Propagation of Light in the Field of Stationary and Radiative Gravitational Multipoles

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Extremely high precision of near-future radio/optical interferometric observatories like SKA, Gaia, SIM and the unparalleled sensitivity of LIGO/LISA gravitational-wave detectors demands more deep theoretical treatment of relativistic effects in the propagation of electromagnetic signals through variable gravitational fields of the solar system, oscillating and precessing neutron stars, coalescing binary systems, exploding supernova, and colliding galaxies. Especially important for future gravitational-wave observatories is the problem of propagation of light rays in the field of multipolar gravitational waves emitted by a localized source of gravitational radiation. Present paper suggests physically-adequate and consistent solution of this problem in the first post-Minkowskian approximation of General Relativity which accounts for all time-dependent multipole moments of an isolated astronomical system. We derive equations of propagation of electromagnetic wave in the retarded gravitational field of the localized source emitting gravitational waves of arbitrary multipolarity and integrate them analytically in closed form. We also prove that the leading terms in the observable relativistic effects (time delay, deflection angle, and rotation of the plane of polarization of light) depend on the instantaneous value of the multipole moments of the isolated system and its time derivatives taken at the retarded instant of time but not on their integrated values. The influence of the multipolar gravitational field of the isolated system on the light propagation is

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examined for a general case when light propagates not only through the wave zone of the system but also through its intermediate and near zones. The gauge freedom of our formalism is carefully studied and all gauge-dependent terms are singled out and separated from observable quantities. We also present a thorough-going analytical treatment of time delay, light-ray bending and polarization of light in the case of large impact parameter corresponding to the approximation of a plane gravitational wave of arbitrary multipolarity. This exploration essentially extends previous results regarding propagation of light rays in the quadrupolar field of a plane monochromatic gravitational wave. Explicit expressions for time delay, deflection angle and rotation of the plane of polarization of light are obtained in terms of the transverse-traceless (TT) part of the space-space components of the metric tensor. We also discuss the relevance of the developed formalism for detection of relativistic effects of multipolar gravitational fields by existing and near-future astronomical techniques as well as the gravitational wave detectors.

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1. INTRODUCTION

1.1. Statement of the Problem and Relation to Previous Work.

We consider propagation of electromagnetic signals through the time-dependent gravitational field of an isolated gravitating system emitting gravitational waves such as, for example, a binary system or oscillating neutron star. Gravitational field is assumed to be weak everywhere along the light ray and a linear (post-Minkowskian) approximation of General Relativity is used. We do not restrict our consideration to a specific system, but rather, following [1], develop a formalism which can be applied to an arbitrary self-gravitating source that is completely characterized by its time-dependent multipole moments of arbitrary order. We assume the wavelength of light to be much smaller than the wavelength of the gravitational waves which allows us to work in the geometrical optics approximation. The relative distances between and positions of the source of light, the isolated system, and the observer are not restricted as well, which makes our mathematical formalism quite general and applicable for most practical situations. With the assumptions made, we show that the equations of light propagation, which in geometrical optics are the equations of null geodesics, can be analytically integrated giving linearized metric perturbations to the unperturbed flat space-time trajectory. We also use the formalism developed in [2] and [3] to consider the propagation of the polarization of light along the light ray and, in particular, the effect of gravity-induced rotation of the polarization plane of an electromagnetic wave (the Skrotskii effect).

This work is essentially a generalization of [1] for the case when the isolated system possesses multipole moments of arbitrary order and type. In [1] the same problem is considered in spin-dipole and mass-quadrupole approximation only. We also generalize the results of [3] for Skrotskii effect on the case of source possessing higher order multipole moments. The results of [3] for isolated system emitting gravitational waves were obtained again only in spin-dipole mass-quadrupole approximation. Formalism of the present paper allows us to extend results of exploration of propagation of electromagnetic waves in random field of plane gravitational waves [4, 5] to more general class of an isolated systems emitting non-planar gravitational waves anisotropically. Additional mathematical extention of the present work with more thorough historical review isgiven in the next publication [6].

