General Relativistic Theory of Light Propagation in the Field of Radiative Gravitational Multipoles

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The extremely high precision of current radio/optical interferometric observations and the unparalleled sensitivity of existing (LIGO) and future (LISA, ASTROD) gravitational-wave detectors demand a much better theoretical treatment of relativistic effects in the propagation of electromagnetic signals through variable gravitational fields. Especially important for future gravitational-wave observatories is the problem of propagation of light rays in the field of multipolar gravitational waves emitted by a localized source of gravitational radiation. A consistent approach giving a complete and exhaustive solution to this problem in the first post-Minkowskian approximation of General Relativity is presented in this paper. We derive a set of equations describing propagation of an electromagnetic wave in the retarded gravitational field of a time-dependent localized source emitting gravitational waves with arbitrary multipolarity and show for the first time that they can be integrated analytically in closed form. We also prove that the leading terms in observable relativistic effects depend exclusively on the values of the multipole moments of the isolated system and its time derivatives taken at the retarded instant of time on the null cone and do not depend on their integrated values. By making use of our integration technique we reproduce the known results of integration of equations of light rays both in a stationary field of a gravitational lens and in that of a plane gravitational wave, thereby establishing a relationship between our formalism and the approximations used by previous researches. The gauge freedom of our formalism is carefully studied and all gauge-dependent terms in the expressions for observable quantities are singled out and used for physically meaningful interpretation of observations. Two limiting cases of small and large values of the light-ray impact parameter, d, are elaborated in more detail. We explicitly show that in the case of small impact parameter the leading order terms for any effect of light propagation in the field of an arbitrary multipole depend neither on its radiative nor on its intermediate zone contributions. The main effect rather comes from the near zone terms. This property makes much more difficult any direct detection of gravitational waves by astronomical techniques if general relativity is correct. We also present an analytical treatment of time delay and light-ray bending in large impact parameter case corresponding to the approximation of a plane gravitational wave of arbitrary multipolarity. Explicit expressions for time delay and deflection angle are obtained in terms of the transverse-traceless (TT) part of the space-space components of the metric tensor.

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1. NOTATIONS AND CONVENTIONS

Metric tensor on the space-time manifold is denoted by $g_{\alpha\beta}$ and its perturbation $h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$. The determinant of the metric tensor is negative and is denoted as $g = \det||g_{\alpha\beta}||$. The four-dimensional fully antisymmetric Levi-Civita symbol $\epsilon_{\alpha\beta\gamma\delta}$ is defined in accordance with the convention $\epsilon_{0123} = +1$.

In the present paper we use a geometrodynamic system of units [1–4] such that the fundamental speed, c, and the universal gravitational constant, G, are equal to unity, that is $c = G = 1$. Space-time is assumed to be globally asymptotically-flat and covered by a single coordinate system $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, where t and (x, y, z) are time and space coordinates respectively. This coordinate system is reduced at infinity to the Minkowskian coordinates defined up to a global Lorentz-Poincare transformation [5]. We shall also use the spherical coordinates (r, θ, ϕ) related to (x, y, z) by the standard transformation

$$x = r\sin\theta\cos\phi\ , \quad y = r\sin\theta\sin\phi\ , \quad z = r\cos\theta\ . \tag{1.1}$$

Greek (spacetime) indices range from 0 to 3, and Latin (space) indices run from 1 to 3. If not specifically stated the opposite, the indices are raised and lowered by means of the Minkowski metric $\eta_{\alpha\beta} \equiv \mathrm{diag}(-1, 1, 1, 1)$. Regarding this rule the following conventions for coordinates hold: $x^i = x_i$ and $x^0 = -x_0$. We also adopt notations, $\delta_{ij} \equiv \mathrm{diag}(1, 1, 1)$, for the Kroneker symbol (a unit matrix), and, ϵ_{ijk}, for the fully antisymmetric 3-dimensional symbol of Levi-Civita with convention $\epsilon_{123} = +1$.

Repeated indices are summed over in accordance with the Einstein's rule [1–3]. In the linearized approximation of general relativity used in the present paper there is no difference between spatial vectors and co-vectors as well as between upper and lower space indices. Therefore, for a dot product of two space vectors we have

$$A^iB_i = A_iB_i \equiv A_1B_1 + A_2B_2 + A_3B_3\ . \tag{1.2}$$

In what follows, we shall commonly use spatial multi-index notations for Cartesian three-dimensional tensors [6], that is

$$\mathcal{I}_{A_l} \equiv \mathcal{I}_{a_1\dots a_l}. \tag{1.3}$$

Tensor product of l identical spatial vectors k^i will be denoted as a three-dimensional tensor having l indices

$$k_{a_1}\dots k_{a_l} \equiv k_{a_1\dots a_l}\ . \tag{1.4}$$

Full symmetrization with respect to a group of spatial indices of a Cartesian tensor will be distinguished with round brackets put around the indices

$$Q_{(a_1\dots a_l)} \equiv \frac{1}{l!}\sum_\sigma Q_{\sigma(1)\dots\sigma(l)}\ , \tag{1.5}$$

where σ is the set of all permutations of $(1, 2, ..., l)$ which makes $Q_{a_1\dots a_l}$ fully symmetrical in $a_1 \dots a_l$.

It is convenient to introduce a special notation for symmetric trace-free (STF) Cartesian tensors by making use of angular brackets around STF indices. The explicit expression of the STF part of a tensor $Q_{a_1\dots a_l}$ is [6]

$$Q_{<a_1\dots a_l>} \equiv \sum_{k=0}^{[l/2]} a_k^l \delta_{(a_1a_2}\cdots\delta_{a_{2k-1}a_{2k}}S_{a_{2k+1}\dots a_l)b_1b_1\dots b_kb_k}\ , \tag{1.6}$$

where $[l/2]$ is the integer part of $l/2$,

$$S_{a_1\dots a_l} \equiv Q_{(a_1\dots a_l)} \tag{1.7}$$

and numerical coefficients

$$a_k^l = \frac{(-1)^k}{(2k)!!}\frac{l!}{(2l-1)!!}\frac{(2l-2k-1)!!}{(l-2k)!}\ . \tag{1.8}$$

We also assume that for any integer $l \geq 0$

$$l! \equiv l(l-1)\dots 2\cdot 1\ , \qquad 0! \equiv 1\ , \tag{1.9}$$

and

$$l!! \equiv l(l-2)(l-4)\dots(2\ \text{or}1)\ , \qquad 0!! \equiv 1\ . \tag{1.10}$$

One has, for example,

$$T_{<abc>} \equiv T_{(abc)} - \frac{1}{5}\delta_{ab}T_{(cjj)} - \frac{1}{5}\delta_{bc}T_{(ajj)} - \frac{1}{5}\delta_{ac}T_{(bjj)}\ . \tag{1.11}$$

Cartesian tensors of the mass-type (mass) multipoles $\mathcal{I}_{<A_l>}$ and spin-type (spin) multipoles $\mathcal{S}_{<A_l>}$ entirely describing gravitational field outside of an isolated astronomical system are always STF objects that can be checked by inspection of the definition following from the multipolar decomposition of the metric tensor perturbation $h_{\alpha\beta}$ [6]. For this reason, to avoid appearance of too complicated index notations we shall omit in the following text the angular brackets around indices of these (and only these) tensors, that is we adopt: $\mathcal{I}_{A_l} \equiv \mathcal{I}_{<A_l>}$ and $\mathcal{S}_{A_l} \equiv \mathcal{S}_{<A_l>}$.

We shall also use transverse (T) and transverse-traceless (TT) Cartesian tensors in our calculations [2, 6]. These objects are defined by making use of the operator of projection $P_{jk} \equiv \delta_{jk} - k_{jk}$ onto the plane orthogonal to a unit vector k_j. Thus, one has [6, 7]

$$Q^{\mathrm{T}}_{a_1\dots a_l} \equiv P_{a_1b_1}P_{a_2b_2}...P_{a_lb_l}Q_{b_1\dots b_l}\ , \tag{1.12}$$

$$Q^{\mathrm{TT}}_{a_1\dots a_l} \equiv \sum_{k=0}^{[l/2]} b^l_k P_{(a_1a_2}\cdots P_{a_{2k-1}a_{2k}}W_{a_{2k+1}\dots a_l)b_1b_1\dots b_kb_k}\ , \tag{1.13}$$

where again $[l/2]$ is the integer part of $l/2$,

$$W_{a_1\dots a_l} \equiv Q^{\mathrm{T}}_{(a_1\dots a_l)} \tag{1.14}$$

and numerical coefficients

$$b^l_k = \frac{(-1)^k}{4^k}\frac{l(l-k-1)!!}{k!(l-2k)!}\ . \tag{1.15}$$

For instance,

$$Q^{\mathrm{TT}}_{ab} \equiv \frac{1}{2}\left(P_{ai}P_{bj}Q_{ij} + P_{bi}P_{aj}Q_{ij}\right) - \frac{1}{2}P_{ab}\left(P_{jk}Q_{jk}\right)\ . \tag{1.16}$$

Polynomial coefficients will be used in some of our equations and they are defined by

$$C_l(p_1,\dots,p_n) \equiv \frac{l!}{p_1!\dots p_n!}\ , \tag{1.17}$$

where l and p_i are positive integers such that $\sum_{i=1}^n p_i = l$. We introduce a Heaviside unit step function $H(p-q)$ such that on the set of whole numbers

$$H(p-q) = \begin{cases} 0, & \text{if } p \le q, \\ 1, & \text{if } p > q. \end{cases} \tag{1.18}$$

For any differentiable function $f = f(t,\boldsymbol{x})$ one uses notations: $f_{,0} = \partial f/\partial t$ and $f_{,i} = \partial f/\partial x^i$ for its partial derivatives. In general, comma standing after the function denotes a partial derivative with respect to a corresponding coordinate: $f_{,\alpha} \equiv \partial f(x)/\partial x^\alpha$. Overdot denotes a total derivative with respect to time $\dot f \equiv df/dt = \partial f/\partial t + \dot x^i \partial f/\partial x^i$ usually taken in this paper along the light ray trajectory $\boldsymbol{x}(t)$. Sometimes the partial derivatives with respect to space coordinate x^i will be also denoted as $\partial_i \equiv \partial/\partial x^i$, and the partial time derivative will be denoted as $\partial_t \equiv \partial/\partial t$. Covariant derivative with respect to the coordinate x^α will be denoted as ∇_α.

We shall use special notations for integrals with respect to time and for those taken along the light-ray trajectory. Specifically, the time integrals from a function $F(t,\boldsymbol{x})$, where $\boldsymbol{x}$ is not specified, are denoted as

$$F^{(-1)}(t,\boldsymbol{x}) \equiv \int_{-\infty}^{t} F(\tau,\boldsymbol{x})d\tau\ , \qquad F^{(-2)}(t,\boldsymbol{x}) \equiv \int_{-\infty}^{t} F^{(-1)}(\tau,\boldsymbol{x})d\tau\ . \tag{1.19}$$

Time integrals from the function $F(t,\boldsymbol{x})$ taken along the light ray so that spatial coordinate $\boldsymbol{x}$ is a function of time $\boldsymbol{x} \equiv \boldsymbol{x}(t)$, are denoted as

$$F^{[-1]}(t,\boldsymbol{r}) \equiv \int_{-\infty}^{t} F(\tau,\boldsymbol{r}(\tau))d\tau, \qquad\qquad F^{[-2]}(t,\boldsymbol{r}) \equiv \int_{-\infty}^{t} F^{[-1]}(\tau,\boldsymbol{r}(\tau))d\tau\ . \tag{1.20}$$

Integrals in equations (1.19) represent functions of time and space coordinates. Integrals in equations (1.20) are defined on the light-ray trajectory and are functions of time only.

Multiple time derivative from function $F(t,\boldsymbol{x})$ is denoted by

$$F^{(p)}(t,\boldsymbol{x}) = \frac{\partial^p F(t,\boldsymbol{x})}{\partial t^p}\ , \tag{1.21}$$

so that its action on the time integrals eliminates integration in the sense that

$$F^{(p)}(t,\boldsymbol{x}) = \frac{\partial^{p+1} F^{(-1)}(t,\boldsymbol{x})}{\partial t^{p+1}} = \frac{\partial^{p+2} F^{(-2)}(t,\boldsymbol{x})}{\partial t^{p+2}}\ , \tag{1.22}$$

and

$$F^{[p]}(t,\boldsymbol{x}) = \frac{d^{p+1} F^{[-1]}(t,\boldsymbol{x})}{dt^{p+1}} = \frac{d^{p+2} F^{[-2]}(t,\boldsymbol{x})}{dt^{p+2}}\ . \tag{1.23}$$

Spatial vectors will be denoted by bold italic letters, for instance $A^i \equiv \boldsymbol{A}$, $k^i \equiv \boldsymbol{k}$, etc. The Euclidean dot product between two spatial vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted with dot between them: $a^i b_i = \boldsymbol{a}\cdot\boldsymbol{b}$. The Euclidean wedge (cross) product between two vectors is denoted with symbol $\times$, that is $\epsilon_{ijk}a^j b^k = (\boldsymbol{a}\times\boldsymbol{b})^i$. Other particular notations will be introduced as they appear in the text.

2. INTRODUCTION

2.1. Statement of the Problem

Direct experimental detection of gravitational waves is a fascinating, yet unsolved problem of modern fundamental physics [8–12]. Enormous efforts are applied to make a progress in its solution both by theorists and experimentalists. The main theoretical efforts are presently focused on calculation of templates of the gravitational waves emitted by coalescing binary systems comprised of neutron stars and/or black holes [13–18] as well as creation of improved filtering technique for gravitational wave detectors [19, 20] which will enable one to extract the gravitational wave signal from all kind of interferences present in the data collected by the gravitational wave observatories. Direct experimental efforts have led to the construction of several ground-based optical interferometers [21–25] with arms reaching few miles. Certain work is under way to build super-sensitive cryogenic-bar gravitational-wave detectors of Weber's type [26–29]. Laser interferometric space projects such as LISA [30] and ASTROD [31] aimed to significantly increase the sensitivity of the gravitational-wave detectors are currently under intensive discussion at NASA, ESA and China.

No doubt, detection of gravitational waves by the specialized gravitational wave antennas would give the most direct evidence of the existence of these as yet elusive ripples of the space-time. At the same time it is conceivable that there exist a number of astronomical events which observation with a significantly improved astronomical technique might be used for indirect detection of gravitational waves as well as the extension of our knowledge of the conditions existing in a very early universe and of physics of the astrophysical systems emitting gravitational waves. It is worth emphasizing that the specialized ground-based and spaceborn gravitational wave detectors are sensitive to a limited band of gravitational wave frequencies ranging from 10^4 Hz down to 1 Hz [31–33]. Hence, they can not tell us anything about the astrophysical effects produced by time-dependent gravitational fields having no radiative character but still playing an important role in the near and/or intermediate zones of isolated gravitating systems. Furthermore, ultra-long gravitational waves with frequencies much less than 1 Hz can be generated by localized gravitational sources in the early universe [34–38] and by ensembles of binary systems in our and other galaxies thus forming a statistical background with a reasonably large energy density [39–41]. Near-zone time-dependent perturbations of the gravitational field and the ultralong gravitational waves can be studied only via astronomical observations (for example, pulsar timing, radiometry of cosmic microwave background radiation, etc.) which provide us with an additional window for exploring different properties of the gravitational field thereby extending the frequency range of the gravitational wave astronomy to all plausible frequencies [32, 33].

Any astronomical observation is done with electromagnetic waves (photons) of different frequencies with an overall spectrum spreading out from radio waves to gamma rays. Therefore, in order to detect the effects of the gravitational field by astronomical technique one has to elaborate on the problem of propagation of electromagnetic waves through this field generated by a localized gravitationally-bounded distribution of masses moving with velocities and accelerations determined from the equations of motion of a particular theory of gravity [42]. Equations of propagation of electromagnetic signals must be derived in the framework of the same theory of gravity in order to retain description of the gravitational and electromagnetic phenomena on a self-compatible ground [43]. In the present paper we rely upon the Einstein theory of general relativity (GR) and assume that light propagates in vacuum, that is the interstellar medium has no impact on the light propagation.

General Relativity is a geometrized theory of gravity and it assumes that both gravity and electromagnetic field propagate locally in vacuum with the same speed which is equal to the fundamental speed of special theory of relativity [1–3]. Gravitational field is found as a solution of the Einstein field equations. The electromagnetic field is obtained from the Maxwell equations on a curved space-time derived in accordance with the principle of equivalence [1–3] which determine how the affine connection $\Gamma^{\alpha}_{\mu\nu}$ is related to the metric tensor $g_{\mu\nu}$. In the approximation of geometric optics [1–3] electromagnetic signals propagate along null geodesics of the metric tensor [1–3].

The problem of finding solutions of the null ray equations in curved space-time attracted many researchers since the time Einstein discovered the field equations of General Relativity. Exact solution of this problem was found in many, particularly simple cases of space-time possessing specific symmetries like the Schwarzschild or Kerr black holes, homogeneous and isotropic Friedman-Robertson-Walker cosmological model, plane gravitational wave, etc. [1–3, 44]. These exact description of null geodesics was (and in some cases still is) satisfactory for purposes of experimental gravitational physics for quite a long time. However, real physical space-time has no symmetries and solution of such current problems as direct detection of gravitational waves, interpretation of an anisotropy and polarization of cosmic microwave background radiation (CMBR), exploration of inflationary models of the early universe, finding new experimental evidences in support of general relativity and quantum gravity, development of higher-precision relativistic algorithms for space missions, and many others, can not fully rely upon the mathematical techniques developed exceptionally for dealing with symmetric space-times. Especially important is to find out a method of integration of equations of light geodesics in space-times having time-dependent gravitational perturbations.

In the present paper we consider a general case of an asymptotically-flat space-time to which an isolated astronomical system is embedded. We assume that gravitational field of this system has a whole set of gravitational multipoles emitting gravitational waves [45]. Precise mathematical definition of the asymptotic flatness of space-time based on the concept of conformal infinity was worked out in [46–51] (see [3] for a comprehensive mathematical introduction to this subject). This definition implies conformal transformation technique and Hansen's multipole moments [52] which work perfectly in stationary space-times but are not applicable for analysis of real astronomical observations and for interpretation of relativistic effects in propagation of electromagnetic signals in time-dependent gravitational fields. We are going to study linear effects of gravitational multipoles on light propagation so that relativistic effects produced by non-linearity of the gravitational field will be ignored. Linearized approximation of general theory of relativity represents a straightforward approach for description of the multipolar structure of the gravitational field of a localized astronomical system. Multipolar formalism was developed by Thorne [6] and further extended and elucidated in [53–57].

Exact formulation of the problem under discussion in this paper is as follows (see Figure 1 for more detail). The space-time is covered by a single coordinate system $x^{\alpha} = (ct, \boldsymbol{x})$, where t is time and $\boldsymbol{x}$ are space coordinates. An electromagnetic wave (photon) is emitted by a source of light at time t_0 from a point $\boldsymbol{x}_0$ towards observer which receives the photon at time t at the point $\boldsymbol{x}$. Photon propagates through the time-dependent gravitational field of an isolated astronomical system emitting gravitational waves. The structure of the gravitational field is described by a set of multipole moments of the system which are functions of the retarded time $t - r/c$ where r is the distance from the isolated system to the field point and c is the fundamental speed. The retardation is due to the finite speed of gravity which coincides in general relativity with the speed of light in vacuum [58]. Our problem is to find out relationships connecting various physical parameters (direction of propagation, frequency, polarization, intensity, etc.) of the electromagnetic signal at the point of emission with those measured by the observer. Gravitational field affects propagation of the electromagnetic signal and changes its parameters along the light ray. Observing these changes allow us to study properties of the gravitational field of the astronomical system and to detect gravitational waves with astronomical technique. The rest of the paper is devoted to the mathematical solution of this problem.

2.2. Historical Background

Propagation of electromagnetic signals through static and/or stationary gravitational field is rather well-known subject. Presently almost any standard textbook on relativity describes solution of the null geodesic equations in the field of a spherically-symmetric and rotating body. This solution is practical since it is used, for example, for interpretation of gravitational lensing events taking place in our galaxy and/or in cosmology [59–61]. However, continually growing accuracy of astronomical observations demands much better treatment of secondary effects in the propagation of light produced by the perturbations of the gravitational field associated with higher-order gravitational multipoles. Time-dependent multipoles emit gravitational waves which perturb propagation of light. Calculation of these perturbations is important for developing correct strategy for searching gravitational waves by making use of astronomical technique.

Among the most interesting sources of periodic gravitational waves with a well-predicted behavior are binary systems comprised of two stars orbiting each other around common barycenter. Indirect evidence in support of existence of gravitational waves emitted by binary pulsars was given by Taylor with collaborators [62, 63]. However, direct observation of gravitational waves remains an unsolved problem of experimental gravitational physics as yet. The expected spectrum of gravitational waves extends from $\sim 10^4$ Hz to 10^{-18} Hz [32, 33]. Within that range, the spectrum of periodic waves from known binary systems extends from about 10^{-3} Hz – the frequency of gravitational radiation from a contact white-dwarf binary [39], through the 10^{-4} to 10^{-6} Hz – the range of radiation from the main-sequence binaries [40], to the 10^{-7} to 10^{-9} Hz – the frequencies emitted by binary supermassive black holes presumably lurking in active galactic nuclei [41]. The dimensionless strain of these waves at the Earth, h, may be as great as 10^{-21} at the highest frequencies, and as great as 3×10^{-15} at the lowest frequencies in the spectrum of gravitational waves [32, 33].

Sazhin [64, 65] first suggested detection of gravitational waves from a binary system using timing observations of a pulsar, the line of sight to which passes near the binary. He had shown that the integrated time delay for propagation of the electromagnetic pulse near the binary is proportional to $1/d^2$ where d is the impact parameter of the unperturbed trajectory of the signal. More recently, Sazhin & Saphonova [66] have made estimates of the probability of observation of this effect for pulsars in globular clusters and showed that the probability can be high, reaching 0.97. Wahlquist [67] proposed another approach to the detection of periodic gravitational waves based on Doppler tracking of spacecraft travelling in space. His approach is restricted by the plane gravitational wave approximation developed earlier by Estabrook & Wahlquist [68]. Tinto ([69], and references therein) made the most recent theoretical contribution in this area. The Doppler tracking technique is used in space missions, by seeking the characteristic triple signature, the presence of which would reveal the influence of a gravitational wave crossing the line of sight from spacecraft to observer [70].

Quite recently, Braginsky *et al.* [71, 72] have raised a question about using very-long baseline radio interferometer as a detector of stochastic gravitational waves. This idea has also been investigated by Kaiser & Jaffe [73] and, in particular, by Pyne *et al.* [74] and Gwinn *et al.* [75] who showed that the overall effect is proportional to the strain of the metric perturbation caused by the plane gravitational wave. They also set an observational limit on the energy density of ultra-long gravitational waves present in early universe. Montanari [76] studied perturbations of polarization of electromagnetic radiation propagating in the field of a plane gravitational wave and found that the effects are exceedingly small.

Fakir ([77, 78], and references therein) has suggested to use astrometry to detect periodic variations in apparent angular separations of appropriate light sources, caused by gravitational waves emitted by isolated sources of gravitational radiation. He was not able to develop a self-consistent approach to tackle the problem with necessary completeness and rigor. For this reason his estimate of the effect is too optimistic. Another attempt to work out a more consistent approach to the calculation of the light deflection angle by the field of arbitrary source of gravitational waves has been undertaken by Durrer [79]. However, the calculations have been done only for the plane wave approximation and the result obtained was extrapolated for the case of the localized source of gravitational waves without justification. For this reason the magnitude of the light deflection angle was overestimated. The same misinterpretation of the effect can be found in the paper by Labeyrie [80] who studied a photometric modulation of background sources by gravitational waves emitted by fast binary stars. Because of this, the expected detection of the gravitational waves from the observations of the radio source GPS QSO 2022+171 undertaken by Pogrebenko *et al.* [81] was not based on firm theoretical ground and did not lead to success.

Damour & Esposito-Farèse [82] have studied the deflection of light and integrated time delay caused by the time-dependent gravitational field generated by a localized material source lying on the null direction being close to the line of sight. They worked in a quadrupole approximation and explicitly calculated effects of the full, retarded gravitational field in the near, intermediate, and wave zones by making use of the Fourier-decomposition technique. Contrary to the claims of Fakir [77, 78] and Durrer [79] and in agreement with Sazhin's [64] calculations, they found that the deflections due to both the wave-zone gravitational wave and the intermediate-zone retarded fields vanish exactly. The leading total time-dependent deflection is given only by the quasi-static, near-zone quadrupolar piece of the gravitational field.

Damour and Esposito-Farese [82] analyzed propagation of light in the quadrupolar field of the isolated system emitting gravitational waves under simplifying condition that the impact parameter of the light ray is small with respect to the distances from both observer and the source of light to the isolated system. We have found [7, 83] another way to solve the problem of propagation of electromagnetic waves in the quadrupolar field of the gravitational waves emitted by the system without making any assumptions on mutual disposition of observer, source of light, and the system, thus, drastically overpowering the result of Damour's paper [84]. At the same time the paper [83] did not answer the question about the impact of other higher-order gravitational multipoles of the isolated system on the process of propagation of electromagnetic signals. This might be important if effective emission of an octupole and/or higher multipolar gravitational radiation is equal or even exceeds that of the quadrupole as it may be in case of, for example, highly asymmetric star collapse [85], nearly head-on collision of two stars, or break-up of a binary system caused by recoil of two black holes [86].

In the present paper we work out a systematic approach to the problem of propagation of light rays in the field of arbitrary gravitational multipole. While the most papers on light propagation consider both a light source and an observer as being located at infinity we do not need these assumptions. For this reason our approach is more general and applicable for any disposition of the source of light and observer with respect to the source of gravitational radiation. The integration technique which we use for finding solution of the equations of propagation of light rays was invented in [7, 83].

The metric tensor and coordinate systems involved in our calculations are described in section 3 along with gauge conditions imposed on the metric tensor. The equations of propagation of electromagnetic waves in the geometric optics approximation are discussed in section 4 and the method of their integration is given in section 5.

Exact solution of the equations of light propagation and the form of relativistic perturbations of the light trajectory are obtained in section 6. Section 7 is devoted to derivation of basic observable relativistic effects - the integrated time delay, the deflection angle, the shift of frequency, and rotation of the plain of polarization. We discuss in sections 8 and 9 two limiting cases of the relative configurations of the source of light, the observer, and the source of gravitational waves – the gravitational-lens configuration and the case of a plane gravitational wave. Appendix gives approximate expressions for the Christoffel symbols, for the metric tensor taken on the light-ray trajectory, and for the gauge functions.

3. THE METRIC TENSOR, GAUGES, AND COORDINATES

3.1. The Canonical Form of the Metric Tensor Perturbation

We consider an isolated astronomical system emitting gravitational waves and assume that gravitational field is weak everywhere so that the metric tensor is expanded in a Taylor series with respect to the powers of gravitational constant G. The metric tensor is a linear combination of the Minkowski metric, $\eta_{\alpha\beta}$, and a small perturbation $h_{\alpha\beta}$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + Gh_{\alpha\beta} + O(G^2) \ , \tag{3.1}$$

$$g^{\alpha\beta} = \eta_{\alpha\beta} - Gh^{\alpha\beta} + O(G^2) \ , \tag{3.2}$$

where $h_{\alpha\beta} \ll 1$ and we use $\eta_{\alpha\beta}$ to rise and lower indices so that, for example, $h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$. For computational convenience and simplification of physical interpretation of our results, we put the origin of the coordinate system to the center of mass of the isolated astronomical system under consideration [87].

The most general expression for the linearized perturbation of the metric tensor outside of the astronomical system under consideration was given by Thorne [6] (see also [53]) in terms of its symmetric and trace-free (STF) mass and spin multipole moments. This perturbation is described by the following expression

$$h_{\alpha\beta} = h^{\rm can.}_{\alpha\beta} + w_{\alpha,\beta} + w_{\beta,\alpha} \ , \tag{3.3}$$

where w_α are, the so-called, gauge functions. The *canonical* perturbation $h^{\rm can.}_{\alpha\beta}$ obeys to vacuum wave equation

$$\Box\, h^{\rm can.}_{\alpha\beta} = 0 \ , \tag{3.4}$$

which solution is chosen as

$$h_{00}^{\rm can.} = \frac{2\mathcal{M}}{r} + 2\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\mathcal{I}_{A_l}(t-r)}{r}\right]_{,A_l} , \tag{3.5}$$

$$h_{0i}^{\rm can.} = -\frac{2\epsilon_{ipq}\mathcal{S}_p(t-r)N_q}{r^2} - 4\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\left[\frac{\epsilon_{ipq}\mathcal{S}_{pA_{l-1}}(t-r)}{r}\right]_{,qA_{l-1}} + \tag{3.6}$$

$$4\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\dot{\mathcal{I}}_{iA_{l-1}}(t-r)}{r}\right]_{,A_{l-1}} ,$$

$$h_{ij}^{\rm can.} = \delta_{ij}h_{00}^{\rm can.} + q_{ij}^{\rm can.} , \tag{3.7}$$

$$q_{ij}^{\rm can.} = 4\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\ddot{\mathcal{I}}_{ijA_{l-2}}(t-r)}{r}\right]_{,A_{l-2}} - 8\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\left[\frac{\epsilon_{pq(i}\dot{\mathcal{S}}_{j)pA_{l-2}}(t-r)}{r}\right]_{,qA_{l-2}} . \tag{3.8}$$

Here $\mathcal{M}$ and $\mathcal{S}_i$ are total mass and spin of the system, and $\mathcal{I}_{A_l}$ and $\mathcal{S}_{A_l}$ are two independent sets of mass-type and spin-type multipole moments, $N^i = x^i/r$ is a unit vector directed from the origin of the coordinate system to the field point. Because the origin of the coordinate system has been chosen at the center of mass, the expansions (3.5) – (3.8) do not depend on the mass-type linear multipole moment $\mathcal{I}_i$ which is equal to zero by definition. We emphasize that in the linearized approximation the total mass $\mathcal{M}$ and spin $\mathcal{S}_i$ of the astronomical system are considered as constant while all other multipoles are arbitrary functions of time. It is well established however [1–3] that gravitational waves emitted away by the system reduce its energy as well as its linear and angular momenta. Hence, in higher order approximations $\mathcal{I}_i \neq 0$, and the mass, spin, and linear momentum of the system must be considered as functions of time like any other multipole.

