and

$$\frac{a_{21}}{a_{11}} t + \frac{b_{12}}{a_{11}} = \beta.$$

Let $E = A \times A$ and let D_t denote the characteristic function of the set

$$\{(a_{11}, b_{11}, a_{12}, b_{12}) \in A \times A \times A \times A : a_{11}b_{11} + a_{12}b_{12} = t\}.$$

Let

$$\mu = D_t(a_{11}, b_{11}, a_{12}, b_{12})E(a_{21}, b_{21})\chi(u(b_{21}t + a_{12} - \alpha b_{11}))\chi(v(a_{21}t + b_{12} - \beta a_{11})$$

where $E(a_{21}, b_{21})$ denotes the characteristic function of the set E.

Then, using orthogonality, the number of six-tuplets satisfying the equations (2.2.1) above equals

$$\begin{aligned}
\nu(t, \alpha, \beta) &= \frac{1}{q^2} \sum_{u,v} \sum_{a_{11}, b_{11}, a_{12}, b_{12}, a_{21}, b_{21}} \mu \\
&= q^{-2}|D_t||E| + q^4 \sum_{\mathbb{F}_q^2 \setminus \{(0,0)\}} \widehat{D}_t(\beta v, \alpha u, -u, -v)\widehat{E}(tv, tu) \\
&= \nu_0(t, \alpha, \beta) + \nu_{main}(t, \alpha, \beta).
\end{aligned}$$

By (2.1.2),

$$\nu_0(t, \alpha, \beta) = q^{-3}|A|^6(1 + o(1)),$$

which implies that

$$\sum_{t,\alpha,\beta} \nu_0^2(t, \alpha, \beta) = q^{-3}|A|^{12}(1 + o(1)).$$

We now estimate $\sum_{t,\alpha,\beta} \nu_{main}^2(t, \alpha, \beta)$. By Cauchy-Schwarz and Plancherel,

$$\begin{aligned}
\nu_{main}^2(t, \alpha, \beta) &\leq q^8 \sum_{u,v} \left|\widehat{D}_t(\beta v, \alpha u, -u, -v)\right|^2 \cdot \sum_{u,v} \left|\widehat{E}(tv, tu)\right|^2 \\
&\leq |E| q^6 \sum_{u,v} \left|\widehat{D}_t(\beta v, \alpha u, -u, -v)\right|^2.
\end{aligned}$$

Now,

$$|E| q^6 \sum_{\alpha,\beta} \sum_{u,v} \left|\widehat{D}_t(\beta v, \alpha u, -u, -v)\right|^2 = |E| q^6 q^{-4} |A|^4 q^{-1}(1 + o(1))$$

as long as $|E|$ is much larger than $q^{\frac{3}{2}}$. It follows that

$$\sum_{t\neq 0,\alpha,\beta} \nu^2_{main}(t,\alpha,\beta) \leq |A|^6 q^2.$$

Hence,

$$\sum_{t,\alpha,\beta} \nu^2(t,\alpha,\beta) \leq C(|A|^{12}q^{-3} + |A|^6 q^2). \tag{2.2.2}$$

Now, by Cauchy-Schwarz and (2.2.2) we have

$$\left(|A|^6 - \sum_{\alpha,\beta} \nu(0,\alpha,\beta)\right)^2 = \left(\sum_{t\neq 0,\alpha,\beta} \nu(t,\alpha,\beta)\right)^2$$
$$\leq C|support(\nu)| \cdot (|A|^{12}q^{-3} + |A|^6 q^2).$$

Suppose that we could show that

$$\sum_{\alpha,\beta} \nu(0,\alpha,\beta) \leq \frac{1}{2}|A|^6. \tag{2.2.3}$$

Then it would follow that

$$|support(\nu)| \gtrsim C \min\left\{q^3, \frac{|A|^6}{q^2}\right\}.$$

This expression is

$$\geq C|SL_2(\mathbb{F}_q)| = q^3(1+o(1))$$

if

$$|A| \geq Cq^{\frac{5}{6}},$$

as desired.

We are left to establish (2.2.3). Observe that if $t=0$, then $\beta = -\alpha^{-1}$. Plugging this into (2.2.1) we see that this forces $a_{11} = -\alpha b_{12}$ and $a_{12} = \alpha b_{11}$ which implies that

$$\nu(0,\alpha,\beta) = \nu(0,\alpha,-\alpha^{-1}) \leq q^4,$$

which implies that

$$\sum_{\alpha,\beta} \nu(0,\alpha,\beta) = \sum_{\alpha} \nu(0,\alpha,-\alpha^{-1}) \leq q^5.$$

We have

$$q^5 \leq \frac{1}{2}|A|^6$$

if

$$|A| \geq Cq^{\frac{5}{6}},$$

which completes the proof.

2.3 Proof of Theorem 4

For completeness, we give the following proof by D. Hart and A. Iosevich ([11]). Observe that

$$\nu(t) = \sum_{x,y\in E} q^{-1} \sum_{s\in\mathbb{F}_q} \chi(s(x\cdot y - t)),$$

where χ is a non-trivial additive character on $\mathbb{F}_q$. It follows that

$$\nu(t) = |E|^2 q^{-1} + \mathcal{D},$$

where

$$\mathcal{D} = \sum_{x,y\in E} q^{-1} \sum_{s\neq 0} \chi(s(x\cdot y - t)).$$

Viewing $\mathcal{D}$ as a sum in x, applying the Cauchy-Schwarz inequality and dominating the sum over $x \in E$ by the sum over $x \in \mathbb{F}_q^d$, we see that

$$\mathcal{D}^2 \leq |E| \sum_{x\in\mathbb{F}_q^d} q^{-2} \sum_{s,s'\neq 0} \sum_{y,y'\in E} \chi(sx\cdot y - s'x\cdot y')\chi(t(s'-s)).$$

Orthogonality in the x variable yields

$$= |E| q^{d-2} \sum_{\substack{sy=s'y' \\ s,s'\neq 0}} \chi(t(s'-s))E(y)E(y').$$

If $s \neq s'$ we may set $a = s/s', b = s'$ and obtain

$$|E|q^{d-2} \sum_{\substack{y \neq y' \\ ay=y' \\ a \neq 1, b}} \chi(tb(1-a))E(y)E(y')$$

$$= -|E|q^{d-2} \sum_{y \neq y', a \neq 1} E(y)E(ay),$$

and the absolute value of this quantity is

$$\leq |E|q^{d-2} \sum_{y \in E} |E \cap l_y|$$
$$\leq |E|^2 q^{d-1}$$

since

$$|E \cap l_y| \leq q$$

by the virtue of the fact that each line contains exactly q points.

If $s = s'$ we get

$$|E|q^{d-2} \sum_{s,y} E(y) = |E|^2 q^{d-1}.$$

It follows that

$$\nu(t) = |E|^2 q^{-1} + \mathcal{D}(t),$$

where

$$\mathcal{D}^2(t) \leq -Q(t) + |E|^2 q^{d-1},$$

with

$$Q(t) \geq 0.$$

It follows that

$$\mathcal{D}^2(t) \leq |E|^2 q^{d-1},$$

so

$$|\mathcal{D}(t)| \leq |E| q^{\frac{d-1}{2}}. \tag{2.3.1}$$

We conclude that

$$\nu(t) = |E|^2 q^{-1} + \mathcal{D}(t)$$

with $|\mathcal{D}(t)|$ bounded as in (2.3.1).

This quantity is strictly positive if $|E| > q^{\frac{d+1}{2}}$ with a sufficiently large constant $C > 0$. This completes the proof of Theorem 4.

