A DECOUPLED SYSTEM OF HYPERBOLIC EQUATIONS FOR LINEARIZED COSMOLOGICAL PERTURBATIONS

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A decoupled system of hyperbolic partial differential equations for linear perturbations around any spatially flat FRW universe is obtained for a wide class of perturbations. The considered perturbing energy momentum-tensors can be expressed as the sum of the perturbation of a minimally coupled scalar field plus an arbitrary (weak) energy-momentum tensor which is covariantly conserved with respect to the background. The key ingredient in obtaining the decoupling of the equations is the introduction of a new covariant gauge which plays a similar role as harmonic gauge does for perturbations around Minkowski space-time. The case of universes satysfying a linear equation of state is discussed in particular, and closed analytic expressions for the retarded Green's functions solving the de Sitter, dust and radiation dominated cases are given.

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1. Introduction

The theory of linearized perturbations around cosmological backgrounds is a cornerstone in the understanding of structure formation in the universe, because it provides the connection between what we see today, and the initial conditions in the early universe [1]. This theory has a long history, beginning (in the context of general relativity) with the pioneering work of E. Lifshitz [2]. An important step was the introduction by J. Bardeen of gauge invariant potentials to describe the metric perturbations [3]. This is the so called standard treatment of cosmological perturbations, and many authors have elaborated this approach and used it to deal with the analysis of observations [4]. The standard treatment is based on the decomposition of the metric perturbations in scalar, vector and tensor parts according to their behaviour under the rotation group of isometries of the FRW background [5]. In the standard approach these three kinds of perturbations are treated independently, and most attention has been devoted to the discussion of scalar perturbations in the particular case in which the two gauge invariant scalar potentials coincide. This particular case is well motivated because it arises naturally when dealing with hydrodynamic or scalar field perturbations. In these cases, the space-space components of the perturbing energy momentum-tensor take the form $\delta T_i^j \propto S\,\delta_i^j$, where S is a scalar, and in turn this implies the equality $\Phi = \Psi$, between the two gauge invariant Bardeen potentials for the scalar sector. An important technical consequence of this equality is that one has to deal only with a second order differential equation for Φ; in contrast with the general case of scalar perturbations, in which the standard formalism yields two coupled equations for Φ and Ψ [4].

In this letter we generalize previous results that we first obtained for a dust filled universe [6], and present a new approach to cosmological perturbations, which allows a unified treatment of all types of perturbations: scalar, vector and tensor. The method follows a very similar path to that used for computing the gravitational radiation emitted by astrophysical sources in Minkowski space-time, using harmonic gauge [8]. As a result of our approach, we obtain a decoupled system of hyperbolic partial differential equations for all components of the metric perturbation $h_{\mu\nu}$. This decoupling of equations holds for a very wide class of perturbations, which we define below, including the presence of seeds like cosmic strings or primordial black holes [7]. In addition the equations for perturbations are formulated in a covariant way, and in our opinion they are best fitted to study linearized quantum gravity on spatially flat cosmological backgrounds.

2. The background model

We choose a spatially flat FRW universe as background. This simplifies matters a little, at not a very high cost concerning physics, because the recent analysis of observations of supernova type Ia [9] and of acoustic peaks in the distribution of CMB temperature versus angular momentum [10], favour a flat universe. In addition, inflationary models of the very early universe also favour flat universes [11]. The metric of such background model written in conformal time η and spatial cartesian coordinates $\vec{x}$, reads [12]

$$ds^2 = \gamma_{\mu\nu}\,dx^\mu dx^\nu = a(\eta)^2\left(-d\eta^2 + d\vec{x}^2\right) \quad , \qquad (1)$$

where the scale factor $a(\eta)$ is an arbitrary function of

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time. Thus, the equations for perturbations that we are going to derive will apply to any flat FRW background, and in particular to multifluid universes, like our own universe. To describe the matter producing this background, we shall use a scalar field ϕ minimally coupled to gravity, i.e. we shall consider the action

$$S\left[g_{\alpha\beta},\phi\right] = -\frac{1}{16\pi\mathcal{G}}\int d^4x\,\sqrt{-g}\,R - \int d^4x\,\sqrt{-g}\left(\frac{1}{2}\,g^{\alpha\beta}\,\phi_{,\alpha}\;\phi_{,\beta}\,+V(\phi)\right) \quad (2)$$