The *canonical* metric tensor (3.5)–(3.8) depends on the multipole moments $\mathcal{I}_{A_l}(t-r)$ and $\mathcal{S}_{A_l}(t-r)$ taken at the retarded instant of time. The retardation is explained by the finite speed of propagation of gravity. In the near zone of the isolated system the retardation due to the propagation of gravity is small and all functions of time in the metric tensor can be expanded in Taylor series around time t [6, 53]. This near-zone expansion of the metric tensor can be smoothly matched to the solution of the linearized Einstein equations in the domain of space being occupied by matter. The matching helps to express the multipole moments in terms of matter variables [54]

$$\sigma \equiv T^{00} + T^{kk} , \qquad \sigma^i \equiv T^{0i} . \tag{3.9}$$

In the first post-Newtonian approximation the multipole moments have a matter-compact support [54]

$$\mathcal{I}_{A_l}^{\rm 1PN} = \int_V d^3\boldsymbol{x}\left\{x_{<A_l>}\sigma + \frac{|\boldsymbol{x}|^2 x_{<A_l>}}{2(2l+3)}\partial_t^2\sigma - \frac{4(2l+1)x_{<iA_l>}}{(l+1)(2l+3)}\partial_t\sigma_i\right\} + O(c^{-4}) , \tag{3.10}$$

$$\mathcal{S}_{A_l}^{\rm 1PN} = \int_V d^3\boldsymbol{x}\epsilon_{pq<a_l}x_{A_{l-1}p>}\sigma_q + O(c^{-2}) . \tag{3.11}$$

In the higher post-Newtonian approximations the multipole moments have no matter-compact support and are expressed by means of more complicated functionals [88]. Radiative approximation of the *canonical* metric tensor reveals that a contribution of tails of gravitational waves must be added to the definitions of the multipole moments (3.10), (3.11). In other words, the multipole moments in the radiative zone of the isolated system read [55, 56, 89]

$$\mathcal{I}_{A_l} = \mathcal{I}_{A_l}^{\rm 1PN} + 2G\mathcal{M}\int_0^{+\infty}d\zeta\dddot{\mathcal{M}}_{A_l}^{\rm 1PN}(t-r-\zeta)\left[\ln\left(\frac{\zeta}{2b}\right) + \frac{2l^2+5l+4}{l(l+1)(l+2)} + \sum_{k=1}^{l-2}\frac{1}{k}\right] , \tag{3.12}$$

$$\mathcal{S}_{A_l} = \mathcal{S}_{A_l}^{\rm 1PN} + 2G\mathcal{M}\int_0^{+\infty}d\zeta\ddot{\mathcal{S}}_{A_l}^{\rm 1PN}(t-r-\zeta)\left[\ln\left(\frac{\zeta}{2b}\right) + \frac{l-1}{l(l+1)} + \sum_{k=1}^{l-1}\frac{1}{k}\right] , \tag{3.13}$$

where b is a normalization constant which value is supposed to be absorbed to the definition of origin of time scale in the radiative zone but this statement has not been checked so far.

3.2. The Harmonic Coordinates

Equation (3.3) holds in an arbitrary gauge imposed on the metric tensor. The harmonic gauge is defined by the condition

$$2h^{\alpha\beta}_{\ \ ,\beta} - h^{,\alpha} = 0\ , \tag{3.14}$$

where $h \equiv h^{\alpha}_{\ \alpha}$. The gauge condition (3.14) reduces the Einstein vacuum field equations to the wave equations on gravitational potentials $h^{\alpha\beta}$ (see equation (3.4)). Harmonic coordinates x^α are defined as solutions of the homogeneous wave equation $\Box x^\alpha = 0$ up to the gauge functions w^α. In particular, the harmonic *canonical* coordinates are defined by the condition that all gauge functions $w_\alpha = 0$. The *canonical* metric tensor (3.5) – (3.8) depends on only two sets of multipole moments [6, 53] which reflects existence of only two degrees of freedom of free (detached from matter) gravitational field in general relativity [1–3]. At the same time one can obtain a generic expression for the harmonic metric tensor by making use of infinitesimal coordinate transformation

$$x'^\alpha = x^\alpha - w^\alpha \tag{3.15}$$

from the *canonical harmonic* coordinates x^α to arbitrary harmonic coordinates x'^α with the harmonic gauge functions w^α which satisfy to a homogeneous wave equation

$$\Box w^\alpha = 0\ . \tag{3.16}$$

The most general solution of this equation contains four sets of STF multipoles [6, 53]

$$w^0 \ = \ \sum_{l=0}^{\infty}\left[\frac{\mathcal{W}_{A_l}(t-r)}{r}\right]_{,A_l}\ , \tag{3.17}$$

$$w^i \ = \ \sum_{l=0}^{\infty}\left[\frac{\mathcal{X}_{A_l}(t-r)}{r}\right]_{,iA_l} + \sum_{l=1}^{\infty}\left\{\left[\frac{\mathcal{Y}_{iA_{l-1}}(t-r)}{r}\right]_{,A_{l-1}} + \left[\epsilon_{ipq}\frac{\mathcal{Z}_{qA_{l-1}}(t-r)}{r}\right]_{,pA_{l-1}}\right\}\ , \tag{3.18}$$

where $\mathcal{W}_{A_l}$, $\mathcal{X}_{A_l}$, $\mathcal{Y}_{iA_{l-1}}$, and $\mathcal{Z}_{qA_{l-1}}$ are Cartesian tensors depending on the retarded time. Their specific form is a matter of computational convenience for derivation and interpretation of observable effects but it does not affect invariant quantities like the phase of electromagnetic wave propagating in the field of the multipoles.

The most convenient choice simplifying the form of the metric tensor perturbations, is given by the following gauge functions

$$w^0 = \sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\mathcal{I}^{(-1)}_{A_l}(t-r)}{r}\right]_{,A_l}\ , \tag{3.19}$$

$$\begin{aligned} w^i = &\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\mathcal{I}^{(-2)}_{A_l}(t-r)}{r}\right]_{,iA_l} - \\ &4\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\mathcal{I}_{iA_{l-1}}(t-r)}{r}\right]_{,A_{l-1}} + \\ &4\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\left[\frac{\epsilon_{iba}\mathcal{S}^{(-1)}_{bA_{l-1}}(t-r)}{r}\right]_{,aA_{l-1}}\ . \end{aligned} \tag{3.20}$$

These functions substituted to equation (3.3) transform the *canonical* metric tensor perturbation to another, remarkably simple form

$$h_{00} \ = \ \frac{2\mathcal{M}}{r}\ , \tag{3.21}$$

$$h_{0i} \ = \ -\frac{2\epsilon_{ipq}\mathcal{S}_p N_q}{r^2}\ , \tag{3.22}$$

$$h_{ij} \ = \ \delta_{ij}h_{00} + h^{TT}_{ij}\ , \tag{3.23}$$

$$h^{TT}_{ij} \ = \ P_{ijkl}q^{\rm can.}_{kl}\ , \tag{3.24}$$

where the TT-projection differential operator P_{ijkl}, applied to symmetric tensors depending on both time and spatial coordinates, is given by

$$P_{ijkl} = (\delta_{ik} - \Delta^{-1}\partial_i\partial_k)(\delta_{jl} - \Delta^{-1}\partial_j\partial_l) - \frac{1}{2}(\delta_{ij} - \Delta^{-1}\partial_i\partial_j)(\delta_{kl} - \Delta^{-1}\partial_k\partial_l) \ , \tag{3.25}$$

and Δ and Δ^{-1} denote the Euclidean Laplacian and the inverse Laplacian respectively.

When comparing the *canonical* metric tensor with that given by equations (3.21)–(3.24) it is instructive to keep in mind that ${}^{(-2)}\ddot{\mathcal{I}}_{A_l}(t-r) = \mathcal{I}_{A_l}(t-r)$ and $\Delta(\mathcal{I}_{A_l}(t-r)/r) = \ddot{\mathcal{I}}_{A_l}(t-r)/r$ for $r \neq 0$. This is a consequence of the fact that function ${}^{(-2)}\ddot{\mathcal{I}}_{A_l}(t-r)$ is a solution of the homogeneous d'Lambert's equation, that is, $\Box\left[{}^{(-2)}\mathcal{I}_{A_l}(t-r)/r\right] = 0$ for $r \neq 0$. We also notice that $\mathcal{I}_{A_l}(t-r)/r = \Delta^{-1}\left[\ddot{\mathcal{I}}_{A_l}(t-r)/r\right]$ and ${}^{(-2)}\mathcal{I}_{A_l}(t-r)/r = \Delta^{-1}\left[\mathcal{I}_{A_l}(t-r)/r\right]$. The metric tensor perturbation (3.21)–(3.24) is similar to the Coulomb gauge in electrodynamics [90].

3.3. The Arnowitt-Deser-Misner Coordinates

The Arnowitt-Deser-Misner (ADM) gauge conditions in the linear approximation read [50, 91]

$$2h_{0i,i} - h_{ii,0} = 0 \ , \qquad 3h_{ij,j} - h_{jj,i} = 0 \ , \tag{3.26}$$

where for any function $f = f(t, \boldsymbol{x})$ one uses notations: $f_{,0} = \partial f/\partial t$ and $f_{,i} = \partial f/\partial x^i$. For comparison, the harmonic gauge conditions (3.14) in the linear approximation, read:

$$2h_{0i,i} - h_{ii,0} = h_{00,0} \ , \qquad 2h_{ij,j} - h_{jj,i} = -h_{00,i} \ . \tag{3.27}$$

The ADM gauge conditions (3.26) brings the space-space component of metric to the form

$$g_{ij} = \delta_{ij}(1 + \frac{1}{3}h_{kk}) + h_{ij}^{TT} \ , \tag{3.28}$$

where h_{ij}^{TT} denotes the transverse-traceless part of h_{ij} and $h_{kk} = 3h_{00}$. The ADM and harmonic gauge conditions are not compatible inside the regions occupied by matter. However, outside of matter they can co-exist. Indeed, it is straightforward to check out that the metric tensor (3.21)–(3.24) satisfies both the harmonic and the ADM gauge conditions in the linear approximation and assumption that $\dot{\mathcal{M}} = 0$ [92]. We call the coordinates in which the metric tensor is given by equations (3.21)–(3.24) as the ADM-harmonic coordinates.

The experimental problem of detection of gravitational waves is reduced to the observation of motion of test particles in the field of the incident gravitational wave. These test particles are photons of the electromagnetic signals and observers (mirrors in ground-based gravitational-wave detectors) which receive these photons. Gravitational wave interacts with both photons and observers and perturbs their motion. These perturbations must be explicitly calculated and clearly separated to avoid possible misinterpretations of observable effects of the gravitational wave. It turns out that the *canonical* form of the metric tensor (3.5) – (3.8) is extremely well-adapted for performing analytic integration of equations of light rays. At the same time freely-falling observers experience influence of gravitational waves emitted by the isolated astronomical system and move with respect to the *canonical harmonic* coordinate system in a complicated way. For this reason, observable effects imposed by the gravitational waves on the light propagation get mixed up with the motion of observer in these coordinates.

Arnowitt, Deser and Misner [50] showed that there exist a *canonical ADM* coordinate system which has a special property such that freely-falling observers are not moving with respect to this coordinates despite that they are perturbed by the gravitational waves. This means that the ADM coordinates themselves are not inertial and, although have nice mathematical properties, should be used with care in the interpretation of physical experiments. Making use of the *canonical ADM* coordinates simplifies analysis of the gravitational wave effects observed at gravitational wave observatories (LIGO, LISA) or by astronomical technique because the motion of observer (proof mass) is excluded from the equations. However, structure of the metric tensor in such *canonical ADM* coordinate system does not allow us to integrate equations for light rays analytically because it contains terms that are instantaneous functions of time [83].

The ADM-harmonic coordinates have advantages of both harmonic and ADM coordinates. Thus, the ADM-harmonic coordinates allow us to get a full analytic solution of the light-ray equations and to eliminate the effects produced by the motion of observers with respect to the coordinate grid caused by influence of gravitational waves. In other words, *all* physical effects of the gravitational waves are contained in the equations of light propagation. This conclusion is, of course, valid in the linear approximation of general relativity and is not extended to the second approximation where gravitational-wave effects on light and motion of observers can not be disentangled and have to be analyzed together.

Similar ideology based on introduction of the TT coordinate system is applied for analysis of the output signal of the gravitational-wave detectors with freely-suspended masses [1, 2, 26] placed to the field of a plane gravitational wave, that is at the distance far away from the localized system emitting gravitational waves where the curvature of the gravitational wave front is negligible. Our ADM-harmonic coordinates are essential generalization of the standard TT coordinates because it can be used at arbitrary distances from the isolated system emitting gravitational waves including its near and intermediate zones.

4. EQUATIONS OF PROPAGATION OF ELECTROMAGNETIC SIGNALS

4.1. Maxwell Equations in Curved Space-Time

In this section all indices are raised and lowered by means of the metric tensor $g_{\alpha\beta}$ with $g^{\alpha\beta}$ defined in accordance with standard rule $g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\beta}$. The general formalism describing the behavior of electromagnetic radiation in an arbitrary gravitational field is well known [93]. An electromagnetic field is defined in terms of the (complex) electromagnetic tensor $F_{\alpha\beta}$ as a solution of the Maxwell equations. In the high-frequency limit one can approximate the electromagnetic tensor $F_{\alpha\beta}$ as [1, 2]

$$F_{\alpha\beta} = \mathrm{Re}\{A_{\alpha\beta}\exp(i\varphi)\}\ , \tag{4.1}$$

where $A_{\alpha\beta}$ is a slowly varying (complex) amplitude and φ is a rapidly varying phase of the electromagnetic wave which is considered as a real function. Both amplitude and phase are functions of time and space coordinates.

The source-free Maxwell equations are given by [1–3]

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0\ , \tag{4.2}$$

$$\nabla_\beta F^{\alpha\beta} = 0\ , \tag{4.3}$$

where ∇_α denotes covariant differentiation [94]. Taking a covariant divergence from equation (4.2), using equation (4.3) and applying the rule of commutation of two covariant derivatives from the tensor field of a second rank [1–3], we obtain the covariant wave equation for the electromagnetic field tensor

$$\Box_g F_{\alpha\beta} + R_{\alpha\beta\mu\nu}F^{\mu\nu} + R_{\mu\alpha}F_\beta{}^\mu - R_{\mu\beta}F_\alpha{}^\mu = 0\ , \tag{4.4}$$

where $\Box_g \equiv \nabla^\alpha\nabla_\alpha$, $R_{\alpha\beta\mu\nu}$ is the Riemann curvature tensor, and $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$ is the Ricci tensor (definitions of the Riemann and Ricci tensors in this paper are the same as in the textbook [3]). We consider the case of the gravitational field in vacuum where $R_{\alpha\beta} = 0$. Hence, in our case the equation (4.4) is reduced to more simple form

$$\Box_g F_{\alpha\beta} + R_{\alpha\beta\mu\nu}F^{\mu\nu} = 0\ . \tag{4.5}$$

Differential operator in equation (4.4) taken along with the Riemann and Ricci tensors is de Rahm's generalized topological d'Alembertian [2, 95] for the electromagnetic field.

4.2. Maxwell Equations in the Geometric Optics Approximation

Let us now assume that the electromagnetic tensor $F_{\alpha\beta}$ shown in equation (4.1) can be expanded with respect to a small dimensionless perturbation parameter $\varepsilon = l/L$ where l is a characteristic wavelength of the electromagnetic wave and L is a characteristic radius of space-time curvature. More specifically, we assume that the expansion of the electromagnetic field given by equation (4.1) has the form [2]

$$F_{\alpha\beta} = \left(a_{\alpha\beta} + \varepsilon b_{\alpha\beta} + \varepsilon^2 c_{\alpha\beta} + ...\right)\exp\left(\frac{i\varphi}{\varepsilon}\right)\ . \tag{4.6}$$

Substituting the expansion (4.6) into equation (4.2), taking into account the definition of the electromagnetic wave vector $l_\alpha = \partial\varphi/\partial x^\alpha$, and arranging terms with similar powers of ε lead to the chain of equations

$$l_\alpha\, a_{\beta\gamma} + l_\beta\, a_{\gamma\alpha} + l_\gamma\, a_{\alpha\beta} \;=\; 0\ , \tag{4.7}$$

$$\nabla_\alpha\, a_{\beta\gamma} + \nabla_\beta\, a_{\gamma\alpha} + \nabla_\gamma\, a_{\alpha\beta} \;=\; -i\left(l_\alpha\, b_{\beta\gamma} + l_\beta\, b_{\gamma\alpha} + l_\gamma\, b_{\alpha\beta}\right)\ , \tag{4.8}$$

where we have neglected the effects of curvature which are of order ε^2 and thus can be considered as negligibly small.

Similarly, equation (4.3) gives a chain of equations

$$l_\beta\, a^{\alpha\beta} \;=\; 0\;, \tag{4.9}$$

$$\nabla_\beta\, a^{\alpha\beta} + i l_\beta\, b^{\alpha\beta} \;=\; 0\;, \tag{4.10}$$

where we again neglected the effects of curvature.

Equation (4.9) implies that the amplitude of the electromagnetic field tensor is orthogonal in the four-dimensional sense to vector l_α at least in the first approximation. Contracting equation (4.7) with l_α and accounting for (4.9), we find that the wave vector l_α is null, that is

$$l_\alpha l^\alpha = 0\;. \tag{4.11}$$

Taking the covariant derivative of this equality and using the fact that $\nabla_{[\beta}\, l_{\alpha]} = 0$ since $l_\alpha = \nabla_\alpha \varphi$, one can show that the vector l_α obeys the null geodesic equation

$$l^\beta \nabla_\beta l^\alpha = 0\;, \tag{4.12}$$

which means that the null vector l^α is parallel transported along itself. Equation (4.12) can be expressed as

$$\frac{dl^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} l^\beta l^\gamma = 0\;, \tag{4.13}$$

where λ is an affine parameter along the light-ray trajectory, and

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma} - \partial_\mu g_{\beta\gamma}\right)\;, \tag{4.14}$$

are the Christoffel symbols.

Finally, equation (4.4) (or (4.5)) can be used to show that in the first approximation

$$D_\lambda a_{\alpha\beta} + \vartheta a_{\alpha\beta} = 0\;, \tag{4.15}$$

where $D_\lambda \equiv l^\mu \nabla_\mu$, and

$$\vartheta \equiv (1/2) \nabla_\alpha l^\alpha\;, \tag{4.16}$$

is the expansion of the light-ray congruence defined at each point of space-time by the derivative of the wave vector l^α.

Equation (4.15) represents the law of propagation for the electromagnetic field tensor along the light ray. In general case of $\vartheta \neq 0$ the electromagnetic field tensor is not parallel-transported along the light ray. It can be shown that the expansion ϑ of the light-ray congruence is defined by the stationary components of the gravitational field of the isolated astronomical system determined by its mass $\mathcal{M}$, and spin $\mathcal{S}^i$, and does not depend on higher-order multipole moments. It means that gravitational waves do not contribute to the expansion of the light-ray congruence in the linearized approximation and their impact on ϑ is postponed to the terms of the second order of magnitude with respect to the universal gravitational constant G.

4.3. Electromagnetic Eikonal and Light-Ray Geodesics

1. The Congruence of Light Rays

We shall assume that geometric optics approximation is a valid approximation and electromagnetic signals propagate in vacuum. This means that we consider electromagnetic signals with wavelengths much smaller than characteristic wavelength of gravitational waves emitted by the isolated astronomical system and that the speed of light is equal to the speed of propagation of gravitational waves. We also neglect all relativistic effects associated with explicit appearance of the curvature tensor of space-time in equations of light propagation. In accordance with consideration given in the previous section 4 4.2, we are allowed to describe electromagnetic signals as massless particles (photons) moving along light-ray geodesics with the initial-boundary condition [96]

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \qquad \frac{d\boldsymbol{x}(-\infty)}{dt} = \boldsymbol{k}\;. \tag{4.17}$$

These conditions determine spatial position $\boldsymbol{x}_0$ of the electromagnetic signal at the time of its emission, t_0, and direction of its propagation given by a unit vector, $\boldsymbol{k}$, at the past null infinity, that is at the infinite distance and at the infinite past [2, 3] where the space-time is assumed to be flat (see Figure 1). We imply that vector $\boldsymbol{k}$ is directed towards observer.

Several functions characterize tensor of the electromagnetic field $F_{\alpha\beta}$. We start in the next section from discussing a differential equation for the phase φ of electromagnetic wave, which is also known as *eikonal* [1, 97]. In the next sections we will derive equations for the null geodesics and the Stokes parameters characterizing polarization properties of the electromagnetic wave. Exact differential equations for these quantities are too complicated and can not be solved exactly. For this reason we shall appeal to the approximations retaining in those equations only those terms which are linear with respect to the metric tensor perturbation $h_{\alpha\beta}$.

At the first iteration one can neglect relativistic perturbation in photon's motion and approximates it by a straight line

$$x^i = x^i_N(t) \equiv x^i_0 + k^i(t - t_0)\ , \tag{4.18}$$

where t_0 is the time of emission of the photon, x^i_0 are space coordinates of the source of the electromagnetic signal taken at the time t_0, and $k^i = \boldsymbol{k}$ is the unit vector along the trajectory of photon's motion defined by equation (4.17).

The bundle of light rays makes 2+1 split of space by projecting any point in space onto the plane being orthogonal to the bundle (see Figure 2). This allows to make a transformation to new independent variables τ and ξ^i defined as follows

$$\tau = k_i x^i_N \qquad , \qquad \xi^i = P^i_{\ j} x^j_N\ , \tag{4.19}$$

where $P^i_{\ j} = \delta^i_j - k^i k_j$ is the operator of projection on the plane orthogonal to $\boldsymbol{k}$. It is easy to see that the parameter τ is equivalent to time

$$\tau \equiv \boldsymbol{k}\cdot\boldsymbol{x} = t - t^*\ , \tag{4.20}$$

where $t^* \equiv \boldsymbol{k}\cdot\boldsymbol{x}_0 - t_0$ is the time of the closest approach of the electromagnetic signal to the coordinate origin which is taken in our case coinciding with the center of mass of the isolated astronomical system. Because for each light ray the time t^* is fixed one concludes that the time differential $d\tau = dt$. The reader may think that results of our calculations will depend on parameters ξ^i and t^*. This is, however, not true since ξ^i and t^* depend on the choice of the origin of the coordinate system and, thus, are unphysical. Inspection of resulting equations shows that parameters ξ^i and t^* vanish from the observed quantities as it must be in agreement with the physical point of view.

The unperturbed light-ray trajectory (4.18) written in terms of the new variables (4.19) reads

$$x^i_N(\tau) = k^i\tau + \xi^i\ , \tag{4.21}$$

so that the new variable $\xi^i \equiv \boldsymbol{\xi} = \boldsymbol{k}\times(\boldsymbol{x}\times\boldsymbol{k})$ should be understood as a vector drawn from the origin of the coordinate system towards the point of the closest approach of the ray to the origin. As vectors k^i and ξ^i are orthogonal, the unperturbed distance $r_N = \sqrt{x_i x^i}$ between the photon and the origin of the coordinate system

$$r_N = \sqrt{\tau^2 + d^2}\ , \tag{4.22}$$

where $d = |\boldsymbol{\xi}|$ is the impact parameter of the unperturbed light-ray trajectory with respect to the coordinate origin.

We introduce two special operators of partial derivatives with respect to τ and ξ^i determined for any smooth function taken on the light ray. These operators are defined as

$$\hat{\partial}_\tau \equiv \frac{\partial}{\partial\tau} \qquad , \qquad \hat{\partial}_i \equiv P_i^{\ j}\frac{\partial}{\partial\xi^j}\ , \tag{4.23}$$

so that, for example,

$$k^i\hat{\partial}_i = 0\ . \tag{4.24}$$

An important consequence of the projective structure of the bundle of light rays is that for any smooth function $F(t,\boldsymbol{x})$ defined on the light-ray trajectory one has

$$\left[\left(\frac{\partial}{\partial x^i} + k_i\frac{\partial}{\partial t}\right)F(t\,,\,\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}(t-t_0)} = \left(\frac{\partial}{\partial\xi^i} + k_i\frac{\partial}{\partial\tau}\right)F(t^*+\tau\,,\,\boldsymbol{\xi}+\boldsymbol{k}\tau)\ , \tag{4.25}$$

$$\left[\left(\frac{\partial}{\partial t} + k^i\frac{\partial}{\partial x^i}\right)F(t\,,\,\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}(t-t_0)} = \frac{d}{d\tau}F(t^*+\tau\,,\,\boldsymbol{\xi}+\boldsymbol{k}\tau)\ , \tag{4.26}$$

$$\left[\frac{\partial}{\partial t}F(t\,,\,\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}(t-t_0)} = \frac{\partial}{\partial t^*}F(t^*+\tau\,,\,\boldsymbol{\xi}+\boldsymbol{k}\tau). \tag{4.27}$$

Here, in the left sides of Eqs. (4.25)–(4.27) one must first calculate partial derivatives and only after that substitute the unperturbed trajectory $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{k}(t - t_0)$, while in the right side one first substitute the unperturbed trajectory parameterized by variables τ and ξ^i and then differentiate. It is worth noticing that making use of equation (4.27) allows us to re-write equation (4.25) as follows

$$\left[\frac{\partial F(t, \boldsymbol{x})}{\partial x^i}\right]_{\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}(t-t_0)} = \left(\frac{\partial}{\partial \xi^i} + k_i \frac{\partial}{\partial \tau} - k_i \frac{\partial}{\partial t^*}\right) F(t^* + \tau\,,\, \boldsymbol{\xi} + \boldsymbol{k}\tau)\;. \tag{4.28}$$

This equation is used later for decomposition of STF spatial derivatives from the retarded Lienard-Wiechert potentials of the gravitational field.

2. *The Eikonal Equation*

The eikonal is related to the wave vector of the electromagnetic wave as $l_\alpha = \partial_\alpha \varphi$. This definition along with equation (4.11) immediately gives us a differential equation for the eikonal

$$g^{\alpha\beta} \frac{\partial \varphi}{\partial x^\alpha} \frac{\partial \varphi}{\partial x^\beta} = 0\;. \tag{4.29}$$

We are interested in an unperturbed solution of this equation which is a plane electromagnetic wave

$$\varphi_N = \varphi_0 + \omega k_\alpha x^\alpha\;, \tag{4.30}$$

where φ_0 is constant, $\omega = 2\pi\nu_\infty$, ν_∞ is (constant) frequency of the electromagnetic wave at infinity and k_α is an unperturbed direction of the co-vector l_α. Equation (4.29) assumes that k_α is a null co-vector with respect to the Minkowski metric in the sense that

$$\eta^{\alpha\beta} k_\alpha k_\beta = 0\;. \tag{4.31}$$

We postulate that the co-vector $k_\alpha = (-1, \boldsymbol{k})$ where the unit Euclidean vector $\boldsymbol{k}$ is defined at past null infinity by equation (4.17).

In the linearized approximation of general relativity the eikonal can be decomposed in a linear combination of unperturbed, φ_N, and perturbed, ψ, parts

$$\varphi = \varphi_N + \omega\psi\;, \tag{4.32}$$

so that the wave co-vector

$$l_\alpha = \omega\left(k_\alpha + \frac{\partial \psi}{\partial x^\alpha}\right)\;, \tag{4.33}$$

Making use of equations (3.2) and (4.29)–(4.33) yields differential equation for the perturbed part of the eikonal

$$k^\alpha \frac{\partial \psi}{\partial x^\alpha} = \frac{1}{2} h_{\alpha\beta} k^\alpha k^\beta\;. \tag{4.34}$$

This equation can be solved in all space by the method of characteristics [98] which, in the case under consideration, are the unperturbed light-ray geodesics given by Eq. (4.21). Hence, after making use of relationship (4.26) one gets an ordinary differential equation for finding the eikonal perturbation

$$\frac{d\psi}{d\tau} = \frac{1}{2} h^{\rm can.}_{\alpha\beta}(\tau, \boldsymbol{\xi}) k^\alpha k^\beta + \hat{\partial}_\tau \left(k^i w^i - w^0\right)\;, \tag{4.35}$$

where the canonical metric tensor perturbation $h^{\rm can.}_{\alpha\beta}$ is taken on the light-ray trajectory as shown in equations (B.1)–(B.9) and the gauge functions w^α can be taken arbitrary. Equation (4.35) can be solved analytically with the mathematical technique shown in section 5.