1.2. Notations and Conventions

Notations in this paper coincide with those of [6]. Metric tensor on the space-time manifold is denoted by $g_{\alpha\beta}$ and its perturbation $h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$. The determinant of the metric tensor is negative and is denoted as $g = \det||g_{\alpha\beta}||$. The four-dimensional fully antisymmetric Levi-Civita symbol $\epsilon_{\alpha\beta\gamma\delta}$ is defined in accordance with the convention $\epsilon_{0123} = +1$. In the present paper we use a geometrodynamic system of units [7, 8] such that $c = G = 1$ where c is the fundamental speed and G is the universal gravitational constant. Space-time is assumed to be asymptotically-flat and covered by a single coordinate system $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, where t and (x, y, z) are time and space coordinates respectively. This coordinate system is reduced at infinity to the Minkowskian coordinates. We shall also use the spherical coordinates (r, θ, ϕ) related to (x, y, z) by the standard transformation

$$x = r\sin\theta\cos\phi\ , \quad y = r\sin\theta\sin\phi\ , \quad z = r\cos\theta\ . \tag{1.1}$$

Greek (spacetime) indices range from 0 to 3, and Latin (space) indices run from 1 to 3. If not specifically stated the opposite, the indices are raised and lowered by means of the Minkowski metric $\eta_{\alpha\beta} \equiv \mathrm{diag}(-1, 1, 1, 1)$. Regarding this rule the following conventions for coordinates hold: $x^i = x_i$ and $x^0 = -x_0$. We also adopt notations, $\delta_{ij} \equiv \mathrm{diag}(1, 1, 1)$, for the Kroneker symbol (a unit matrix), and, ϵ_{ijk}, for the fully antisymmetric 3-dimensional symbol of Levi-Civita with convention $\epsilon_{123} = +1$.

Repeated indices are summed over in accordance with the Einstein's rule [7, 8]. In the linearized approximation of general relativity used in this work there is no difference between spatial vectors and co-vectors or between upper and lower space indices. Therefore, for a dot product of two space vectors we have

$$A^iB_i = A_iB_i \equiv A_1B_1 + A_2B_2 + A_3B_3\ . \tag{1.2}$$

In what follows, we shall commonly use spatial multi-index notations for Cartesian three-dimensional tensors [9], that is

$$\mathcal{I}_{A_l} \equiv \mathcal{I}_{a_1 \ldots a_l}. \tag{1.3}$$

Tensor product of l identical spatial vectors k^i will be denoted as a three-dimensional tensor having l indices

$$k_{a_1}\ldots k_{a_l} \equiv k_{a_1\ldots a_l}\ . \tag{1.4}$$

Full symmetrization with respect to a group of spatial indices of a Cartesian tensor will be distinguished with round brackets put around the indices

$$Q_{(a_1 \dots a_l)} \equiv \frac{1}{l!} \sum_{\sigma} Q_{\sigma(1) \dots \sigma(l)} \ , \tag{1.5}$$

where σ is the set of all permutations of $(1, 2, ..., l)$ which makes $Q_{a_1 \dots a_l}$ fully symmetrical in $a_1 \dots a_l$.

It is convenient to introduce a special notation for symmetric trace-free (STF) Cartesian tensors by making use of angular brackets around STF indices. The explicit expression of the STF part of a tensor $Q_{a_1 \dots a_l}$ is [9]

$$Q_{<a_1 \dots a_l>} \equiv \sum_{k=0}^{[l/2]} a_k^l \delta_{(a_1 a_2} \cdots \delta_{a_{2k-1} a_{2k}} S_{a_{2k+1} \dots a_l) b_1 b_1 \dots b_k b_k} \ , \tag{1.6}$$

where $[l/2]$ is the integer part of $l/2$,

$$S_{a_1 \dots a_l} \equiv Q_{(a_1 \dots a_l)} \tag{1.7}$$

and numerical coefficients

$$a_k^l = \frac{(-1)^k}{(2k)!!} \frac{l!}{(2l-1)!!} \frac{(2l-2k-1)!!}{(l-2k)!} \ . \tag{1.8}$$

We also assume that for any integer $l \geq 0$

$$l! \equiv l(l-1) \dots 2 \cdot 1 \ , \qquad 0! \equiv 1 \ , \tag{1.9}$$

and

$$l!! \equiv l(l-2)(l-4) \dots (2 \text{ or} 1) \ , \qquad 0!! \equiv 1 \ . \tag{1.10}$$