Chapter 3

Finite Point Configurations

3.1 Statement of Results

Let P_k denote a k-simplex, that is $k+1$ points spanning a k dimensional subspace. Given another k-simplex P'_k we write $P'_k \sim P_k$ if there exists a $\tau \in \mathbb{F}_q^d$ and an $O \in SO_d(\mathbb{F}_q)$, the set of d-by-d orthogonal matrices over $\mathbb{F}_q$ such that

$$P'_k = O(P_k) + \tau.$$

For $E \subset \mathbb{F}_q^d$ define

$$\mathcal{T}_k(E) = \{P_k \in E \times \cdots \times E\} \;/ \sim .$$

Under this equivalence relation one may specify a simplex by the distances determined by its vertices. This follows from the following simple lemma from [10].

Lemma 5. *Let P_k be a simplex with vertices $V_0, V_1, \dots, V_k$, $V_j \in \mathbb{F}_q^d$. Let P' be another simplex with vertices $V'_0, V'_1, \dots, V'_k$. Suppose that*

$$||V_i - V_j|| = ||V'_i - V'_j|| \tag{3.1.1}$$

for all i, j. Then there exists $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$ such that $\tau + O(P) = P'$.

We will specify simplices by specifying the distances determining them piece by piece. With this in mind denote a k-star by

$$S_k(t_1, \ldots, t_k) = \{(x, y^1 \ldots y^k) : \|x - y^1\| = t_1, \ldots \|x - y^k\| = t_k\},$$

where $t_1, \ldots, t_k \in \mathbb{F}_q$.

Define $\Delta_{y^1, y^2, \ldots, y^k}(E) = \{(\|x - y^1\|, \ldots, \|x - y^k\|) \in \mathbb{F}_q^k : x \in E\}$ where y^1, y^2,. . . ,$y^k \in E$. In [3] we have the following results, the first of which is a projection theorem involving distance sets.

Theorem 6. *Let $E \subset \mathbb{F}_q^d$. If $|E| \gtrsim q^{\frac{d+k}{2}}$ then*

$$\frac{1}{|E|^k} \sum_{y^1, \ldots, y^k \in E} |\Delta_{y^1, \ldots, y^k}(E)| \gtrsim q^k.$$

A pigeon-holing argument using Theorem 6 will allow us to move from sets of k-stars to sets of k-simplices.

Theorem 7. *Let $E \subset \mathbb{F}_q^d$. If $|E| \gtrsim q^{\frac{d+k}{2}}, k \leq d$ then $|\mathcal{T}_k(E)| \gtrsim q^{\binom{k+1}{2}}$, in other words E determines a positive proportion of all k-simplices.*

Similarly, define $\Pi_{y^1, y^2, \ldots, y^k}(E) = \{(x \cdot y^1, x \cdot y^2, \ldots, x \cdot y^k) \in \mathbb{F}_q^k : x \in E\}$ where y^1, y^2,. . . ,$y^k \in E$. Then we have the following projection theorem involving dot product sets.

Theorem 8. *Let $E \subset \mathbb{F}_q^d$. If $|E| \gtrsim q^{\frac{d+k}{2}}$ then*

$$\frac{1}{|E|^k} \sum_{y^1, \ldots, y^k \in E} |\Pi_{y^1, \ldots, y^k}(E)| \gtrsim q^k.$$

If E is subset of a sphere S where $S = \{x \in \mathbb{F}_q^d : \|x\| = 1\}$ then one has for $x, y \in E$ that $\|x - y\| = 2 - 2x \cdot y$. Therefore in this case determining distances is the same as determining dot products. Under this assumption on E the proof of Theorem 8 may be modified improving the exponent in Theorem 6 to $\frac{d+k-1}{2}$ as follows.

Theorem 9. *Let $E \subset S$. If $|E| \gtrsim q^{\frac{d+k-1}{2}}$ then*

$$\frac{1}{|E|^k} \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,\dots,y^k}(E)| \gtrsim q^k.$$

This in turn yields the following result.

Theorem 10. *Let $E \subset S$. If $|E| \gtrsim q^{\frac{d+k-1}{2}}, k \leq d-1$ then $|\mathcal{T}_k(E)| \gtrsim q^{\binom{k+1}{2}}$, in other words E determines a positive proportion of all k-simplices.*

3.2 Proof of Theorem 6: k-star distance sets

We begin by defining the counting function,

$$\nu_{y^1,\dots,y^k}(t_1,\dots,t_k) = \sum_{\|x-y^1\|=t_1,\dots,\|x-y^k\|=t_k} E(x)$$

where $E(x)$ is the characteristic function of the set E. The proof of Theorem 6 is based on the following lemma.

Lemma 11. *Let $E \subset \mathbb{F}_q^d$. Then*

$$\sum_{y^1,\dots,y^k \in E} \sum_{t_1,t_2,\dots,t_k \in \mathbb{F}_q} |\nu_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k)|^2 \lesssim \frac{|E|^{k+2}}{q^k} + q^d|E|^k.$$

Proof. We proceed by induction. To prove the initial case we start by squaring $\nu_{y^1}(t)$. We have

$$\nu^2_{y^1}(t) = \sum_{\|x-y^1\|=\|x'-y^1\|=t} E(x)E(x').$$

Summing over $y^1 \in E$ and $t \in \mathbb{F}_q$, we see

$$\sum_{y^1 \in E} \sum_{t \in \mathbb{F}_q} \nu^2_{y^1}(t) = \sum_{\|x-y^1\|=\|x'-y^1\|} E(y^1)E(x)E(x'),$$

applying orthogonality,

$$= q^{-1} \sum_{s \in \mathbb{F}_q} \sum_{y^1,x,x' \in \mathbb{F}_q^d} \chi(s(||x-y^1|| - ||x'-y^1||))E(y^1)E(x)E(x'),$$

and extracting the $s = 0$ term,

$$= q^{-1}|E|^3 + q^{-1} \sum_{s \neq 0} \sum_{y^1, x, x' \in \mathbb{F}_q^d} \chi(s(||x - y^1|| - ||x' - y^1||)) E(y^1) E(x) E(x') = I + II.$$

Here

$$II = q^{-1} \sum_{s \neq 0} \sum_{y^1 \in E} \left| \sum_{x \in E} \chi(s(||x|| - 2y^1 \cdot x)) \right|^2,$$

since

$$\|x - y^1\| - \|x' - y^1\| = (||x|| - 2y^1 \cdot x) - (||x'|| - 2y^1 \cdot x').$$

It follows by extending the sum over $y^1 \in E$ to over $y^1 \in \mathbb{F}_q^d$ that

$$0 \leq II \leq q^{-1} \sum_{s \neq 0} \sum_{y^1 \in \mathbb{F}_q^d} \sum_{x, x' \in E} \chi(-2sy^1 \cdot (x - x')) \chi(s(||x|| - ||x'||)),$$

and from orthogonality in the variable $y^1 \in \mathbb{F}_q^d$,

$$= q^{d-1} \sum_{s \neq 0} \sum_{x \in E} 1,$$

which is less than the quantity $q^d|E|$. It therefore follows that

$$\sum_{y^1 \in E} \sum_{t \in \mathbb{F}_q} \nu_{y^1}^2(t) = I + II < q^{-1}|E|^3 + q^d|E|.$$

This proves the initial step. For the induction hypothesis, suppose that

$$\sum_{y^1, \ldots, y^{k-1} \in E} \sum_{t_1, \ldots, t_{k-1} \in \mathbb{F}_q} \nu_{y^1, \ldots, y^{k-1}}^2(t_1, \ldots, t_{k-1}) \lesssim \frac{|E|^{k+1}}{q^{k-1}} + q^d|E|^{k-1}.$$

Now,

$$\sum_{y^1, \ldots, y^{k-1}, y^k \in E} \sum_{t_1, \ldots, t_k \in \mathbb{F}_q} \nu_{y^1, \ldots, y^{k-1}, y^k}^2(t_1, \ldots, t_k) =$$

$$\sum_{\substack{\|x - y^1\| = \|x' - y^1\|, \ldots, \|x - y^{k-1}\| = \|x' - y^{k-1}\| \\ \|x - y^k\| = \|x' - y^k\|}} \cdots \sum E(y^1) \ldots E(y^{k-1}) E(y^k) E(x) E(x').$$