The full metric $g_{\alpha\beta}$ and the scalar field ϕ are to be developed up to first order in perturbations, in the form $g_{\alpha\beta}=\gamma_{\alpha\beta}+h_{\alpha\beta}$, and $\phi=\bar{\phi}+\delta\phi$. The zero order parts describe the background. It is important to emphasize that this description of the background, by means of the scalar field ϕ, does not place any restriction on the scale factor $a(\eta)$. Any desired function $a(\eta)$ can be obtained by an appropriate choice of the potential $V(\phi)$. In this sense, the scalar field ϕ is not necessarily to be thought of as a fundamental field filling the universe, but as a convenient way of parametrizing the matter content of the universe at cosmological scales; as for example cold matter plus a cosmological constant, or whatever model future observations will support.

Using this scalar field parametrization of the matter, the zeroth order energy-momentum tensor can be written in the perfect fluid covariant form

$$\bar{T}_{\mu}{}^{\nu} = \left[\frac{1}{2}\,\bar{u}^{\alpha}\bar{u}^{\beta}\bar{\phi}_{,\alpha}\,\bar{\phi}_{,\beta}\,-V(\bar{\phi})\right]\delta^{\nu}_{\mu}\,+\,\bar{u}^{\alpha}\bar{u}^{\beta}\bar{\phi}_{,\alpha}\,\bar{\phi}_{,\beta}\;\bar{u}_{\mu}\bar{u}^{\nu}\quad , \quad (3)$$

where $\bar{u}^{\nu}$ is the background velocity field of the fluid. The values of the background pressure and density can be derived from (3), and the covariant conservation of $\bar{T}^{\nu}_{\mu}$, with respect to the background metric, yields the zeroth order equation of motion for the field $\bar{\phi}$

$$\bar{\phi}_{|\alpha}{}^{|\alpha} = \frac{\partial V\left(\bar{\phi}\right)}{\partial\bar{\phi}}\quad , \quad (4)$$

where the vertical bar | denotes the covariant derivative with respect to the background metric.

In coordinates $(\eta,\vec{x})$, and using a dot for derivatives with respect to conformal time η, the background values of pressure, density, and velocity take the form:

$$\bar{p} = \frac{1}{2}\,a^{-2}\,\dot{\bar{\phi}}^{\,2} - V\left(\bar{\phi}\right)\quad , \quad (5)$$

$$\bar{\rho} = \frac{1}{2}\,a^{-2}\,\dot{\bar{\phi}}^{\,2} + V\left(\bar{\phi}\right)\quad , \quad (6)$$

and

$$\bar{u}^{\nu} = a^{-1}\,\delta^{\nu}_{0}\quad . \quad (7)$$

Writting $a=\exp\Omega$, the Hubble parameter is given by

$$H = \frac{\dot{a}}{a^2} = \dot{\Omega}\,\exp(-\Omega)\quad , \quad (8)$$

the Einstein equations for the background are

$$\ddot{\Omega} - \dot{\Omega}^2 = -4\pi\mathcal{G}\,\dot{\bar{\phi}}^{\,2}\quad , \quad (9)$$

$$\ddot{\Omega} + \frac{1}{2}\,\dot{\Omega}^2 = -4\pi\mathcal{G}\left[\frac{1}{2}\,\dot{\bar{\phi}}^{\,2} - a^2 V\left(\bar{\phi}\right)\right]\quad , \quad (10)$$

and the integrability condition (4) reads

$$\ddot{\bar{\phi}} + 2\,\dot{\Omega}\,\dot{\bar{\phi}} + a^2\,\frac{\partial V\left(\bar{\phi}\right)}{\partial\bar{\phi}} = 0\quad . \quad (11)$$