One has, for example,

$$T_{<abc>} \equiv T_{(abc)} - \frac{1}{5} \delta_{ab} T_{(cjj)} - \frac{1}{5} \delta_{bc} T_{(ajj)} - \frac{1}{5} \delta_{ac} T_{(bjj)} \ . \tag{1.11}$$

Cartesian tensors of the mass-type $\mathcal{I}_{<A_l>}$ and spin-type multipoles $\mathcal{S}_{<A_l>}$ entirely describing gravitational field outside of an isolated astronomical system are always STF objects that can be checked by inspection of the definition following from the multipolar decomposition of the metric tensor perturbation $h_{\alpha\beta}$ [9]. For this reason, to avoid appearance of too complicated index notations we shall omit in the following text the angular brackets around indices of these (and only these) tensors, that is we adopt: $\mathcal{I}_{A_l} \equiv \mathcal{I}_{<A_l>}$ and $\mathcal{S}_{A_l} \equiv \mathcal{S}_{<A_l>}$.

We shall also use transverse (T) and transverse-traceless (TT) Cartesian tensors in our calculations [8, 9]. These objects are defined by making use of the operator of projection $P_{jk} \equiv \delta_{jk} - k_{jk}$ onto the plane orthogonal to a unit vector k_j. Thus, one has [9]

$$Q^{\mathrm{T}}_{a_1 \ldots a_l} \equiv P_{a_1 b_1} P_{a_2 b_2} \ldots P_{a_l b_l} Q_{b_1 \ldots b_l} \ , \tag{1.12}$$

$$Q^{\mathrm{TT}}_{a_1 \ldots a_l} \equiv \sum_{k=0}^{[l/2]} b^l_k P_{(a_1 a_2} \cdots P_{a_{2k-1} a_{2k}} W_{a_{2k+1} \ldots a_l) b_1 b_1 \ldots b_k b_k} \ , \tag{1.13}$$

where again $[l/2]$ is the integer part of $l/2$,

$$W_{a_1 \ldots a_l} \equiv Q^{\mathrm{T}}_{(a_1 \ldots a_l)} \tag{1.14}$$

and numerical coefficients

$$b^l_k = \frac{(-1)^k}{4^k} \frac{l(l-k-1)!!}{k!(l-2k)!} \ . \tag{1.15}$$

For instance,

$$Q^{\mathrm{TT}}_{ab} \equiv \frac{1}{2} \left(P_{ai} P_{bj} Q_{ij} + P_{bi} P_{aj} Q_{ij} \right) - \frac{1}{2} P_{ab} \left(P_{jk} Q_{jk} \right) \ . \tag{1.16}$$

Polynomial coefficients will be used in some of our equations and they are defined by [10]

$$C_l(p_1, \ldots, p_n) \equiv \frac{l!}{p_1! \ldots p_n!} \ , \tag{1.17}$$

where l and p_i are positive integers such that $\sum_{i=1}^n p_i = l$. We introduce a Heaviside unit step function $H(p-q)$ such that on the set of whole numbers

$$H(p-q) = \begin{cases} 0, & \text{if } p \le q, \\ 1, & \text{if } p > q. \end{cases} \tag{1.18}$$

For any differentiable function $f = f(t, \boldsymbol{x})$ one uses notations: $f_{,0} = \partial f / \partial t$ and $f_{,i} = \partial f / \partial x^i$ for its partial derivatives. In general, comma standing after the function denotes a partial derivative with respect to a corresponding coordinate: $f_{,\alpha} \equiv \partial f(x) / \partial x^\alpha$. Overdot denotes a total derivative with respect to time $\dot{f} \equiv df/dt = \partial f / \partial t + \dot{x}^i \partial f / \partial x^i$ usually taken in this paper along the light ray trajectory $\boldsymbol{x}(t) \equiv (x^i)$. Sometimes the partial derivatives with respect to space coordinate x^i will be also denoted as $\partial_i \equiv \partial / \partial x^i$, and the partial time derivative will be denoted as $\partial_t \equiv \partial / \partial t$. Covariant derivative with respect to the coordinate x^α will be denoted as ∇_α.