Then applying orthogonality,

$$= q^{-1} \sum_{s \in \mathbb{F}_q} \sum_{\substack{\|x-y^1\|=\|x'-y^1\|,\dots,\|x-y^{k-1}\|=\|x'-y^{k-1}\| \\ x,x',y^1,\dots,y^{k-1},y^k \in E}} \cdots \sum \chi(s(||x|| - 2y^k \cdot x))\chi(-s(||x'|| - 2y^k \cdot x')).$$

since

$$\|x - y^k\| - \|x' - y^k\| = (||x|| - 2y^k \cdot x) - (||x'|| - 2y^k \cdot x').$$

Extracting the $s = 0$ term and applying the induction hypothesis gives

$$\lesssim \frac{|E|^{k+2}}{q^k} + q^{d-1}|E|^k + R,$$

where

$$R = q^{-1} \sum_{s \in \mathbb{F}_q^*} \sum_{\substack{\|x-y^1\|=\|x'-y^1\|,\dots,\|x-y^{k-1}\|=\|x'-y^{k-1}\| \\ x,x',y^1,\dots,y^{k-1},y^k \in E}} \cdots \sum \chi(s(||x|| - 2y^k \cdot x))\chi(-s(||x'|| - 2y^k \cdot x')).$$

Then R may be expressed as

$$q^{-1} \sum_{s \in \mathbb{F}_q^*} \sum_{t_1,\dots,t_{k-1} \in \mathbb{F}_q} \sum_{\substack{y^1,\dots y^{k-1} \in E \\ y^k \in E}} \left| \sum_{\substack{\|x-y^1\|=t_1,\dots,\|x-y^{k-1}\|=t_{k-1} \\ x \in E}} \cdots \sum \chi(s(||x|| - 2y^k \cdot x)) \right|^2.$$

Then extending sum over $y^k \in E$ to over $y^k \in \mathbb{F}_q^d$, expanding the square, and applying orthogonality in y^k gives

$$R \leq q^{d-1} \sum_{s \in \mathbb{F}_q^*} \sum_{y^1,\dots y^{k-1}, x \in E} 1$$

which in turn is less than $q^d|E|^k$.

Therefore we have

$$\sum_{y^1,\dots,y^k \in E} \sum_{t_1,\dots,t_k \in \mathbb{F}_q} \nu^2_{y^1,\dots,y^k}(t_1,\dots,t_k) \lesssim \frac{|E|^{k+2}}{q^k} + q^d|E|^k,$$

which completes the proof of Lemma 11. □

We are ready to complete the proof of Theorem 6. By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|E|^{2k+2} &= \left(\sum_{y^1,\dots,y^k \in E} \sum_{t_1,t_2,\dots,t_k \in \mathbb{F}_q} \nu_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k) \right)^2 \\
&\leq \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \cdot \sum_{y^1,\dots,y^k \in E} \sum_{t_1,t_2,\dots,t_k \in \mathbb{F}_q} |\nu_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k)|^2.
\end{aligned}
$$

By Lemma 11 it follows that

$$|E|^{2k+2} \lesssim \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \cdot \left(\frac{|E|^{k+2}}{q^k} + q^d |E|^k \right).$$

Therefore,

$$\sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim \frac{|E|^{2k+2}}{\frac{|E|^{k+2}}{q^k} + q^d |E|^k}.$$

Normalize to obtain

$$\frac{1}{|E|^k} \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim \frac{|E|^{k+2}}{\frac{|E|^{k+2}}{q^k} + q^d |E|^k},$$

which for $|E| \gtrsim q^{\frac{d+k}{2}}$ gives

$$\frac{1}{|E|^k} \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim q^k.$$

Thus the proof of Theorem 6 is complete.

Remark 2. *Since*

$$\frac{1}{|E|^k} \sum_{y^1,\dots,y^k \in E} |\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim q^k$$

is an average and $|\Delta_{y^1,y^2,\dots,y^k}(E)|$ *is bounded above by* q^k*, it follows that there exists* $\mathcal{E} \subseteq E \times \cdots \times E = E^k$, $|\mathcal{E}| \gtrsim |E|^k$ *such that*

$$|\Delta_{y^1,y^2,\dots,y^k}(E)| \gtrsim q^k$$

for all $(y_1,\ldots,y_k)\in\mathcal{E}$*. In other words, for a positive proportion of* $(y_1,\ldots,y_k)$ *we have* $|\Delta_{y^1,y^2,\ldots,y^k}(E)|\gtrsim q^k$*.*

Proof. Define $\mathcal{E}=\{(y_1,\ldots,y_k)\in E^k:|\Delta_{y^1,y^2,\ldots,y^k}(E)|\geq c_g q^k\}$ and $B=\{(y_1,\ldots,y_k)\in E^k:|\Delta_{y^1,y^2,\ldots,y^k}(E)|<c_g q^k\}$. Then it follows that

$$\begin{aligned} cq^k &\leq \frac{1}{|E|^k}\sum_{y^1,\ldots,y^k\in E}|\Delta_{y^1,y^2,\ldots,y^k}(E)| \\ &= \frac{1}{|E|^k}\sum_{(y^1,\ldots,y^k)\in\mathcal{E}}|\Delta_{y^1,y^2,\ldots,y^k}(E)|+\frac{1}{|E|^k}\sum_{(y^1,\ldots,y^k)\in B}|\Delta_{y^1,y^2,\ldots,y^k}(E)| \\ &\leq \frac{1}{|E|^k}|\mathcal{E}|q^k+\frac{1}{|E|^k}|B|c_g q^k. \end{aligned}$$

Solving for $|\mathcal{E}|$ and setting $c_g=\frac{c}{2}$ we obtain $|\mathcal{E}|\geq\frac{c}{2}|E|^k$. □

3.3 General Version of Theorem 6

In order to prove the k-simplices result, we need the following theorem which is a more general version of Theorem 6.

Theorem 12. *Given* $E\subset\mathbb{F}_q^d$*, let* $\mathcal{E}\subset E\times\cdots\times E=E^s, s\geq 2$*, with* $|\mathcal{E}|\sim|E|^s$*. Define*

$$\mathcal{E}'=\{(y^1,\ldots,y^{s-1})\in E^{s-1}:(y^1,\ldots,y^{s-1},y^s)\in\mathcal{E}\ \ \textit{for some}\ \ y^s\in E\}.$$

In addition, for each $(y^1,\ldots,y^{s-1})\in\mathcal{E}'$ *we define*

$$\mathcal{E}(y^1,\ldots,y^{s-1})=\{y^s\in E:(y^1,\ldots,y^{s-1},y^s)\in\mathcal{E}\}.$$

If $|E|\gtrsim q^{\frac{d+s-1}{2}}$*, then we have*

$$\frac{1}{|\mathcal{E}'|}\sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'}\left|\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1,\ldots,y^{s-1})\right)\right|\gtrsim q^{s-1}, \tag{3.3.1}$$

where

$$\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1,\ldots,y^{s-1})\right) = \{\left(\|y^s - y^1\|,\ldots,\|y^s - y^{s-1}\|\right) \in (\mathbb{F}_q)^{s-1} : y^s \in \mathcal{E}(y^1,\ldots,y^{s-1})\}.$$

Proof. For each $t_1,\ldots,t_s \in \mathbb{F}_q$, the incidence function on $\Delta_{y^1,\ldots,y^{s-1}}(\mathcal{E}(y^1,\ldots,y^{s-1}))$ is given by

$$\nu_{y^1,\ldots,y^{s-1}}^{\mathcal{E}(y^1,\ldots,y^{s-1})}(t_1,\ldots,t_{s-1}) = |\{y^s \in \mathcal{E}(y^1,\ldots,y^{s-1}) : \|y^s - y^1\| = t_1,\ldots,\|y^s - y^{s-1}\| = t_{s-1}\}|.$$