For example, in the particular case of a flat FRW background with a linear equation of state $p=\alpha\,\rho$, the corresponding potential for the scalar field is

$$V(\phi) = \frac{3(1-\alpha)\,H_0^2}{16\pi\mathcal{G}}\,\exp\left(-2\sqrt{6(1+\alpha)\pi\mathcal{G}}\,(\phi-\phi_0)\right)\quad , \quad (12)$$

and the solutions of the Einstein equations for the Hubble parameter, the background scalar field $\bar{\phi}$, and the conformal Hubble parameter $\dot{\Omega}$, are

$$H(\eta) = H_0\left(\frac{\eta_0}{\eta}\right)^{\frac{3+3\alpha}{1+3\alpha}}\quad , \quad (13)$$

$$\bar{\phi} = \bar{\phi}_0 + \frac{1}{1+3\alpha}\sqrt{\frac{3+3\alpha}{2\pi\mathcal{G}}}\,\ln\frac{\eta}{\eta_0}\quad , \quad (14)$$

and

$$\dot{\Omega} = \frac{2}{(1+3\alpha)\,\eta}\quad , \quad (15)$$

the 0 label refers, as usual, to the present epoch values of the corresponding quantities, and the evolution of the density follows the critical value law $\rho=3H^2/8\pi\mathcal{G}$.

3. Structure of perturbations and gauge invariance

Now we are going to specify the class of perturbations that we will allow on the background. The equations for perturbations come out, as usual, by developing the Einstein tensor up to first order in perturbations $G_\mu^\nu = \bar{G}_\mu^\nu + \delta G_\mu^\nu$, and splitting the Einstein equations into a zeroth order part

$$\bar{G}_\mu{}^\nu = -8\pi\mathcal{G}\,\bar{T}_\mu{}^\nu \quad , \tag{16}$$

which describes how the background is created by $\bar{T}_\mu^\nu$, and a first order part

$$\delta G_\mu{}^\nu = -8\pi\mathcal{G}\,\delta T_\mu{}^\nu \quad , \tag{17}$$

which gives the equation governing perturbations.

Expansion of the Einstein tensor up to first order in peturbations yields the covariant expression

$$\begin{aligned} 2\,\delta G_\mu{}^\nu \;=\; & \psi_\mu{}^\nu{}_{|\lambda}{}^{|\lambda} - \psi_\mu{}^\lambda{}_{|\lambda}{}^{|\nu} - \psi^\nu{}_\lambda{}^{|\lambda}{}_{|\mu} + \delta_\mu^\nu\,\psi_\alpha{}^\beta{}_{|\beta}{}^{|\alpha} \\ & + 2\,\bar{R}^\nu{}_{\alpha\beta\mu}\,\psi^{\alpha\beta} + \bar{R}_\alpha{}^\nu\,\psi_\mu{}^\alpha - \bar{R}_\mu{}^\alpha\,\psi_\alpha{}^\nu \\ & + \bar{R}_\mu{}^\nu\,\psi + \delta_\mu^\nu\left[\bar{R}_{\alpha\beta}\,\psi^{\alpha\beta} - \frac{1}{2}\,\bar{R}\,\psi\right] \quad , \end{aligned} \tag{18}$$

where $\psi_\mu^\nu = h_\mu^\nu - \frac{1}{2}\delta_\mu^\nu\,h$ is the so called trace reversed graviton field. On the other hand the structure of the perturbing energy-momentum tensor $\delta T_\mu{}^\nu$ is constrained by the Bianchi identity. Expanding the full Bianchi identity to first order in perturbations, and taking into account the covariant conservation of $\bar{T}_\mu{}^\nu$ with respect to the background metric, one obtains the constraint

$$\delta T_\mu{}^\nu{}_{|\nu} + \delta\Gamma^\nu_{\nu\alpha}\,\bar{T}_\mu{}^\alpha - \delta\Gamma^\alpha_{\nu\mu}\,\bar{T}_\alpha{}^\nu = 0 \quad , \tag{19}$$

where $\delta\Gamma^\lambda_{\mu\nu}$ is the first order part of the full metric connection. The inhomogeneous equation (19) must be fulfilled by any perturbing energy momentum tensor. Thus, any perturbing δT_μ^ν can be decomposed in the form

$$\delta T_\mu^\nu = \delta T_\mu^{(I)\nu} + \delta T_\mu^{(F)\nu} \quad . \tag{20}$$

where $\delta T_\mu^{(I)\nu}$ is a particular solution of the constraint equation (19), and $\delta T_\mu^{(F)\nu}$ is any solution of the homogeneous equation

$$\delta T^{(F)\nu}_{\mu|\nu} = 0 \quad . \tag{21}$$

In addition, we will assume that $\delta T_\mu^{(F)\nu}$ does not functionally depend on the metric perturbation $h_{\mu\nu}$. We shall call $\delta T_\mu^{(I)\nu}$ the *intrinsic* perturbation, and $\delta T_\mu^{(F)\nu}$ the *free* perturbation [13].