We shall use special notations for integrals with respect to time and for those taken along the light-ray trajectory. Specifically, the time integrals from a function $F(t, \boldsymbol{x})$, where $\boldsymbol{x}$ is not specified, are denoted as

$$F^{(-1)}(t,\boldsymbol{x}) \equiv \int\limits_{-\infty}^{t} F(\tau,\boldsymbol{x})d\tau\ , \qquad\qquad F^{(-2)}(t,\boldsymbol{x}) \equiv \int\limits_{-\infty}^{t} F^{(-1)}(\tau,\boldsymbol{x})d\tau\ . \tag{1.19}$$

Time integrals from the function $F(t, \boldsymbol{x})$ taken along the light ray so that spatial coordinate $\boldsymbol{x}$ is a function of time $\boldsymbol{x} \equiv \boldsymbol{x}(t)$, are denoted as

$$F^{[-1]}(t,\boldsymbol{x}) \equiv \int\limits_{-\infty}^{t} F(\tau,\boldsymbol{x}(\tau))d\tau, \qquad\qquad F^{[-2]}(t,\boldsymbol{x}) \equiv \int\limits_{-\infty}^{t} F^{[-1]}(\tau,\boldsymbol{x}(\tau))d\tau\ . \tag{1.20}$$

Integrals in equations (1.19) represent functions of time and space coordinates. Integrals in equations (1.20) are defined on the light-ray trajectory and are functions of time only.

Multiple time derivative from function $F(t, \boldsymbol{x})$ is denoted by

$$F^{(p)}(t,\boldsymbol{x}) = \frac{\partial^p F(t,\boldsymbol{x})}{\partial t^p}\ , \tag{1.21}$$

so that its action on the time integrals eliminates integration in the sense that

$$F^{(p)}(t,\boldsymbol{x}) = \frac{\partial^{p+1} F^{(-1)}(t,\boldsymbol{x})}{\partial t^{p+1}} = \frac{\partial^{p+2} F^{(-2)}(t,\boldsymbol{x})}{\partial t^{p+2}}\ , \tag{1.22}$$

and

$$F^{[p]}(t,\boldsymbol{x}) = \frac{d^{p+1} F^{[-1]}(t,\boldsymbol{x})}{dt^{p+1}} = \frac{d^{p+2} F^{[-2]}(t,\boldsymbol{x})}{dt^{p+2}}\ . \tag{1.23}$$

Spatial vectors will be denoted by bold italic letters, for instance $A^i \equiv \boldsymbol{A}$, $k^i \equiv \boldsymbol{k}$, etc. The Euclidean dot product between two spatial vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted with dot between them: $a^i b_i = \boldsymbol{a}\cdot\boldsymbol{b}$. The Euclidean wedge (cross) product between two vectors is denoted with symbol $\times$, that is $\epsilon_{ijk}a^j b^k = (\boldsymbol{a}\times\boldsymbol{b})^i$. Other particular notations will be introduced as they appear in the text.

2. METRIC TENSOR AND COORDINATE SYSTEMS

2.1. Metric Tensor

We assume that the gravitational field is weak everywhere along the light-ray path. This allows us to work in the linear (with respect to G) approximation of General Relativity which is sometimes called the post-Minkowskian approximation [11]. Thus, the metric tensor can be expressed as the sum of the Minkowski metric and a small perturbation $h_{\alpha\beta} \ll 1$ being proportional to the universal gravitational constant G:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \tag{2.1}$$

We consider propagation of light in the space-time around an isolated gravitating system. The most general expressions for metric perturbations in this case were given by Thorne [9] in a special coordinate system which belongs to the class of the harmonic coordinate systems. In what follows we shall call these special coordinates *canonical harmonic* coordinates. The expressions given by Thorne read

$$h_{00}^{\text{can.}} = \frac{2\mathcal{M}}{r} + 2\sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}_{A_l}(t-r)}{r}\right]_{,A_l}, \tag{2.2}$$

$$h_{0i}^{\text{can.}} = -\frac{2\epsilon_{ipq}\mathcal{S}_p N_q}{r^2} - 4\sum_{l=2}^{\infty} \frac{(-1)^l l}{(l+1)!} \left[\frac{\epsilon_{ipq}\mathcal{S}_{pA_{l-1}}(t-r)}{r}\right]_{,qA_{l-1}} + 4\sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\dot{\mathcal{I}}_{iA_{l-1}}(t-r)}{r}\right]_{,A_{l-1}}, \tag{2.3}$$