3. Light-Ray Geodesic Equations

General form of the geodesic equation for light rays is obtained from equation (4.13) after substitution of the approximate expressions (A.1)–(A.6) for the Christoffel symbols. It gives

$$\ddot{x}^i(t) = \frac{1}{2}h_{00,i} - h_{0i,0} - \frac{1}{2}h_{00,0}\dot{x}^i - h_{ik,0}\dot{x}^k - (h_{0i,k} - h_{0k,i})\dot{x}^k - h_{00,k}\dot{x}^k\dot{x}^i - \left(h_{ik,j} - \frac{1}{2}h_{kj,i}\right)\dot{x}^k\dot{x}^j + \left(\frac{1}{2}h_{kj,0} - h_{0k,j}\right)\dot{x}^k\dot{x}^j\dot{x}^i, \tag{4.36}$$

where overdot denotes an ordinary time derivative and h_{00}, h_{0i}, h_{ij} are components of the metric tensor taken on the unperturbed light-ray trajectory as shown in equations (B.1)–(B.9), that is $h_{\alpha\beta} = h_{\alpha\beta}(t, \boldsymbol{x}_N(t))$.

Equation (4.36) can be further simplified after substituting the unperturbed light-ray trajectory (4.21) to the right side of Eq. (4.36) and making use of equation (4.25). Working in arbitrary coordinates one obtains

$$\frac{d^2x^i(\tau)}{d\tau^2} = \frac{1}{2}k^\alpha k^\beta\,\hat{\partial}_i h^{\text{can.}}_{\alpha\beta} - \hat{\partial}_\tau\left(k^\alpha h^{\text{can.}}_{i\alpha} - \frac{1}{2}\,k^i k^j k^p q^{\text{can.}}_{jp}\right) - \hat{\partial}_{\tau\tau}(w^i - k^i w^0). \tag{4.37}$$

where all functions in equation (4.37) are taken (before any differentiation) on the unperturbed light-ray trajectory given by equation (4.21) and the gauge functions w^α have not yet been specified [99] which means that equations (4.37) are gauge-invariant.

The main advantage of Eq. (4.37) is that when one integrates along the light-ray path the following rules, applied to any smooth function $F(\tau, \boldsymbol{\xi})$, can be used

$$\int \frac{\partial}{\partial\tau}F(\tau, \boldsymbol{\xi})\,d\tau = F(\tau, \boldsymbol{\xi}) + C(\boldsymbol{\xi})\ , \tag{4.38}$$

$$\int \frac{\partial}{\partial\xi^i}F(\tau, \boldsymbol{\xi})\,d\tau = \frac{\partial}{\partial\xi^i}\int F(\tau, \boldsymbol{\xi})\,d\tau\ . \tag{4.39}$$

This means that terms which are represented as partial derivatives with respect to the parameter τ can be immediately integrated by making use of (4.38). At the same time Eq. (4.39) shows that one can change the order of operations of integration and differentiation with respect to the parameter ξ^i which definitely simplifies the problem of finding solution of Eq. (4.37) as it will be evident in next sections.

The gauge functions w^α are not the only gauge functions which appear in the equations of motion (4.37) of the light ray. In order to see it more evidently, it is worth noticing that Eq. (4.37) is linear with respect to the perturbations of the metric tensor $h_{\alpha\beta}$. Hence, it can be linearly decomposed in two equations for perturbations of the light-ray trajectory caused separately by mass and spin multipole moments. Substitution of the metric tensor (3.5) – (3.8) to Eq. (4.37) and replacement of spatial derivatives with respect to x^i with those with respect to parameters ξ^i and τ by making use of (4.27) yield

$$\ddot{x}^i = \ddot{x}^i_{(G)} + \ddot{x}^i_{(M)} + \ddot{x}^i_{(S)}\ , \tag{4.40}$$

where $\ddot{x}^i_{(M)}$ and $\ddot{x}^i_{(S)}$ are components of the photon's coordinate acceleration caused by mass and spin multipoles of the metric tensor respectively, and $\ddot{x}^i_{(G)}$ is the gauge-dependent acceleration. These components read

$$\ddot{x}^i_{(G)} = \hat{\partial}_{\tau\tau}\left[(k^i\varphi^0_{(M)} - \varphi^i_{(M)}) + (k^i w^0 - w^i)\right]\ , \tag{4.41}$$

$$\begin{aligned}\ddot{x}^i_{(M)} = {}& 2(\hat{\partial}_i - k_i\hat{\partial}_\tau)\frac{\mathcal{M}}{r} + 2\hat{\partial}_i\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p, p-q, q)H(2-q)\times\\ &\left(1 - \frac{p-q}{l}\right)\left(1 - \frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^{p-q}_{t^*}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}_{A_l}(t-r)}{r}\right] -\\ &2\hat{\partial}_\tau\sum_{l=2}^{\infty}\sum_{p=0}^{l}\frac{(-1)^{l+p}}{l!}C_l(l-p, p)\left(1 - \frac{p}{l}\right)\left\{\left(1 + \frac{p}{l-1}\right)k_{i<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^p_{t^*}\left[\frac{\mathcal{I}_{A_l}(t-r)}{r}\right] -\right.\\ &\left.\frac{2p}{l-1}\,k_{<a_1\ldots a_{p-1}}\hat{\partial}_{a_p\ldots a_{l-1}>}\hat{\partial}^p_{t^*}\left[\frac{\mathcal{I}_{iA_{l-1}}(t-r)}{r}\right]\right\}\ ,\end{aligned} \tag{4.42}$$

and

$$
\begin{aligned}
\ddot{x}^i_{(S)} = 2\left(k_j\hat{\partial}_{ia} - \delta_{ij}\hat{\partial}_{a\tau}\right)\frac{\epsilon_{jba}\mathcal{S}_b}{r} - 4k_j\hat{\partial}_{ia}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)H(2-q)\times \qquad (4.43)\\
\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{p-q}_{t^*}\hat{\partial}^q_\tau\left[\frac{\epsilon_{jba}\mathcal{S}_{bA_{l-1}}(t-r)}{r}\right]+\\
4\left(\hat{\partial}_a - k_a\hat{\partial}_{t^*}\right)\hat{\partial}_\tau\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\times\\
\left(1-\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{p}_{t^*}\left[\frac{\epsilon_{iba}\mathcal{S}_{bA_{l-1}}(t-r)}{r}\right]+\\
4k_j\hat{\partial}_\tau\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{p}_{t^*}\left[\frac{\epsilon_{jba_{l-1}}\dot{\mathcal{S}}_{\hat{i}bA_{l-2}}(t-r)}{r}\right]\ ,
\end{aligned}
$$

where overdot denotes a time derivative which is equivalent to a partial derivative with respect to the parameter t^* [100], w^α are gauge functions generated by arbitrary post-Minkowskian coordinate transformations, and φ^α are the gauge functions generated by the multipolar structure of the metric tensor in the course of reshuffling terms in the right side of equations (4.37) for a light-ray geodesic.

More specifically, functions φ^α have been obtained after we singled out in the equations of light propagation all terms containing second and higher-order time derivatives with respect to the parameter τ. These terms enter the equations of motion in the form of combination $k^i\varphi^0 - \varphi^i$ given in Appendix C. It is worthwhile to emphasize that we do not intend to separate $k^i\varphi^0 - \varphi^i$ in two functions φ^i and φ^0 because such a separation is not unique while the linear combination $k^i\varphi^0 - \varphi^i$ does. We have not combined functions φ^α with the gauge functions w^α for two reasons:

1. to indicate that solution of equations of light-ray geodesic, performed in one specific coordinate system, leads to generation of terms which can be eliminated by gauge transformation,

2. to simplify the final form of the result of the integration as all terms with second and higher order time derivatives are immediately integrated in accordance with Eq. (4.38).

Gauge functions w^α are arbitrary making our equations gauge-invariant. However, for the sake of physical interpretation of the result of integration of equations of light-ray geodesic, we shall chose function w^α to make our coordinate system both harmonic and ADM. Coordinate description of motion of free-falling particles in these coordinates postpones effects of perturbations by gravitational waves emitted by multipoles to higher orders of approximation. Specific form of the gauge functions w^α at arbitrary field point is shown in equations (3.17), (3.18) and their form at any point on the light-ray trajectory is given in Appendix C.

It is important to realize that for each multipole of order l the right sides of Eqs. (4.42), (4.43) have such structure that all terms, which are proportional to $l-th$ time derivative $\hat{\partial}^l_{t^*}$ from this multipole, mutually cancel out. Specifically for this reason all time integrals, which can not be performed explicitly, vanish from the relativistic perturbations of the light-ray trajectory irrespectively of the order of the multipole moment. This fact was established in [83] in a quadrupole approximation only. Present paper proves that this result is valid for arbitrary multipole. As one will see later this property of the null geodesic prevents to amplify effects of gravitational waves acting on a light ray propagating near a gravitational lens emitting multipolar gravitational radiation (binary star, etc.).

4.4. Equations for Polarization of Light and the Stokes Parameters

1. *Reference Tetrad Field*

Algebraic classification of electromagnetic fields and studying their physical properties are conveniently performed with the help of the Newman-Penrose formalism [101, 102]. This formalism is based on introducing at each point of space-time a null tetrad of four vectors associated with the bundle of light rays defined by the electromagnetic wave vector l^α. The Newman-Penrose tetrad consists of two real and two complex null vectors $(l^\alpha, n^\alpha, m^\alpha, \overline{m}^\alpha)$, where the bar indicates complex conjugation. The tetrad vectors are normalized in such a way that $n_\alpha l^\alpha = -1$ and $m_\alpha\overline{m}^\alpha = +1$ are the only nonvanishing products among the four tetrad vectors.

The vectors of the null tetrad are not uniquely determined by specifying l^α. Indeed, for a fixed direction l^α the normalization conditions for tetrad's vectors are preserved under the linear transformations (null rotation) [101, 102]

$$l'^\alpha = Al^\alpha , \tag{4.44}$$
$$n'^\alpha = A^{-1}\left(n^\alpha + \overline{B}m^\alpha + B\overline{m}^\alpha + B\overline{B}l^\alpha\right) , \tag{4.45}$$
$$m'^\alpha = e^{-i\Theta}\left(m^\alpha + \overline{B}l^\alpha\right) , \tag{4.46}$$
$$\overline{m}'^\alpha = e^{i\Theta}\left(\overline{m}^\alpha + Bl^\alpha\right) , \tag{4.47}$$

where A, Θ are real and B is a complex scalar. These transformations form a four-parameter subgroup of the Lorentz group.

For analysis of intensity and polarization of electromagnetic waves it is useful to introduce a local orthonormal reference frame of observer moving with four velocity u^α and seeing the electromagnetic wave travelling in the positive direction of z axis of the reference frame. It means that at each point of space-time the observer uses a tetrad frame $e^\alpha_{\ (\beta)} = \left(e^\alpha_{\ (0)}, e^\alpha_{\ (1)}, e^\alpha_{\ (2)}, e^\alpha_{\ (3)}\right)$ defined in such a way that

$$e^\alpha_{\ (0)} = u^\alpha \qquad , \qquad e^\alpha_{\ (3)} = (-l_\alpha u^\alpha)^{-1}\left[l^\alpha + (l_\beta u^\beta)u^\alpha\right] , \tag{4.48}$$

and two other tetrad vectors, $e^\alpha_{\ (1)}$ and $e^\alpha_{\ (2)}$, are unit space-like vectors being orthogonal to each other as well as to $e^\alpha_{\ (0)}$ and $e^\alpha_{\ (3)}$ [103]. In other words, vectors of the tetrad are subject to the following normalization conditions

$$g_{\alpha\beta} e^\alpha_{\ (\mu)} e^\beta_{\ (\nu)} = \eta_{\mu\nu} , \qquad \eta^{\mu\nu} e^\alpha_{\ (\mu)} e^\beta_{\ (\nu)} = g^{\alpha\beta} . \tag{4.49}$$

Let us define at each space-time point a coordinate basis

$$\partial^\alpha_{(0)} = (1,0,0,0) , \quad \partial^\alpha_{(1)} = (0,a^1,a^2,a^3) , \quad \partial^\alpha_{(2)} = (0,b^1,b^2,b^3) , \quad \partial^\alpha_{(3)} = (0,k^1,k^2,k^3) . \tag{4.50}$$

where the spatial vectors $\boldsymbol{a} = (a^1,a^2,a^3)$, $\boldsymbol{b} = (b^1,b^2,b^3)$, and $\boldsymbol{k} = (k^1,k^2,k^3)$ are orthonormal in the Euclidean sense, and the vector $\boldsymbol{k}$ defines the spatial direction of propagation of the light ray at infinity (see equation (4.17)). In a particular case of a linearized gravity field and static observer, when $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and $u^0 = 1 - (1/2)h_{00}$, $u^i = 0$, the locally-orthonormal tetrad $E^\alpha_{\ (\beta)}$ is given by

$$E^\alpha_{\ (0)} = \left(1 + \frac{1}{2}h_{00}\ ,\ 0\ ,\ 0\ ,\ 0\right) , \tag{4.51}$$
$$E^\alpha_{\ (1)} = \left(h_{0j}a^j\ ,\ a^i - \frac{1}{2}h_{ij}a^j\right) , \tag{4.52}$$
$$E^\alpha_{\ (2)} = \left(h_{0j}b^j\ ,\ b^i - \frac{1}{2}h_{ij}b^j\right) , \tag{4.53}$$
$$E^\alpha_{\ (3)} = \left(h_{0j}k^j\ ,\ k^i - \frac{1}{2}h_{ij}k^j\right) . \tag{4.54}$$

which is a direct consequence of equations (4.48)–(4.50). The local tetrad $e^\alpha_{\ (\beta)}$ of the moving observer relates to the tetrad $E^\alpha_{\ (\beta)}$ of the static observer by means of the Lorentz transformation

$$e^\alpha_{\ (\beta)} = \lambda^\gamma_{\ \beta} E^\alpha_{\ (\gamma)} , \qquad E^\alpha_{\ (\beta)} = \tilde{\lambda}^\gamma_{\ \beta} e^\alpha_{\ (\gamma)} , \tag{4.55}$$

where the matrix of the Lorentz transformation is [2]

$$\lambda^0_{\ 0} = u^0 \equiv \gamma \tag{4.56}$$
$$\lambda^\alpha_{\ 0} = \Lambda^{(0)}_{\ (\alpha)} = u^\alpha , \tag{4.57}$$
$$\lambda^i_{\ j} = \delta^{ij} + \frac{u^i u^j}{1+\gamma} , \tag{4.58}$$

and the inverse matrix of the Lorentz transformation $\tilde{\lambda}^\alpha_{\ \beta}$ is obtained from $\lambda^\alpha_{\ \beta}$ by replacing $u^i \to -u^i$, as it follows from its definition.

The connection between the null tetrad and the frame $e^\alpha_{(\beta)}$ is given by equations

$$l^\alpha = -(l_\gamma u^\gamma)\left(e^\alpha_{(0)} + e^\alpha_{(3)}\right) , \tag{4.59}$$

$$n^\alpha = -\frac{1}{2}(l_\gamma u^\gamma)\left(e^\alpha_{(0)} - e^\alpha_{(3)}\right) , \tag{4.60}$$

$$m^\alpha = \frac{1}{\sqrt{2}}\left(e^\alpha_{(1)} + ie^\alpha_{(2)}\right) , \tag{4.61}$$

$$\overline{m}^\alpha = \frac{1}{\sqrt{2}}\left(e^\alpha_{(1)} - ie^\alpha_{(2)}\right) . \tag{4.62}$$

If vectors $e^\alpha_{(0)}$ and $e^\alpha_{(3)}$ are fixed the vectors $e^\alpha_{(1)}$ and $e^\alpha_{(2)}$ are defined up to an arbitrary rotation in their own subspace. Transformations (4.46), (4.47) with $B = 0$ yield

$$e'^\alpha_{(1)} = \cos\Theta\, e^\alpha_{(1)} + \sin\Theta\, e^\alpha_{(2)} , \tag{4.63}$$

$$e'^\alpha_{(2)} = -\sin\Theta\, e^\alpha_{(1)} + \cos\Theta\, e^\alpha_{(2)} , \tag{4.64}$$

where Θ is the rotation angle in the $\left\{e^\alpha_{(1)}, e^\alpha_{(2)}\right\}$ plane. Vectors $e^\alpha_{(1)}$ and $e^\alpha_{(2)}$ play a significant role in the discussion of polarized radiation.

2. *Propagation Laws for Reference Tetrad*

Discussion of the rotation of the polarization plane and the change of the Stokes parameters of electromagnetic radiation is not conceivable without an unambiguous definition of a local reference frame (tetrad) constructed along the light-ray geodesic. We start construction of this tetrad at the point of observation and prolongate it backward along the light-ray geodesic. By definition, the tetrad frames $e^\alpha_{(\beta)}$ and $(l^\alpha, n^\alpha, m^\alpha, \overline{m}^\alpha)$ are parallel transported along the light ray. The propagation equations for these vectors are thus obtained by applying the operator $D_\lambda = l^\alpha\nabla_\alpha$ of the parallel transport. For example,

$$\frac{de^\alpha_{(\mu)}}{d\lambda} + \Gamma^\alpha_{\beta\gamma}\, l^\beta\, e^\gamma_{(\mu)} = 0 , \tag{4.65}$$

where λ is an affine parameter along the light ray. Using the definition of the Christoffel symbols (4.14) and changing over to the variable τ with $dx^\alpha/d\tau = k^\alpha + O(h)$, one can recast equation (4.65) in the approximate form

$$\frac{d}{d\tau}\left[e^\alpha_{(\mu)} + \frac{1}{2}h^\alpha_\beta e^\beta_{(\mu)}\right] = \frac{1}{2}\eta^{\alpha\nu}\left(\partial_\nu h_{\gamma\beta} - \partial_\gamma h_{\nu\beta}\right) k^\beta e^\gamma_{(\mu)} . \tag{4.66}$$

The law of propagation of the null vectors m^α and $\overline{m}^\alpha$ are

$$\frac{dm^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma}\, l^\beta\, m^\gamma = 0 , \tag{4.67}$$

$$\frac{d\overline{m}^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma}\, l^\beta\, \overline{m}^\gamma = 0 , \tag{4.68}$$

and the same laws are valid for n^α and l^α (see equation (4.13)). Equations (4.66)–(4.68) are the main equations for the discussion of the rotation of the plane of polarization and variation of the Stokes parameters.

3. *Relativistic Description of Polarized Electromagnetic Radiation*

We consider propagation of plane electromagnetic waves. Such wave has an electromagnetic tensor $F_{\alpha\beta} = \mathcal{F}_{\alpha\beta} + O(\epsilon)$ defined in the first approximation by equation [102, 104]

$$\mathcal{F}_{\alpha\beta} = a_{\alpha\beta}\exp\left(\frac{i\varphi}{\varepsilon}\right) , \tag{4.69}$$

$$a_{\alpha\beta} = \Phi\, l_{[\alpha}m_{\beta]} + \overline{\Phi}\, l_{[\alpha}\overline{m}_{\beta]} \tag{4.70}$$

where Φ is a complex scalar amplitude of the electromagnetic wave with two real components which are independent of each other in the most general case of incoherent radiation. In the proper frame of an observer with 4-velocity u^α the components of the electric and magnetic field vectors are defined respectively as $E^\alpha = -F^{\alpha\beta}u_\beta$ and $H^\alpha = (-1/2\sqrt{-g})\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}u_\beta$. The electric field is a product of slowly-changing amplitude $\mathcal{E}_\alpha = -a_{\alpha\beta}u^\beta$ and fast-oscillating phase exponent

$$E_\alpha = \mathcal{E}_\alpha \exp\left(\frac{i\varphi}{\varepsilon}\right) . \tag{4.71}$$

The polarization properties of electromagnetic radiation consisting of an ensemble of waves with equal frequencies but different phases are defined in terms of the electric field measured by an observer. In the rest frame of an observer with 4-velocity u^α, the intensity and polarization properties of the radiation are described in terms of the *polarization* tensor [1]

$$J_{\alpha\beta} =< E_\alpha \overline{E}_\beta >=< \mathcal{E}_\alpha \overline{\mathcal{E}}_\beta > , \tag{4.72}$$

where the angular brackets represent an average with respect to an ensemble of electromagnetic waves with randomly distributed and fast-oscillating phases. This averaging eliminates all fast-oscillating terms from $J_{\alpha\beta}$. One has to notice [1] that the polarization tensor $J_{\alpha\beta}$ is not fully symmetric since, in general, $\mathcal{E}_\alpha \neq \overline{\mathcal{E}}_\alpha$. Furthermore, the tensor $J_{\alpha\beta}$ satisfies the equalities $J_{\alpha\beta}u^\beta = J_{\alpha\beta}l^\beta = 0$ which follows directly from its definition (4.72) and antisymmetry of $a_{\alpha\beta}$.

The amplitude $\mathcal{E}_\alpha$ of the electric field can be decomposed in two components in the plane of polarization. Two tetrad vectors, $m^\alpha, \overline{m}^\alpha$, form the circular polarization basis in this plane, and vectors $e^\alpha_{(1)}, e^\alpha_{(2)}$ form a linear polarization basis. The decomposition reads

$$\mathcal{E}^\alpha = \mathcal{E}_L m^\alpha + \mathcal{E}_R \overline{m}^\alpha , \tag{4.73}$$

$$\mathcal{E}^\alpha = \mathcal{E}_{(1)} e^\alpha_{(1)} + \mathcal{E}_{(2)} e^\alpha_{(2)} , \tag{4.74}$$

where

$$\mathcal{E}_L = \frac{1}{2}\omega\Phi , \qquad \mathcal{E}_R = \frac{1}{2}\omega\overline{\Phi} , \tag{4.75}$$

are left and right circularly-polarized components of the electric field, and

$$\mathcal{E}_{(1)} = \frac{\omega}{\sqrt{8}}\left(\Phi + \overline{\Phi}\right) , \qquad \mathcal{E}_{(2)} = \frac{i\omega}{\sqrt{8}}\left(\Phi - \overline{\Phi}\right) , \tag{4.76}$$

are linearly polarized components of the electric field, and $\omega = -l_\alpha u^\alpha$ is the frequency of the electromagnetic wave.

The set $S = (I, Q, U, V)$ of electromagnetic Stokes parameters is defined with respect to two of the four vectors of the observer's tetrad $e^\alpha_{(\beta)}$ introduced in (4.48). We have [1, 105]

$$I = J_{\alpha\beta}\left[e^\alpha_{(1)} e^\beta_{(1)} + e^\alpha_{(2)} e^\beta_{(2)}\right] , \tag{4.77}$$

$$Q = J_{\alpha\beta}\left[e^\alpha_{(1)} e^\beta_{(1)} - e^\alpha_{(2)} e^\beta_{(2)}\right] , \tag{4.78}$$

$$U = J_{\alpha\beta}\left[e^\alpha_{(1)} e^\beta_{(2)} + e^\alpha_{(2)} e^\beta_{(1)}\right] , \tag{4.79}$$

$$V = iJ_{\alpha\beta}\left[e^\alpha_{(1)} e^\beta_{(2)} - e^\alpha_{(2)} e^\beta_{(1)}\right] , \tag{4.80}$$

where I is intensity, Q and U – linear polarizations, and V – circular polarization of the electromagnetic wave. It is important to emphasize that though the Stokes parameters have four components, they do not form a 4-dimensional vector. This is because they do not behave like a vector under the Lorentz transformations [1].

Using equation (4.72), the Stokes parameters can be expressed in the standard way [1] in a linear polarization basis $e^\alpha_{(1)}, e^\alpha_{(2)}$ as

$$I = < |\mathcal{E}_{(1)}|^2 + |\mathcal{E}_{(2)}|^2 > , \tag{4.81}$$

$$Q = < |\mathcal{E}_{(1)}|^2 - |\mathcal{E}_{(2)}|^2 > , \tag{4.82}$$

$$U = < \mathcal{E}_{(1)}\overline{\mathcal{E}}_{(2)} + \overline{\mathcal{E}}_{(1)}\mathcal{E}_{(2)} > , \tag{4.83}$$

$$V = i < \mathcal{E}_{(1)}\overline{\mathcal{E}}_{(2)} - \overline{\mathcal{E}}_{(1)}\mathcal{E}_{(2)} > , \tag{4.84}$$

where $\mathcal{E}_{(n)} = \mathcal{E}_\alpha e^\alpha_{(n)}$ for $n = 1, 2$. Under the gauge subgroup of the little group of l^α, the Stokes parameters remain invariant. However, for a constant rotation of angle Θ in the $(e^\alpha_{(1)}, e^\alpha_{(2)})$ polarization plane, one has

$$I' = I \; , \tag{4.85}$$
$$Q' = Q \cos 2\Theta + U \sin 2\Theta \; , \tag{4.86}$$
$$U' = U \cos 2\Theta - Q \sin 2\Theta \; , \tag{4.87}$$
$$V' = V \; . \tag{4.88}$$

This is what would be expected for a spin-1 field. That is, under a duality rotation of $\Theta = \pi/4$, one linear polarization state turns into the other, while the circular polarization state remains the same. The transformation properties (4.85)–(4.88) of the Stokes parameters point out that the Stokes parameters Q, U represent the linearly polarized components, and V represents the circularly polarized component.

The polarization vector $\boldsymbol{P} = (P_1, P_2, P_3)$ and the degree of polarization $P = |\boldsymbol{P}|$ of the electromagnetic radiation can be defined in terms of the normalized Stokes parameters by $\boldsymbol{P} = (Q/I, U/I, V/I)$. Any partially polarized wave may be thought of as an incoherent superposition of a completely polarized wave with the degree of polarization P and the polarization vector $\boldsymbol{P}$, and a completely unpolarized wave with the degree of polarization $1 - P$ and zero polarization vector, $\boldsymbol{P} = 0$, so that for arbitrary polarized radiation one has: $(I, Q, U, V) = (PI, Q, U, V) + (I - PI, 0, 0, 0)$. For completely polarized waves, vector $\boldsymbol{P}$ describes the surface of the unit sphere introduced by Poincaré [1]. The center of Poincaré sphere corresponds to unpolarized radiation and the interior to partially polarized radiation. Orthogonally polarized waves represent any two conjugate points on the Poincaré sphere. In particular, $(P_1 = \pm 1, P_2 = 0, P_3 = 0)$, and $(P_1 = 0, P_2 = 0, P_3 = \pm 1)$ represent orthogonally polarized waves corresponding to the linear and circular polarization bases, respectively.

4. Propagation Laws for the Stokes Parameters

Taking definition (4.70) of the electromagnetic tensor and accounting for the parallel transport of the null vectors l^α, m^α, $\overline{m}^\alpha$ along the light ray and the laws of propagation of the electromagnetic tensor given by equations (4.15), yield the law of propagation of the complex scalar functions Φ and $\overline{\Phi}$

$$\frac{d\Phi}{d\lambda} + \vartheta\Phi = 0 \; , \tag{4.89}$$
$$\frac{d\overline{\Phi}}{d\lambda} + \vartheta\overline{\Phi} = 0 \; , \tag{4.90}$$

where λ is an affine parameter along the ray and $\vartheta \equiv (1/2)\nabla_\alpha l^\alpha$ as given by equation (4.16). One notices that if $\mathcal{A}$ is a two-dimensional area of the cross section of the congruence of light rays lying in a surface of constant phase ϕ, then [2, 59, 102]

$$\frac{d\mathcal{A}}{d\lambda} - 2\vartheta\mathcal{A} = 0 \; . \tag{4.91}$$

Thus, the product, $\mathcal{A}|\Phi|^2$, remains constant along the congruence of light rays:

$$D_\lambda \left(\mathcal{A}|\Phi|^2\right) = 0 \; . \tag{4.92}$$

This law of propagation for the product $\mathcal{A}|\Phi|^2$ corresponds to the conservation of photon's flux [2, 59].

The law of conservation of the number of photons propagating along the light ray, corresponds to the propagation law of vector $|\Phi|^2 l^\alpha$. Indeed, taking covariant divergence of this quantity and making use of the equations (4.89), (4.90) along with definition (4.16) for the expansion ϑ of the bundle of light rays, yields [2, 59]

$$\nabla_\alpha \left(|\Phi|^2 l^\alpha\right) = 0 \; . \tag{4.93}$$

This equation assumes that the scalar amplitude Φ of the electromagnetic wave must be interpreted in terms of the number density of photons in phase space, $\mathcal{N}$, and the energy of one photon, $E_{\rm photon} = \hbar\omega$, as follows [2]

$$|\Phi| = \sqrt{8\pi}\hbar \left(\frac{\mathcal{N}}{E_{\rm photon}}\right)^{1/2} \; , \tag{4.94}$$

where the Planck reduced constant $\hbar = h/2\pi$ has been introduced to achieve consistency between the classical and quantum definitions of the energy of an electromagnetic wave.