Observe that

$$\begin{aligned}\nu_{y^1,\ldots,y^{s-1}}^{\mathcal{E}(y^1,\ldots,y^{s-1})}(t_1,\ldots,t_{s-1}) &\leq \nu_{y^1,\ldots,y^{s-1}}(t_1,\ldots,t_{s-1}) \\ &= |\{(y^s \in E : ||y^s - y^1|| = t_1,\ldots,||y^s - y^{s-1}|| = t_{s-1}\}|.\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$|\mathcal{E}|^2 = \left(\sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'} \sum_{t_1\ldots,t_{s-1}\in\mathbb{F}_q} \nu_{y^1,\ldots,y^{s-1}}^{\mathcal{E}(y^1,\ldots,y^{s-1})}(t_1,\ldots,t_{s-1})\right)^2$$

$$\leq \left(\sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'} |\Delta_{y^1,\ldots,y^{s-1}}(\mathcal{E}(y^1,\ldots,y^{s-1}))|\right)\cdot\left(\sum_{y^1,\ldots,y^{s-1}\in E} \sum_{t_1,\ldots,t_{s-1}\in\mathbb{F}_q} |\nu_{y^1,\ldots,y^{s-1}}(t_1,\ldots,t_{s-1})|^2\right).$$

Using Lemma 11, we therefore have

$$|\mathcal{E}|^2 \leq \sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'} |\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1,\ldots,y^{s-1})\right)| \cdot \left(\frac{|E|^{s+1}}{q^{s-1}} + q^d|E|^{s-1}\right).$$

Observe that $|\mathcal{E}'| \sim |E|^{s-1}$ because otherwise $|\mathcal{E}| \leq |\mathcal{E}'||E| << |E|^s$ which contradicts $|\mathcal{E}| \sim |E|^s$. Therefore, if $|E| \gtrsim q^{(d+s-1)/2}$, then it follows that

$$\frac{1}{|\mathcal{E}'|} \sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'} \left|\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1,\ldots,y^{s-1})\right)\right| \gtrsim q^{s-1}.$$

Thus the proof of Theorem 12 is complete. □

When a pigeon-holing argument similar to Remark 2 is applied to the inequality (3.3.1) in Theorem 12, the following corollary immediately follows.

Corollary 13. *Let* $E \subset \mathbb{F}_q^d$ *and* $\mathcal{E} \subset E \times \cdots \times E = E^s, s \geq 2$, *with* $|\mathcal{E}| \sim |E|^s$. *If* $|E| \gtrsim q^{\frac{d+s-1}{2}}$, *then there exists* $\mathcal{E}^{(1)} \subset \mathcal{E}' \subset E^{s-1}$ *with* $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|^{s-1}$ *such that for every* $(y^1, \ldots, y^{s-1}) \in \mathcal{E}^{(1)}$,

$$\left|\Delta_{y^1,\ldots,y^{s-1}}(\mathcal{E}(y^1, \ldots, y^{s-1}))\right| \gtrsim q^{s-1}.$$

Namely, the elements in $\mathcal{E}$ *determine a positive proportion of all* $(s-1)$*-simplices whose bases are fixed as a* $(s-2)$*-simplex given by any element* $(y^1, \ldots, y^{s-1}) \in \mathcal{E}^{(1)}$.

3.4 Exposition of k = 2

To help make the proof of Theorem 7 as clear as possible, we first prove the result for $k = 2$. We want to show that if $|E| \gtrsim q^{\frac{d+2}{2}}$, then the set E determines a positive proportion of all triangles.

Using Remark 2 together with Theorem 6, we see that for $|E| \gtrsim q^{\frac{d+2}{2}}$, there exists a set $\mathcal{E} \subset E \times E = E^2$ with $|\mathcal{E}| \gtrsim |E|^2$ such that for every $(y^1, y^2) \in \mathcal{E}$, we have $|\Delta_{y^1,y^2}(E)| \gtrsim q^2$. Notice that this implies that if $|E| \gtrsim q^{\frac{d+2}{2}}$, then the set E determines a positive proportion of all 2-simplices whose bases are given by any fixed 1-simplex determined by $(y^1, y^2) \in \mathcal{E}$. It therefore suffices to show that a positive proportion of all 1-simplices can be constructed by the elements of $\mathcal{E}$. Since $|E| \gtrsim q^{\frac{d+2}{2}} \gtrsim q^{\frac{d+1}{2}}$ and $|\mathcal{E}| \subset |E|^2$ with $|\mathcal{E}| \sim |E|^2$, we can apply Corollary 13 where s is replaced by 2. Then we see that there exists a set $\mathcal{E}^{(1)} \subset \mathcal{E}' \subset E$ with $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|$ such that for every $y^1 \in \mathcal{E}^{(1)}$, we have $|\Delta_{y^1}(\mathcal{E}(y^1))| \gtrsim q$. Since we have constructed a positive proportion of all 1-simplices from the elements of $\mathcal{E}$, the proof is complete.

3.5 Exposition of k = 3

The case $k = 3$ encompasses all of the necessary ideas needed to prove Theorem 7. We want to show that if $|E| \gtrsim q^{\frac{d+3}{2}}$, then the set E determines a positive proportion of all 3-simplices.

Using Remark 2 together with Theorem 6, we see that for $|E| \gtrsim q^{\frac{d+3}{2}}$, there exists a set $\mathcal{E} \subset E \times E \times E = E^3$ with $|\mathcal{E}| \gtrsim |E|^3$ such that for every $(y^1, y^2, y^3) \in \mathcal{E}$, we have $|\Delta_{y^1,y^2,y^3}(E)| \gtrsim q^3$. Notice that this implies that if $|E| \gtrsim q^{\frac{d+3}{2}}$, then the set E determines a positive proportion of all 3-simplices whose bases are given by any fixed 2-simplex determined by $(y^1, y^2, y^3) \in \mathcal{E}$. It therefore suffices to show that a positive proportion of all 2-simplices can be constructed by the elements of $\mathcal{E}$. Since $|E| \gtrsim q^{\frac{d+3}{2}} \gtrsim q^{\frac{d+2}{2}}$ and $|\mathcal{E}| \subset |E|^3$ with $|\mathcal{E}| \sim |E|^3$, we can apply Corollary 13 where s is replaced by 3. Then we see that there exists a set $\mathcal{E}^{(1)} \subset \mathcal{E}' \subset E^2$ with $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|^2$ such that for every $(y^1, y^2) \in \mathcal{E}^{(1)}$, we have $|\Delta_{y^1,y^2}(\mathcal{E}(y^1, y^2))| \gtrsim q^2$. Namely, the elements in $\mathcal{E}$ determine a positive proportion of all 2-simplices whose bases are fixed as a 1-simplex given by any element $(y^1, y^2) \in \mathcal{E}^{(1)}$. It therefore suffices to show that a positive proportion of all 1-simplices can be constructed by the elements of $\mathcal{E}^{(1)}$. Since $|E| \gtrsim q^{\frac{d+3}{2}} \gtrsim q^{\frac{d+1}{2}}$ and $\mathcal{E}^{(1)} \subset E^2$ with $|\mathcal{E}^{(1)}| \sim |E|^2$, we see that we can apply Corollary 13 where $\mathcal{E}$ is replaced by $\mathcal{E}^{(1)}$ and $s = 2$. Then we see that there exists $\mathcal{E}^{(2)} \subset \left(\mathcal{E}^{(1)}\right)' \subset E$ with $|\mathcal{E}^{(2)}| \sim \left|\left(\mathcal{E}^{(1)}\right)'\right| \sim |E|$ such that for every $y^1 \in \mathcal{E}^{(2)}$, we have $\left|\Delta_{y^1}(\mathcal{E}^{(1)}(y^1))\right| \gtrsim q$. Namely, the elements in $\mathcal{E}^{(1)}$ determine a positive proportion of all 1-simplices. Therefore, the proof for the case $k = 3$ is complete.