$$h_{ij}^{\text{can.}} = \delta_{ij} h_{00}^{\text{can.}} + q_{ij}^{\text{can.}}, \tag{2.4}$$

$$q_{ij}^{\text{can.}} = 4\sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\ddot{\mathcal{I}}_{ijA_{l-2}}(t-r)}{r}\right]_{,A_{l-2}} - 8\sum_{l=2}^{\infty} \frac{(-1)^l l}{(l+1)!} \left[\frac{\epsilon_{pq(i}\dot{\mathcal{S}}_{j)pA_{l-2}}(t-r)}{r}\right]_{,qA_{l-2}}. \tag{2.5}$$

Here $\mathcal{M}$ and $\mathcal{S}_i$ are the total mass and angular momentum of the system, $\mathcal{I}_{A_l}$ and $\mathcal{S}_{A_l}$ are two independent sets of mass-type and spin-type multipole moments, and $N^i = x^i/r$ is a unit vector directed from the origin of the coordinate system to the field point. Since the origin of the coordinate system has been chosen at the center of mass, the expansions (2.2) – (2.5) do not depend on the mass-type linear multipole moment $\mathcal{I}_i = 0$.

The expressions for the metric perturbations in a different coordinate system can be obtained by means of an infinitesimal coordinate transformation

$$x'^{\alpha} = x^{\alpha} - w^{\alpha} \tag{2.6}$$

from the *canonical harmonic* coordinates x^{α} to coordinates x'^{α} where w^{α} are gauge functions. Then in the new coordinate system one has [8]

$$h_{\alpha\beta} = h_{\alpha\beta}^{can.} + w_{\alpha,\beta} + w_{\beta,\alpha}\ . \tag{2.7}$$

We note that if functions w^α satisfy the homogeneous wave equation

$$\Box\, w^\alpha = 0, \tag{2.8}$$

the transformation (2.6) leaves us within the class of the harmonic coordinate systems, since the harmonic gauge conditions can be formulated as $\Box\, x^\alpha = 0$ [12].

The most general solution of the equation (2.8) is given in [9] (see also [13]) and reads

$$w^0 = \sum_{l=0}^{\infty} \left[\frac{\mathcal{W}_{A_l}(t-r)}{r}\right]_{,A_l} , \tag{2.9}$$

$$w^i = \sum_{l=0}^{\infty} \left[\frac{\mathcal{X}_{A_l}(t-r)}{r}\right]_{,iA_l} + \sum_{l=1}^{\infty} \left\{ \left[\frac{\mathcal{Y}_{iA_{l-1}}(t-r)}{r}\right]_{,A_{l-1}} + \left[\epsilon_{ipq} \frac{\mathcal{Z}_{qA_{l-1}}(t-r)}{r}\right]_{,pA_{l-1}} \right\} , \tag{2.10}$$

where $\mathcal{W}_{A_l}$, $\mathcal{X}_{A_l}$, $\mathcal{Y}_{iA_{l-1}}$, and $\mathcal{Z}_{qA_{l-1}}$ are four sets of STF multipole moments depending on the retarded time.

We shall use the gauge freedom in choosing functions w^α to simplify the problems of integration of the equations of light propagation and analysis of the observable effects. It turns out that two coordinate systems possess properties useful for solution of these problems. These are the harmonic and the Arnowitt-Deser-Misner (ADM) coordinates.

2.2. The Harmonic Coordinate System

The harmonic coordinate system, analogous to the Lorentz gauge in electrodynamics, is defined in linear approximation by the conditions [12]

$$2h^{\alpha\beta}_{\ \ ,\beta} - h^{,\alpha} = 0 \tag{2.11}$$

imposed on the metric perturbations, where $h \equiv h^{\alpha}_{\ \alpha}$. The linearized Einstein field equations in harmonic coordinates reduce to wave equations for metric perturbations $h^{\alpha\beta}$.