From the physical point of view, the intrinsic perturbation $\delta T_\mu^{(I)\nu}$ corresponds to the irregularities in the matter that creates the background, i.e. to deviations of homogeneity and isotropy in the matter which produces the background metric. On the other hand, what we call the free perturbation corresponds to additional matter (more or less exotic) that can exist in the universe in addition to the main component, and which moves as test matter on the background, i. e. it fulfills equation (21). This extra matter could be, for instance, topological defects as cosmic strings or domain walls produced by phase transitions in the early universe [14], or any other seed perturbation as for example primordial black holes. But also, ordinary baryonic matter, amounting only to two or three percent of the total energy density in the present universe, can be considered as a free perturbation too. In the last case the intrinsic perturbation $\delta T_\mu^{(I)\nu}$ would correspond entirely to nonbaryonic dark matter or energy.

Now the expression for $\delta T_\mu^{(I)\nu}$ is obtained from the action (2), and in turn, it can be decomposed as

$$\delta T_\mu^{(I)\nu} = \delta T_\mu^{(h)\nu} + \delta T_\mu^{(\phi)\nu} \quad , \tag{22}$$

where $\delta T_\mu^{(h)\nu}$ and $\delta T_\mu^{(\phi)\nu}$ are the pieces linear in the metric perturbation $h_{\mu\nu}$, and in the scalar field perturbation $\delta\phi$ respectively. The covariant form of these pieces is

$$\delta T_\mu^{(h)\nu} = (\bar{p} + \bar{\rho})\left[\frac{1}{2}\,\bar{u}_\alpha\bar{u}_\beta\,h^{\alpha\beta}\,\delta_\mu^\nu - \bar{u}_\mu\bar{u}_\alpha\,h^{\alpha\nu}\right] \quad , \tag{23}$$

and

$$\begin{aligned} \delta T_\mu^{(\phi)\nu} \;=\; & \gamma^{\nu\alpha}\left[\bar{\phi}_{,\mu}\;\delta\phi_{,\alpha} + \bar{\phi}_{,\alpha}\;\delta\phi_{,\mu}\right] \\ & - \delta_\mu^\nu\left[\gamma^{\alpha\beta}\,\bar{\phi}_{,\alpha}\;\delta\phi_{,\beta} - \frac{\partial V\left(\bar{\phi}\right)}{\partial\bar{\phi}}\,\delta\phi\right] \quad . \end{aligned} \tag{24}$$

Let us discuss now how the perturbations behave under gauge transformations induced by infinitesimal coordinate transformations. Given an infinitesimal coordinate transformation $x^\mu \longrightarrow x'^\mu = x^\mu - \xi^\mu(x)$, the corresponding gauge transformations of the metric and scalar field perturbations are

$$h_{\mu\nu} \longrightarrow h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu|\nu} + \xi_{\nu|\mu} \quad , \tag{25}$$

and

$$\delta\phi \longrightarrow \delta\phi' = \delta\phi + \xi^\mu\bar{\phi}_{,\mu} \quad . \tag{26}$$

Then, the Einstein tensor perturbation and the intrinsic energy-momentum perturbation transform accordingly as

$$\delta G_\mu{}^\nu \longrightarrow \delta G'_\mu{}^\nu = \delta G_\mu{}^\nu + \mathcal{L}_\xi\,\bar{G}_\mu{}^\nu \quad . \tag{27}$$

and

$$\delta T_\mu^{(I)\nu} \longrightarrow \delta T_\mu^{(I)\prime\nu} = \delta T_\mu^{(I)\nu} + \mathcal{L}_\xi\,\bar{T}_\mu{}^\nu \quad , \tag{28}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the infinitesimal vector field ξ^μ. We have assumed, as usual, that the zeroth order parts of all tensors remain invariant in a fixed coordinate system on the background space-time manifold (for example $(\eta, \vec{x})$, but not necessarily this one), and that the effect of infinitesimal coordinate transformations is fully charged to the perturbations of tensors. Also notice that the transformation laws (27) and (28), can be directly checked by replacing the transformation laws (25) and (26) into the functional expressions (18), (23) and (24).