3.6 Proof of Theorem 7: k-simplices

After looking specifically at the cases $k = 2$ and $k = 3$ in the previous two sections, we now give a proof for general k. As stated in the introduction, in order to specify a k-simplex up to isometry it is enough to specify the distances determined by the points. Here we will specify our k-simplices using Theorem 6 as one set of distances at a time.

First, using Remark 2 together with Theorem 6, we see that for $|E| \gtrsim q^{\frac{d+k}{2}}$, there exists a set $\mathcal{E} \subset E \times \cdots \times E = E^k$ with $|\mathcal{E}| \gtrsim |E|^k$ such that for every $(y^1, \ldots, y^k) \in \mathcal{E}$, we have

$$|\Delta_{y^1,\ldots,y^k}(E)| = |\{(\|y^0 - y^j\|)_{1\leq j\leq k} \in (\mathbb{F}_q)^k : y^0 \in E\}| \gtrsim q^k.$$

Notice that this implies that if $|E| \gtrsim q^{\frac{d+k}{2}}$, then the set E determines a positive proportion of all k-simplices whose bases are given by any fixed $(k-1)-$simplex determined by $(y^1, \ldots, y^k) \in \mathcal{E}$. It therefore suffices to show that a positive proportion of all $(k-1)$-simplices can be constructed by the elements of $\mathcal{E}$. Since $|E| \gtrsim q^{\frac{d+k}{2}} \gtrsim q^{\frac{d+k-1}{2}}$ and $|\mathcal{E}| \sim |E|^k$, we can apply Corollary 13 where s is replaced by k. Then we see that there exists a set $\mathcal{E}^{(1)} \subset \mathcal{E}'$ with $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|^{k-1}$ such that for every $(y^1, \ldots, y^{k-1}) \in \mathcal{E}^{(1)}$, we have

$$\left|\Delta_{y^1,\ldots,y^{k-1}}(\mathcal{E}(y^1, \ldots, y^{k-1}))\right| \gtrsim q^{k-1}.$$

Observe that this estimation implies that the elements in $\mathcal{E}$ determines a positive proportion of all possible $(k-1)$-simplices where their bases are fixed by a $(k-2)-$simplex given by any $(y^1, \ldots, y^{k-1}) \in \mathcal{E}^{(1)}$. Thus, it is enough to show that the elements in $\mathcal{E}^{(1)}$ can determine a positive proportion of all $(k-2)-$simplices. Putting

$\mathcal{E}^{(0)} = \mathcal{E}$ and using Corollary 13, if we repeat above process p-times, then we see that there exists a set $\mathcal{E}^{(p)} \subset \left(\mathcal{E}^{(p-1)}\right)' \subset E^{k-p}$ with $|\mathcal{E}^{(p)}| \sim |\left(\mathcal{E}^{(p-1)}\right)'| \sim |E|^{k-p}$ such that for each $(y^1, \ldots, y^{k-p}) \in \mathcal{E}^{(p)}$, we have

$$\left|\Delta_{y^1,\ldots,y^{k-p}}(\mathcal{E}^{(p-1)}(y^1, \ldots, y^{k-p}))\right| \gtrsim q^{k-p},$$

and so it suffices to show that the elements in $\mathcal{E}^{(p)} \subset E^{k-p}$ determine a positive proportion of all $(k-p-1)-$simplices. Taking $p = k-2$, we reduce our problem to showing that the elements in $\mathcal{E}^{(k-2)} \subset E \times E$ determine a positive proportion of all $1-$simplices. However, it is clear by applying Corollary 13 after setting $s = 2$ and $\mathcal{E} = \mathcal{E}^{(k-2)}$. To see this, first notice from our repeated process that $\mathcal{E}^{(k-2)} \subset E \times E$ and $|\mathcal{E}^{(k-2)}| \sim |E|^2$. Since $|E| \gtrsim q^{\frac{d+k}{2}} \gtrsim q^{\frac{d+1}{2}}$, Corollary 13 yields the desired result. Therefore, the proof of Theorem 7 is complete.

3.7 Proof of Theorem 8: k-star dot product sets

Define $\eta_{y^1,y^2,\ldots,y^k}(s_1, s_2, \ldots, s_k)$ by the relation

$$\sum_{s_1,s_2,\ldots,s_k \in \mathbb{F}_q} g(s_1, s_2, \ldots, s_k)\eta_{y^1,y^2,\ldots,y^k}(s_1, s_2, \ldots, s_k) = \sum_{x \in \mathbb{F}_q^d} g(x\cdot y^1, x\cdot y^2, \ldots, x\cdot y^k)E(x),$$

where g is a complex-valued function on $\mathbb{F}_q^k$, and $y^j \in \mathbb{F}_q^d$ for $j = 1, 2, \ldots, k$. The proof of Theorem 8 is based on the following lemma.

Lemma 14. *Let $E \subset \mathbb{F}_q^d$. Then*

$$\sum_{y^1,\ldots,y^k \in E}\ \sum_{s_1,s_2,\ldots,s_k \in \mathbb{F}_q} |\eta_{y^1,y^2,\ldots,y^k}(s_1, s_2, \ldots, s_k)|^2 \lesssim \frac{|E|^{k+2}}{q^k} + q^d|E|^k.$$

Proof. We proceed by induction. To prove the initial case, take $g(s) = q^{-1}\chi(-ts)$. Then we see that

$$\widehat{\eta}_{y^1}(t) = q^{d-1}\widehat{E}(ty^1).$$

It follows that

$$\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}|\widehat{\eta}_{y^1}(t)|^2 = q^{2(d-1)}\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}\left|\widehat{E}(ty^1)\right|^2,$$

and extracting $t=0$ we have that

$$\begin{aligned}\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}|\widehat{\eta}_{y^1}(t)|^2 &= |E|^3q^{-2} + q^{2(d-1)}\sum_{t\neq 0}\sum_{y^1\in E}\left|\widehat{E}(ty^1)\right|^2\\ &= |E|^3q^{-2} + q^{2(d-1)}\sum_{x\in\mathbb{F}_q^d}\left|\widehat{E}(x)\right|^2\cdot n(x)\end{aligned}$$

where

$$n(x) = |\{(t,y^1)\in\mathbb{F}_q^*\times E : ty^1 = x\}|.$$

Observe that $n(x)\leq q$ since every line in $\mathbb{F}_q^d$ contains exactly q points. Therefore, it follows by the Plancherel theorem that

$$\begin{aligned}\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}|\widehat{\eta}_{y^1}(t)|^2 &\leq |E|^3q^{-2} + q^{2d-1}(|E|q^{-d})\\ &= |E|^3q^{-2} + q^{d-1}|E|,\end{aligned}$$

and applying the Plancherel theorem once again, we see that

$$\begin{aligned}q\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}|\widehat{\eta}_{y^1}(t)|^2 &= \sum_{s\in\mathbb{F}_q}\sum_{y^1\in E}|\eta_{y^1}(s)|^2\\ &\leq |E|^3q^{-1} + q^d|E|.\end{aligned}$$

This proves the initial step. Now, suppose that

$$\sum_{y^1,\dots,y^{k-1}\in E}\ \sum_{s_1,s_2,\dots,s_{k-1}\in\mathbb{F}_q} |\eta_{y^1,y^2,\dots,y^{k-1}}(s_1,s_2,\dots,s_{k-1})|^2 \lesssim \frac{|E|^{k+1}}{q^{k-1}} + q^d|E|^{k-1}.$$