The metric tensor perturbations in harmonic coordinates (2.2) - (2.5) possess a property that can be used to simplify the problem of analytical integration of the equations of light propagation. Specifically, the method of integration of equations of light geodesics, developed first in [15] for stationary fields, and extended in [1] to time-dependent fields, is utilized in the present work (Section 4 4.1). This method takes advantage of the fact that the metric perturbations are functions of the retarded time. Not all coordinates in general relativity have this property. For instance, metric perturbations in the canonical ADM coordinate system, considered in the next section, contain terms that depend on instantaneous time, which does not allow us to use the abovementioned method of integration with the retarded time.

2.3. The Arnowitt-Deser-Misner Coordinate System

The ADM gauge conditions [14] in the linear approximation read

$$2h_{0i,i} - h_{ii,0} = 0 \;, \qquad 3h_{ij,j} - h_{jj,i} = 0 \;. \tag{2.12}$$

For comparison, the harmonic gauge conditions (2.11) in the linear approximation have the form

$$2h_{0i,i} - h_{ii,0} = h_{00,0} \;, \qquad 2h_{ij,j} - h_{jj,i} = -h_{00,i} \;. \tag{2.13}$$

The ADM coordinates have a property that test particles perturbed by gravitational waves stay at rest with respect to these coordinates, that is the ADM coordinate system is co-moving with test particles. This property can be used to simplify analysis of observable effects. For example, when one considers detection of a photon by an observer in the field of gravitational wave, then normally one would have to solve both the equations of motion of the photon and observer [12]. Using ADM coordinates allows us to exclude the problem of motion of the observer from the coordinate description of the gravitational-wave effect on the photon.

The disadvantage of using the ADM coordinates for analysis of the effects in propagation of light through gravitational field of an isolated source is that, as it was mentioned above, the ADM metric perturbations normally contain terms that are functions of instantaneous time. This does not allow one to use mathematical advantages of the method of integration of the equations of light propagation (Section 4 4.1).

Fortunately, the classes of ADM and harmonic coordinates overlap. In [1] an *ADM-harmonic* coordinate system was constructed in the case when isolated system possesses constant mass and angular momentum and time-dependent quadrupole moment. We generalize this result to the case of isolated system possessing the whole multipole structure

(2.2)-(2.5). The gauge functions

$$w^0 = \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}^{(-1)}_{A_l}(t-r)}{r} \right]_{,A_l} , \tag{2.14}$$

$$w^i = \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}^{(-2)}_{A_l}(t-r)}{r} \right]_{,iA_l} - 4 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left[\frac{\mathcal{I}_{iA_{l-1}}(t-r)}{r} \right]_{,A_{l-1}} + \tag{2.15}$$

$$4 \sum_{l=2}^{\infty} \frac{(-1)^l l}{(l+1)!} \left[\frac{\epsilon_{iba} \mathcal{S}^{(-1)}_{bA_{l-1}}(t-r)}{r} \right]_{,aA_{l-1}}$$

bring the metric perturbations $h_{\alpha\beta}$ to the form

$$h_{00} = \frac{2\mathcal{M}}{r} , \tag{2.16}$$

$$h_{0i} = -\frac{2\epsilon_{ipq}\mathcal{S}_p N_q}{r^2} , \tag{2.17}$$

$$h_{ij} = \delta_{ij} h_{00} + h^{TT}_{ij} , \tag{2.18}$$

$$h^{TT}_{ij} = P_{ijkl} q^{\text{can.}}_{kl} , \tag{2.19}$$

which satisfies both harmonic and ADM gauge conditions. Here the TT-projection differential operator P_{ijkl}, applied to symmetric tensors depending on both time and spatial coordinates, is given by

$$P_{ijkl} = (\delta_{ik} - \Delta^{-1}\partial_i\partial_k)(\delta_{jl} - \Delta^{-1}\partial_j\partial_l) - \frac{1}{2}(\delta_{ij} - \Delta^{-1}\partial_i\partial_j)(\delta_{kl} - \Delta^{-1}\partial_k\partial_l) , \tag{2.20}$$

and Δ and Δ^{-1} denote the Laplacian and the inverse Laplacian respectively.

The *ADM-harmonic* coordinates combine the advantages of both coordinate systems and will be used in what follows for analysis of observable effects in propagation of electromagnetic signals through the gravitational field of the isolated system.