It is important to observe that the free perturbation $\delta T_\mu^{(F)\nu}$ is gauge invariant. This is compulsory because the total perturbation $\delta T_\mu{}^\nu$ has the same transformation law (28) as the intrinsic perturbation $\delta T_\mu^{(I)\nu}$. This happens due to the fact that the gauge transformation of $\delta T_\mu^{(F)\nu}$ is of second order in perturbations, since this energy-momentum tensor has no zeroth order counterpart. Moreover, the zeroth order Einstein equations (16) imply that the combination

$$\delta\hat{G}_\mu{}^\nu \equiv \delta G_\mu + 8\pi\mathcal{G}\,\delta T_\mu^{(I)\nu} \quad , \tag{29}$$

is gauge invariant. Therefore, by sending the intrinsic perturbation to the l.h.s. of (17), we obtain the equations

$$\delta\hat{G}_\mu{}^\nu = -\,8\pi\mathcal{G}\,\delta T_\mu^{(F)\nu} \tag{30}$$

in which both sides are gauge invariant. Thus equations (30) for cosmological perturbations are covariant with respect to finite general coordinate transformations, and gauge invariant with respect to infinitesimal coordinate changes around a fixed coordinate system on the background space-time.

Also notice that both sides of (30), are covariantly conserved with respect to the background metric.

4. Choosing the gauge and decoupling of the equations

Now we are finally going to give the covariant and gauge invariant equations (30) a nice definite form. This entails a combination of two things: choosing a convenient coordinate system on the background, and fixing the gauge. Although all gauges contain the same physics, most of them contain it in a complicated way, meaning that the independent functions which solve the equations that remain after the gauge fixing, have spurious complications related to the gauge, but not related to real complications in the physics. The situation here is similar to that of the simple system of gauge invariant Maxwell equations for the photon field $\Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = J_\mu$. In Lorentz gauge ($\partial_\nu A^\nu = 0$), one has nice solutions for these equations, but one could complicate the functional form of the solutions by choosing an inappropriate (but possible) gauge fixing condition.

In the case of the equations for cosmological perturbations (30), the appropriate gauge simplifying the equations is not so obvious as in the case of electrodynamics. The strategy that we follow is inspired by the way in which harmonic gauge works to simplify the equations for perturbations around Minkowski space-time. First, we set the covariant gauge fixing condition

$$\psi_\mu{}^\nu{}_{|\nu} = B_\mu \quad , \tag{31}$$

where the field B_μ is yet unspecified. Then, if B_μ does not depend on the derivatives of the metric perturbation $\psi_\mu{}_{|\nu}$, this condition eliminates all terms in second derivatives coming from the second, third and fourth terms in the expression (18) for $\delta G_\mu{}^\nu$. Thus, only the flat d'Alembertian remains as second order differential operator in the left hand side of (30). In addition, working in coordinates $(\eta, \vec{x})$, we fix the B^μ field to simplify the equations (30) as much as possible. After the long but straightforward exercise of writing $\delta\hat{G}_\mu{}^\nu$ given by the covariant expressions (18), (23), (24) and (29) in coordinates $(\eta, \vec{x})$, the analysis of the resulting expression shows that a great simplification is achieved by the choice

$$B_\mu = -\,2H\,\bar{u}_\nu\,\psi_\mu{}^\nu + 16\pi\mathcal{G}\,\bar{\phi}_{,\mu}\,\delta\phi \quad , \tag{32}$$

which has also been written in covariant form. Then, using the choice (32) to specify the gauge (31), the cosmological perturbations equations (30) in this gauge, and in coordinates $(\eta, \vec{x})$, take the simple form

$$\Box\psi_\mu{}^\nu - 2\,\dot{\Omega}\,\partial_\eta\psi_\mu{}^\nu + 2\ddot{\Omega}\left[\delta_\mu^0\,\delta_0^\nu\,\psi - \delta_\mu^0\,\psi_0{}^\nu - \delta_0^\nu\,\psi_\mu{}^0\right] + 32\pi\mathcal{G}\,\delta_\mu^0\,\delta_0^\nu\,\ddot{\bar{\phi}}\,\delta\phi = -16\pi\mathcal{G}\,a^2\,\delta T_\mu^{(F)\nu} \quad , \tag{33}$$

where $\Box = -\partial^2/\partial\eta^2 + \vec{\nabla}^2$ is the Minkowski d'Alembertian.

