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Collective modes and electronic spectral function in smooth edges of quantum hall systems

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We present a microscopic theory of the collective modes of a ''smooth'' edge of a quantum Hall system, showing under what conditions these modes can be described as a set of independent bosons. We then calculate the electronic spectral function in an independent-boson model—a procedure that reduces to standard bosonization in the limit of ''sharp'' edge. The I-V tunneling characteristics deduced from this model exhibit, for low voltage, a power-law behavior, with exponents that differ significantly from those of the sharp edge model. [S0163-1829(96)51744-3]

Understanding the character of edge excitations is crucial to the theory of the transport properties of two-dimensional systems in the presence of a strong perpendicular magnetic field, such as quantum Hall bars, quantum wires, and dots.[1] An effective theory of edge excitations was first derived by Wen.[2] He showed that a ''sharp'' edge (see below) is a realization of the one-dimensional chiral Luttinger liquid (CLL) model, where the electronic spectral function exhibits, in general, a nontrivial behavior, leading to a density of states that vanishes, at low energy, as a power law. This theory has been well confirmed by detailed microscopic calculations[3] and by recent experiments.[4,5] The effect of the long range of the Coulomb interaction, which was initially ignored, has been recently included by Zülicke and MacDonald.[6]

All the above papers assumed the validity of the so-called ''sharp edge'' model, in which the density of the system drops sharply from the bulk value ρ_0 to zero within a few magnetic lengths $l=(\hbar c/eB)^{1/2}$. There are numerous indications that this is not always the correct model for the edge. On one hand, Hartree-Fock calculations[7] for strongly confined systems predict that, at sufficiently strong magnetic field, the edge undergoes a reconstruction, taking up a more extended shape. On the other hand, in the case of *smooth* confinement, such as can be realized by gate electrodes, the electronic density is expected to have a smoothly varying profile (on the scale of l), determined by classical electrostatic equilibrium.[8,9] Detailed calculations using density functional theory, Thomas-Fermi theory, and other methods[10–14] have confirmed the theoretical validity of this ''smooth edge'' picture. Edge imaging experiments[15] have confirmed the relevancy of this description for gate-confined Hall bars.

In this paper we want to investigate the spectral properties of the smooth edge model described above. The mapping to a one-dimensional chiral electron liquid is not justified in this case. In fact, a recent study by Aleiner and Glazman (AG),[16] based on the classical hydrodynamics approach, has shown that a smooth edge, in contrast to a sharp edge, supports *multiple branches* of edge waves. Of these, one is the usual edge magnetoplasmon,[17] and the others (infinitely many in the classical approach of AG) are phononlike and lower in energy than the magnetoplasmon. We shall show that, under the assumption of smooth density variation, only a finite number of these phononlike modes are correctly described as independent bosons.

In order to calculate the electronic spectral function we rely on the strong analogy between this problem and that of a uniform electron gas in the partially filled lowest Landau level (LLL). The similarity between these two problems arises from the fact that, in both cases, in a mean field approximation, the self-consistent potential is uniform, so that the electrons are distributed among a large number of degenerate orbitals at the Fermi energy. In a smooth edge this occurs because the nonuniform electronic density perfectly screens the field due to the external confinement potential.[8,9] Recently, Johansson and Kinaret[18] have shown that a qualitatively correct description of the spectral function[19] of the uniform electron gas in the LLL at general filling factor is given by an independent boson model[20] (IBM) in which a single localized electron interacts with the density fluctuations of the system. An essentially equivalent procedure has been applied by Aleiner, Baranger, and Glazman[21] to study the spectral function of the two-dimensional electron liquid in a weak magnetic field. Finally, a formal justification of the IBM from diagrammatic many-body theory has been provided by Haussmann.[22] Encouraged by these successes, here we apply the independent boson model to the problem of the smooth edge. The resulting theory reduces to standard bosonization in the limit of a sharp edge, i.e., when there is only one branch of edge waves. When multiple branches are present, our results for the low energy behavior of the tunneling density of states are significantly different from those of the sharp edge model. The actual number of modes that must be included depends on the width of the edge, as explained below.