Each of the Stokes parameter is proportional to the square of frequency of light, $\omega = -u^\alpha l_\alpha$, as directly follows from equations (4.81)–(4.84) and (4.76). Therefore, the variation of the Stokes parameters along the light ray can be obtained directly from their definitions (4.77)–(4.80) along the light ray and making use of the laws of propagation (4.89)–(4.91). However, the set of the Stokes parameters $S = (I, Q, U, V)$ is not directly observed in astronomy. Instead the set of four other polarization parameters $(F_\omega, P_1, P_2, P_3)$ is practically observed [59, 106]. Here $\boldsymbol{P} = (P_1, P_2, P_3)$ is the polarization vector as defined at the end of the preceding section, and F_ω is the *specific flux* of radiation (also known as the *monochromatic flux* of a light source [106]) entering a telescope from a given source. The specific flux is defined as an integral of the *specific intensity* (also known as the *surface brightness* [106]) of the radiation, $I_\omega \equiv I_\omega(\omega, \boldsymbol{l})$, over the total solid angle (assumed $\ll 4\pi$) subtended by the source on the observer's sky:

$$F_\omega = \int I_\omega(\omega, \boldsymbol{l}) d\Omega(\hat{\boldsymbol{l}}) \ , \tag{4.95}$$

where $\hat{\boldsymbol{l}} = (\sin\hat\theta\cos\hat\phi, \sin\hat\theta\sin\hat\phi, \cos\hat\theta)$ is the unit vector in the direction of the radiation flow and $d\hat\Omega(\boldsymbol{l}) = \sin\hat\theta d\hat\theta d\hat\phi$ is the element of the solid angle formed by light rays from the source and measured in the observer's local Lorentz frame.

The *specific intensity* I_ω of radiation at a given frequency $\omega = 2\pi\nu$, flowing in a given direction, $\hat{\boldsymbol{l}}$, as measured in a specific local Lorentz frame, is defined by

$$I_\omega = \frac{d(\text{energy})}{d(\text{time})d(\text{area})d(\text{frequency})d(\text{solid angle})} \ . \tag{4.96}$$

A simple calculation (see, for instance, the problem 5.10 in [107]), reveals that

$$\mathcal{N} = \frac{8\pi^3}{h^4}\frac{I_\omega}{\omega^3} \ , \tag{4.97}$$

where h is the Planck's constant. The number density $\mathcal{N}$ is invariant along the light ray and does not change under the Lorentz transformation. Invariance of $\mathcal{N}$ is a consequence of the kinetic equation for photons (radiative transfer equation) which in the case of gravitational field and without any other scattering processes, assumes the following form [2]

$$\frac{d\mathcal{N}}{d\lambda} = 0 \ . \tag{4.98}$$

Equations (4.97), (4.98) tell us that the ratio I_ω/ω^3 is invariant along the light-ray trajectory, that is

$$\frac{I_\omega}{I_{\omega_0}} = \left(\frac{\omega}{\omega_0}\right)^3 \ , \tag{4.99}$$

where ω_0, ω are frequency of light at the point of emission and observation respectively, I_{ω_0} is the *surface brightness* of the source of light at the point of emission, and I_ω is the *surface brightness* of the source of light at the point of observation.

Equations (4.89)–(4.99) make it evident that in the geometric optics approximation the gravitational field does not mix up the linear and circular polarizations of the electromagnetic radiation but can change its *surface brightness* I_ω due to gravitational (and Doppler) shift of the light frequency caused by the time-dependent part of the gravitational field of the isolated system emitting gravitational waves. Furthermore, the *monochromatic flux* from the source of radiation changes due to the distortion of the domain of integration in equation (4.95) caused by the gravitational light-bending effect. Taking into account that the gravitationally-unperturbed solid angle $d\Omega(\boldsymbol{k}) = \sin\theta d\theta d\phi$, and introducing the Jacobian, $J(\theta, \phi)$, of transformation between the spherical coordinates $(\hat\theta, \hat\phi)$ and (θ, ϕ) at the point of observation, one obtains that the measured monochromatic flux is

$$F_\omega = \int d\phi \int I_{\omega_0}(\theta, \phi) J(\theta, \phi)(\omega/\omega_0)^3 \sin\theta d\theta \ . \tag{4.100}$$

Equation (4.100) tells us that the monochromatic flux of the source of light can vary due to:

1. the gravitational Doppler shift of the electromagnetic frequency of light when it travels from the point of emission to the point of observation;

2. the change in the solid angle at the point of observation caused by the gravitational light deflection. The "magnification" matrix is the Jacobian of the transformation of the null directions on the celestial sphere generated by the bending of the light-ray trajectories by the gravitational field of the isolated system.

Orientation of two components, P_1 and P_2, of the polarization vector $\boldsymbol{P}$ at the point of observation differs from that taken at the point of emission of the electromagnetic signal due to the parallel transport of the reference tetrad, which changes its orientation with respect to the reference tetrad at infinity as it propagates along the light ray. This can be seen after taking into account equations (4.63), (4.64) where the rotational angle Θ is determined as a solution of the parallel transport equation (4.66) for the reference tetrad. Thus, the gravitational field changes the tilt angle of the polarization ellipse as light propagates along the light-ray trajectory. This effect was predicted by Skrotskii [108, 109] and discussed independently in [110, 111]. Its observation will play a significant role in a foreseeable future for detection of gravitational waves of cosmological origin by CMBR-radiometry space missions [112]. The third component of the polarization vector, P_3, remains the same along the light-ray trajectory because it represents the circularly polarized component of the radiation and is not affected by the rotation of the reference tetrad as it propagates along the light-ray path.

5. THE MATHEMATICAL TECHNIQUE FOR ANALYTIC INTEGRATION OF THE LIGHT-RAY EQUATIONS

This section provides mathematical technique for performing integration of equations of propagation of various characteristics of electromagnetic wave from the point of emission of light $\boldsymbol{x}_0$ to the point of its observation $\boldsymbol{x}$. The basic element, which is to be integrated, is the metric tensor perturbation $h_{\alpha\beta}(t,\boldsymbol{x})$ generated by the localized astronomical system and taken at the point lying on the light-ray trajectory. The metric tensor depends on the multipole moments taken at the retarded instant of time $s = t - r$ divided by the distance from the system r, and its time and spatial derivatives. These derivatives are handled by the method shown in this section. The main integrals are taken from the quantity $F(s)/r$, wherein $F(s)$ denotes a multipole moment of the gravitating system depending on the retarded time $s = t - r$. Calculation of the integrals and their derivatives is slightly different for stationary and non-stationary components of the metric tensor and are treated in the next two sections.

5.1. Light-ray Integrals from Monopole and Dipole

The monopole and dipole parts of the metric tensor (3.5)–(3.8) are formed by terms being proportional to the mass $\mathcal{M}$ and spin $\mathcal{S}^i$ of the isolated system. These terms as they appear in the solution of the light-ray equations, are given by integrals $[\mathcal{M}(t-r)/r]^{[-1]}$, $[\mathcal{M}(t-r)/r]^{[-2]}$ and $[\mathcal{S}^i(t-r)/r]^{[-1]}$, $[\mathcal{S}^i(t-r)/r]^{[-2]}$ respectively. In this section we shall assume for simplicity that mass $\mathcal{M}$ and spin $\mathcal{S}^i$ of the isolated system are constant during the time of propagation of light from the source of light to observer. The assumption about constancy of the mass $\mathcal{M}$ and spin $\mathcal{S}^i$ of the isolated system is valid as long as one neglects the energy emitted in the form of the gravitational waves by the isolated system under consideration. For light sources at the edge of our visible universe the characteristic time for emission of gravitational waves by an isolated system can be comparable with time the light takes to travel from the point of emission to observer. In this case the time evolution of the mass-monopole and spin-dipole is to be taken into account for correct calculation of the perturbations in propagation of light ray (see section 6.1 for more detail). Mass and spin of the isolated system can also change due to catastrophic disruption of the isolated system, for example, as a result of a supernova explosion. Specific details of how this process affects the light-ray propagation are not considered in the present paper but perhaps are worthwhile to studying.

In the case when mass and spin are constant, the integrals that we need to carry out are reduced to $[1/r]^{[-1]}$ and $[1/r]^{[-2]}$ which are formally divergent at the lower limit of integration at the past null infinity when time $t \to -\infty$. However, one must bear in mind that these integrals do not enter equations of light-ray geodesics (4.37) alone but appear in these equations after taking at least one partial derivative with respect to either ξ^i or τ parameters. This differentiation effectively eliminates divergent parts of the integrals from the final result. Hence, in what follows, we drop out the formally divergent terms so that the integrals under discussion assume the following form

$$\left[\frac{1}{r}\right]^{[-1]} \equiv \int_{-\infty}^{t} \frac{d\tau}{r} = \int \frac{d\tau}{\sqrt{d^2+\tau^2}} = -\ln\left[\frac{r(\tau)-\tau}{r_{\mathrm{E}}}\right] , \tag{5.1}$$

$$\left[\frac{1}{r}\right]^{[-2]} \equiv \int_{-\infty}^{t} \left[\frac{1}{r}\right]^{[-1]} d\tau = -\tau\ln\left[\frac{r(\tau)-\tau}{r_{\mathrm{E}}}\right] - r(\tau) , \tag{5.2}$$

where $\tau = t - t^*$ and we used equation (4.22) explicitly while expressing the distance $r = r_N(\tau)$ as a function of time τ. The constant distance $r_{\rm E}$ was introduced to make the argument of the logarithmic function dimensionless. This constant is not important for calculations as it always cancel out in final formulas. However, in the case of gravitational lensing it is convenient to identify the scale constant $r_{\rm E}$ with the radius of the Einstein ring [59–61]

$$r_{\rm E} = \left(\frac{4G\mathcal{M}}{c^2}\frac{rr_0}{R}\right)^{1/2} , \tag{5.3}$$

where r, r_0 are distances from the isolated system (the deflector of light) to observer and the source of light, and $R = \tau + \tau_0$. The radius of the Einstein ring is a characteristic distance separating naturally the case of weak gravitational lensing ($d > r_{\rm E}$) from the strong one ($d < r_{\rm E}$). The Einstein ring as visible in the sky, has an angular size that is given by

$$\theta_{\rm E} = \frac{r_{\rm E}}{r} = \left(\frac{4G\mathcal{M}}{c^2}\frac{r_0}{Rr}\right)^{1/2} . \tag{5.4}$$

The angular Einstein radius $\theta_{\rm E}$ defines the angular scale for a lensing situation. For cosmology, typical mass $M \simeq 10^{12} M_\odot$ and distances r, r_0, and R are of order 1 gigaparsec. Consequently, the angular Einstein radius $\theta_{\rm E}$ is of order one arcsecond and the linear radius $r_{\rm E}$ is of order 1 kiloparsec. A star within our galaxy having $M \simeq M_\odot$ and distances r, r_0, and R of order 10 kiloparsec has an angular Einstein radius $\theta_{\rm E}$ of order a milli-arcsecond and the linear radius $r_{\rm E}$ of order 10 astronomical units (10 AU).

5.2. Light-ray Integrals from Quadrupole and Higher-Order Multipoles

Time-dependent terms in the metric tensor (3.5) – (3.8) result from the multipole moments which can be either periodic (a binary system) or aperiodic (a supernova explosion) functions of time. The most straightforward way to calculate the impact of the gravitational field of such a source on the propagation of light is to decompose its multipole moments $F(s) \Leftrightarrow (\mathcal{I}_L(s), \mathcal{S}_L(s))$ in the Fourier series [83, 113]

$$F(t-r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{F}(\tilde{\omega}) e^{i\tilde{\omega}(t-r)} d\tilde{\omega} , \tag{5.5}$$

substitute this decomposition to the light-ray propagation equations and integrate term by term. This method makes an impression that in order to obtain the final result of the integration the (complex-valued) Fourier image $\tilde{F}(\tilde{\omega})$ of the multipole moments of the isolated system must be specified explicitly, otherwise the convolution of the integrated Fourier series will be impossible. However, this is not true, at least in general relativity, and, as we shall show later, the explicit structure of $\tilde{F}(\tilde{\omega})$ is irrelevant for subsequent general-relativistic calculations. This is explained, if one recollects that in general relativity gravitational field propagates with the same speed as light. In alternative theories of gravity the speed of gravity and light may be different, so it can lead to appearance of terms proportional to the difference between the speed of gravity and light that will drastically complicates calculations which will require the Fourier images of gravitational-wave sources as explicit functions of $\tilde{\omega}$ [114]. All such terms are, however, cancelled out in general relativity exactly, thus making calculations manageable and applicable for any source of gravitational waves.

We notice that the integration of the light-ray equations is effectively reduced to the calculation of only two types of integrals along the light ray: $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$, where $F(t-r)$ denotes any type of the time-dependent multipole moments of the gravitational field of the localized astronomical system. These integrals can be performed after introducing a new variable [83]

$$y \equiv s - t^* = \tau - r(\tau) = \tau - \sqrt{d^2 + \tau^2} , \tag{5.6}$$

which has a unique physical meaning of the retarded interval of time between two space-time events: position of photon $x^\alpha = (\tau, \boldsymbol{x})$ on the light ray and the center of mass of the isolated system $z^\alpha = (y, \boldsymbol{0})$. It is important to realize that the equation (5.6) is a retarded solution of the null cone equation in flat space-time

$$\eta_{\alpha\beta}(x^\alpha - z^\alpha)(x^\beta - z^\beta) = 0 , \tag{5.7}$$

describing propagation of gravity from the isolated system to the photon. This retardation of gravity effect has its origin in the time argument of the solution (3.5)–(3.8) of the linearized Einstein equations and is due to the finite speed of propagation of gravity. The retardation of gravity effect was predicted in [115] and experimentally tested with the precision 20% in the jovian light-deflection experiment in September 2002 [116]. This replacement of the time argument from τ to the retarded time $y = \tau - r(\tau)$ allows us to perform integration of equations of light-ray geodesics completely without making specific assumptions about time dependence of the multipole moments. It is worth noting that while the parameter τ runs from $-\infty$ to $+\infty$, the retarded time y runs from $-\infty$ to 0; that is, y is always negative, $y \le 0$.

Equation (5.6) leads to other useful transformations

$$\tau = \frac{y^2 - d^2}{2y}\ , \qquad \sqrt{d^2+\tau^2} = -\frac{1}{2}\frac{d^2+y^2}{y}\ , \qquad d\tau = \frac{1}{2}\frac{d^2+y^2}{y^2}dy. \tag{5.8}$$

Making use of the new variable y and relationships (5.8) the integrals under discussion can be displayed as follows:

$$\left[\frac{F(t-r)}{r}\right]^{[-1]} = -\int_{-\infty}^{y} \frac{F(t^*+\zeta)}{\zeta}d\zeta\ , \tag{5.9}$$

$$\left[\frac{F(t-r)}{r}\right]^{[-2]} = -\frac{1}{2}\int_{-\infty}^{y}\int_{-\infty}^{\eta} \frac{F(t^*+\zeta)}{\zeta}d\zeta d\eta - \frac{d^2}{2}\int_{-\infty}^{y}\frac{1}{\eta^2}\int_{-\infty}^{\eta} \frac{F(t^*+\zeta)}{\zeta}d\zeta d\eta\ , \tag{5.10}$$

where ζ, η are new variables of the integration replacing integration along the light-ray trajectory by that along a null characteristic of the gravitational field, and t^* is the time of the closest approach of photon to the origin of the coordinate system. The time t^* has no special physical meaning in general case and appears in calculations as an auxiliary (constant) parameter which vanishes in the final result. It can be used as an approximation of the retarded time $s = t - r$ in the case of a small impact parameter d of the light ray to the isolated astronomical system, that is $s \simeq t^* + O(d^2/cr)$. Notice that for observer at spatial infinity $r = \infty$ and the retarded time $s = t^*$ which justifies dependence of the total angle of the deflection of light on, otherwise unphysical, value of t^* adopted, for example, in [82].

A remarkable property of the integrals in the right side of equations (5.9), (5.10) is that they depend on the parameters ξ^i and τ only through either the upper limit of the integrals which is the variable $y = \tau - \sqrt{d^2+\tau^2}$, or the square of the impact parameter, $d^2 = (\boldsymbol{\xi}\cdot\boldsymbol{\xi})$ standing in front of the second integral in the right side of equation (5.10). For this reason, differentiation of the integrals in the left part of equations (5.9), (5.10) with respect to either ξ^i or τ will effectively eliminate the integration along the light ray trajectory. For example,

$$\hat{\partial}_i\left\{\left[\frac{F(t-r)}{r}\right]^{[-1]}\right\} = -\frac{F(t^*+y)}{y}\,\hat{\partial}_i y = \frac{\xi^i}{yr}\,F(t-r)\ . \tag{5.11}$$

Similar calculation can be easily performed in case of differentiation of integral $[F(t-r)/r]^{[-1]}$ with respect to τ. It results in

$$\hat{\partial}_\tau\left\{\left[\frac{F(t-r)}{r}\right]^{[-1]}\right\} = -\frac{F(t^*+y)}{y}\,\hat{\partial}_\tau y = -\frac{F(t^*+y)}{y}\left(1-\frac{\tau}{r}\right) = \frac{F(t-r)}{r}\ , \tag{5.12}$$

as it could be expected because the partial differentiation with respect to τ keeps ξ^i and t^* fixed and, hence, is equivalent to taking a total time derivative with respect to t along the light ray.

Since all terms in the solution of the light geodesic equation are represented as derivatives from the integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$ with respect to the parameters ξ^i and/or τ, it is clear from equations (5.11), (5.12) that the solution will not contain any single integral $[F(t-r)/r]^{[-1]}$ at all; only integrands of this integral will appear. We notice that taking first and second derivatives of the double integral $[F(t-r)/r]^{[-2]}$ do not eliminate the integration along the light ray trajectory

$$-\hat{\partial}_k\left\{\left[\frac{F(t-r)}{r}\right]^{[-2]}\right\} = \xi^k\left\{\frac{1}{y}\left[\frac{F(t-r)}{r}\right]^{[-1]} - \int_{-\infty}^{y}\frac{1}{\eta^2}\int_{-\infty}^{\eta}\frac{F(t^*+\zeta)}{\zeta}\,d\zeta d\eta\right\}\ , \tag{5.13}$$

$$\hat{\partial}_{jk}\left\{\left[\frac{F(t-r)}{r}\right]^{[-2]}\right\} = \frac{\xi^k\xi^j}{y^2}\frac{F(t-r)}{r} + P^{jk}\left\{\frac{1}{y}\left[\frac{F(t-r)}{r}\right]^{[-1]} - \int_{-\infty}^{y}\frac{1}{\eta^2}\int_{-\infty}^{\eta}\frac{F(t^*+\zeta)}{\zeta}\,d\zeta d\eta\right\}\ , \tag{5.14}$$

However, taking one more (a third) derivative eliminates all integrals from equation (5.14) completely. More specifically, we have

$$\hat{\partial}_{ijk}\left\{\left[\frac{F(t-r)}{r}\right]^{[-2]}\right\} = \frac{1}{y}\left\{\left(P^{ij}+\frac{\xi^{ij}}{yr}\right)\hat{\partial}_k + P^{jk}\hat{\partial}_i + \xi^j\hat{\partial}_{ik}\right\}\left[\frac{F(t-r)}{r}\right]^{[-1]} , \tag{5.15}$$

and making use of equations (5.11), (5.14) for explicit calculation of partial derivatives in the right part of equation (5.15) proves that all integrations disappear, that is

$$\hat{\partial}_{ijk}\left\{\left[\frac{F(t-r)}{r}\right]^{[-2]}\right\} = \frac{P^{ij}\xi^k + P^{jk}\xi^i + P^{ik}\xi^j}{y^2 r}F(t-r) + \frac{\xi^i\xi^j\xi^k}{y^2r^2}\left[\left(\frac{2}{y}-\frac{1}{r}\right)F(t-r) - \dot{F}(t-r)\right] , \tag{5.16}$$

where $\dot{F}(t-r) \equiv \partial_t F(t-r)$. The same kind of reasoning works for the third derivatives with respect to the parameter τ and to the mixed derivatives taken with respect to both ξ^i and τ.

We shall obtain solution of the light geodesic in terms of STF derivatives with respect to the impact parameter ξ^i of the light ray acting on the single and double integrals having a symbolic form $\hat{\partial}_{<a_1 \dots a_k>}[F(t-r)/r]^{[-1]}$ and $\hat{\partial}_{<a_1 \dots a_k>}[F(t-r)/r]^{[-2]}$. Explicit expression for this STF derivative can be obtained directly by applying differentiation rules shown in equations (5.13) – (5.15). In what follows, we shall see that solutions of equations of light rays will always contain three or more derivatives acting on the integrals from the higher-order multipole moments (multipolarity $l \geq 2$) which have the same structure as that shown in equations (5.9), (5.10). It means that the final result of the integration of the light-ray propagation equations depending on the higher-order multipoles of the localized astronomical system can be expressed completely in terms of these multipoles taken at the retarded instant of time $s = t - r$. In other words, motion of photon in *general relativity* does not depend on the past history of its propagation and the effects caused by temporal variations of the gravitational field (gravitational waves) are not accumulating [117]. On the other hand, the past history of the isolated system can affect the propagation of the light ray through the tails of the gravitational waves contributing to the multipole moments of the system as given by equations (3.12), (3.13).

6. RELATIVISTIC PERTURBATIONS OF THE LIGHT-RAY PROPAGATION

6.1. Relativistic Perturbation of the Electromagnetic-Wave Eikonal

Perturbation of eikonal (phase) of electromagnetic signal propagating from the point $\boldsymbol{x}_0$ to the point $\boldsymbol{x}$ are obtained by solving equation (4.35). This solution is found by integrating the metric tensor perturbation along the light-ray trajectory and can be written down in the linearized approximation as an algebraic sum of three separate terms

$$\psi = \psi_{(G)} + \psi_{(M)} + \psi_{(S)} , \tag{6.1}$$

where $\psi_{(G)}$ represents the gauge-dependent part of the eikonal, and $\psi_{(M)}$, $\psi_{(S)}$ are eikonal's perturbations caused by the mass and spin multipole moments correspondingly. Their explicit expressions are as follows

$$\psi_{(G)} = (k^i\varphi^i - \varphi^0) + (k^i w^i - w^0) \;, \tag{6.2}$$

$$\begin{aligned}\psi_{(M)} &= 2\left[\frac{\mathcal{M}(t-r)}{r}\right]^{[-1]} + 2\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\\ &\times\left\{\left(1+\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\left[\frac{\mathcal{I}^{(p)}_{A_l}(t-r)}{r}\right]^{[-1]}\right.\\ &\left.-\frac{2p}{l-1}k_{<a_1\ldots a_{p-1}}\hat{\partial}_{a_p\ldots a_{l-1}>}\left[\frac{k^i\mathcal{I}^{(p)}_{iA_{l-1}}(t-r)}{r}\right]^{[-1]}\right\}\end{aligned} \tag{6.3}$$

$$\begin{aligned}\psi_{(S)} &= 2\epsilon_{abi}k^a\mathcal{S}^b\hat{\partial}_i\ln(r-\tau) + 4\hat{\partial}_a\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\\ &\times\left(1-\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{k^i\epsilon_{iba}\mathcal{S}^{(p)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]} \;,\end{aligned} \tag{6.4}$$

where $H(p-q)$ is the Heaviside function defined by the expression (1.18), $C(p,q)$ is the polynomial coefficient given by equation (1.17) and $F^{(p)}(t-r)$ denotes $\partial^p F(t-r)/\partial t^p$ where r is considered as constant.

We note that the gauge-dependent part of the eikonal, $\psi_{(G)}$, contains combination of terms $k^i\varphi^i - \varphi^0$ defined by equations (C.7)–(C.9) which can be, in principle, eliminated with an appropriate choice of the gravitational field gauge functions w^0 and w^i. However, such a procedure will introduce a reference frame in the sky with a coordinate grid being sensitive to the direction to a source of light rays; that is, to the unit vector k^i. The coordinate frame obtained in this way will have the direction-dependent distortions leading to its trifling practical usage. For this reason we do not recommend the elimination of functions φ^0 and φ^i from equation (6.2) and give preference to the ADM-harmonic coordinate system which admits a much simpler and unique treatment of observable relativistic effects. Thus, we leave the functions φ^0 and φ^i in equation (6.2) where the gauge functions w^0 and w^i are defined by formulas (C.2), (C.3).

Expressions (6.3), (6.4) for the mass- and spin-multipoles of the eikonal contains integrals. However, one can easily prove that the integrals from all high-order multipoles are eliminated. Indeed, scrutiny inspection of equations (6.3), (6.4) elucidates that all the integrals from the high-order multipole moments enter the equation in combination with at least one derivative with respect to the impact parameter vector ξ^i. Hence, the differentiation rule (5.11) is applied which eliminates the integrals from the high-order multipoles in the eikonal.

The integral, which remains unperformed, is that from the mass monopole in equation (6.3). It can be taken by parts as follows

$$\left[\frac{\mathcal{M}(t-r)}{r}\right]^{[-1]} = -\mathcal{M}(t-r)\ln(r-\tau) + \int_{-\infty}^{y}\dot{\mathcal{M}}(t^*+\zeta)\ln\zeta d\zeta \;. \tag{6.5}$$

If one assumes that the mass $\mathcal{M}$ is constant, then the second term in the right side of equation (6.5) vanishes and the eikonal does not contain any integral dependence on the past history of the light propagation. This assumption is usually implied, for example, in the theory of gravitational lensing [59–61] and other numerous applications of the relativistic theory of light propagation. Here, we extend our approach to take into account the case of $\dot{\mathcal{M}} \neq 0$. In the most general case the lost of energy by the isolated system caused by emission of gravitational waves is given by [1, 2]

$$\dot{\mathcal{M}}(t) = -\frac{G}{c^7}\mathcal{I}^{(3)}_{ij}(t)\mathcal{I}^{(3)}_{ij}(t) + O\left(c^{-9}\right) \;, \tag{6.6}$$

where $\mathcal{I}^{(3)}_{ij}$ represents a third time derivative from the mass quadrupole moment, and terms of order $O\left(c^{-9}\right)$ describe contribution of higher-order mass and spin multipoles. Accounting for the contribution of the gravitational waves given by equation (6.6) yields

$$\left[\frac{\mathcal{M}(t-r)}{r}\right]^{[-1]} = -\mathcal{M}(t-r)\ln(r-\tau) - \frac{G}{c^7}\int_{-\infty}^{y}\mathcal{I}^{(3)}_{ij}(t^*+\zeta)\mathcal{I}^{(3)}_{ij}(t^*+\zeta)\ln\zeta d\zeta + O\left(c^{-9}\right) \;. \tag{6.7}$$

It is instructive to evaluate contribution of the second term in the right side of equation (6.5) in the case of light propagating in gravitational field of a binary system consisting of two stars with masses m_1 and m_2 orbiting each other along a circular orbit. The total mass $\mathcal{M}$ of the system is defined in accordance with equation (3.10) which takes into account the first post-Newtonian correction

$$\mathcal{M}(t) = m_1 + m_2 - \frac{1}{2}\frac{G\mu\mathcal{M}}{c^2 a(t)} + O\left(c^{-4}\right) \ , \tag{6.8}$$

where $\mu = m_1 m_2/\mathcal{M}$ is the reduced mass of the system and $a = a(t)$ is the orbital radius of the system. Assuming that the lost of the orbital energy is due to the emission of gravitational waves from the system in accordance with the quadrupole formula approximation [1, 2], the time evolution of the orbital radius is defined by equation [1, 2]

$$\dot{a} = -\frac{64G^3}{5c^5}\frac{\mu\mathcal{M}^2}{a^3} \ . \tag{6.9}$$

It has a simple solution [2]

$$a(t) \ = \ a_0\left(1 - \frac{\tau}{T}\right)^{1/4} \ , \tag{6.10}$$

$$T \ = \ \frac{5c^5}{256G^3}\frac{a_0^4}{\mu\mathcal{M}^2} \ , \tag{6.11}$$

where $\tau = t - t^*$, t^* is the time of the closest approach of light ray to the binary system, $a_0 = a(t^*)$ is the orbital radius of the binary system at the time of the closest approach, and T is the spiral time of the binary. The lost of the total mass due to the emission of the gravitational waves is

$$\dot{\mathcal{M}} = -\frac{G}{8c^2}\frac{\mu\mathcal{M}}{a_0 T}\left(1 - \frac{\tau}{T}\right)^{-5/4} \ . \tag{6.12}$$

Substituting equation (6.12) to the right side of equation (6.5) and performing integration yields

$$\int_{-\infty}^{y}\dot{\mathcal{M}}(t^* + \zeta)\ln\zeta d\zeta = -\frac{1}{2}\frac{G\mu\mathcal{M}}{c^2 a(s)}\ln(r-\tau) + \frac{1}{2}\frac{G\mu\mathcal{M}}{c^2 a_0}\left\{\ln\left[\frac{a(s)-a_0}{a(s)+a_0}\right] + 2\arctan\left[\frac{a(s)}{a_0}\right]\right\} \ , \tag{6.13}$$

where $a(s) \equiv a(t-r)$ denotes the orbital radius of the binary system taken at the retarded time $s = t - r$. Putting all terms in equations (6.5), (6.8), (6.13) together yields

$$\left[\frac{\mathcal{M}(t-r)}{r}\right]^{[-1]} = -(m_1 + m_2)\ln(r-\tau) + \frac{1}{2}\frac{G\mu\mathcal{M}}{c^2 a_0}\left\{\ln\left[\frac{a(s)-a_0}{a(s)+a_0}\right] + 2\arctan\left[\frac{a(s)}{a_0}\right]\right\} \ , \tag{6.14}$$

where the first term in the right side of this equation is the standard Shapiro time delay in the gravitational field of the binary system with constant total mass $m_1 + m_2$, and the second term represents relativistic correction due to the emission of gravitational waves by the system causing the overall loss of its orbital energy.