Let $g(s_1,s_2,\dots,s_k) = q^{-k}\chi(-s_1t_1 - s_2t_2 - \cdots - s_kt_k)$. It follows that

$$\widehat{\eta}_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k) = q^{d-k}\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k).$$

Then substituting in,

$$\sum_{t_1,\dots,t_k\in\mathbb{F}_q}\ \sum_{y^1,\dots,y^k\in E} |\widehat{\eta}_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k)|^2$$

$$= q^{2(d-k)} \sum_{t_1,\dots,t_k\in\mathbb{F}_q}\ \sum_{y^1,\dots,y^k\in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2,$$

and extracting the case when $t_k = 0$ we have

$$= q^{2(d-k)}|E| \sum_{t_1,\dots,t_{k-1}\in\mathbb{F}_q}\ \sum_{y^1,\dots,y^{k-1}\in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_{k-1}y^{k-1})|^2$$

$$+q^{2(d-k)} \sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q \\ t_k\neq 0}}\ \sum_{y^1,\dots,y^k\in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2 = I + II.$$

For the first term we apply Plancherel and the induction hypothesis to get

$$I \lesssim \frac{|E|^{k+2}}{q^{2k}} + q^{d-k-1}|E|^k.$$

For the second term we write,

$$II = q^{2(d-k)} \sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q \\ t_k\neq 0}}\ \sum_{y^1,\dots,y^k\in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2$$

$$= q^{2(d-k)} \sum_{y^1,\dots,y^{k-1}\in E}\ \sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q \\ t_k\neq 0}} \left(\sum_{y^k\in\mathbb{F}_q^d} E(y^k)|\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2\right),$$

and changing variables gives

$$\lesssim q^{2(d-k)} \sum_{y^1,\dots,y^{k-1}\in E} \sum_{\substack{t_1,\dots,t_{k-1}\in\mathbb{F}_q \\ t_k\neq 0}} \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 E(t_1y^1+\cdots+t_{k-1}y^{k-1}+mt_k^{-1}),$$

which summing in $t_1,\dots,t_k$ gives

$$= q^{2(d-k)} \sum_{y^1,\dots,y^{k-1}\in E} \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 |E\cap H_{y^1,\dots,y^{k-1},m}|,$$

where $H_{y^1,\dots,y^{k-1},m}$ is k dimensional hyperplane running through the origin. Since $|E\cap H_{y^1,\dots,y^{k-1},m}|\leq q^k$, it follows that

$$\lesssim q^{2(d-k)}|E|^{k-1}q^k \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{d-k}|E|^k.$$

Therefore we have that

$$\sum_{t_1,\dots,t_k\in\mathbb{F}_q} \sum_{y^1,\dots,y^k\in E} |\widehat{\eta}_{y^1,y^2,\dots,y^k}(t_1,t_2,\dots,t_k)|^2 \lesssim \frac{|E|^{k+2}}{q^{2k}} + q^{d-k}|E|^k.$$

Applying Plancherel in $t_1,\dots,t_k$ we obtain

$$\sum_{y^1,\dots,y^k\in E} \sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} |\eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k)|^2 \lesssim \frac{|E|^{k+2}}{q^k} + q^d|E|^k.$$

□

We are ready to complete the proof of Theorem 8 . By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |E|^{2(k+1)} &= \left(\sum_{y^1,\dots,y^k\in E} \sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} \eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k) \right)^2 \\ &\lesssim \sum_{y^1,\dots,y^k\in E} |\Pi_{y^1,y^2,\dots,y^k}(E)| \cdot \sum_{y^1,\dots,y^k\in E} \sum_{s_1,s_2,\dots,s_k\in\mathbb{F}_q} |\eta_{y^1,y^2,\dots,y^k}(s_1,s_2,\dots,s_k)|^2. \end{aligned}$$

By Lemma 14 it follows that

$$|E|^{2k+2} \lesssim \sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \cdot \left(\frac{|E|^{k+2}}{q^k} + q^d |E|^k \right).$$

Therefore,

$$\sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \gtrsim \frac{|E|^{2k+2}}{\frac{|E|^{k+2}}{q^k} + q^d|E|^k}.$$

Normalize to obtain

$$\frac{1}{|E|^k} \sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \gtrsim \frac{|E|^{k+2}}{\frac{|E|^{k+2}}{q^k} + q^d|E|^k},$$

which for $|E| \gtrsim q^{\frac{d+k}{2}}$ gives

$$\frac{1}{|E|^k} \sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \gtrsim q^k.$$

Thus the proof of Theorem 8 is complete.

3.8 Proof of Theorem 9: k-star distance sets on a sphere

Here we will need the following lemma whose proof is very similar to the proof of Lemma 14.

Lemma 15. *Let $E \subset S$. Then*

$$\sum_{y^1,\ldots,y^k \in E} \sum_{s_1,s_2,\ldots,s_k \in \mathbb{F}_q} |\eta_{y^1,y^2,\ldots,y^k}(s_1, s_2, \ldots, s_k)|^2 \lesssim \frac{|E|^{k+2}}{q^k} + q^{d-1}|E|^k.$$

Proof. We proceed by induction. To prove the initial case, take $g(s) = q^{-1}\chi(-ts)$. Then we see that

$$\widehat{\eta}_{y^1}(t) = q^{d-1}\widehat{E}(ty^1).$$

It follows that

$$\sum_{t \in \mathbb{F}_q} \sum_{y^1 \in E} |\widehat{\eta}_{y^1}(t)|^2 = q^{2(d-1)} \sum_{t \in \mathbb{F}_q} \sum_{y^1 \in E} \left|\widehat{E}(ty^1)\right|^2,$$

and extracting $t=0$ we have that

$$\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}|\widehat{\eta}_{y^1}(t)|^2 = |E|^3q^{-2}+q^{2(d-1)}\sum_{t\neq 0}\sum_{y^1\in E}\left|\widehat{E}(ty^1)\right|^2$$
$$= |E|^3q^{-2}+q^{2(d-1)}\sum_{x\in\mathbb{F}_q^d}\left|\widehat{E}(x)\right|^2\cdot n(x)$$

where

$$n(x)=|\{(t,y^1)\in\mathbb{F}_q^*\times E : ty^1=x\}|.$$

Since $E\subset S$, it does not contain the origin and $n(x)\leq 2$ as seen in [12]. Therefore, it follows by the Plancherel theorem that

$$\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}|\widehat{\eta}_{y^1}(t)|^2 \leq |E|^3q^{-2}+2q^{2(d-1)}(|E|q^{-d})$$
$$= |E|^3q^{-2}+2q^{d-2}|E|$$
$$\lesssim |E|^3q^{-2}+q^{d-2}|E|,$$

and applying the Plancherel theorem once again, we see that

$$q\sum_{t\in\mathbb{F}_q}\sum_{y^1\in E}|\widehat{\eta}_{y^1}(t)|^2 = \sum_{s\in\mathbb{F}_q}\sum_{y^1\in E}\eta_{y^1}^2(s)$$
$$\lesssim |E|^3q^{-1}+q^{d-1}|E|.$$

This proves the initial step. Now, suppose that

$$\sum_{y^1,\dots,y^{k-1}\in E}\ \sum_{s_1,s_2,\dots,s_{k-1}\in\mathbb{F}_q}|\eta_{y^1,y^2,\dots,y^{k-1}}(s_1,s_2,\dots,s_{k-1})|^2 \lesssim \frac{|E|^{k+1}}{q^{k-1}}+q^{d-1}|E|^{k-1}.$$

Let $g(s_1, s_2, \ldots, s_k) = q^{-k}\chi(-s_1t_1 - s_2t_2 - \cdots - s_kt_k)$. It follows that

$$\widehat{\eta}_{y^1,y^2,\ldots,y^k}(t_1, t_2, \ldots, t_k) = q^{d-k}\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k).$$