Let us begin by writing down the microscopic Hamiltonian, within the lowest Landau level, in terms of density fluctuations relative to the equilibrium density profile $\rho_0(y)$:

$$H=\frac{1}{2}\int_{\text{edge}}\frac{e^2}{|\vec{r}-\vec{r}'|}\,\delta\rho(\vec{r})\,\delta\rho(\vec{r}')\,d^2r\,d^2r', \qquad (1)$$

where the density operator (projected in the LLL) has been written as

$$\rho(\vec{r})=\rho_0(y)+\delta\rho(\vec{r}). \tag{2}$$

The integral in Eq. (1) extends over the edge region, which we take to be $0<x<L$, $0<y<d$, with $L\gg d\gg l$, and translationally invariant along x [$\rho_0(y)=0$ for $y<0$]. The projected density fluctuation operator is given by

$$\delta\rho(\vec{r})=\frac{1}{\sqrt{\pi}lL}\sum_{h\neq k} c_k^\dagger c_h e^{i(h-k)x} e^{-[(y-y_k)^2+(y-y_h)^2]/2l^2}, \tag{3}$$

where $y_h=hl^2$, h and k are integral multiples of $2\pi/L$, and $c_k^\dagger$ is the creation operator of a Landau gauge orbital centered at y_k with wave vector k in the x direction. Note the restriction $h\neq k$, which excludes the equilibrium component of the density. The kinetic energy is absent in Eq. (1) due to projection on the LLL, and we have assumed that density fluctuations vanish in the bulk of the system, i.e., the bulk is incompressible. The terms linear in $\delta\rho$ have vanished because of the equilibrium condition $\int\rho_0(\vec{r})v(\vec{r}-\vec{r}')d^2r' + V_{\rm ext}(\vec{r})=\text{const}$ where $V_{\rm ext}(\vec{r})$ is the confinement potential. Therefore the problem is formally similar to that of a translationally invariant electron gas: the nonuniformity enters only through the restricted region of integration in Eq. (1).

The normal mode operators $\delta\rho_{nk}$ are now introduced according to the definition

$$\delta\rho_{nk}=\int_0^L\frac{dx}{L}e^{-ikx}\int_0^d dy f_{nk}(y)\,\delta\rho(x,y), \tag{4}$$

where $f_{nk}(y)$ are the solutions of the equation

$$\int_0^d K_0(k|y-y'|)f_{nk}(y')\,\frac{\rho_0'(y)}{\overline{\rho}}\,dy'=\frac{1}{\lambda_{nk}}f_{nk}(y), \tag{5}$$

where $K_0(y)$ is the modified Bessel function. They satisfy the orthonormality condition $\int_0^d f_{nk}(y)f_{mk}(y)[\rho_0'(y)/\overline{\rho}]dy=\delta_{nm}$, and vanish outside the interval $[0,d]$. Equation (5) is the eigenvalue problem solved by AG. The nth eigenfunction has n nodes in the y direction. In terms of the normal modes, the Hamiltonian (1) takes the form

$$H=\sum_{nk>0}\hbar\omega_{nk}b_{nk}^\dagger b_{nk}, \tag{6}$$

where the operators b_{nk} are defined via $\delta\rho_{nk}\equiv\sqrt{kl^2\overline{\rho}/L}\,b_{nk}^\dagger$ and the eigenfrequencies ω_{nk} are given by $k\overline{\nu}e^2/\lambda_{nk}\pi$, and $\overline{\nu}=2\pi l^2\overline{\rho}$ is the usual filling factor in the bulk.