Eikonal describes propagation of a wave front of the electromagnetic wave. Light rays are orthogonal to the wave front and their trajectories can be easily calculated as soon as the eikonal is known. In the present paper we do not use this technique and obtain solution for the light rays directly from the light-ray geodesic equations that gives identical results.

6.2. Relativistic Perturbation of the Light-Particle Velocity

Integration of the light-ray propagation equations (4.40) – (4.43) is fairly straightforward. Performing one integration of these equations with respect to time yields

$$\dot{x}^i(\tau) \ = \ k^i + \dot{\Xi}^i(\tau, \boldsymbol{\xi}) \ , \tag{6.15}$$

$$\dot{\Xi}^i(\tau, \boldsymbol{\xi}) \ = \ \underset{(G)}{\dot{\Xi}^i}(\tau, \boldsymbol{\xi}) + \underset{(M)}{\dot{\Xi}^i}(\tau, \boldsymbol{\xi}) + \underset{(S)}{\dot{\Xi}^i}(\tau, \boldsymbol{\xi}) \ , \tag{6.16}$$

where the relativistic perturbations of photon's trajectory are given by

$$\underset{(G)}{\dot{\Xi}^i}(\tau, \boldsymbol{\xi}) = \hat{\partial}_\tau\left[(\varphi^i - k^i\varphi^0) + (w^i - k^i w^0)\right] \ , \tag{6.17}$$

$$\dot{\underset{(M)}{\Xi}}{}^{i}(\tau,\boldsymbol{\xi}) = 2(\hat{\partial}_i - k_i\hat{\partial}_\tau)\left[\frac{\mathcal{M}}{r}\right]^{[-1]} + 2\hat{\partial}_i\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)H(2-q)\times \tag{6.18}$$
$$\left(1-\frac{p-q}{l}\right)\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right]^{[-1]} -$$
$$2\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\left\{\left(1+\frac{p}{l-1}\right)k_{i<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\left[\frac{\mathcal{I}^{(p)}_{A_l}(t-r)}{r}\right] -\right.$$
$$\left.\frac{2p}{l-1}k_{<a_1\ldots a_{p-1}}\hat{\partial}_{a_p\ldots a_{l-1}>}\left[\frac{\mathcal{I}^{(p)}_{iA_{l-1}}(t-r)}{r}\right]\right\},$$

$$\dot{\underset{(S)}{\Xi}}{}^{i}(\tau,\boldsymbol{\xi}) = 2k_j\hat{\partial}_{ia}\left[\frac{\epsilon_{jba}\mathcal{S}_b}{r}\right]^{[-1]} - 2\hat{\partial}_a\frac{\epsilon_{iba}\mathcal{S}_b}{r} - \tag{6.19}$$
$$4k_j\hat{\partial}_{ia}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)H(2-q)\times$$
$$\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^q_\tau\left[\frac{\epsilon_{jba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]} +$$
$$4\left(\hat{\partial}_a - k_a\hat{\partial}_{t^*}\right)\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\times$$
$$\left(1-\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{iba}\mathcal{S}^{(p)}_{bA_{l-1}}(t-r)}{r}\right] +$$
$$4k_j\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{jba_{l-1}}\mathcal{S}^{(p+1)}_{ibA_{l-2}}(t-r)}{r}\right].$$

Here $H(p-q)$ is a Heaviside function defined by the expression (1.18) and C_l are the polynomial coefficients (1.17). The gauge functions are given in appendix C. Mass monopole and spin dipole terms are written down in equations (6.18), (6.19) in symbolic form and after taking derivatives are simplified

$$2(\hat{\partial}_i - k_i\hat{\partial}_\tau)\left[\frac{\mathcal{M}}{r}\right]^{[-1]} = \frac{2\mathcal{M}}{r}\left(\frac{\xi^i}{y} - k^i\right), \tag{6.20}$$

$$2k_j\hat{\partial}_{ia}\left[\frac{\epsilon_{jba}\mathcal{S}_b}{r}\right]^{[-1]} = 2k^j\epsilon_{jba}\hat{\partial}_a\left(\frac{\mathcal{S}_b\xi^i}{yr}\right). \tag{6.21}$$

Remaining integrals shown in the right side of equations (6.18), (6.19) are convenient for presentation of the result of integration in the symbolic form. The integrals are actually eliminated because they are always appear in combination with, at least, one operator of derivative $\hat{\partial}_i$ with respect to the unperturbed value of the impact parameter of the light ray. The derivative operator acts on the integrals in accordance with equation (5.11) converting the integrals into functions of the retarded time $y = \tau - r(\tau)$

$$\hat{\partial}_i\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right]^{[-1]} = \mathcal{I}^{(p-q)}_{A_l}(t^*+y)\hat{\partial}_i\ln(-y), \tag{6.22}$$

$$\hat{\partial}_i\left[\frac{\epsilon_{jba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]} = \epsilon_{jba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t^*+y)\hat{\partial}_i\ln(-y). \tag{6.23}$$

We conclude that at each point of the wave front of the electromagnetic wave the relativistic perturbation of the direction of propagation of light ray (wave vector) caused by the time-dependent gravitational field of the isolated system depends only on the value of its multipole moments taken at the retarded instant of time and it does not depend on the past history of the light propagation.

6.3. Perturbation of the Light-Ray Trajectory

Integration of equation (6.15) with respect to time yields the relativistic perturbation of the trajectory of the light ray

$$x^i(\tau) = x^i_N + \Delta \underset{(G)}{\Xi^i} + \Delta \underset{(M)}{\Xi^i} + \Delta \underset{(S)}{\Xi^i} \,, \tag{6.24}$$

where

$$\Delta \underset{(G)}{\Xi^i} \equiv \underset{(G)}{\Xi^i}(\tau,\boldsymbol{\xi}) - \underset{(G)}{\Xi^i}(\tau_0,\boldsymbol{\xi}) \,, \qquad \Delta \underset{(M)}{\Xi^i} \equiv \underset{(M)}{\Xi^i}(\tau,\boldsymbol{\xi}) - \underset{(M)}{\Xi^i}(\tau_0,\boldsymbol{\xi}) \,, \qquad \Delta \underset{(S)}{\Xi^i} \equiv \underset{(S)}{\Xi^i}(\tau,\boldsymbol{\xi}) - \underset{(S)}{\Xi^i}(\tau_0,\boldsymbol{\xi}) \,. \tag{6.25}$$

Herein the term

$$\underset{(G)}{\Xi^i}(\tau,\boldsymbol{\xi}) = (\varphi^i - k^i\varphi^0) + (w^i - k^i w^0) \,, \tag{6.26}$$

is the gauge-dependent part of the trajectory's perturbation, and the physically meaningful perturbations due to the mass and spin multipoles

$$\begin{aligned}\underset{(M)}{\Xi^i}(\tau,\boldsymbol{\xi}) = {} & 2(\hat{\partial}_i - k_i\hat{\partial}_\tau)\left[\frac{\mathcal{M}}{r}\right]^{[-2]} + 2\hat{\partial}_i\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)H(2-q)\times \\ & \left(1-\frac{p-q}{l}\right)\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right]^{[-2]} - \\ & 2\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\left\{\left(1+\frac{p}{l-1}\right)k_{i<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\left[\frac{\mathcal{I}^{(p)}_{A_l}(t-r)}{r}\right]^{[-1]} - \right. \\ & \left.\frac{2p}{l-1}k_{<a_1\ldots a_{p-1}}\hat{\partial}_{a_p\ldots a_{l-1}>}\left[\frac{\mathcal{I}^{(p)}_{iA_{l-1}}(t-r)}{r}\right]^{[-1]}\right\} \,,\end{aligned} \tag{6.27}$$

$$\begin{aligned}\underset{(S)}{\Xi^i}(\tau,\boldsymbol{\xi}) = {} & 2k_j\hat{\partial}_{ia}\left[\frac{\epsilon_{jba}\mathcal{S}_b}{r}\right]^{[-2]} - 2\hat{\partial}_a\left[\frac{\epsilon_{iba}\mathcal{S}_b}{r}\right]^{[-1]} - \\ & 4k_j\hat{\partial}_{ia}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)H(2-q)\times \\ & \left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^q_\tau\left[\frac{\epsilon_{jba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]^{[-2]} + \\ & 4\left(\hat{\partial}_a - k_a\hat{\partial}_{t^*}\right)\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\times \\ & \left(1-\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{iba}\mathcal{S}^{(p)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]} + \\ & 4k_j\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{jba_{l-1}}\mathcal{S}^{(p+1)}_{\hat{i}bA_{l-2}}(t-r)}{r}\right]^{[-1]} \,.\end{aligned} \tag{6.28}$$

Here $H(p-q)$ is again a Heaviside function defined by equation (1.18) and C_l are the polynomial coefficients (1.17).

Relativistic perturbations (6.27), (6.28) contain two types of integrals along the light ray – $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$. In section 5 we have shown that taking a derivative from the integral $[F(t-r)/r]^{[-1]}$ effectively reduces the integral to the ordinary function of the retarded time $s = t - r$ with no dependence on the past history of the propagation of light ray. In order to eliminate second order integrals $[F(t-r)/r]^{[-2]}$ one needs to take three or more derivatives. One can notice that in (6.27), (6.28) there are some terms in which the number of derivatives is not sufficient to eliminate the integrals. Nevertheless all such terms in (6.27), (6.28) are cancelled out because the numerical factors standing in front of these terms get nullified. As an example let us examine equation (6.27) for $\underset{(M)}{\Xi^i}(\tau, \boldsymbol{\xi})$. In this expression the integral $\left[\mathcal{I}^{(p-q)}_{A_l}(t-r)/r\right]^{[-2]}$ is differentiated $l-p+1$ times with respect to the impact parameter (derivative $\hat{\partial}_a$) and q times with respect to the parameter τ (derivative $\hat{\partial}_\tau$). Expression $\hat{\partial}_{<a_1 \ldots a_{l-p+1}>}\hat{\partial}^q_\tau\left[\mathcal{I}^{(p-q)}_{A_l}(t-r)/r\right]^{[-2]}$ can contain the integral dependence if either (a) $q = 0$, $p \in (l-1, l)$ or (b) $q = 1$, $p = l$. In any case either factor $1-(p-q)/l$ or that $1-(p-q)/(l-1)$ will be zero, thus, annihilating the integrals from the mass multipoles in equation (6.27). Similar consideration can be done for the spin-dependent perturbation $\underset{(S)}{\Xi^i}(\tau, \boldsymbol{\xi})$ which shows that there is no integral dependence of the spin multipole moments on the past history of the light propagation.

We emphasize, however, that the integral dependence of the light-ray perturbation on the past history remains in equation (6.26) which contains the gauge functions φ^α and w^α used later in this paper for interpretation of observable effects of the gravitational waves. The past-history dependence of the light-ray trajectory comes into play also through the integrals from mass monopole and spin-dipole terms, $[\mathcal{M}/r]^{[-2]}$ and $[\mathcal{S}/r]^{[-2]}$, if mass and/or spin are not conserved and change as time passes. The non-conservation can be caused, for example, by emission of gravitational waves carrying away the orbital energy and angular momentum of the system. The past-history contribution of these integrals can be calculated similarly to eikonal perturbation (see equations (6.6)–(6.14)) but we do not carry out this calculation in the present paper.

Solution of the light-ray equation with the initial-boundary conditions (4.17) depends on the unit vector $\boldsymbol{k}$ defining direction of the light-ray propagation extrapolated backward in time to the past null infinity. In real practice the light ray is emitted at the point $\boldsymbol{x}_0$ where the source of light is located at time t_0, and it arrives to observer at time t to the point $\boldsymbol{x}$ separated from the source of light by finite coordinate distance $R = |\boldsymbol{x} - \boldsymbol{x}_0|$. Therefore, solution of the light-ray equations must be expressed in this case in terms of the integrals of the boundary-value problem. It is formulated in terms of initial $\boldsymbol{x}_0$ and final $\boldsymbol{x}$ positions of the photon

$$\boldsymbol{x}(t) = \boldsymbol{x}\,, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0\,, \tag{6.29}$$

and assumes that the solution of the light-ray equation depends on the unit Euclidean vector

$$K^i = -\frac{x^i - x^i_0}{|\boldsymbol{x} - \boldsymbol{x}_0|} \tag{6.30}$$

that defines the direction from the observer towards the source of light and may be interpreted as a coordinate direction in the Euclidean space.

In what follows it is convenient to make use of the astronomical coordinates $\boldsymbol{x} \equiv x^i = (x^1, x^2, x^3)$ based on a triad of the unit vectors $(\boldsymbol{I}_0, \boldsymbol{J}_0, \boldsymbol{K}_0)$ (see Figure 2). Vector $\boldsymbol{K}_0$ points from the observer towards the deflector, and vectors $\boldsymbol{I}_0$ and $\boldsymbol{J}_0$ lie in the *plane of the sky* defined as the plane being orthogonal to vector $\boldsymbol{K}_0$. The unit vector $\boldsymbol{I}_0$ is directed to the east, and that $\boldsymbol{J}_0$ points towards the north celestial pole. The origin of the coordinate system is chosen to lie at the barycenter of the isolated system which emits gravitational waves.

Another reference frame based on a triad of the unit vectors $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ rotated with respect to vectors $(\boldsymbol{I}_0, \boldsymbol{J}_0, \boldsymbol{K}_0)$ is useful as well. Vector $\boldsymbol{K}$ points from the observer towards the source of light, and vectors $\boldsymbol{I}$ and $\boldsymbol{J}$ lie in the *plane of the sky* defined to be orthogonal to vector $\boldsymbol{K}$. We emphasize that this plane is different from the *plane of the sky* being orthogonal to vector $\boldsymbol{K}_0$. This is because the *plane of the sky* is actually a tangent plane to the unit sphere, and vectors $\boldsymbol{K}$ and $\boldsymbol{K}_0$ point in different directions on it. Mutual orientation of one triad with respect to another one is determined by the following orthogonal transformation

$$\boldsymbol{I}_0 = \boldsymbol{I}\cos\Omega + \boldsymbol{J}\sin\Omega\,, \tag{6.31}$$

$$\boldsymbol{J}_0 = -\boldsymbol{I}\cos\theta\sin\Omega + \boldsymbol{J}\cos\theta\cos\Omega + \boldsymbol{K}\sin\theta\,, \tag{6.32}$$

$$\boldsymbol{K}_0 = \boldsymbol{I}\sin\theta\sin\Omega - \boldsymbol{J}\sin\theta\cos\Omega + \boldsymbol{K}\cos\theta\,, \tag{6.33}$$

where the rotational angles Ω and θ are constant.

It is rather straightforward to obtain solution of the boundary value problem for light propagation in terms of the unit vector $\boldsymbol{K}$. All what one needs is to convert the unit vector $\boldsymbol{k}$ to $\boldsymbol{K}$ written in terms of spatial coordinates of the points of emission, $\boldsymbol{x}_0$, and observation, $\boldsymbol{x}$, of the light ray. From formula (6.24) one has

$$k^i \;=\; -K^i - \beta^i(\tau,\boldsymbol{\xi}) \;, \tag{6.34}$$

where the relativistic correction $\beta^i(\tau,\boldsymbol{\xi})$ to the vector K^i is defined as follows

$$\beta^i(\tau,\boldsymbol{\xi}) \;=\; \frac{P^i_{\;j}\left[\Delta \underset{(G)}{\Xi^j} + \Delta \underset{(M)}{\Xi^j} + \Delta \underset{(S)}{\Xi^j}\right]}{|\boldsymbol{x}-\boldsymbol{x}_0|} \;, \tag{6.35}$$

and $P^i_{\;j} = \delta^i_{\;j} - k^i k_j$ is the operator of projection on the plane being orthogonal to vector k^i. Equation (6.35) contains in its denominator the distance $R = |\boldsymbol{x}-\boldsymbol{x}_0|$ between observer and source of light which can be very large making impression that the relativistic correction $\beta^i(\tau,\boldsymbol{\xi})$ is negligibly small. However, the difference $\Delta\Xi^j(\tau,\boldsymbol{\xi})$ in the numerator of (6.35) is proportional either to the distance $r = |\boldsymbol{x}|$ between observer and the isolated system or to the distance $r_0 = |\boldsymbol{x}_0|$ between the source of light and the isolated system. Both distances can be comparable with R so that the relativistic correction $\beta^i(\tau,\boldsymbol{\xi})$ can be observable and must be taken into account, in general, for calculation of light deflection in the cases of finite distances of observer and/or source of light from the localized source of gravitational waves. Only in the case where observer and source of light reside at extremely large distances on opposite sides of the source of gravitational waves can the relativistic correction β^i be neglected.

7. OBSERVABLE RELATIVISTIC EFFECTS OF THE MULTIPOLAR GRAVITATIONAL FIELDS

In this section we derive general expressions for four observable relativistic effects – time delay, light bending, frequency shift, and rotation of the polarization plane.

7.1. Gravitational Time Delay of Light

Relativistic time delay can be obtained either directly from the expression (6.24) or from the electromagnetic eikonal (4.32), (6.1) by observing that the eikonal is constant not only on the null hypersurface of the wave front of the electromagnetic wave but also along the light rays. Both derivations lead to the same result

$$t - t_0 = |\boldsymbol{x}-\boldsymbol{x}_0| + \Delta(\tau,\tau_0) \;, \tag{7.1}$$

$$\Delta(\tau,\tau_0) = \underset{(G)}{\Delta}(\tau,\tau_0) + \underset{(M)}{\Delta}(\tau,\tau_0) + \underset{(S)}{\Delta}(\tau,\tau_0) \;, \tag{7.2}$$

where $x_0^\alpha = (t_0,\boldsymbol{x}_0)$ are coordinates of the point of emission of light and $x^\alpha = (t,\boldsymbol{x})$ are coordinates of the point of observation, $\underset{(G)}{\Delta}(\tau,\tau_0)$, $\underset{(M)}{\Delta}(\tau,\tau_0)$ and $\underset{(S)}{\Delta}(\tau,\tau_0)$ are functions describing the delay of the electromagnetic signal due to the gauge-freedom, mass and spin multipoles of the gravitational field of the isolated system correspondingly. These functions are expressed as follows

$$\underset{(G)}{\Delta}(\tau,\tau_0) = -\,k_i\left[\underset{(G)}{\Xi^i}(\tau,\boldsymbol{\xi}) - \underset{(G)}{\Xi^i}(\tau_0,\boldsymbol{\xi})\right] , \tag{7.3}$$

$$\underset{(M)}{\Delta}(\tau,\tau_0) = -\,k_i\left[\underset{(M)}{\Xi^i}(\tau,\boldsymbol{\xi}) - \underset{(M)}{\Xi^i}(\tau_0,\boldsymbol{\xi})\right] , \tag{7.4}$$

$$\underset{(S)}{\Delta}(\tau,\tau_0) = -\,k_i\left[\underset{(S)}{\Xi^i}(\tau,\boldsymbol{\xi}) - \underset{(S)}{\Xi^i}(\tau_0,\boldsymbol{\xi})\right] . \tag{7.5}$$

We also notice a relationship between the relativistic perturbations of the eikonal and the light-ray trajectory

$$\psi(\tau,\boldsymbol{\xi}) = -k_i\Xi^i(\tau,\boldsymbol{\xi}) \;, \tag{7.6}$$

where the eikonal perturbation ψ reads off the equations (6.1)–(6.4). From equations (7.3)–(7.5) one can infer that in the linear approximation the functions describing the time delay are just the projections of the coordinate perturbation of the light-ray trajectory onto the unperturbed direction k^i from the source of light to observer.

It is worth to remark that equation (7.1) defines the time delay effect with respect to the global ADM-harmonic reference frame. In order to convert it to observable proper time, we assume for simplicity that the observer is in a state of free fall and moves with velocity V^i with respect to the reference frame of the isolated system. Transformation from the ADM-harmonic coordinate time t to the proper time T is made with the help of the standard formula [1, 2]

$$dT^2 = -g_{\alpha\beta}dx^\alpha dx^\beta \; . \tag{7.7}$$

Substituting the metric tensor expansion $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, where $h_{\alpha\beta}$ is given by equations (3.21)–(3.24), to equation (7.7) yields

$$\frac{dT}{dt} = \sqrt{1 - \boldsymbol{V}^2 - h_{00}(1+\boldsymbol{V}^2) - 2h_{0i}V^i - h^{TT}_{ij}V^iV^j} \; . \tag{7.8}$$

In the most simple case, when observer is at rest ($\boldsymbol{V} = 0$) with respect to the ADM-harmonic reference frame equation (7.8) is drastically simplified and depends only on h_{00} component of the metric tensor. Implementation of formula (3.21) for h_{00} and subsequent integration of (7.8) with respect to time yields then

$$T = \left(1 - \frac{\mathcal{M}}{r}\right)(t - t_i) \; , \tag{7.9}$$

where t_i is the initial epoch of observation. Another simple case of equation (7.8) is obtained for observer located at the distance r so large that one can neglect h_{00} and h_{0i} quasi-static perturbations of the metric tensor. Then, the difference between the observer's proper time T and coordinate time t is

$$\frac{dT}{dt} = \sqrt{1 - \boldsymbol{V}^2 - h^{TT}_{ij}V^iV^j} \; , \tag{7.10}$$

leading in the case of a small velocity to

$$T = \sqrt{1 - \boldsymbol{V}^2}(t - t_i) - V^iV^j \int_{t_i}^{t} h^{TT}_{ij}\, dt \; . \tag{7.11}$$

The last term in the right side of this equation describes the effect of the gravitational waves on the rate of the proper time of observer which may be important under some specific circumstances but is enormously small in real practice and, as a rule, can be neglected.

7.2. Gravitational Deflection of Light

The coordinate direction to the source of light measured at the point of observation $\boldsymbol{x}$ is defined by the four-vector of photon l^α. This vector depends on the energy (frequency) of photon which is irrelevant for the present section. Normalization of l^α to its frequency brings about four-vector $p^\alpha = (1, p^i)$ where $p^i = -\dot{x}^i$, which is proportional to l^α but is only direction-dependent. Spatial part of this vector can be decomposed around direction k^i of unperturbed photon's trajectory

$$p^i = -k^i - \dot{\Xi}^i \; , \tag{7.12}$$

where overdot denotes derivative with respect to coordinate time t and the minus sign indicates that the tangent vector p^i directs from observer to the source of light.

The coordinate direction p^i is not a directly observable quantity as it is defined with respect to coordinate grid on the curved space-time manifold. A real observable vector towards the source of light, $s^\alpha = (1, s^i)$, is defined with respect to the local inertial frame co-moving with the observer. In this frame $s^i = -dX^i/dT$, where T is the observer's proper time and X^i are spatial coordinates of the local inertial frame with observer being located at its origin. We shall assume for simplicity that observer is at rest with respect to the global ADM-harmonic coordinate system (t, x^i). The case of observer moving with respect to the ADM-harmonic system with velocity V^i may be considered after completing the additional Lorentz transformation which is pretty straightforward procedure so that we do not discuss it from now on. In case of the static observer the infinitesimal transformation from the global ADM-harmonic coordinates (t, x^i) to the local coordinates (T, X^i) is given by the formulas

$$dT = \Lambda^0_{\;0}\, dt + \Lambda^0_{\;j}\, dx^j \qquad , \qquad dX^i = \Lambda^i_{\;0}\, dt + \Lambda^i_{\;j}\, dx^j \; , \tag{7.13}$$

where the matrix of transformation Λ^α_β is defined by the requirements of orthonormality

$$g_{\alpha\beta} = \eta_{\mu\nu}\Lambda^\mu_{\;\alpha}\Lambda^\nu_{\;\beta} \; . \tag{7.14}$$

In particular, the orthonormality condition (7.14) assumes that spatial angles and lengths at the point of observations are measured with the Euclidean metric δ_{ij}. Because the vector s^α is isotropic (null) with respect to the Minkowski metric $\eta_{\alpha\beta}$, we conclude that the Euclidean length $|\boldsymbol{s}|$ of the vector s^i is equal to 1. Indeed, one has

$$\eta_{\alpha\beta}s^\alpha s^\beta \;=\; -1+\boldsymbol{s}^2=0\;. \tag{7.15}$$

Hence, $|\boldsymbol{s}|=1$.

In the linear approximation with respect to the universal gravitational constant G, the matrix of the transformation is as follows [118]

$$\begin{aligned}
\Lambda^0_{\ 0} &= 1-\frac{1}{2}h_{00}(t,\boldsymbol{x})\;,\\
\Lambda^0_{\ i} &= -h_{0i}(t,\boldsymbol{x})\;,\\
\Lambda^i_{\ 0} &= 0\;,\\
\Lambda^i_{\ j} &= \left[1+\frac{1}{2}h_{00}(t,\boldsymbol{x})\right]\delta_{ij}+\frac{1}{2}h^{TT}_{ij}(t,\boldsymbol{x})\;.
\end{aligned}\tag{7.16}$$

Using transformation (7.13) we obtain relationship between the observable vector s^i and the coordinate direction p^i

$$s^i \;=\; -\frac{\Lambda^i_{\ 0}-\Lambda^i_{\ j}\,p^j}{\Lambda^0_{\ 0}-\Lambda^0_{\ j}\,p^j}\;. \tag{7.17}$$

In the linear approximation this takes the form

$$s^i \;=\; \left(1+h_{00}-h_{0j}p^j\right)p^i+\frac{1}{2}h^{TT}_{ij}p^j\;. \tag{7.18}$$

Remembering that vector $|\boldsymbol{s}|=1$, we find the Euclidean norm of the vector p^i from the relationship

$$|\boldsymbol{p}| \;=\; 1-h_{00}+h_{0j}p^j-\frac{1}{2}h^{TT}_{ij}p^ip^j\;, \tag{7.19}$$

which brings equation (7.18) to the form

$$s^i \;=\; m^i+\frac{1}{2}P^{ij}m^qh^{TT}_{jq}(t,\boldsymbol{x})\;, \tag{7.20}$$

where $P^{ij}=\delta^{ij}-k^ik^j$, and the Euclidean unit vector $m^i=p^i/|\boldsymbol{p}|$.

Let us now denote by α^i the dimensionless vector describing the total angle of deflection of the light ray measured at the point of observation, and calculated with respect to vector k^i given at past null infinity. It is defined according to the relationship [119]

$$\alpha^i(\tau,\boldsymbol{\xi}) \;=\; k^i[\boldsymbol{k}\cdot\dot{\boldsymbol{\Xi}}(\tau,\boldsymbol{\xi})]-\dot{\Xi}^i(\tau,\boldsymbol{\xi})\;, \tag{7.21}$$

or

$$\alpha^i(\tau,\boldsymbol{\xi}) \;=\; -\,P^i_{\ j}\,\dot{\Xi}^j(\tau,\boldsymbol{\xi})\;. \tag{7.22}$$

As a consequence of the definitions (7.12) and (7.22) we conclude that

$$m^i \;=\; -k^i+\alpha^i(\tau,\boldsymbol{\xi})\;. \tag{7.23}$$

Accounting for expressions (7.20), (7.23), and (6.34) we obtain for the observed direction to the source of light

$$s^i(\tau,\boldsymbol{\xi}) \;=\; K^i+\alpha^i(\tau,\boldsymbol{\xi})+\beta^i(\tau,\boldsymbol{\xi})+\gamma^i(\tau,\boldsymbol{\xi})\;, \tag{7.24}$$

where the unit vector K^i is given by equation (6.30), relativistic correction β^i is defined by equation (6.35), and the perturbation

$$\gamma^i(\tau,\boldsymbol{\xi}) \;=\; -\frac{1}{2}P^{ij}k^qh^{TT}_{jq}(t,\boldsymbol{x}) \tag{7.25}$$

determines deformation of the local coordinates of observer with respect to the global ADM-harmonic frame caused by the transverse-traceless part of the gravitational field of the isolated system at the point of observation.

If two sources of light (quasars, stars, etc.) are observed along the directions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ corresponding to two different light rays passing near the isolated system with impact parameters, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, the measured angle Ψ between them in the local inertial frame is

$$\cos\Psi \;=\; \boldsymbol{s}_1\cdot\boldsymbol{s}_2\;, \tag{7.26}$$

where the dot between the two vectors denotes the usual Euclidean scalar product. It is worth emphasizing that the observed direction s^i to the source of light includes relativistic deflection of the light ray in the form of three perturbations. Two of them, α^i and γ^i, depend only on quantities taken at the point of observation but β^i, according to equation (6.35), is also sensitive to the strength of the gravitational field taken at the point of emission of light. This remark reveals that according to relation (7.24) a single gravitational wave signal may cause different angular displacements and/or time delays for different sources of light located at different distances from the source of gravitational waves even if the directions to the light sources are the same.