Then substituting in,

$$\sum_{t_1,\ldots,t_k \in \mathbb{F}_q} \sum_{y^1,\ldots,y^k \in E} |\widehat{\eta}_{y^1,y^2,\ldots,y^k}(t_1, t_2, \ldots, t_k)|^2$$

$$= q^{2(d-k)} \sum_{t_1,\ldots,t_k \in \mathbb{F}_q} \sum_{y^1,\ldots,y^k \in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2,$$

and extracting the case when $t_k = 0$ we have

$$= q^{2(d-k)}|E| \sum_{t_1,\ldots,t_{k-1} \in \mathbb{F}_q} \sum_{y^1,\ldots,y^{k-1} \in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_{k-1}y^{k-1})|^2$$

$$+q^{2(d-k)} \sum_{\substack{t_1,\ldots,t_{k-1} \in \mathbb{F}_q \\ t_k \neq 0}} \sum_{y^1,\ldots,y^k \in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2 = I + II.$$

For the first term we apply Plancherel and the induction hypothesis to get

$$I \lesssim \frac{|E|^{k+2}}{q^{2k}} + q^{d-k-1}|E|^k.$$

For the second term we write,

$$II = q^{2(d-k)} \sum_{\substack{t_1,\ldots,t_{k-1} \in \mathbb{F}_q \\ t_k \neq 0}} \sum_{y^1,\ldots,y^k \in E} |\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2$$

$$= q^{2(d-k)} \sum_{y^1,\ldots,y^{k-1} \in E} \sum_{\substack{t_1,\ldots,t_{k-1} \in \mathbb{F}_q \\ t_k \neq 0}} \left(\sum_{y^k \in \mathbb{F}_q^d} E(y^k)|\widehat{E}(t_1y^1 + t_2y^2 + \cdots + t_ky^k)|^2 \right),$$

and changing variables gives

$$\lesssim q^{2(d-k)} \sum_{y^1,\ldots,y^{k-1} \in E} \sum_{\substack{t_1,\ldots,t_{k-1} \in \mathbb{F}_q \\ t_k \neq 0}} \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 E(t_1y^1 + \ldots t_{k-1}y^{k-1} + mt_k^{-1}),$$

which summing in $t_1, \ldots, t_k$ gives

$$= q^{2(d-k)} \sum_{y^1,\ldots,y^{k-1}\in E} \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 |E \cap H_{y^1,\ldots,y^{k-1},m}|,$$

where $H_{y^1,\ldots,y^{k-1},m}$ is k dimensional hyperplane running through the origin. Since E is a subset of a sphere, we see that $|E \cap H_{y^1,\ldots,y^{k-1},m}| \leq q^{k-1}$. Then the quantity is

$$\lesssim q^{2(d-k)} |E|^{k-1} q^{k-1} \sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{d-k-1}|E|^k.$$

Therefore we have that

$$\sum_{t_1,\ldots,t_k\in\mathbb{F}_q} \sum_{y^1,\ldots,y^k\in E} |\widehat{\eta}_{y^1,y^2,\ldots,y^k}(t_1,t_2,\ldots,t_k)|^2 \lesssim \frac{|E|^{k+2}}{q^{2k}} + q^{d-k-1}|E|^k.$$

Applying Plancherel in $t_1, \ldots, t_k$ we obtain

$$\sum_{y^1,\ldots,y^k\in E} \sum_{s_1,s_2,\ldots,s_k\in\mathbb{F}_q} |\eta_{y^1,y^2,\ldots,y^k}(s_1,s_2,\ldots,s_k)|^2 \lesssim \frac{|E|^{k+2}}{q^{k}} + q^{d-1}|E|^k.$$

□

We are now ready to complete the proof of Theorem 9.

By the Cauchy-Schwarz inequality, we have

$$|E|^{2(k+1)} = \left(\sum_{y^1,\ldots,y^k\in E} \sum_{s_1,s_2,\ldots,s_k\in\mathbb{F}_q} \eta_{y^1,y^2,\ldots,y^k}(s_1,s_2,\ldots,s_k) \right)^2$$
$$\lesssim \sum_{y^1,\ldots,y^k\in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \cdot \sum_{y^1,\ldots,y^k\in E} \sum_{s_1,s_2,\ldots,s_k\in\mathbb{F}_q} |\eta_{y^1,y^2,\ldots,y^k}(s_1,s_2,\ldots,s_k)|^2.$$

By Lemma 15 it follows that

$$|E|^{2k+2} \lesssim \sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \cdot \left(\frac{|E|^{k+2}}{q^k} + q^{d-1}|E|^k \right).$$

Therefore,

$$\sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \gtrsim \frac{|E|^{2k+2}}{\frac{|E|^{k+2}}{q^k} + q^{d-1}|E|^k}.$$

Normalize to obtain

$$\frac{1}{|E|^k} \sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \gtrsim \frac{|E|^{k+2}}{\frac{|E|^{k+2}}{q^k} + q^{d-1}|E|^k},$$

which for $|E| \gtrsim q^{\frac{d+k-1}{2}}$ gives

$$\frac{1}{|E|^k} \sum_{y^1,\ldots,y^k \in E} |\Pi_{y^1,y^2,\ldots,y^k}(E)| \gtrsim q^k.$$

Since E is a subset of S, recall that determining distances is equivalent to determining dot products. Therefore, for $|E| \gtrsim q^{\frac{d+k-1}{2}}$ we have

$$\frac{1}{|E|^k} \sum_{y^1,\ldots,y^k \in E} |\Delta_{y^1,y^2,\ldots,y^k}(E)| \gtrsim q^k.$$

Thus the proof of Theorem 9 is complete.

3.9 Proof of Theorem 10: k-simplices on a sphere

For this proof, we will follow the same basic outline as the proof of Theorem 7. If $k = 1$, then the statement of Theorem 10 immediately follows when a pigeon-holing argument similar to Remark 2 is applied to Theorem 9. We therefore assume that $k \geq 2$. As stated in the introduction, in order to specify a k-simplex up to isometry it is enough to specify the distances determined by the points. Here we will specify our k-simplices using Theorem 9 as one set of distances at a time. In addition, we need the following theorem which is more general version of Theorem 9.

Theorem 16. *Given $E \subset S$, let $\mathcal{E} \subset E \times \cdots \times E = E^s$, $s \geq 2$, with $|\mathcal{E}| \sim |E|^s$. Define*

$$\mathcal{E}' = \{(y^1, \ldots, y^{s-1}) \in E^{s-1} : (y^1, \ldots, y^{s-1}, y^s) \in \mathcal{E} \quad \text{for some} \quad y^s \in E\}.$$

In addition, for each $(y^1, \ldots, y^{s-1}) \in \mathcal{E}'$ we define

$$\mathcal{E}(y^1, \ldots, y^{s-1}) = \{y^s \in E : (y^1, \ldots, y^{s-1}, y^s) \in \mathcal{E}\}.$$

If $|E| \gtrsim q^{\frac{d+s-2}{2}}$, then we have

$$\frac{1}{|\mathcal{E}'|} \sum_{(y^1,\ldots,y^{s-1}) \in \mathcal{E}'} \left|\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1, \ldots, y^{s-1})\right)\right| \gtrsim q^{s-1}, \tag{3.9.1}$$

where

$$\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1, \ldots, y^{s-1})\right) = \{\left(\|y^s - y^1\|, \ldots, \|y^s - y^{s-1}\|\right) \in (\mathbb{F}_q)^{s-1} : y^s \in \mathcal{E}(y^1, \ldots, y^{s-1})\}.$$