In addition the covariant conservation of $\delta T_\mu^{(F)\nu}$, when applied to equations (33) yields an equation for the perturbation $\delta\phi$, which can also be obtained by perturbing the full equation of motion for the field ϕ. This last equation can be skipped since it is implicit in the equations (33), as can be checked by direct computation and taking into account the gauge fixing condition. Instead the scalar field perturbation $\delta\phi$ can be expressed in terms of the metric perturbations through the gauge fixing condition. In coordinates $(\eta,\vec{x})$, we obtain

$$\delta\phi = \frac{1}{16\pi\mathcal{G}\,\dot{\bar{\phi}}}\left[\partial_\nu {\psi_0}^\nu + \dot{\Omega}\,(2\,\psi_0^0 - \psi)\right] \qquad (34)$$

Finally, replacing (34) in (33), these ten equations can be recast in the form

$$\left[\Box + \dot{\Omega}^2 + \ddot{\Omega}\right]\left(a\,{\psi_i}^j\right) = -\,16\pi\mathcal{G}\,a^3\,\delta T_i^{(F)j} \quad , \qquad (35)$$

$$\left[\Box + \dot{\Omega}^2 - \ddot{\Omega}\right]\left(a\,{\psi_0}^j\right) = -\,16\pi\mathcal{G}\,a^3\,\delta T_0^{(F)j} \quad , \qquad (36)$$

and

$$\left[\Box + \dot{\Omega}^2 - \ddot{\Omega} + \left(\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}}\right)^2 - \partial_\eta\left(\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}}\right)\right]\left(\frac{a}{\dot{\bar{\phi}}}\,(2{\psi_0}^0 - \psi)\right) = -16\pi\mathcal{G}\,\frac{a^3}{\dot{\bar{\phi}}}\,(2\,\delta T_0^{(F)0} - \delta T^{(F)}) - 2\,\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}^2}\,a\,(\partial_\eta{\psi_i}^i + \partial_i{\psi_0}^i) \quad . \qquad (37)$$

Equations (35), (36) and (37) are a decoupled system of hyperbolic partial differential equations for the ten components of the metric perturbation. They can be exactly integrated and their solutions expressed as retarded potentials using the retarded Green's functions for the three differential operators appearing on the left hand sides. Equations (35) and (36), for the ψ_i^j and ψ_0^i components, must be solved first because some combinations of the spatial components of the metric perturbation enter as a source in equation (37).

The decoupled system of hyperbolic equations (35), (36) and (37) is the central result of this paper. They give a unified description of all three types of metric perturbations: scalar, vector and tensor, and they have a high degree of generality since they are valid for any spatially flat FRW background and any perturbation that can be decomposed as a scalar field perturbation plus an arbitrary (weak) perturbation corresponding to geodesically moving matter on the background. The key ingredient in obtaining this system has been the appropriate selection of the gauge fixing condition given by (31) and (32). The residual gauge invariance remaining under this gauge fixing condition has been imposed is given by the vector fields ξ^μ fulfilling the equation

$$\Box\xi_\mu + 2\,\dot{\Omega}\,\partial_\eta\xi_\mu + 2\ddot{\Omega}\,\left(\xi_\mu - \delta_\mu^0\,\xi_0\right) = 0 \quad , \qquad (38)$$

that can be decomposed in the two simple equations

$$\left[\Box + \dot{\Omega}^2 - \ddot{\Omega}\right]\left(a\,\xi^0\right) = 0 \quad , \qquad (39)$$

and

$$\left[\Box + \dot{\Omega}^2 + \ddot{\Omega}\right]\left(a\,\xi^i\right) = 0 \quad , \qquad (40)$$

involving the same differential operators (and hence the same mode solutions) as (36) and (35).

In the case in which the *free* perturbation $\delta T_\mu^{(F)\nu}$ vanishes, the gauge fixing condition plus the residual gauge invariance reduce the number of physical degrees of freedom contained in the metric perturbation ${\psi_\mu}^\nu$ plus the scalar field perturbation $\delta\phi$, down two three: a scalar perturbation plus two polarizations for gravitational waves on the background.