Without going into further details of the observational procedure we give an explicit expression for the total angle of the light deflection α^i. We have

$$\alpha^i(\tau,\boldsymbol{\xi}) = \underset{(G)}{\alpha^{\mathrm{i}}}\,(\tau,\boldsymbol{\xi}) + \underset{(M)}{\alpha^{\mathrm{i}}}\,(\tau,\boldsymbol{\xi}) + \underset{(S)}{\alpha^{\mathrm{i}}}\,(\tau,\boldsymbol{\xi})\;, \tag{7.27}$$

where

$$\underset{(G)}{\alpha^{\mathrm{i}}}\,(\tau,\boldsymbol{\xi}) \;=\; -P^{ij}\hat{\partial}_\tau\left(\varphi^j+w^j\right)\;, \tag{7.28}$$

$$\begin{aligned}\underset{(M)}{\alpha^{\mathrm{i}}}\,(\tau,\boldsymbol{\xi}) \;=\;& -\frac{2\mathcal{M}}{r}\frac{\xi^i}{y} - 2\hat{\partial}_i\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)H(2-q)\times\\ &\left(1-\frac{p-q}{l}\right)\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right]^{[-1]}\\ &-4P^{ij}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}}{l!}C_{l-2}(l-p-1,p-1)k_{<a_1\ldots a_{p-1}}\hat{\partial}_{a_p\ldots a_{l-1}>}\left[\frac{\mathcal{I}^{(p)}_{jA_{l-1}}(t-r)}{r}\right]\;,\end{aligned} \tag{7.29}$$

$$\begin{aligned}\underset{(S)}{\alpha^{\mathrm{i}}}\,(\tau,\boldsymbol{\xi}) \;=\;& -2k^j\epsilon_{jba}\hat{\partial}_a\left(\frac{\mathcal{S}_b\xi^i}{yr}\right)+2P^{ij}\hat{\partial}_a\frac{\epsilon_{jba}\mathcal{S}_b}{r}\\ &+4\hat{\partial}_{ia}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)H(2-q)\times\\ &\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^q_\tau\left[\frac{k^j\epsilon_{jba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]}\\ &-4\left(\hat{\partial}_a-k_a\hat{\partial}_{t^*}\right)\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-2}\left(l-p-2,p\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[P^{ij}\frac{\epsilon_{jba}\mathcal{S}^{(p)}_{bA_{l-1}}(t-r)}{r}\right]\\ &-4\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{P^{iq}k^j\epsilon_{jba_{l-1}}\mathcal{S}^{(p+1)}_{qbA_{l-2}}(t-r)}{r}\right]\;.\end{aligned} \tag{7.30}$$

7.3. Gravitational Shift of Frequency

The exact calculation of the gravitational shift of electromagnetic frequency between emitted and observed photons plays a crucial role for the adequate interpretation of measurements of radial velocities of astronomical objects [120], anisotropy of electromagnetic cosmic microwave background radiation (CMBR), and other spectral astronomical investigations. In the last several years, for instance, radial velocity measuring technique has reached unprecedented accuracy and is approaching to the precision of about 10 cm/sec [121]. In the near future there is a hope to improve the accuracy up to 1 cm/sec [122] when measurement of the post-Newtonian relativistic effects in spectroscopic binaries and/or multiple star systems will be possible [120].

Let a source of light move with respect to the ADM-harmonic coordinate frame (t, x^i) with velocity $\boldsymbol{V}_0 = \dot{\boldsymbol{x}}_0(t_0)$ and emit electromagnetic radiation with frequency $\nu_0 = 1/(\delta T_0)$, where t_0 and T_0 are the coordinate and proper time of the source of light, respectively. We denote by $\nu = 1/(\delta T)$ the observed frequency of the electromagnetic radiation measured at the point of observation by an observer moving with velocity $\boldsymbol{V} = \dot{\boldsymbol{x}}(t)$ with respect to the ADM-harmonic coordinate frame. In the geometric optics approximation we can consider the increments of the proper time, δT_0 and δT, as infinitesimally small which allows us to operate with them as with differentials [123]. Time delay equation (7.1) can be considered as an implicit function of the emission time $t_0 = t_0(t)$ having the time of observation t as its argument. Because the coordinate and proper time at the points of emission of light and its observation are connected through the metric tensor we conclude that the observed gravitational shift of frequency $1 + z = \nu/\nu_0$ can be defined through the consecutive differentiation of the proper time of the source of light, T_0, with respect to the proper time of the observer, T, [123, 124]

$$1 + z \ = \ \frac{dT_0}{dT} = \frac{dT_0}{dt_0}\frac{dt_0}{dt}\frac{dt}{dT} \, . \tag{7.31}$$

Synge calls the relationship (7.31) as the *Doppler effect in terms of frequency* (see [124], page 122). It is fully consistent with definition of the *Doppler shift in terms of energy* (see [124], page 231) when one compares the energy of photon at the points of emission and observation of light. The *Doppler shift in terms of energy* is given by

$$1 + z \ = \ \frac{\nu}{\nu_0} = \frac{\eta_{\alpha\beta}u^\alpha l^\beta}{\eta_{\alpha\beta}u_0^\alpha l_0^\beta} \, , \tag{7.32}$$

where u_0^α, u^α and l_0^α, l^α are the 4-velocities of the source of light and observer and the 4-momenta of photon at the points of emission and observation respectively. It is quite easy to see that both mentioned formulations of the Doppler shift effect are equivalent. Indeed, taking into account that $u^\alpha = dx^\alpha/dT$ and $l_\alpha = \partial\varphi/\partial x^\alpha$, where φ is the phase of the electromagnetic wave (eikonal), we obtain $\eta_{\alpha\beta}u^\alpha l^\beta = d\varphi/dT$. Thus, equation (7.32) yields

$$1 + z \ = \ \frac{d\varphi}{d\varphi_0}\frac{dT_0}{dT} \, . \tag{7.33}$$

The phase of electromagnetic wave remains constant along the light ray trajectory. For this reason, $d\varphi/d\varphi_0 = 1$ and, hence, equation (7.33) is reduced to equation (7.31) as expected on the ground of physical intuition. Detailed comparison of the two definitions of the Doppler shift and the proof of their identity in general theory of relativity is thoroughly discussed in our paper [125].

We emphasize that in equation (7.31) the time derivative

$$\frac{dT_0}{dt_0} \ = \ \left[1 - \boldsymbol{V}_0^2 - (1 + \boldsymbol{V}_0^2)h_{00}(t_0, \boldsymbol{x}_0) - 2V_0^i h_{0i}(t_0, \boldsymbol{x}_0) - V_0^i V_0^j h_{ij}^{TT}(t_0, \boldsymbol{x}_0)\right]^{1/2} \, , \tag{7.34}$$

is taken at the time t_0 at point of emission of light $\boldsymbol{x}_0$, and the time derivative

$$\frac{dt}{dT} \ = \ \left[1 - \boldsymbol{V}^2 - (1 + \boldsymbol{V}^2)h_{00}(t, \boldsymbol{x}) - 2V^i h_{0i}(t, \boldsymbol{x}) - V^i V^j h_{ij}^{TT}(t, \boldsymbol{x})\right]^{-1/2} \, , \tag{7.35}$$

is calculated at the time of observation t at the position of observer.

The time derivative dt_0/dt along the light-ray trajectory is calculated from the time delay equation (7.1) where we have to take into account that function $\Delta(\tau, \tau_0)$ depends on times t_0 and t indirectly through the retarded times $s_0 = t_0 - r_0$ and $s = t - r$ that are arguments of the multipole moments of the isolated system and wherein $r_0 \equiv |\boldsymbol{x}_0| = |\boldsymbol{x}(t_0)|$, $r = |\boldsymbol{x}| = |\boldsymbol{x}(t)|$ are functions of time t_0 and t respectively. Function $\Delta(\tau, \tau_0)$ also depends on the time of the closest approach of light ray, t^*, through variables $\tau = t - t^*$, $\tau_0 = t_0 - t^*$, and on the unit vector $\boldsymbol{k}$. Both t^* and $\boldsymbol{k}$ should be considered as parameters depending on t_0 and t because of the relative motion of observer with respect to the source of light which causes variation in the relative position of the source of light and observer and, consequently, to the corresponding change in the parameters characterizing trajectory of the light ray, that is in t^* and $\boldsymbol{k}$. Therefore, the function $\Delta(\tau, \tau_0)$ must be viewed as depending on four variables $\Delta = \Delta(s, s_0, t^*, \boldsymbol{k})$. Accounting for these remarks the derivative along the light ray reads as follows

$$\frac{dt_0}{dt} \ = \ \frac{1 + \boldsymbol{K}\cdot\boldsymbol{V} - \left\{\dfrac{\partial s}{\partial t}\dfrac{\partial}{\partial s} + \dfrac{\partial s_0}{\partial t}\dfrac{\partial}{\partial s_0} + \dfrac{\partial t^*}{\partial t}\dfrac{\partial}{\partial t^*} + \dfrac{\partial k^i}{\partial t}\dfrac{\partial}{\partial k^i}\right\}\Delta(s, s_0, t^*, \boldsymbol{k})}{1 + \boldsymbol{K}\cdot\boldsymbol{V}_0 + \left\{\dfrac{\partial s}{\partial t_0}\dfrac{\partial}{\partial s} + \dfrac{\partial s_0}{\partial t_0}\dfrac{\partial}{\partial s_0} + \dfrac{\partial t^*}{\partial t_0}\dfrac{\partial}{\partial t^*} + \dfrac{\partial k^i}{\partial t_0}\dfrac{\partial}{\partial k^i}\right\}\Delta(s, s_0, t^*, \boldsymbol{k})} \, , \tag{7.36}$$

where the unit vector $\boldsymbol{K}$ is defined in (6.30) and where we explicitly show the dependence of function $\Delta(\tau,\tau_0)$ on all parameters which implicitly depend on time.

The time derivative of vector $\boldsymbol{k}$ is calculated using the approximation $\boldsymbol{k}=-\boldsymbol{K}$ and formula (6.30) where the coordinates of the source of light, $\boldsymbol{x}_0(t_0)$, and of the observer, $\boldsymbol{x}(t)$, are functions of time. These derivatives are

$$\frac{\partial k^i}{\partial t}=\frac{(\boldsymbol{k}\times(\boldsymbol{V}\times\mathbf{k}))^i}{R}\;,\qquad\qquad \frac{\partial k^i}{\partial t_0}=-\frac{(\boldsymbol{k}\times(\boldsymbol{V}_0\times\mathbf{k}))^i}{R}\;, \tag{7.37}$$

where $R=|\boldsymbol{x}-\boldsymbol{x}_0|$ is the distance between the observer and the source of light.

Time derivatives of retarded times s and s_0 with respect to t and t_0 are calculated from their definitions, $s=t-r$ and $s_0=t_0-r_0$, where we have to take into account that the spatial position of the point of observation is connected to the point of emission of light by the unperturbed trajectory of light, $\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}\,(t-t_0)$. More explicitly, we use for the calculations the following relationships

$$s \;=\; t-|\boldsymbol{x}_0(t_0)+\boldsymbol{k}\,(t-t_0)|\;, \tag{7.38}$$

$$s_0 \;=\; t_0-|\boldsymbol{x}_0(t_0)|\;, \tag{7.39}$$

where the unit vector $\boldsymbol{k}=\boldsymbol{k}(t,t_0)$ must be also considered as a function of two arguments t, t_0 with its derivatives given by equation (7.37). We notice that relationship (7.38) combines two null cones related to propagation of gravitational field from the isolated system and to that of an electromagnetic wave from the source of light to observer. Equation (7.39) is just a null cone corresponding to propagation of the gravitational field from the isolated system to the point of emission of light. Calculation of infinitesimal variations of equations (7.38), (7.39) immediately gives a set of partial derivatives

$$\frac{\partial s}{\partial t} \;=\; 1-\boldsymbol{k}\cdot\boldsymbol{N}-(\boldsymbol{k}\times\boldsymbol{V})\cdot(\boldsymbol{k}\times\boldsymbol{N})\;, \tag{7.40}$$

$$\frac{\partial s}{\partial t_0} \;=\; (1-\boldsymbol{k}\cdot\boldsymbol{V}_0)(\boldsymbol{k}\cdot\boldsymbol{N})\;, \tag{7.41}$$

$$\frac{\partial s_0}{\partial t_0} \;=\; 1-\boldsymbol{V}_0\cdot\boldsymbol{N}_0\;, \tag{7.42}$$

$$\frac{\partial s_0}{\partial t} \;=\; 0\;, \tag{7.43}$$

where $N^i=x^i/r$ and $N_0^i=x_0^i/r_0$ are the unit vectors directed from the isolated system to observer and to the source of light respectively..

Time derivatives of the parameter $t^*=t_0-\boldsymbol{k}\cdot\boldsymbol{x}_0(t_0)$, where again $\boldsymbol{k}=\boldsymbol{k}(t,t_0)$, read

$$\frac{\partial t^*}{\partial t_0}=1-\boldsymbol{k}\cdot\boldsymbol{V}_0+\frac{\boldsymbol{V}_0\cdot\boldsymbol{\xi}}{R}\;,\qquad\qquad \frac{\partial t^*}{\partial t}=-\frac{\boldsymbol{V}\cdot\boldsymbol{\xi}}{R}\;. \tag{7.44}$$

The last terms of order $|\boldsymbol{\xi}|/R$ in both formulas are generated by the time derivatives of the vector $\boldsymbol{k}$.

In what follows we shall restrict ourselves with the case of static observer and the source of light, that is velocities $\boldsymbol{V}=\boldsymbol{V}_0=0$. Taking into account this restriction in equations (7.37)–(7.44) and leaving only linear with respect to the universal gravitational constant G terms, allow us to simplify equation (7.36)

$$\frac{dt_0}{dt} \;=\; 1-\left\{\frac{\partial}{\partial s}+\frac{\partial}{\partial s_0}+\frac{\partial}{\partial t^*}\right\}\Delta(s,s_0,t^*,\boldsymbol{k})\;, \tag{7.45}$$

Function $\Delta=\Delta(s,s_0,t^*,\boldsymbol{k})$ is defined by the integral (7.2) which (in the case of static observer and the source of light) depends on the retarded times s, s_0 and the time of the closest approach t^* through the only arguments $y=s-t^*$, $y_0=s_0-t^*$ and t^*, that is

$$\Delta(s,s_0,t^*,\boldsymbol{k})\equiv\Delta(y,y_0,t^*)=\psi(y,t^*)-\psi(y_0,t^*)\;, \tag{7.46}$$

where the relativistic perturbation of the eikonal, $\psi=-k^i\Xi$, is defined in accordance with equation (7.6). Therefore, the partial derivatives with respect to s and t^* from the argument y, and those with respect to s_0 and t^* from y_0 in equation (7.45) are nullified and it is reduced to simpler form

$$\frac{dt_0}{dt} \;=\; 1-\frac{\partial\psi(y,t^*)}{\partial t^*}+\frac{\partial\psi(y_0,t^*)}{\partial t^*}\;. \tag{7.47}$$

The partial time derivative from $\psi(y,t^*)$ is found by differentiation of relationships (6.2)–(6.4)

$$\frac{\partial\psi(y,t^*)}{\partial t^*}=\frac{\partial\psi_{(G)}(y,t^*)}{\partial t^*}+\frac{\partial\psi_{(M)}(y,t^*)}{\partial t^*}+\frac{\partial\psi_{(S)}(y,t^*)}{\partial t^*} \tag{7.48}$$

where

$$\frac{\partial\psi_{(G)}(y,t^*)}{\partial t^*} = (k^i\frac{\partial\varphi^i}{\partial t^*}-\frac{\partial\varphi^0}{\partial t^*})+(k^i\frac{\partial w^i}{\partial t^*}-\frac{\partial w^0}{\partial t^*})\,, \tag{7.49}$$

$$\begin{aligned}\frac{\partial\psi_{(M)}(y,t^*)}{\partial t^*} &= 2\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\\ &\times\left\{\left(1+\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\left[\frac{\mathcal{I}^{(p+1)}_{A_l}(t-r)}{r}\right]^{[-1]}\right.\\ &\left.-\frac{2p}{l-1}k_{<a_1\ldots a_{p-1}}\hat{\partial}_{a_p\ldots a_{l-1}>}\left[\frac{k^i\mathcal{I}^{(p+1)}_{iA_{l-1}}(t-r)}{r}\right]^{[-1]}\right\}\end{aligned} \tag{7.50}$$

$$\begin{aligned}\frac{\partial\psi_{(S)}(y,t^*)}{\partial t^*} &= 4\hat{\partial}_a\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\\ &\times\left(1-\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{k^i\epsilon_{iba}\mathcal{S}^{(p+1)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]}\,.\end{aligned} \tag{7.51}$$

When deriving these equations we assumed that the time evolution of both the mass $\mathcal{M}$ and the angular momentum $\mathcal{S}^i$ of the isolated system can be neglected. In opposite case, the right side of equations (7.50) and (7.51) would contain also time derivatives from the mass and the angular momentum. Partial derivative from function $\psi(y_0,t^*)$ is obtained from equations (7.48)–(7.51) after replacements $y\to y_0$, $t\to t_0$, and $r\to r_0$.

Provided that the frequency shift z is given by equation (7.31) we find that in the case when both observer and the source of light are static with respect to the ADM-harmonic reference frame, the frequency (energy) of a photon propagating through gravitational potential changes in accordance with (exact) relationship

$$\frac{\nu}{\nu_0}=\sqrt{\frac{1-h_{00}(t_0,\boldsymbol{x}_0)}{1-h_{00}(t,\boldsymbol{x})}}\left[1-\frac{\partial\psi(y,t^*)}{\partial t^*}+\frac{\partial\psi(y_0,t^*)}{\partial t^*}\right]\,, \tag{7.52}$$

In the linear with respect to G approximation equation (7.52) yields

$$\frac{\delta\nu}{\nu_0}=\frac{1}{2}\Big[h_{00}(t,\boldsymbol{x})-h_{00}(t_0,\boldsymbol{x}_0)\Big]-\frac{\partial\psi(y,t^*)}{\partial t^*}+\frac{\partial\psi(y_0,t^*)}{\partial t^*}\,, \tag{7.53}$$

where $\delta\nu\equiv\nu-\nu_0$. One notices that the time of the closest approach t^* enters equation (7.53) explicitly. At the first glance the reader may think that it plays a significant role in calculations and must be known in order to calculate the gravitational shift of frequency. However, the partial derivative with respect to t^* must be understood in the sense of equation (4.27) which makes it evident that the time derivative with respect to t^* taken on the light-ray path is, in fact, the time derivative with respect to time t before using the light-ray substitution. Thus, the gravitational shift of frequency can be recast to the next practical form

$$\frac{\delta\nu}{\nu_0}=\frac{1}{2}\Big[h_{00}(t,\boldsymbol{x})-h_{00}(t_0,\boldsymbol{x}_0)\Big]-\frac{\partial\psi(t,\boldsymbol{x})}{\partial t}+\frac{\partial\psi(t_0,\boldsymbol{x}_0)}{\partial t_0}\,, \tag{7.54}$$

where $\psi(t,\boldsymbol{x})$ and $\psi(t_0,\boldsymbol{x}_0)$ are relativistic perturbations of the electromagnetic eikonal taken at the points of observation and emission of light respectively. These time derivatives are, of course, the same as shown in equations (7.48)–(7.51). Equation (7.54) has been derived by making use of the definition (7.31). It is straightforward to prove that the definition (7.32) brings about the same result. Indeed, in the case of the static observer and the source of light their 4-velocities are simply $u^\alpha=(dt/dT,0,0,0)$ and $u^\alpha_0=(dt_0/dT_0,0,0,0)$, and the wave vector $l_\alpha=\omega(k_\alpha+\partial_\alpha\psi)$ (see equation (4.33)) where $\omega = 2\pi\nu_\infty$ is a constant frequency which is conserved along light ray because of the equation of the parallel transport (4.13). Substituting these definitions to equation (7.32) leads immediately to equation (7.54) as expected

Physical interpretation of the relativistic frequency shift given by equation (7.54) is straightforward although the calculation of different components of the time derivatives from the eikonal are tedious. The term in the square brackets in the left side of equation (7.54) reads

$$\frac{1}{2}\Big[h_{00}(t,\boldsymbol{x}) - h_{00}(t_0,\boldsymbol{x}_0)\Big] = \frac{G\mathcal{M}}{r} - \frac{G\mathcal{M}}{r_0}\;, \tag{7.55}$$

and represents the difference between the values of the spherically-symmetric part of the Newtonian gravitational potential of the isolated system taken at the point of observation and emission of light. The time derivatives of the eikonal depending on the mass and spin multipole moments of the isolated system are given in equations (7.50), (7.51). Scrutiny examination of these equations reveal that these components of the frequency shift depend on the first and higher order time derivatives of the multipole moments and would vanish in the stationary case. The gravitational frequency shift contains the gauge-dependent contribution as well. This contribution is given by equation (7.49) and can be calculated by making use of equations from appendix C. The result is as follows

$$\begin{aligned}
k^i\frac{\partial w^i}{\partial t^*} - \frac{\partial w^0}{\partial t^*} &= k^i\nabla_i\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\mathcal{I}^{(-1)}_{A_l}(t-r)}{r}\right]_{,A_l} - \sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{\mathcal{I}_{A_l}(t-r)}{r}\right]_{,A_l} - \\
&\quad 4\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left[\frac{k^i\dot{\mathcal{I}}_{iA_{l-1}}(t-r)}{r}\right]_{,A_{l-1}} + 4\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\left[\frac{k^i\epsilon_{iba}\mathcal{S}_{bA_{l-1}}(t-r)}{r}\right]_{,aA_{l-1}}\;,
\end{aligned} \tag{7.56}$$

$$\begin{aligned}
k^i\frac{\partial \varphi^i}{\partial t^*} - \frac{\partial \varphi^0}{\partial t^*} &= -2\sum_{l=2}^{\infty}\sum_{p=1}^{l}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)\left(1-\frac{p-q}{l}\right)\times \\
&\quad \left\{\left(1+\frac{p-q}{l-1}\right)k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>}\hat{\partial}_\tau^{q-1}\left[\frac{\mathcal{I}^{(p-q+1)}_{A_l}(t-r)}{r}\right] - \right. \\
&\quad \left. 2\frac{p-q}{l-1}k_{<a_1\dots a_{p-1}}\hat{\partial}_{a_p\dots a_{l-1}>}\hat{\partial}_\tau^{q-1}\left[\frac{k^i\mathcal{I}^{(p-q+1)}_{iA_{l-1}}(t-r)}{r}\right]\right\} - \\
&\quad 4\hat{\partial}_a\sum_{l=2}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times \\
&\quad \left(1-\frac{p-q}{l-1}\right)k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_{l-1}>}\hat{\partial}_\tau^{q-1}\left[\frac{k^i\epsilon_{iab}\mathcal{S}^{(p-q+1)}_{bA_{l-1}}(t-r)}{r}\right]\;.
\end{aligned} \tag{7.57}$$

Time derivatives $\hat{\partial}_{t_0^*}(k^iw^i - w^0)$ and $\hat{\partial}_{t_0^*}(k^i\varphi^i - \varphi^0)$ can be obtained from equations (7.56), (7.57) after making replacements $t \to t_0$, $r \to r_0$, and $\tau \to \tau_0$.

7.4. Gravity-Induced Rotation of the Plane of Polarization of Light

Any stationary or time-dependent axisymmetric gravitational field in general induces a relativistic effect of the rotation of the polarization plane of an electromagnetic wave propagating in this field. This effect is similar to Faraday's effect in electrodynamics [126]. The Faraday effect is caused by the presence of magnetic field while the gravity-induced rotation of the plane of polarization of light is caused by the presence of the, so-called, gravito-magnetic field [127]. This gravito-magnetic effect was first discussed by Skrotskii ([108, 109] and shortly afterwards by many other researches (see, for example, [110, 111, 128–132] and references therein). Recently, we have studied the Skrotskii effect caused by a spinning body moving arbitrarily fast and derived Lorentz-invariant expression for this outstanding effect [133] in the linearized approximation of general relativity containing all terms of order v/c coupled with the universal gravitational constant G. In the present paper we further generalize our theoretical results along with those of previous authors regarding the case of an isolated system emitting gravitational waves of arbitrary multipolarity.

In order to simplify description of the Skrotskii effect we shall consider the parallel transport of the reference tetrad $e^{\alpha}_{\ (\beta)}$ defined by equations (4.55) at each point along the light ray. We shall assume for simplicity that at the past null infinity the tetrad has the property that $e^{0}_{\ (i)}(-\infty)\partial^{0}_{\ (i)} = 0$ in accordance with equation (4.50). The parallel transport of this tetrad is defined by equation (4.66) which makes clear that in an arbitrary point on the light-ray trajectory $e^{0}_{\ (i)}(t, \boldsymbol{x}) = O(h_{\alpha\beta})$. We are interested only in solving equation (4.66) for the vectors $e^{\alpha}_{\ (n)}$ $(n = 1, 2)$ that are used in the description of the polarization of light. The propagation equation for the spatial components $e^{i}_{\ (n)}$ of these vectors are directly obtained from equation (4.66) which yields

$$\frac{d}{d\tau}\left[e^{i}_{\ (n)} + \frac{1}{2}h_{ij}e^{p}_{\ (n)}\right] + \epsilon_{ijp}e^{j}_{\ (n)}\Omega^{p} = 0\ , \qquad\qquad (n = 1, 2.) \tag{7.58}$$

Here we have defined the angular velocity of gravito-magnetic drag by Ω^i defined as

$$\Omega^{i} = -\frac{1}{2}\epsilon_{ijp}\partial_{j}\left(h_{p\alpha}k^{\alpha}\right)\ . \tag{7.59}$$

This quantity describes the rate of the rotation of the plane of polarization of electromagnetic wave caused by the presence of the gravito-magnetic field. As soon as equation (7.58) is solved the time component $e^{0}_{\ (n)}$ is obtained from the orthogonality condition, $l_{\alpha}e^{\alpha}_{\ (n)} = 0$, which implies that

$$e^{0}_{\ (n)} = k_{i}e^{i}_{\ (n)} + h_{0i}e^{i}_{\ (n)} + h_{ij}k^{i}e^{j}_{\ (n)} + \delta_{ij}\dot{\Xi}^{i}e^{j}_{\ (n)}\ , \tag{7.60}$$

where the relativistic perturbation $\dot{\Xi}^{i}$ of the light-ray trajectory is given in equation (6.15).

Let us decompose vector Ω^i into three components that are parallel and perpendicular to the unit vector k^i. We can use in the first approximation the well-known decomposition of the Kroneker symbol in the orthonormal basis

$$\delta^{ij} = a^{i}a^{j} + b^{i}b^{j} + k^{i}k^{j}\ , \tag{7.61}$$

where $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{k})$ are three orthogonal unit vectors of the reference tetrad at infinity. Decomposition of Ω^i is, then, given by

$$\Omega^{i} = (\mathbf{a}\cdot\boldsymbol{\Omega})a^{i} + (\mathbf{b}\cdot\boldsymbol{\Omega})b^{i} + (\boldsymbol{k}\cdot\boldsymbol{\Omega})k^{i}\ . \tag{7.62}$$

Taking into account that vectors $e^{i}_{\ (1)} = a^{i} + O(h_{\alpha\beta})$, $e^{i}_{\ (2)} = b^{i} + O(h_{\alpha\beta})$ one obtains the equations of the parallel transport of these vectors in the next, more convenient for integration, form

$$\frac{d}{d\tau}\left[e^{i}_{\ (1)} + \frac{1}{2}h_{ij}e^{j}_{\ (1)}\right] - (\boldsymbol{k}\cdot\boldsymbol{\Omega})e^{i}_{\ (2)} \ = \ 0\ , \tag{7.63}$$

$$\frac{d}{d\tau}\left[e^{i}_{\ (2)} + \frac{1}{2}h_{ij}e^{j}_{\ (2)}\right] + (\boldsymbol{k}\cdot\boldsymbol{\Omega})e^{i}_{\ (1)} \ = \ 0\ , \tag{7.64}$$

where equalities $\varepsilon_{ijl}e^{i}_{\ (1)}k^{l} = -e^{i}_{\ (2)} + O(h_{\alpha\beta})$ and $\varepsilon_{ijl}e^{i}_{\ (2)}k^{l} = e^{i}_{\ (1)} + O(h_{\alpha\beta})$ have been used.