Proof. For each $t_1, \ldots, t_s \in \mathbb{F}_q$, recall that the incidence function on $\Delta_{y^1,\ldots,y^{s-1}}(\mathcal{E}(y^1, \ldots, y^{s-1}))$ is given by

$$\nu_{y^1,\ldots,y^{s-1}}^{\mathcal{E}(y^1,\ldots,y^{s-1})}(t_1, \ldots, t_{s-1}) = |\{y^s \in \mathcal{E}(y^1, \ldots, y^{s-1}) : \|y^s - y^1\| = t_1, \ldots, \|y^s - y^{s-1}\| = t_{s-1}\}|.$$

Observe that

$$\begin{aligned} \nu_{y^1,\ldots,y^{s-1}}^{\mathcal{E}(y^1,\ldots,y^{s-1})}(t_1, \ldots, t_{s-1}) &\leq \nu_{y^1,\ldots,y^{s-1}}(t_1, \ldots, t_{s-1}) \\ &= |\{(y^s \in E : ||y^s - y^1|| = t_1, \ldots, ||y^s - y^{s-1}|| = t_{s-1}\}|. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$|\mathcal{E}|^2 = \left(\sum_{(y^1,\ldots,y^{s-1}) \in \mathcal{E}'} \sum_{t_1 \ldots, t_{s-1} \in \mathbb{F}_q} \nu_{y^1,\ldots,y^{s-1}}^{\mathcal{E}(y^1,\ldots,y^{s-1})}(t_1, \ldots, t_{s-1}) \right)^2$$

$$\leq \left(\sum_{(y^1,\ldots,y^{s-1}) \in \mathcal{E}'} |\Delta_{y^1,\ldots,y^{s-1}}(\mathcal{E}(y^1, \ldots, y^{s-1}))| \right) \cdot \left(\sum_{y^1,\ldots,y^{s-1} \in E} \sum_{t_1,\ldots,t_{s-1} \in \mathbb{F}_q} |\nu_{y^1,\ldots,y^{s-1}}(t_1, \ldots, t_{s-1})|^2 \right).$$

Using Lemma 15 and remembering that determining distances and dot products are equivalent in this situation, it follows that

$$|\mathcal{E}|^2 \leq \sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'} |\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1,\ldots,y^{s-1})\right)| \cdot \left(\frac{|E|^{s+1}}{q^{s-1}} + q^{d-1}|E|^{s-1}\right).$$

Observe that $|\mathcal{E}'| \sim |E|^{s-1}$ because otherwise $|\mathcal{E}| \leq |\mathcal{E}'||E| << |E|^s$ which contradicts $|\mathcal{E}| \sim |E|^s$. Therefore, if $|E| \gtrsim q^{(d+s-2)/2}$, then

$$\frac{1}{|\mathcal{E}'|} \sum_{(y^1,\ldots,y^{s-1})\in\mathcal{E}'} \left|\Delta_{y^1,\ldots,y^{s-1}}\left(\mathcal{E}(y^1,\ldots,y^{s-1})\right)\right| \gtrsim q^{s-1}.$$

Thus the proof of Theorem 16 is complete. □

When a pigeon-holing argument similar to Remark 2 is applied to the inequality (3.9.1) in Theorem 16, the following corollary immediately follows.

Corollary 17. *Let* $E \subset S$ *and* $\mathcal{E} \subset E \times \cdots \times E = E^s, s \geq 2,$ *with* $|\mathcal{E}| \sim |E|^s$. *If* $|E| \gtrsim q^{\frac{d+s-2}{2}}$, *then there exists* $\mathcal{E}^{(1)} \subset \mathcal{E}' \subset E^{s-1}$ *with* $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|^{s-1}$ *such that for every* $(y^1,\ldots,y^{s-1}) \in \mathcal{E}^{(1)}$,

$$\left|\Delta_{y^1,\ldots,y^{s-1}}(\mathcal{E}(y^1,\ldots,y^{s-1}))\right| \gtrsim q^{s-1}.$$

Namely, the elements in $\mathcal{E}$ *determine a positive proportion of all* $(s-1)$*-simplices whose bases are fixed as a* $(s-2)$*-simplex given by any element* $(y^1,\ldots,y^{s-1}) \in \mathcal{E}^{(1)}$.

We are now ready to prove Theorem 10. First, using a pigeon-holing argument together with Theorem 9, we see that for $|E| \gtrsim q^{\frac{d+k-1}{2}}$, there exists a set $\mathcal{E} \subset E \times \cdots \times E = E^k$ with $|\mathcal{E}| \gtrsim |E|^k$ such that for every $(y^1,\ldots,y^k) \in \mathcal{E}$, we have

$$|\Delta_{y^1,\ldots,y^k}(E)| = |\{(\|y^0 - y^j\|)_{1\leq j\leq k} \in (\mathbb{F}_q)^k : y^0 \in E\}| \gtrsim q^k.$$

Notice that this implies that if $|E| \gtrsim q^{\frac{d+k-1}{2}}$, then the set E determines a positive proportion of all k-simplices whose bases are given by any fixed $(k-1)-$simplex determined by $(y^1,\ldots,y^k) \in \mathcal{E}$. It therefore suffices to show that a positive proportion of all $(k-1)$-simplices can be constructed by the elements of $\mathcal{E}$. Since $|E| \gtrsim q^{\frac{d+k-1}{2}} \gtrsim q^{\frac{d+k-2}{2}}$ and $|\mathcal{E}| \sim |E|^k$, we can apply Corollary 17 where s is replaced by k. Then we see that there exists a set $\mathcal{E}^{(1)} \subset \mathcal{E}'$ with $|\mathcal{E}^{(1)}| \sim |\mathcal{E}'| \sim |E|^{k-1}$ such that for every $(y^1, \ldots, y^{k-1}) \in \mathcal{E}^{(1)}$, we have

$$\left|\Delta_{y^1,\ldots,y^{k-1}}(\mathcal{E}(y^1, \ldots, y^{k-1}))\right| \gtrsim q^{k-1}.$$

Observe that this estimation implies that the elements in $\mathcal{E}$ determines a positive proportion of all possible $(k-1)$-simplices where their bases are fixed by a $(k-2)-$simplex given by any $(y^1, \ldots, y^{k-1}) \in \mathcal{E}^{(1)}$. Thus, it is enough to show that the elements in $\mathcal{E}^{(1)}$ determine a positive proportion of all $(k-2)-$simplices. Putting $\mathcal{E}^{(0)} = \mathcal{E}$ and using Corollary 17 we see that if we repeat the above process p-times then there exists a set $\mathcal{E}^{(p)} \subset \left(\mathcal{E}^{(p-1)}\right)' \subset E^{k-p}$ with $|\mathcal{E}^{(p)}| \sim |\left(\mathcal{E}^{(p-1)}\right)'| \sim |E|^{k-p}$ such that for each $(y^1, \ldots, y^{k-p}) \in \mathcal{E}^{(p)}$, we have

$$\left|\Delta_{y^1,\ldots,y^{k-p}}(\mathcal{E}^{(p-1)}(y^1, \ldots, y^{k-p}))\right| \gtrsim q^{k-p},$$

and so it suffices to show that the elements in $\mathcal{E}^{(p)} \subset E^{k-p}$ determine a positive proportion of all $(k-p-1)-$simplices. Taking $p = k-2$, we reduce our problem to showing that the elements in $\mathcal{E}^{(k-2)} \subset E \times E$ determine a positive proportion of all $1-$simplices. However, it is clear by applying Corollary 17 after setting $s = 2, \mathcal{E} = \mathcal{E}^{(k-2)}$. To see this, first notice from our repeated process that $\mathcal{E}^{(k-2)} \subset E \times E$ and $|\mathcal{E}^{(k-2)}| \sim |E|^2$. Since $|E| \gtrsim q^{\frac{d+k-1}{2}} \gtrsim q^{\frac{d}{2}}$, Corollary 17 yields the desired result. Therefore, we complete the proof of Theorem 10.

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