It remains to be determined under what conditions the operators b_{nk} are good boson operators. To this end we substitute Eq. (3) in Eq. (4), noting that when $n\ll d/l$ the Gaussian factors in the integral can be replaced by δ functions on the scale of variation of f_{nk}. We obtain

$$\delta\rho_{nk}\simeq e^{-k^2l^2/4}\frac{1}{L}\sum_h c_{h-k/2}^\dagger c_{h+k/2} f_{nk}(hl^2)\quad(n\ll d/l). \tag{7}$$

The commutator of two density fluctuations can now be easily calculated to be .

$$\begin{aligned}[\delta\rho_{nk},\delta\rho_{m-k}]&=e^{-k^2l^2/2}\frac{1}{L^2}\sum_h (n_{h-k/2}-n_{h+k/2})\\&\quad\times f_{nk}(hl^2)f_{mk}(hl^2)\\&\simeq-\frac{kl^2\overline{\rho}}{L}\delta_{nm},\end{aligned} \tag{8}$$

in agreement with the commutation rules for bosons. In arriving at Eq. (8) we have assumed $kl\ll1$, and we have replaced the occupation number operators by their ground-state expectation values n_k, which amounts to a linearization of the equations of motion around the equilibrium state. For the commutator $[\delta\rho_{nk},\delta\rho_{m-k'}]$, with $k\neq k'$, we find, at the same level of approximation, zero. This is because the commutator in question contains terms of the form $c_h^\dagger c_{h'}$ with $h\neq h'$, which vanish upon averaging in a translationally invariant (along x) state. Recently, Han and Thouless (unpublished) have argued that the Hamiltonian (6) should include an additional term describing the dynamics of the boundary between the compressible edge and the incompressible bulk. However, since the additional term is quadratic in the boson operators, the modified Hamiltonian can be recast (after a unitary transformation) in the form of Eq. (6), with modified, but qualitatively similar, eigenfrequencies.

Having thus completed the bosonization of the Hamiltonian, we proceed to the calculation of the spectral function within the independent boson model.[18,20] The model describes a single electron, localized at point $\vec{r}$, electrostatically coupled to density fluctuations:

$$\begin{aligned}H_{\rm IBM}&=\sum_{nk>0}\hbar\omega_{nk}b_{nk}^\dagger b_{nk}\\&\quad+\psi^\dagger(\vec{r})\psi(\vec{r})\sum_{nk>0}M_{nk}(y)[b_{nk}^\dagger e^{ikx}+b_{nk}e^{-ikx}],\end{aligned} \tag{9}$$

where the matrix element $M_{nk}(y)$ is given by

$$M_{nk}(y)=\frac{2e^2}{\lambda_{nk}}f_{nk}(y)\sqrt{\frac{kl^2\overline{\rho}}{L}}, \tag{10}$$

and $\psi^\dagger(\vec{r})$ is the field operator that creates an electron in the LLL coherent state (Gaussian) orbital centered at $\vec{r}$. The Hamiltonian (9) can be solved by standard methods,[20] within the one-electron Hilbert space. The fermionic Green's function is obtained as

$$\begin{aligned}G_>(y;t)&\equiv-i\langle\psi(\vec{r},t)\psi^\dagger(\vec{r},0)\rangle\\&=[1-\nu_0(y)]\exp\left(\sum_{nk>0}\frac{M_{nk}^2(y)}{\omega_{nk}^2}[e^{-i\omega_{nk}t}-1]\right),\end{aligned} \tag{11}$$

where $\nu_0(y)\equiv2\pi l^2\rho_0(y)$, and the sum over n and k in the exponent is restricted by the conditions $n\ll d/l$ and $k\ll1/l$, which define the regime of validity of the hydrodynamic approximation. This result can also be obtained from direct bosonization of the electron field operator, as in Ref. 21. The

Fourier transform of $G(y,t)/2\pi$ is the spectral function $A_>(y,\omega)$ and gives the local density of states, which controls the tunneling current from a point contact located at position y into the edge. From Eq. (11) it can be easily shown[23] that $A_>(y,\omega)$ satisfies the integral equation

$$\omega A_>(y,\omega)=\int_0^\omega g(\Omega)A_>(y,\omega-\Omega)d\Omega, \tag{12}$$

where

$$g(y,\Omega)\equiv\sum_{nk}\frac{M_{nk}(y)^2}{\omega_{nk}}\delta(\Omega-\omega_{nk}). \tag{13}$$