5. Universes with a linear equation of state

To obtain a definite expression for the retarded Green's functions solving equations (35), (36) and (37), a particular background model has to be chosen. In the case of universes obeying a linear equation of state $p = \alpha\rho$, these equations take a particular simple form, and in addition the operators on the left hand side of equations (36) and (37) coincide. Thus, in this case only two Green's functions for the two differential operators

$$\Box + \dot{\Omega}^2 + \ddot{\Omega} \equiv \Box + \frac{2-6\alpha}{(1+3\alpha)^2}\,\frac{1}{\eta^2} \quad , \qquad (41)$$

and

$$\Box + \dot{\Omega}^2 - \ddot{\Omega} \equiv \Box + \frac{6+6\alpha}{(1+3\alpha)^2}\,\frac{1}{\eta^2} \quad , \qquad (42)$$

are required.

The Green's functions for these operators can be expressed as superposition of homogeneous mode solutions, which are given by Bessel functions of indices $\nu_1 = \pm(3-3\alpha)/(2+6\alpha)$ and $\nu_2 = \pm(3\alpha+5)/(2+6\alpha)$. Moreover in the particular cases of de Sitter ($\alpha = -1$) [15], dust ($\alpha = 0$) [6], and radiation ($\alpha = 1/3$) dominated cosmological models, the Bessel indices are the simplest half integers ($\nu_1 = \pm 3/2, \nu_2 = \pm 1/2$), ($\nu_1 = \pm 3/2, \nu_2 = \pm 5/2$), and ($\nu_1 = \pm 1/2, \nu_2 = \pm 3/2$) respectively.

Thus, to integrate the perturbations equations for de Sitter, dust, and radiation backgrounds we only require the three retarded Green's functions solving

$$\left(\Box + \frac{A}{\eta^2}\right) G_R^{(A)}(x,x') = -\delta^{(4)}(x-x') \quad , \qquad (43)$$

for $A = 0, 2, 6$ and where $x = (\eta, \vec{x})$.

Then $G_R^{(0)}(x,x')$ is the well known retarded Green's function for Minkowski, and $G_R^{(2)}(x,x')$ and $G_R^{(6)}(x,x')$ can be obtained by elementary QFT methods as explained in [15], with the result

$$\begin{aligned} G_R^{(2)}(x,x') &= \frac{1}{4\pi\,|\vec{x}-\vec{x'}|}\,\delta(\eta-\eta'-|\vec{x}-\vec{x'}|) \\ &\quad + \frac{1}{4\pi\,\eta\eta'}\,\theta(\eta-\eta'-|\vec{x}-\vec{x'}|) \quad , \qquad (44) \end{aligned}$$

and

$$\begin{aligned} G_R^{(6)}(x,x') &= \frac{1}{4\pi\,|\vec{x}-\vec{x'}|}\,\delta(\eta-\eta'-|\vec{x}-\vec{x'}|) \\ &\quad + \frac{3}{8\pi}\left[\frac{1}{\eta^2}+\frac{1}{\eta'^2}-\frac{|\vec{x}-\vec{x'}|^2}{\eta^2\eta'^2}\right] \\ &\quad \times\theta(\eta-\eta'-|\vec{x}-\vec{x'}|) \quad . \qquad (45) \end{aligned}$$

Thus, $G_R^{(2)}(x,x')$ and $G_R^{(6)}(x,x')$ are given by the Minkowski retarded Green's function with support on the past light cone plus an additional term with support in the interior of the past light cone. Then the metric perturbation ${\psi_\mu}^\nu$ can be expressed as retarded potentials by means of these Green's functions.

6. Concluding renarks

We have analyzed cosmological perturbations on a spatially flat FRW background given by a scalar field perturbation plus an arbitrary (weak) perturbation corresponding to test matter moving geodesically on this background. We have shown that by an apropriate choice of the gauge the cosmological perturbations obey decoupled hyperbolic equations. The class of perturbations considered have a high degree of generality, although it is not fully general since a covariantly conserved tensor, linearly dependent in the metric perturbation, could still be added to the perturbing energy-momentum tensor.

On the other hand since the treatment and the gauge fixing condition that we have introduced is covariant with respect to finite coordinate transformations on the background, it provides a very suitable framework to develop linear quantum gravity around a spatially flat FRW background.

It is important to fully elucidate the relationships of the present treatment with the standard Lifshitz-Bardeen treatment in terms of gauge invariant perturbations and the decomposition of the metric perturbation in scalar, vector and tensor parts. This task together with more details of calulations and applications are being worked out and will be the subject of a forthcomming paper.

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