Solutions of equations (7.63), (7.64) in the linear approximation with respect to the angular velocity Ω^i reads

$$e^{i}_{\ (1)} \ = \ a^{i} - \frac{1}{2}h_{ij}a^{j} + \left(\int_{-\infty}^{\tau}\boldsymbol{k}\cdot\boldsymbol{\Omega}(\sigma)\ d\sigma\right)b^{i}\ , \tag{7.65}$$

$$e^{i}_{\ (2)} \ = \ b^{i} - \frac{1}{2}h_{ij}b^{j} - \left(\int_{-\infty}^{\tau}\boldsymbol{k}\cdot\boldsymbol{\Omega}(\sigma)\ d\sigma\right)a^{i}\ , \tag{7.66}$$

where the second terms in the right sides of equations (7.65), (7.66) preserve orthogonality of vectors $e^{i}_{\ (1)}$ and $e^{i}_{\ (2)}$ in the presence of gravitational field while the last terms describe a small rotation (the Skrotskii effect [108, 109]) of each of the vectors at the angle

$$\phi(\tau) = \int_{-\infty}^{\tau}\boldsymbol{k}\cdot\boldsymbol{\Omega}\ d\sigma \tag{7.67}$$

about the direction of light propagation, $\boldsymbol{k}$, in the local plane of vectors $\boldsymbol{e}_{(1)}$ and $\boldsymbol{e}_{(2)}$. It is worth noting that the Euclidean dot product $\boldsymbol{k}\cdot\boldsymbol{\Omega}$ can be expressed in terms of partial differentiation with respect to the vector ξ^i of the impact parameter. This can be done by making use of equation (4.28) and noting that $\varepsilon_{ijp}k^jk^p\equiv 0$, so that

$$\boldsymbol{k}\cdot\boldsymbol{\Omega}=\frac{1}{2}k^\alpha k^j\varepsilon_{j\hat{p}\hat{q}}\hat{\partial}_q h_{\alpha\hat{p}}\ , \tag{7.68}$$

where the hat over spatial indices denotes the projection onto the plane orthogonal to the propagation of light ray, for instance, $A^{\hat{i}}\equiv P^i{}_jA^j$. Hence, the transport equation for the angle ϕ assumes the following form

$$\frac{d\phi}{d\tau}=\frac{1}{2}k^\alpha k^j\epsilon_{j\hat{p}\hat{q}}\hat{\partial}_q h_{\alpha\hat{p}}\ . \tag{7.69}$$

This equation can be split in three linearly-independent parts corresponding to contributions of the gauge, $\phi_{(G)}$, the mass, $\phi_{(M)}$, and the spin, $\phi_{(S)}$, modes of the gravitational multipolar field to the overall value of the Skrotskii effect. Specifically, we have

$$\phi=\phi_{(G)}+\phi_{(M)}+\phi_{(S)}+\phi_0\ , \tag{7.70}$$

where ϕ_0 is constant angle characterizing initial orientation of the polarization ellipse in the plane formed by the $\boldsymbol{e}_{(1)}$ and $\boldsymbol{e}_{(2)}$ vectors.

The gauge-dependent part of the Skrotskii effect is easily integrated so that we obtain

$$\phi_{(G)}=\frac{1}{2}k^j\epsilon_{jpq}\hat{\partial}_q\left(w^p+\chi^p\right)\ , \tag{7.71}$$

where the gauge functions $w^i=w^p_{(M)}+w^p_{(S)}$ and $\chi^i=\chi^p_{(M)}+\chi^p_{(S)}$ are given in equations (3.20) and (C.10) respectively. More specific calculations reveal that the gauge-dependent terms enter the right side of equation (7.71) in the following form

$$\frac{1}{2}k^j\epsilon_{j\hat{m}n}\hat{\partial}_n\chi^{\hat{m}}_{(M)} = 2\sum_{l=2}^{\infty}\sum_{p=1}^{l}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}}{l!}\frac{l-p+q}{(l-1)}\frac{p-q}{p}C_l(l-p,p-q,q)$$
$$\times\, k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>j}\hat{\partial}_{\tau}^{q-1}\left[\frac{\epsilon_{j\hat{b}a_l}\mathcal{I}^{(p-q)}_{\hat{b}L-1}(t-r)}{r}\right]^{[-1]}, \tag{7.72}$$

$$\frac{1}{2}k^j\epsilon_{j\hat{m}n}\hat{\partial}_n\chi^{\hat{m}}_{(S)} = 2\sum_{l=1}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_l(l-p,p-q,q)$$
$$\times\left\{H(q)\left(1-\frac{p}{l}\right)\left[1+H(l-1)\left(\frac{l-p-1}{l-1}-\frac{p-q}{l-1}\right)\right]\hat{\partial}^2_{t^*}\right.$$
$$-\left[\frac{l-p}{l}+2\frac{p-q}{l}-H(l-1)\frac{p-q}{l}\left(\frac{l-p}{l-1}+2\frac{p-q-1}{l-1}\right)\right]\hat{\partial}_{t^*\tau}$$
$$\left.+\frac{p-q}{l}\left[1-H(l-1)\frac{p-q-1}{l-1}\right]\hat{\partial}^2_{\tau}\right\}$$
$$\times\, k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>}\hat{\partial}_{\tau}^{q-1}\left[\frac{\mathcal{S}^{(p-q-1)}_{L}(t-r)}{r}\right]^{[-1]}, \tag{7.73}$$

$$\frac{1}{2}k^j\epsilon_{j\hat{m}n}\hat{\partial}_n w^{\hat{m}}_{(M)} = -2\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}}{l!}\frac{p-q}{p}C_l(l-p,p-q,q)$$
$$\times\, k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>j}\hat{\partial}_{\tau}^{q}\left[\frac{\epsilon_{j\hat{b}a_l}\mathcal{I}^{(p-q-1)}_{\hat{b}L-1}(t-r)}{r}\right], \tag{7.74}$$

$$\frac{1}{2}k^j\epsilon_{j\hat{m}n}\hat{\partial}_n w^{\hat{m}}_{(S)} = -2\sum_{l=1}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_l(l-p,p-q,q)$$
$$\times\left\{\left(1-\frac{p}{l}\right)\hat{\partial}^2_{t^*}-\left(\frac{l-p}{l}+2\frac{p-q}{l}\right)\hat{\partial}_{t^*\tau}+\frac{p-q}{l}\hat{\partial}^2_{\tau}\right\}$$
$$\times\, k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>}\hat{\partial}_{\tau}^{q}\left[\frac{\mathcal{S}^{(p-q-2)}_{L}(t-r)}{r}\right]. \tag{7.75}$$

The reader can notice the integrals in the right side of equations (7.72) and (7.73). The integrals are actually should not be calculated explicitly since there is sufficient number of partial derivatives in front of them which makes the integration unnecessary in correspondence with the rules of differentiation of such integrals worked out in section 5.

The Skrotskii effect due to the mass multipoles of the isolated system is given by

$$\phi_{(M)}(\tau) = 2\sum_{l=2}^{\infty}\sum_{p=0}^{l}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\frac{l-p}{l-1}k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>j}\left[\frac{\epsilon_{j\hat{b}a_l}\mathcal{I}^{(p)}_{\hat{b}A_{l-1}}(t-r)}{r}\right]^{[-1]}. \tag{7.76}$$

The gravitational field of the spin multipoles rotates the polarization plane of the electromagnetic wave at the following angle

$$\phi_{(S)}(\tau) = 2\sum_{l=1}^{\infty}\sum_{p=0}^{l}\frac{(-1)^{l+p}l}{(l+1)!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\left[1+H(l-1)\left(1-\frac{2p}{l-1}\right)\right]k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>}\left[\frac{\mathcal{S}^{(p+1)}_{A_l}(t-r)}{r}\right]^{[-1]} \tag{7.77}$$

Integrals in equations (7.76) and (7.77) are eliminated after taking one partial derivative so that we do not need to integrate.

8. OBSERVABLE RELATIVISTIC EFFECTS IN THE GRAVITATIONAL LENS APPROXIMATION

Here we will consider a limiting case of gravitational lens approximation when the impact parameter d of the light ray with respect to an isolated system is much smaller than distance r_0 from the isolated system to the source of light and distance r from the isolated system to observer (see Figs. 3).

8.1. Useful Asymptotic Expansions

In the case of a small impact parameter of the light ray with respect to the isolated system the near-zone gravitational field of the system strongly affects propagation of the light ray when the light particle (photon) moves in the vicinity of the system so that the effects due to the gravitational waves are suppressed [82, 83]. In what follows, we assume that the impact parameter d of the light ray is small comparatively with both distances r and r_0, that is, $d \ll min[r, r_0]$ (see Fig. 3). This assumption allows us to introduce two small parameters: $\varepsilon \equiv d/r$ and $\varepsilon_0 \equiv d/r_0$. If the light ray is propagated through the near-zone of the isolated system one more small parameter can be introduced, specifically, the parameter $\varepsilon_\lambda \equiv d/\lambda$, where λ is the characteristic wavelength of the gravitational radiation emitted by the system proportional to the product of the speed of gravity and the characteristic period of oscillations of the isolated system. One emphasize that the parameters ε, ε_0 and ε_λ are not physically correlated. Parameter ε_λ is used for the post-Newtonian expansion of the observed effects – deflection of light, time delay, etc. This expansion can be also viewed as a Taylor expansion with respect to the parameter v/c, where v is the characteristic speed of motion of matter composing the isolated system and c is the speed of propagation of gravity (equal to the speed of light . If the light ray does not enter the near zone of the system the parameter ε_λ is not small and the post-Newtonian expansion can not be performed.

Small-impact-parameter expansions for variables y and y_0 yield

$$y = \sqrt{r^2 - d^2} - r = -d\left(\frac{\varepsilon}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon^{2k-1}\right) , \tag{8.1}$$

$$y_0 = -\sqrt{r_0^2 - d^2} - r_0 = -2r_0 + d\left(\frac{\varepsilon_0}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon_0^{2k-1}\right) , \tag{8.2}$$

and

$$\frac{1}{yr} = -\frac{1}{d^2}\left(2 + \sum_{k=1}^{\infty} \mathcal{C}_k \varepsilon^{2k}\right) , \tag{8.3}$$

$$\frac{1}{y_0 r_0} = \frac{1}{d^2}\sum_{k=1}^{\infty} \mathcal{C}_k \varepsilon_0^{2k} , \tag{8.4}$$

where the numerical coefficient of the expansions is

$$\mathcal{C}_k = (-1)^k \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right) \cdot \ldots \cdot \left(\frac{1}{2} - k + 1\right)}{k!} = \frac{(2k-1)!}{(2k)!!} , \tag{8.5}$$

Time variables t and t_0 are expanded as follows

$$t = t^* + r - d\left(\frac{\varepsilon}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon^{2k-1}\right) , \tag{8.6}$$

$$t_0 = t^* - r_0 + d\left(\frac{\varepsilon_0}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon_0^{2k-1}\right) , \tag{8.7}$$

where $\mathcal{C}_k$ is given by equation (8.5) and t^* is the time of the closest approach of light ray to the barycenter of the isolated system.

Using (8.6) and (8.7) we can write down the post-Newtonian expansions for functions of the retarded time $t-r$ as follows

$$F(t-r) = \sum_{k=0}^{\infty} \frac{(-1)^k d^k}{k!} \left(\frac{\varepsilon}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon^{2k-1} \right)^k F^{(k)}(t^*) = F(t^*) - \varepsilon \frac{d}{2} \dot{F}(t^*) + O\left(\varepsilon^2 \varepsilon_\lambda^2\right) , \tag{8.8}$$

$$F(t_0-r_0) = \sum_{k=0}^{\infty} \frac{d^k}{k!} \left(\frac{\varepsilon_0}{2} - \sum_{k=2}^{\infty} \mathcal{C}_k \varepsilon_0^{2k-1} \right)^k F^{(k)}(t^*-2r_0) = F(t^*-2r_0) + \varepsilon_0 \frac{d}{2} \dot{F}(t^*-2r_0) + O\left(\varepsilon_0^2 \varepsilon_\lambda^2\right) . \tag{8.9}$$

We notice that convergence of the post-Newtonian time series given above depends, in fact, not just on one parameter ε_λ which is typical in the post-Newtonian celestial mechanics of extended bodies, but on the product of two parameters. Therefore, the convergence requires satisfaction of two conditions

$$\varepsilon \varepsilon_\lambda \ll 1 , \tag{8.10}$$

$$\varepsilon_0 \varepsilon_\lambda \ll 1 , \tag{8.11}$$

where the numerical value of the parameter ε_λ must be taken for the smallest wavelength, $\lambda_{\rm min}$, in the spectrum of the gravitational radiation emitted by the isolated system. Conditions (8.10), (8.11) ensure convergence of the post-Newtonian series in (8.8) and (8.9) respectively. If the source of light rays and observer are at infinite distances from the isolated system then $\varepsilon \simeq 0$ and $\varepsilon_0 \simeq 0$, and the requirements (8.10), (8.11) are satisfied automatically, irrespective of the structure of the Fourier spectrum (5.5) of the multipole moments of the isolated system. In real astronomical practice such an assumption may not be always satisfied. In such cases it is more natural to avoid the post-Newtonian expansions of the metric tensor and/or observable effects.

If we assume that the mass and angular momentum of the isolated system are conserved the asymptotic expansions of integrals (5.1), (5.2) of the stationary part of the metric tensor have the following form

$$\left[\frac{1}{r}\right]^{[-1]} = -\ln\left(\frac{r}{2r_{\rm E}}\right) - 2\ln\varepsilon - \sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k}(k!)^2} \varepsilon^{2k} = -2\ln d + \ln r + \ln(2r_{\rm E}) + O(\varepsilon^2), \tag{8.12}$$

$$\left[\frac{1}{r_0}\right]^{[-1]} = -\ln\left(\frac{2r_0}{r_{\rm E}}\right) - \sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k}(k!)^2} \varepsilon_0^{2k} = -\ln\left(\frac{2r_0}{r_{\rm E}}\right) + O(\varepsilon_0^2), \tag{8.13}$$

$$\begin{aligned}\left[\frac{1}{r}\right]^{[-2]} &= -r\left\{1 + \left(1 + \sum_{k=1}^{\infty} \mathcal{C}_k \varepsilon^{2k}\right) \left[\ln\left(\frac{\varepsilon^2 r}{2r_{\rm E}}\right) + \sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k}(k!)^2} \varepsilon^{2k}\right]\right\} \\ &= -r - 2r\ln d + r\ln(2rr_{\rm E}) - \varepsilon \frac{d}{2}\left[\frac{1}{2} - \ln\left(\frac{d^2}{2rr_{\rm E}}\right)\right] + O(\varepsilon^2),\end{aligned} \tag{8.14}$$

$$\begin{aligned}\left[\frac{1}{r_0}\right]^{[-2]} &= -r_0\left\{1 - \left(1 + \sum_{k=1}^{\infty} \mathcal{C}_k \varepsilon_0^{2k}\right) \left[\ln\left(\frac{2r_0}{r_{\rm E}}\right) + \sum_{k=1}^{\infty} \frac{(2k-1)!}{2^{2k}(k!)^2} \varepsilon_0^{2k}\right]\right\} \\ & -r_0 + r_0\ln\left(\frac{2r_0}{r_{\rm E}}\right) - \varepsilon_0 \frac{d}{2}\left[\frac{1}{2} + \ln\left(\frac{2r_0}{r_{\rm E}}\right)\right] + O(\varepsilon_0^2).\end{aligned} \tag{8.15}$$

Some estimates for the derivatives with respect to ξ^i and τ of some functions:

$$\hat{\partial}_{<a_1 \ldots a_l>} \frac{F(t-r)}{r} = O\left(\varepsilon^m \varepsilon_\lambda^n \frac{F}{d^{l+1}}\right), \tag{8.16}$$

where $m \geq 2, n \geq 0$. Here and in the subsequent expressions (8.17) - (8.23) m and n depend on l; we give the estimates from below for m and n for $l \geq 1$.

$$\hat{\partial}_{<a_1 \ldots a_l>} \frac{F(t_0-r_0)}{r_0} = O\left(\varepsilon^m \varepsilon_\lambda^n \frac{F}{d^{l+1}}\right), \tag{8.17}$$

where $m \geq 2, n \geq 0$;

$$\hat{\partial}_\tau^l \left[\frac{F(t-r)}{r}\right] = O\left(\varepsilon^m \varepsilon_\lambda^n \frac{F}{d^{l+1}}\right), \tag{8.18}$$

where $m \geq 2, n \geq 0$;

$$\hat{\partial}^l_\tau \left[\frac{F(t_0 - r_0)}{r_0}\right] = O\left(\varepsilon^m \varepsilon^n_\lambda \frac{F}{d^{l+1}}\right), \tag{8.19}$$

where $m \geq 1, n \geq 0$;

$$\hat{\partial}_{<a_1 \ldots a_l>} \left[\frac{F(t-r)}{r}\right]^{[-1]} = -2F(t-r)\hat{\partial}_{<a_1 \ldots a_l>} \ln d + O\left(\varepsilon\varepsilon_\lambda \frac{F}{d^l}\right), \tag{8.20}$$

$$\hat{\partial}_{<a_1 \ldots a_l>} \left[\frac{F(t_0 - r_0)}{r_0}\right]^{[-1]} = O\left(\varepsilon^m \varepsilon^n_\lambda \frac{F}{d^l}\right), \tag{8.21}$$

where $m \geq 2, n \geq 0$;

$$\hat{\partial}_{<a_1 \ldots a_l>} \left[\frac{F(t-r)}{r}\right]^{[-2]} = -2rF(t-r)\hat{\partial}_{<a_1 \ldots a_l>} \ln d + O\left(\varepsilon_\lambda \frac{F}{d^{l-1}}\right), \tag{8.22}$$

$$\hat{\partial}_{<a_1 \ldots a_l>} \left[\frac{F(t_0 - r_0)}{r_0}\right]^{[-2]} = O\left(\varepsilon^m \varepsilon^n_\lambda \frac{F}{d^{l-1}}\right), \tag{8.23}$$

where $m \geq 3$ and $n \geq 0$.

Having written down the auxiliary asymptotic expansions we can consider the observable relativistic effects in the gravitational lens approximation.

8.2. Asymptotic expressions for observable effects

Here we write out the expressions for relativistic effects neglecting all the first order terms with respect to ε.

$$\Delta(\tau, \tau_0) = \underset{(M)}{\Delta}(\tau, \tau_0) + \underset{(S)}{\Delta}(\tau, \tau_0), \tag{8.24}$$

where

$$\begin{aligned}\underset{(M)}{\Delta}(\tau, \tau_0) = &-4\mathcal{M}\ln d + 2\mathcal{M}\ln(4rr_0) - \\ &4\sum_{l=2}^{\infty}\sum_{p=0}^{l-2}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\left(1-\frac{p}{l-1}\right)\times \\ &\hat{\partial}^p_{t^*}\mathcal{I}_{A_l}(t-r)k_{<a_1 \ldots a_p}\hat{\partial}_{a_{p+1} \ldots a_l>}\ln d + O(\varepsilon\varepsilon_\lambda),\end{aligned} \tag{8.25}$$

$$\begin{aligned}\underset{(S)}{\Delta}(\tau, \tau_0) = &-4\epsilon_{iba}k_i\mathcal{S}_b\hat{\partial}_a \ln d + \\ &8\epsilon_{iba}k_i\hat{\partial}_a\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}(l-p-1,p)\left(1-\frac{p}{l-1}\right)\times \\ &\hat{\partial}^p_{t^*}\mathcal{S}_{bA_{l-1}}(t-r)k_{<a_1 \ldots a_p}\hat{\partial}_{a_{p+1} \ldots a_{l-1}>}\ln d + O(\varepsilon\varepsilon_\lambda).\end{aligned} \tag{8.26}$$

The observable unit vector in the direction "observer - source of light" is given by the expression

$$s^i(\tau, \boldsymbol{\xi}) = K^i + \alpha^i(\tau, \boldsymbol{\xi}) + \beta^i(\tau, \boldsymbol{\xi}), \tag{8.27}$$

where we have neglected the quantities $\beta^i(\tau_0,\boldsymbol{\xi})$ and $\gamma^i(\tau,\boldsymbol{\xi})$. The vector $\alpha^i(\tau,\boldsymbol{\xi})$, characterizing the deflection of light, and the relativistic correction $\beta^i(\tau,\boldsymbol{\xi})$ are given by the expressions

$$\alpha^i(\tau,\boldsymbol{\xi}) = \alpha^i_{(M)}(\tau,\boldsymbol{\xi}) + \alpha^i_{(S)}(\tau,\boldsymbol{\xi}), \tag{8.28}$$

where

$$\begin{aligned}\alpha^i_{(M)}(\tau,\boldsymbol{\xi}) =& 4\mathcal{M}\hat{\partial}_i \ln d + \\ & 4\hat{\partial}_i \sum_{l=2}^{\infty}\sum_{p=0}^{l-2} \frac{(-1)^{l+p}}{l!} C_l(l-p,p)\left(1-\frac{p}{l}\right)\left(1-\frac{p}{l-1}\right)\times \\ & \hat{\partial}^p_{t^*}\mathcal{I}_{A_l}(t-r)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\ln d + O(\varepsilon\varepsilon_\lambda),\end{aligned} \tag{8.29}$$

$$\begin{aligned}\alpha^i_{(S)}(\tau,\boldsymbol{\xi}) =& 4\epsilon_{jba}k_j\mathcal{S}_b\hat{\partial}_{ia}\ln d - \\ & 8\epsilon_{iba}k_i\hat{\partial}_{ia}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}(l-p-1,p)\left(1-\frac{p}{l-1}\right)\times \\ & \hat{\partial}^p_{t^*}\mathcal{S}_{bA_{l-1}}(t-r)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\ln d + O(\varepsilon\varepsilon_\lambda).\end{aligned} \tag{8.30}$$

$$\beta^i(\tau,\boldsymbol{\xi}) = \beta^i_{(M)}(\tau,\boldsymbol{\xi}) + \beta^i_{(S)}(\tau,\boldsymbol{\xi}), \tag{8.31}$$

where

$$\beta^i_{(M)}(\tau,\boldsymbol{\xi}) = -\frac{r}{R}\alpha^i_{(M)} + O(\varepsilon_R), \tag{8.32}$$

$$\beta^i_{(S)}(\tau,\boldsymbol{\xi}) = -\frac{r}{R}\alpha^i_{(S)} + O(\varepsilon_R). \tag{8.33}$$

We note that using the expressions (8.8) and (8.9), we can rewrite (8.25), (8.26), (8.29) and (8.30) expressing functions at the moment of time $(t-r)$ in terms of their values at the moment of the closest approach of the photon and the gravitating system t^*. This is accomplished simply by substituting the value t^* instead of $(t-r)$ in the expressions (8.25), (8.26), (8.29) and (8.30) because the corrections will be of the first order with respect to the parameter ε (see Eq. (8.8)).

$$\Delta(\tau,\tau_0) = 2\mathcal{M}\ln(4rr_0) - 4\psi, \tag{8.34}$$

$$\alpha^i(\tau,\boldsymbol{\xi}) = 4\hat{\partial}_i\psi, \tag{8.35}$$

where ψ – is the gravitational lens potential, which has the form

$$\begin{aligned}\psi = \Bigg\{ & \mathcal{M} + \sum_{l=2}^{\infty}\sum_{p=0}^{l-2}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\left(1-\frac{p}{l-1}\right)\times \\ & \hat{\partial}^p_{t^*}\mathcal{I}_{A_l}(t^*)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>} + \\ & \epsilon_{jba}k_j\mathcal{S}_b\hat{\partial}_a - \\ & 2\epsilon_{jba}k_j\hat{\partial}_a\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}(l-p-1,p)\left(1-\frac{p}{l-1}\right)\times \\ & \hat{\partial}^p_{t^*}\mathcal{S}_{bA_{l-1}}(t^*)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\Bigg\}\ln d + O(\varepsilon\varepsilon_\lambda).\end{aligned} \tag{8.36}$$

This expression takes into account multipole moments of all orders in the geostationary case and presents a generalization of the corresponding results obtained in [7], [84] and [83].

The expressions for the angle of rotation of the polarization plane of light in gravitational lens approximation have the form

$$\begin{aligned}\phi_{(M)}(\tau) - \phi_{(M)}(\tau_0) =& -4\sum_{l=2}^{\infty}\sum_{p=0}^{l}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\frac{l-p}{l-1}\times \\ & \times\hat{\partial}^p_{t^*}\epsilon_{j\hat{b}a_l}\mathcal{I}_{\hat{b}A_{l-1}}(t^*)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>j}\ln d + O(\varepsilon\varepsilon_\lambda),\end{aligned} \tag{8.37}$$

$$\phi_{(S)}(\tau) - \phi_{(S)}(\tau_0) = -4\sum_{l=1}^{\infty}\sum_{p=0}^{l}\frac{(-1)^{l+p}l}{(l+1)!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\times$$
$$\times\left[1+H(l-1)\left(1-\frac{2p}{l-1}\right)\right]\times$$
$$\times\,\hat{\partial}^{p+1}_{t^*}\mathcal{S}_{A_l}(t^*)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\ln d + O(\varepsilon\varepsilon_\lambda). \tag{8.38}$$

9. OBSERVABLE RELATIVISTIC EFFECTS IN THE PLANE GRAVITATIONAL-WAVE APPROXIMATION

9.1. Plane-wave Asymptotic Expansions

In the plane gravitational wave approximation we assume that $\lambda \ll min[r, r_0]$ (wave zone condition) and $R \ll min[r, r_0]$ (plane wave condition) (Fig. 2). We introduce small parameters $\delta_\lambda = \lambda/r$, $\delta = R/r$ and $\delta_0 = R/r_0$, where λ is the characteristic wavelength of the gravitational radiation emitted by the isolated astronomical system. The relationship between r, r_0 and R can be written as follows

$$r_0^2 = r^2 - 2rR\cos\theta + R^2 = r^2(1-2\delta\cos\theta+\delta^2)\ , \tag{9.1}$$

where θ – is the angle between the directions "observer – the source of light" and "observer – the gravitating system" (Fig .2). From the previous expression it follows that

$$r_0 \ = \ r(1-\delta\cos\theta+O(\delta^2))\ , \tag{9.2}$$
$$\frac{1}{r_0} \ = \ \frac{1}{r}(1+\delta\cos\theta+O(\delta^2))\ . \tag{9.3}$$

For the variables τ and τ_0 we have

$$\tau = r\cos\theta, \qquad \tau_0 = \tau - R = r\cos\theta - R. \tag{9.4}$$

Quantities d and y can be expressed as follows

$$d \ = \ r\sin\theta\ , \tag{9.5}$$
$$y \ = \ \tau - r = -r(1-\cos\theta)\ . \tag{9.6}$$

The instants of time $(t-r)$ and $(t_0 - r_0)$ are related to each other as follows

$$t_0 - r_0 = t - r - R(1-\cos\theta)\ . \tag{9.7}$$

Using this expression we can write the Taylor expansion for the functions of retarded time $s = t - r$

$$F(t_0 - r_0) = F(t-r) - R(1-\cos\theta)\dot{F}(t-r) + O\left((R/\lambda)^2\right)\ . \tag{9.8}$$

Let us now write out the asymptotic expressions for the derivatives with respect to ξ^i and τ of some functions

$$\hat{\partial}^n_\tau\left[\frac{F(t-r)}{r}\right] = (1-\cos\theta)^n\hat{\partial}^n_{t^*}\left[\frac{F(t-r)}{r}\right] + O(\delta^2)\ , \tag{9.9}$$
$$\hat{\partial}_{a_1\ldots a_n}\left[\frac{F(t-r)}{r}\right] = (-1)^n\frac{\xi_{a_1\ldots a_n}}{r^n}\hat{\partial}^n_{t^*}\left[\frac{F(t-r)}{r}\right] + O(\delta^2)\ , \tag{9.10}$$
$$\hat{\partial}^n_\tau\left[\frac{F(t-r)}{r}\right]^{[-1]} = (1-\cos\theta)^{n-1}\hat{\partial}^{n-1}_{t^*}\left[\frac{F(t-r)}{r}\right] + O(\delta^2)\ , \tag{9.11}$$
$$\hat{\partial}_{a_1\ldots a_n}\left[\frac{F(t-r)}{r}\right]^{[-1]} = \frac{(-1)^n}{1-\cos\theta}\frac{\xi_{a_1\ldots a_n}}{r^n}\hat{\partial}^{n-1}_{t^*}\left[\frac{F(t-r)}{r}\right] + O(\delta^2)\ . \tag{9.12}$$

$$\left\{\left[\frac{F(t-r)}{r}\right]_{,<A_l>}\right\}^{[-1]} = \sum_{p=0}^{l}\sum_{q=0}^{p}(-1)^{p-q}C_l(l-p,p-q,q)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^{p-q}_{t^*}\hat{\partial}^{q}_{\tau}\left[\frac{F(t-r)}{r}\right]^{[-1]} = \tag{9.13}$$
$$\frac{(-1)^l}{1-\cos\theta}[N_{<A_l>}-k_{<A_l>}]\hat{\partial}^{l-1}_{t^*}\frac{F(t-r)}{r}+(-1)^l k_{<A_l>}\hat{\partial}^{l}_{t^*}\left[\frac{F(t-r)}{r}\right]^{[-1]}+O(\delta_\lambda) =$$
$$\frac{1}{1-\cos\theta}\left[\frac{{}^{(-1)}F(t-r)}{r}\right]_{,<A_l>}-\frac{(-1)^l}{1-\cos\theta}k_{<A_l>}\hat{\partial}^{l-1}_{t^*}\frac{F(t-r)}{r}+$$
$$+(-1)^l k_{<A_l>}\hat{\partial}^{l}_{t^*}\left[\frac{F(t-r)}{r}\right]^{[-1]}+O(\delta_\lambda)\;,$$

where $N^i = x^i/r$. Expressions (9.13) can be proved by induction. Expressions for functions taken at the retarded instant of time $s_0 = t_0 - r_0$ can be obtained from equations (9.13) by substitution t_0, r_0 and θ_0 instead of t, r and θ respectively.