Equation (12), together with the conditions $A_>(y,\omega)=0$ for $\omega<0$ and $\int_0^\infty A_>(y,\omega)d\omega=1-\nu_0(y)$, completely determines the spectral function. This equation further implies that, at sufficiently small ω, $A_>(y,\omega)$ will have a power-law behavior

$$A(y,\omega)\sim\omega^{g(y,0)-1} \tag{14}$$

if and only if the function $g(y,\Omega)$ has a finite limit for $\Omega\to 0$. The tunneling current I, as a function of voltage V, will then behave as $V^{g(y,0)}$ for sufficiently low voltage. Notice that this conclusion is completely general, and does not depend on the specific (hydrodynamic) model that led to the definition of $g(y,\Omega)$ in Eq. (13). In the general case, $g(y,\Omega)$ could be computed from the microscopic density-density response function $\chi(\vec{r},\vec{r}',\Omega)$ of the edge as follows:

$$g(y,\Omega)=\int d^2r'd^2r''v(\vec{r}-\vec{r}')v(\vec{r}-\vec{r}'')\mathrm{Im}\chi(\vec{r}',\vec{r}'',\Omega)/\Omega, \tag{15}$$

where $v(\vec{r}-\vec{r}')$ has the Fourier transform $v(\vec{k})=(2\pi e^2/k)\exp(-(kl)^2/4)$.[22] An important advantage of this microscopic formulation is that the finite lifetime of the collective modes (which is assumed to be infinite in the hydrodynamic model) would be taken into account through the width of the peaks in $\mathrm{Im}\chi$.

The calculation of the exponent $g(y,0)$ is easily performed within the hydrodynamic model. Neglecting the weak nonlinearity of the $n=0$ mode[24] we obtain $g(y,0)=\Sigma_n\beta_n(y)$, where

$$\beta_n(y)=\frac{1}{\overline{\nu}}f_{n0}^2(y). \tag{16}$$

Although the cutoff at $n=d/l$ introduces an uncertainty in the evaluation of the exponent at any given d, we emphasize that there would be no uncertainty if one used the microscopic formula (15) for $g(y,\Omega)$. Our approximate hydrodynamic evaluation of the exponent should be in good qualitative agreement with the results of the more accurate microscopic calculation.

We observe that independently of the shape of the density profile $\beta_0(y)=1/\overline{\nu}$, with negligible corrections arising from the weak nonlinearity of the dispersion of the $n=0$ mode. Therefore in the sharp edge limit, when only one branch of edge waves exists, we recover the familiar result of Wen's theory $A_>(\omega)\sim\omega^{1/\overline{\nu}-1}$. For $n>1$ $\beta_n(y)$ fluctuates around an average value of $1/\overline{\nu}$ in a way dependent on the form of the equilibrium density profile of the edge. This can be confirmed by explicit calculation in the special case $\rho_0(y)=(2/\pi)\overline{\rho}\arctan\sqrt{y/d}$ considered by AG for a gate-confined electron gas, leading to the result $\beta_n(y)=(1/\overline{\nu})T_{2n}^2[\sqrt{d/(y+d)}](2-\delta_{n0})$, where $T_n(y)$ is the nth Chebyschev polynomial. We conclude that the exponent in Eq. (14) increases linearly with d and therefore that in the limit $d\to\infty$ (limit of infinitely smooth edge) the tunneling density of states vanishes at low energy faster than any power law, that is, a ''hard'' gap develops. However, it is easy to see that the power-law behavior of Eq. (14) only holds for $\omega\ll\overline{\nu}e^2/d\pi$—an interval that shrinks to zero for $d\to\infty$.