Finally, we notice that the impact vector ξ^i can be decomposed as follows

$$\xi^i = r\left(N^i - k^i\cos\theta\right)\;. \tag{9.14}$$

This form of the impact vector ξ^i is useful in subsequent approximations.

9.2. Asymptotic Expressions for Observable Effects

In this section we give expressions for the relativistic effects of the time delay and bending of light in the gravitational plane-wave approximation. In this approximation we neglect all terms of order δ^2, δ_0^2 and δ_λ^2. For the time delay one has

$$\Delta = \underset{(M)}{\Delta}(\tau,\tau_0) + \underset{(S)}{\Delta}(\tau,\tau_0)\,, \tag{9.15}$$

where

$$\underset{(M)}{\Delta}(\tau,\tau_0) = 2\mathcal{M}\frac{R}{r}+\frac{2k_{ij}}{1-\cos\theta}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\left\{\left[\frac{\dot{\mathcal{I}}_{ijA_{l-2}}(t-r)}{r}\right]^{TT}_{,A_{l-2}}-\left[\frac{\dot{\mathcal{I}}_{ijA_{l-2}}(t_0-r_0)}{r_0}\right]^{TT}_{,A_{l-2}}\right\}, \tag{9.16}$$

$$\underset{(S)}{\Delta}(\tau,\tau_0) = -2\frac{\epsilon_{iba}k_i\xi_a\mathcal{S}_b}{1-\cos\theta}\left(\frac{1}{r}-\frac{1}{r_0}\right)- \tag{9.17}$$
$$\frac{4k_{ij}}{1-\cos\theta}\sum_{l=2}^{\infty}\frac{(-1)^l l}{(l+1)!}\left\{\left[\frac{\epsilon_{ba(i}\dot{\mathcal{S}}_{j)bA_{l-2}}(t-r)}{r}\right]^{TT}_{,aA_{l-2}}-\left[\frac{\epsilon_{ba(i}\dot{\mathcal{S}}_{j)bA_{l-2}}(t_0-r_0)}{r_0}\right]^{TT}_{,aA_{l-2}}\right\}\;,$$

Here the transverse - traceless (TT) part of the tensors depending on the multipole moments is taken with respect to the direction N^i. Taking into account expression (3.8) for the components of the metric tensor h_{ij} we can re-write expressions (9.15) – (9.17) for the time delay as follows

$$\Delta = \underset{(M)}{\Delta}(\tau,\tau_0) + \underset{(S)}{\Delta}(\tau,\tau_0) = \tag{9.18}$$
$$2\mathcal{M}\frac{R}{r}-2\frac{\epsilon_{iba}k_i\xi_a\mathcal{S}_b}{1-\cos\theta}\left(\frac{1}{r}-\frac{1}{r_0}\right)-\frac{1}{2}\frac{k_{ij}}{1-\cos\theta}\left({}^{(-1)}h^{TT}_{ij}(t-r)-{}^{(-1)}h^{TT}_{ij}(t_0-r_0)\right).$$

In a similar way we obtain expressions for the observable direction from observer to the source of light perturbed by the gravitational wave

$$s^i(\tau,\boldsymbol{\xi}) = K^i + \alpha^i(\tau,\boldsymbol{\xi}) + \gamma^i(\tau,\boldsymbol{\xi})\,, \tag{9.19}$$

where

$$\alpha^i(\tau,\boldsymbol{\xi}) = \frac{1}{2}\frac{k_{pq}}{1-\cos\theta}\left[(\cos\theta-2)k^i+N^i\right]h^{TT}_{pq}(t-r)+k^q h^{TT}_{ip}(t-r), \tag{9.20}$$

$$\gamma^i(\tau,\boldsymbol{\xi}) = -\frac{1}{2}P^{ij}k^q h^{TT}_{jq}(t,\boldsymbol{x}). \tag{9.21}$$

In expression (9.19) we have neglected the quantities $\beta^i(\tau, \boldsymbol{\xi})$. Expressions (9.18) – (9.20) were obtained in [83] in the spin-dipole mass-quadrupole approximation only. This paper has generalized this result for the case when h_{ij}^{TT} contains all mass and spin multipole moments of arbitrary orders.

In the case when the distance between observer and the source of light is much smaller than the wavelength of the gravitational waves $R \ll \lambda$, expression (9.18) is reduced to a well known result [1, 2] for gravitational wave detectors located in a wave-zone of an isolated system

$$\frac{\Delta R}{R} = \frac{1}{2} k_{ij} h_{ij}^{TT}(t-r) + O(\delta^2) + O\left(\frac{R}{\lambda}\right) , \tag{9.22}$$

where $\Delta R = c\,\Delta(\tau, \tau_0)$.

Relativistic rotation of the polarization plane of light (the Skrotskii effect) in the gravitational plane-wave approximation assumes the next form

$$\phi(\tau) - \phi(\tau_0) = \frac{1}{2}\frac{(\boldsymbol{k}\times N)^i k^j}{1-\boldsymbol{k}\cdot N} h_{ij}^{TT}(t-r) - \frac{1}{2}\frac{(\boldsymbol{k}\times N_0)^i k^j}{1-\boldsymbol{k}\cdot N_0} h_{ij}^{TT}(t_0-r_0) . \tag{9.23}$$

APPENDIX A: APPROXIMATE EXPRESSIONS FOR THE CHRISTOFFEL SYMBOLS

Making use of the general definition of the Christoffel symbols given in equation (4.14) and applying the expansion of the metric tensor (3.1) result in the approximate linearized post-Minkowskian expressions for the Christoffel symbols

$$\Gamma^0_{00} = -\frac{1}{2}\partial_t h_{00}(t,\boldsymbol{x}) , \qquad \partial_t \equiv \frac{\partial}{\partial t} , \tag{A.1}$$

$$\Gamma^0_{0i} = -\frac{1}{2}\partial_i h_{00}(t,\boldsymbol{x}) , \qquad \partial_i \equiv \frac{\partial}{\partial x^i} , \tag{A.2}$$

$$\Gamma^0_{ij} = -\frac{1}{2}\left[\partial_i h_{0j}(t,\boldsymbol{x}) + \partial_j h_{0i}(t,\boldsymbol{x}) - \partial_t h_{ij}(t,\boldsymbol{x})\right] , \tag{A.3}$$

$$\Gamma^i_{00} = \partial_t h_{0i}(t,\boldsymbol{x}) - \frac{1}{2}\partial_i h_{00}(t,\boldsymbol{x}) , \tag{A.4}$$

$$\Gamma^i_{0j} = \frac{1}{2}\left[\partial_j h_{0i}(t,\boldsymbol{x}) - \partial_i h_{0j}(t,\boldsymbol{x}) + \partial_t h_{ij}(t,\boldsymbol{x})\right] , \tag{A.5}$$

$$\Gamma^i_{jp} = \frac{1}{2}\left[\partial_j h_{ip}(t,\boldsymbol{x}) + \partial_p h_{ij}(t,\boldsymbol{x}) - \partial_i h_{jp}(t,\boldsymbol{x})\right] . \tag{A.6}$$

These expressions are used for the calculation of the right-hand side of the null geodesic equation (4.13).

APPENDIX B: THE METRIC TENSOR IN TERMS OF THE LIGHT-RAY PARAMETERS $\boldsymbol{\xi}$ AND τ

Expressions for perturbations of the metric tensor (3.5) – (3.8) in terms of the parameters $\boldsymbol{\xi}^i$ and τ have the following form follows

$$h^{\text{can.}}_{00} = \frac{2\mathcal{M}}{r} + 2\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p} \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(t^*,\tau,\boldsymbol{\xi}) , \tag{B.1}$$

$$h^{\text{can.}}_{0i} = -\frac{2\epsilon_{iba}\mathcal{S}^b N^a}{r^2} + 4\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\left[\underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) + \underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi})\right] , \tag{B.2}$$

$$h^{\text{can.}}_{ij} = \delta_{ij} h^{\text{can.}}_{00} + q^{\text{can.}}_{ij} , \tag{B.3}$$

$$q^{\text{can.}}_{ij} = 4\sum_{l=2}^{\infty}\sum_{p=0}^{l-2}\sum_{q=0}^{p} \underset{(M)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi}) - 8\sum_{l=2}^{\infty}\sum_{p=0}^{l-2}\sum_{q=0}^{p} \underset{(S)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi}) , \tag{B.4}$$

where

$$\underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}}{l!} C_l(l-p,p-q,q) k_{<a_1\dots a_p} \hat{\partial}_{a_{p+1}\dots a_l>} \hat{\partial}^q_\tau \left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right] , \tag{B.5}$$

$$\underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}}{l!} C_{l-1}(l-p-1,p-q,q) k_{<a_1\dots a_p} \hat{\partial}_{a_{p+1}\dots a_{l-1}>} \hat{\partial}^q_\tau \left[\frac{\mathcal{I}^{(p-q+1)}_{iA_{l-1}}(t-r)}{r}\right] , \tag{B.6}$$

$$\underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}\, l}{(l+1)!} C_{l-1}(l-p-1,p-q,q) (\hat{\partial}_a + k_a\hat{\partial}_\tau - k_a\hat{\partial}_{t^*}) k_{<a_1\dots a_p} \hat{\partial}_{a_{p+1}\dots a_{l-1}>} \hat{\partial}^q_\tau \left[\frac{\epsilon_{iab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right] , \tag{B.7}$$

$$\underset{(M)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}}{l!} C_{l-2}(l-p-2,p-q,q) k_{<a_1\dots a_p} \hat{\partial}_{a_{p+1}\dots a_{l-2}>} \hat{\partial}^q_\tau \left[\frac{\mathcal{I}^{(p-q+2)}_{ijA_{l-2}}(t-r)}{r}\right] , \tag{B.8}$$

$$\underset{(S)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}\, l}{(l+1)!} C_{l-2}(l-p-2,p-q,q) (\hat{\partial}_a + k_a\hat{\partial}_\tau - k_a\hat{\partial}_{t^*}) k_{<a_1\dots a_p} \hat{\partial}_{a_{p+1}\dots a_{l-2}>} \hat{\partial}^q_\tau \left[\frac{\epsilon_{ba(i}\mathcal{S}^{(p-q+1)}_{j)bA_{l-2}}(t-r)}{r}\right] . \tag{B.9}$$

All quantities in the right side of expressions (B.5)–(B.9), which are explicitly shown as functions of x^i, $r = |\boldsymbol{x}|$ and t, must be understood as taken on the unperturbed light-ray trajectory and expressed in terms of ξ^i, $d = |\boldsymbol{\xi}|$, τ and t^* in accordance with equations (4.20), (4.22). For example, the ratio $\mathcal{I}^{(p-q)}_{A_l}(t-r)/r$ in equation (B.5) must be understood as

$$\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r} \equiv \frac{\mathcal{I}^{(p-q)}_{A_l}(t^*+\tau-\sqrt{\tau^2+d^2})}{\sqrt{\tau^2+d^2}} , \tag{B.10}$$

and the same replacement rule is applied to the other equations.

APPENDIX C: THE GAUGE FUNCTIONS

Gauge functions w^α, generating the coordinate transformation from the canonical harmonic coordinate system to the ADM-harmonic one are given by equations (3.19), (3.20). They transform the metric tensor as follows

$$h^{\text{can.}}_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha w_\beta - \partial_\beta w_\alpha , \tag{C.1}$$

where $h^{\text{can.}}_{\alpha\beta}$ is the canonical form of the metric tensor in harmonic coordinates given by equations (3.5)–(3.8) and $h_{\alpha\beta}$ is the metric tensor given in the ADM-harmonic coordinates by equations (3.21)–(3.24).

The gauge functions taken on the light-ray trajectory and expressed in terms of the variables ξ and τ can be written down in the next form

$$w^0 = \sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\int_{-\infty}^{\tau+t^*} du\, \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi}), \tag{C.2}$$

$$\begin{aligned} w^i = & (\hat{\partial}_i + k_i\hat{\partial}_\tau - k_i\hat{\partial}_{t^*}) \sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\int_{-\infty}^{\tau+t^*} dv \int_{-\infty}^{v} du\, \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi}) \\ & -4\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\int_{-\infty}^{\tau+t^*} du \left[\underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi}) + \underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi})\right] , \end{aligned} \tag{C.3}$$

where $\underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi})$, $\underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi})$ and $\underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi})$ are defined by the Eqs. (B.5), (B.6) and (B.7) after making use of the substitution $t^* \to u$. It is worth noting the following relationships

$$\frac{\partial w^0}{\partial t^*} = \sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p} \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(t^*,\tau,\boldsymbol{\xi}), \tag{C.4}$$

$$\frac{\partial w^i}{\partial t^*} = \sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p} \int_{-\infty}^{\tau+t^*} du(\hat{\partial}_i + k_i\hat{\partial}_\tau)\, \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi}) - 4\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\left[\underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) + \underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi})\right], \tag{C.5}$$

and

$$\begin{aligned} k^i\frac{\partial w^i}{\partial t^*} - \frac{\partial w^0}{\partial t^*} =& \sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\left[\int_{-\infty}^{\tau+t^*} du\hat{\partial}_\tau\, \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi}) - \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(t^*,\tau,\boldsymbol{\xi})\right] \\ &- 4\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\left[k^i\, \underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) + k^i\, \underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi})\right], \end{aligned} \tag{C.6}$$

which are helpful in calculation of the gravitational frequency shift.

Linear combination $k^i\varphi^0 - \varphi^i$ of the gauge-dependent functions φ^α introduced in equation (4.41) is given by the expressions

$$k^i\varphi^0 - \varphi^i = (k^i\varphi^0_{(M)} - \varphi^i_{(M)}) + (k^i\varphi^0_{(S)} - \varphi^i_{(S)}), \tag{C.7}$$

$$\begin{aligned} k^i\varphi^0_{(M)} - \varphi^i_{(M)} = 2\hat{\partial}_i &\sum_{l=2}^{\infty}\sum_{p=2}^{l}\sum_{q=2}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)\times \\ &\left(1-\frac{p-q}{l}\right)\left(1-\frac{p-q}{l-1}\right)k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>}\hat{\partial}_\tau^{q-2}\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right] + \\ &2\sum_{l=2}^{\infty}\sum_{p=1}^{l}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)\times \\ &\left(1-\frac{p-q}{l}\right)\left\{\left(1+\frac{p-q}{l-1}\right)k_{i<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>}\hat{\partial}_\tau^{q-1}\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right] - \right. \\ &\left. 2\frac{p-q}{l-1}\,k_{<a_1\dots a_{p-1}}\hat{\partial}_{a_p\dots a_{l-1}>}\hat{\partial}_\tau^{q-1}\left[\frac{\mathcal{I}^{(p-q)}_{iA_{l-1}}(t-r)}{r}\right]\right\}, \end{aligned} \tag{C.8}$$

$$k^i\varphi^0_{(S)} - \varphi^i_{(S)} = 2\frac{\epsilon_{iab}k^a\mathcal{S}^b}{r} + 4k_j\hat{\partial}_{ia}\sum_{l=3}^{\infty}\sum_{p=2}^{l-1}\sum_{q=2}^{p}\frac{(-1)^{l+p-q}\,l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times$$
$$\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{q-2}_{\tau}\left[\frac{\epsilon_{jab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]-$$
$$4\left(\hat{\partial}_a - k_a\hat{\partial}_{t^*}\right)\sum_{l=2}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times$$
$$\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{q-1}_{\tau}\left[\frac{\epsilon_{iab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]-$$
$$4k_a\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times$$
$$\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{q}_{\tau}\left[\frac{\epsilon_{iab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]+$$
$$4k_j\sum_{l=2}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times$$
$$k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{q-1}_{\tau}\left[\frac{\epsilon_{jba_{l-1}}\mathcal{S}^{(p-q+1)}_{\hat{i}bA_{l-2}}(t-r)}{r}\right] \tag{C.9}$$

Gauge-dependent term generated by equation (7.69) for the rotation of the plane of polarization of electromagnetic wave is a pure spatial vector χ^i that can be decomposed in two linear parts corresponding to the mass and spin multipoles:

$$\chi^i = \chi^i_{(M)} + \chi^i_{(S)}\ , \tag{C.10}$$

where

$$+ =4\sum_{l=2}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}}{l!}C_{l-1}(l-p-1,p-q,q)\left(1-\frac{p-q}{l-1}\right)$$
$$\times\, k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{q-1}_{\tau}\left[\frac{\mathcal{I}^{(p-q+1)}_{iA_{l-1}}(t-r)}{r}\right]\ , \tag{C.11}$$

$$\chi^i_{(S)} = -\,4\sum_{l=1}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}(l-p-1,p-q,q)\left[1-\frac{p-q}{l-1}\,H(l-1)\right]$$
$$\times\left[H(q)(\hat{\partial}_a - k_a\hat{\partial}_{t^*}) + k_a\hat{\partial}_{\tau}\right]k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^{q-1}_{\tau}\left[\frac{\epsilon_{iba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]$$
$$+\,4\sum_{l=3}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}(l-p-1,p-q,q)\left(1-\frac{p}{l}\right)\left(1-\frac{q}{p}\right)$$
$$\times\, k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>a}\hat{\partial}^{q-1}_{\tau}\left[\frac{\epsilon_{baa_{l-1}}\mathcal{S}^{(p-q)}_{ibA_{l-2}}(t-r)}{r}\right]\ . \tag{C.12}$$

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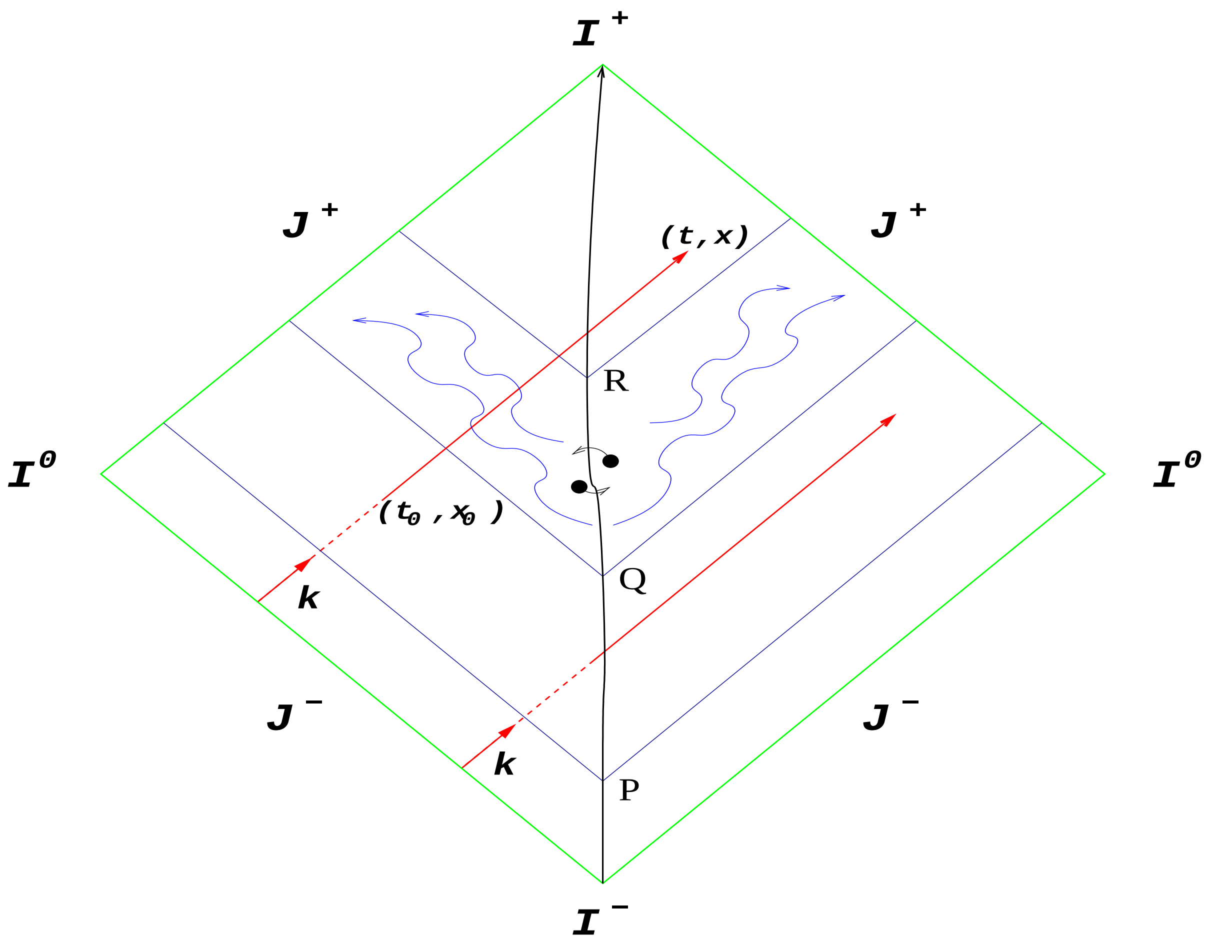

FIG. 1: Penrose's diagram shows a worldline of the isolated system (binary star) originating at past timelike infinity I^- and ending at the future timelike infinity I^+. Both light and gravity propagate along null geodesics going from the past null infinity J^- to the future null infinity J^+. A particular light geodesic is emanating from the event $(t_0, \boldsymbol{x}_0)$ and ending at the event $(t, \boldsymbol{x})$. Extrapolation of this light geodesic to the past null infinity defines the null wave vector $k^\alpha = (k^0, \boldsymbol{k})$ of the electromagnetic wave under consideration.

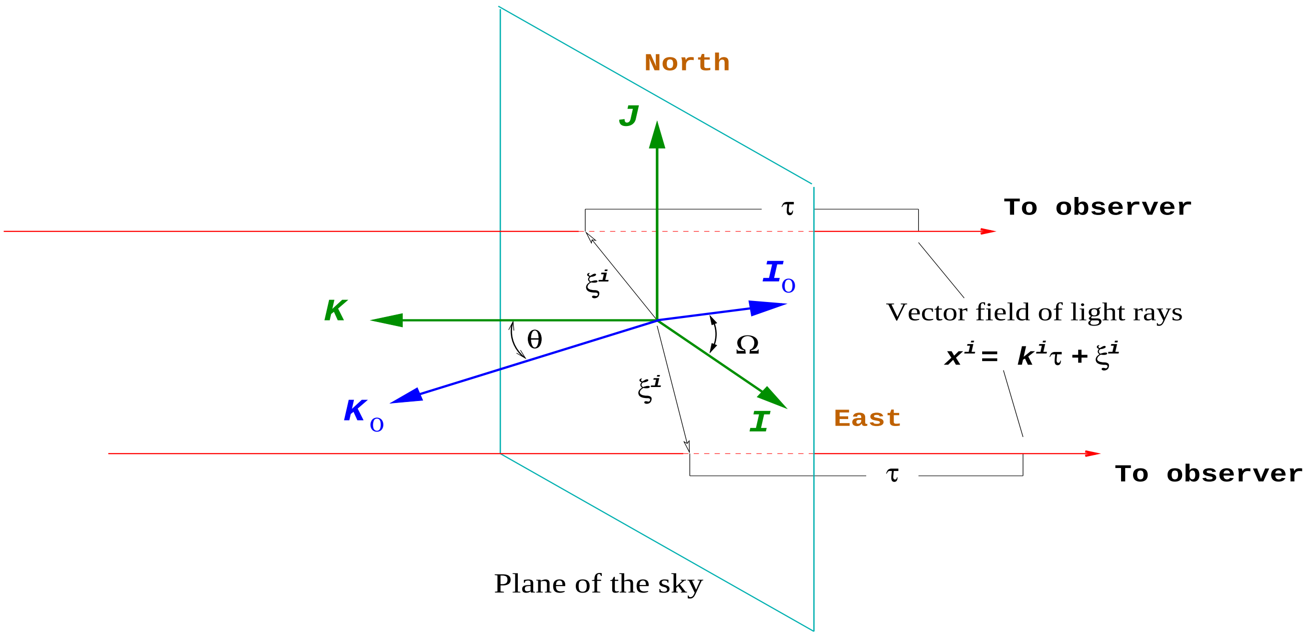

FIG. 2: Astronomical coordinate system used for calculations. The origin of the coordinate system is at the center-of-mass of the source of gravitational waves. The bundle of light rays is defined by the vector field k^i. The vector $K^i = -k^i + O(c^{-2})$ is directed from observer towards the source of light. The vector K_0^i is directed from the observer towards the source of gravitational waves. We use in the paper the equalities $K_0^i = -N^i = -x^i/r$, where x^i are the coordinates of the observer with respect to the source of gravitational waves, and $r = |\boldsymbol{x}|$. The plane of the sky to the vector K_0^i is not shown.

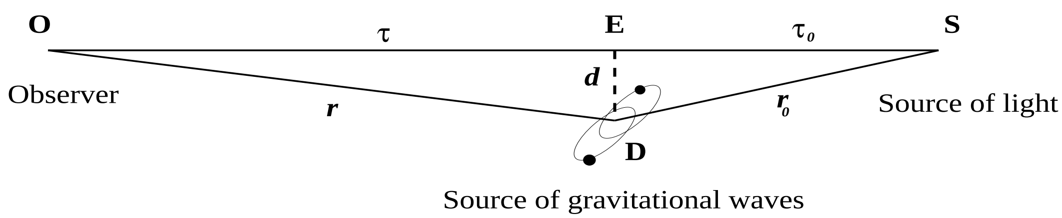

FIG. 3: Relative configuration of observer (O), source of light (S), and a localized source of gravitational waves (D). The source of gravitational waves deflects light rays which are emitted at the moment t_0 at the point S and received at the moment t at the point O. The point E on the line OS corresponds to the moment of the closest approach of light ray to the deflector D. Distances are $OS = R$, $DO = r$, $DS = r_0$, the impact parameter $DE = d$, $OE = \tau > 0$, $ES = \tau_0 = \tau - R < 0$. The impact parameter d is small in comparison to all other distances.

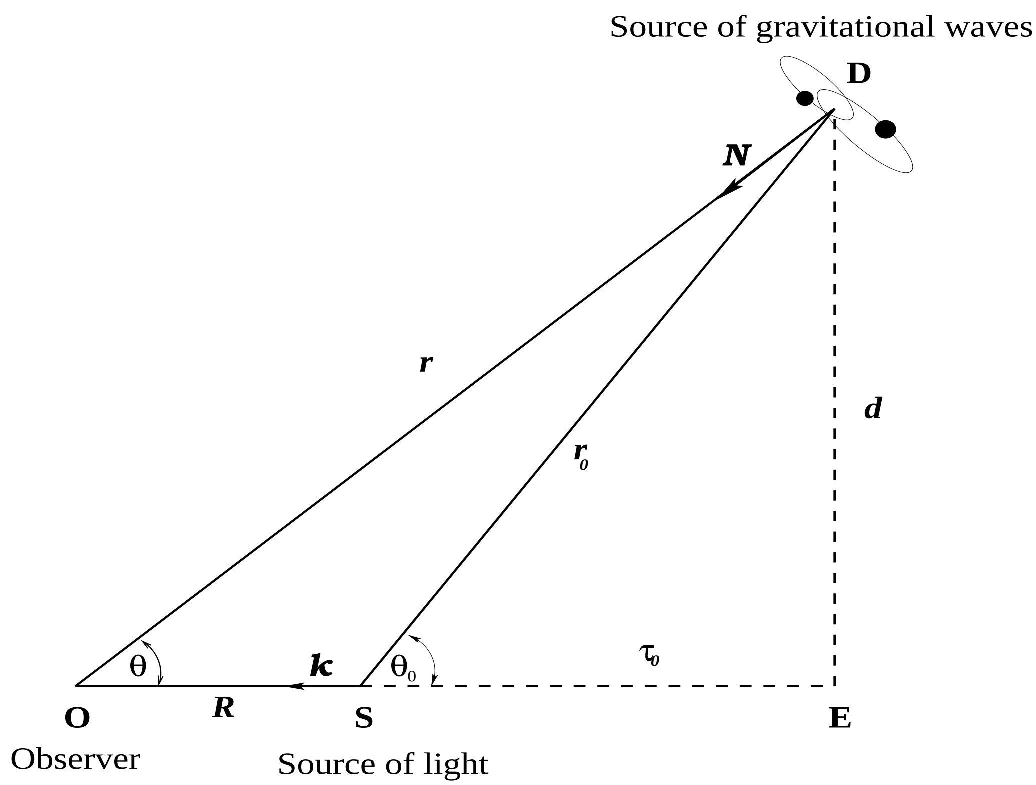

FIG. 4: Relative configuration of observer (O), source of light (S), and a localized source of gravitational waves (D). The source of gravitational waves deflects light rays which are emitted at the moment t_0 at the point S and received at the moment t at the point O. The point E on the line OS corresponds to the moment of the closest approach of light ray to the deflector D. Distances are $OS = R$, $DO = r$, $DS = r_0$, the impact parameter $DE = d$, $OE = \tau = r\cos\theta$, $ES = \tau_0 = \tau - R$. The distance R is much smaller than both r and r_0. The impact parameter d is, in general, not small in comparison to all other distances.