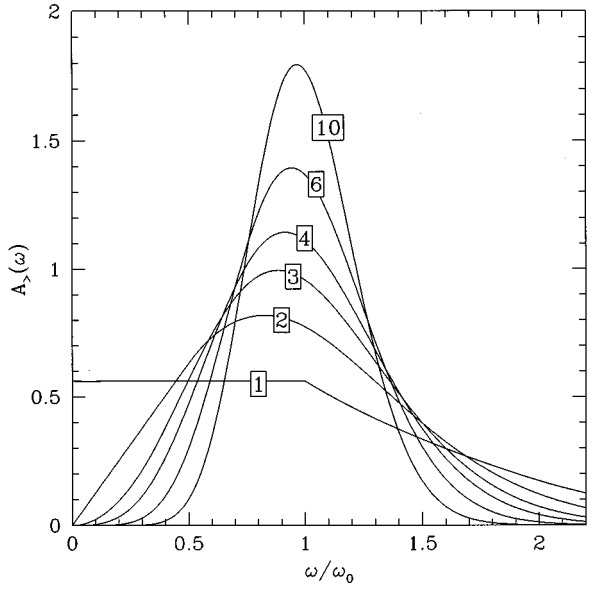

FIG. 1. Electronic spectral function $A_>(\omega)$ as a function of ω/ω_0, where $\omega_0=\overline{\nu}e^2/\pi l$, for edges of a $\overline{\nu}=1$ quantum Hall system with 1, 2, 3, 4, 6, and 10 modes. Logarithmic corrections to the edge magnetoplasmon dispersion are neglected. The dependence on y and $\rho_0(y)$ have been eliminated neglecting the constant $1-\nu_0(y)$ and using $\beta_n=1/\overline{\nu}$, $\omega_{nk}=\overline{\nu}e^2k/\pi n$. The integral over k has been cut off as explained in the text.

In Fig. 1 we present our numerical results for the full electronic spectral function, calculated from Eq. (12) within the hydrodynamic model for different edge widths d. In contrast to the analysis of the low-frequency behavior, this calculation depends on the detailed form of the eigenfunctions $f_{nk}(y)$ and eigenfrequencies ω_{nk}. From a detailed study of the solutions of the eigenvalue equation (5) we have found that the f_{nk}'s can be treated as being independent of k and the ω_{nk}'s to be linear functions of k up to a maximum wave vector $k_c=n/d$ for which the wavelength along the edge equals the wavelength perpendicular to the edge. For $k>k_c$ the mode dispersion becomes approximately constant, and the wave function f_{nk} becomes localized near the boundaries of the edge region, giving negligible contribution to the spectral function. The results presented in Fig. 1 have been obtained using the double cutoff $n<d/l$ and $k<k_c$: the results are found to be largely independent of the details of the cutoff procedure.

Figure 1 shows clearly how the low energy pseudogap

becomes more and more pronounced with increasing d (and, therefore, increasing number of branches of edge waves). For very large d the spectral function is found to converge to a δ function centered at $\omega_0=\overline{\nu e}^2/\pi l$, which coincides with the simplest estimate of the potential energy cost for the insertion of an electron into a frozen liquid.[22]

Our results for $d\rightarrow\infty$ are in qualitative agreement with those obtained in Refs. 18 and 22 for the spectral function of the homogeneous electron gas, except that the latter is found to have a finite width. This happens because our hydrodynamic approach is unable to give the gapful collective modes of the homogeneous fluid phase,[25] and hence our spectral function does not reduce to that of the homogeneous phase.

In conclusion, we have performed an independent boson model calculation of the tunneling density of states for a smooth edge, and we have found that it vanishes at low frequency as a power, with an exponent that differs significantly from the one found in the sharp edge case. Recent experiments by Chang[5] have apparently confirmed the predictions of the CLL for the exponent of the tunneling density of states in a sharp edge. It should be interesting to extend these studies to see if and how the exponents change as the smoothness of the edge is varied.

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