3. EQUATIONS FOR PROPAGATION OF ELECTROMAGNETIC SIGNALS

3.1. Geometrical Optics in Curved Space-Time

In this section following [3] and [8] (see §22.5) we provide a derivation of the main laws of geometrical optics in curved space-time from the Maxwell's equations.

In absence of sources the Maxwell's equations for the electromagnetic field tensor $F_{\alpha\beta}$ in curved space-time take the well known form [7, 8]

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \ , \tag{3.1}$$

$$\nabla_\beta F^{\alpha\beta} = 0 \ , \tag{3.2}$$

where ∇_α denotes a covariant derivative. The wave equation for $F_{\alpha\beta}$ can be derived from (3.1) and (3.2) and in vacuum ($R_{\alpha\beta} = 0$, where $R_{\alpha\beta}$ is the Ricci tensor) it assumes the form

$$\Box_g F_{\alpha\beta} + R_{\alpha\beta\mu\nu} F^{\mu\nu} = 0 \ , \tag{3.3}$$

where $\Box_g \equiv \nabla^\alpha \nabla_\alpha$ and $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor of the space-time.

The solution to the Maxwell's equations corresponding to a high frequency wave is given by [4, 5]

$$F_{\alpha\beta} = \mathrm{Re}\{A_{\alpha\beta} \exp(i\varphi)\} \ , \tag{3.4}$$

where $A_{\alpha\beta}$ is the complex amplitude, which is a slowly varying function of position and time, and φ is the phase, rapidly changing with position and time.

The criterion for applicability of the geometrical optics approximation in curved space-time is formulated [8] as follows [8]. Let $\mathcal{R}$ be the characteristic radius of curvature of the space-time through which the electromagnetic wave propagates and $\mathcal{L}$ – the characteristic length, over which the amplitude, wavelength and polarization of the electromagnetic wave change significantly. Geometrical optics approximation can be applied whenever the wavelength of the electromagnetic radiation satisfies the following two conditions: 1)$\lambda \ll \mathcal{R}$; and 2)$\lambda \ll \mathcal{L}$. Then a small parameter $\varepsilon \equiv \lambda / min\{\mathcal{L}, \mathcal{R}\}$ can be introduced, and the expansion of the electromagnetic field of a wave (3.4) in powers of ε is assumed to have the form

$$F_{\alpha\beta} = \left(a_{\alpha\beta} + \varepsilon b_{\alpha\beta} + \varepsilon^2 c_{\alpha\beta} + ...\right) \exp\left(\frac{i\varphi}{\varepsilon}\right) \ . \tag{3.5}$$

For further discussion of this procedure see [16] and [19]. Substituting this expansion into (3.1), taking into account the definition of the electromagnetic wave vector $l_\alpha \equiv \partial\varphi/\partial x^\alpha$ and rearranging the terms with the same powers of ε lead to the chain of equations

$$l_\alpha \ a_{\beta\gamma} + l_\beta \ a_{\gamma\alpha} + l_\gamma \ a_{\alpha\beta} = 0 \ , \tag{3.6}$$

$$\nabla_\alpha \ a_{\beta\gamma} + \nabla_\beta \ a_{\gamma\alpha} + \nabla_\gamma \ a_{\alpha\beta} = - \, i \left(l_\alpha \ b_{\beta\gamma} + l_\beta \ b_{\gamma\alpha} + l_\gamma \ b_{\alpha\beta}\right) \ , \tag{3.7}$$

where the effects of curvature appears in the next approximation and, hence, have been neglected. Another chain of equations is obtained by substituting (3.5) into (3.2):

$$l_\beta \, a^{\alpha\beta} \; = \; 0 \; , \tag{3.8}$$

$$\nabla_\beta \, a^{\alpha\beta} + i l_\beta \, b^{\alpha\beta} \; = \; 0 \; . \tag{3.9}$$

The equation (3.8) shows that in the lowest order approximation (terms of the order $1/\varepsilon$) the amplitude of the electromagnetic field tensor is perpendicular to the wave vector in the four dimensional sense. Contracting (3.6) with l_α and accounting for (3.8) implies that l_α is a null vector

$$l_\alpha l^\alpha = 0. \tag{3.10}$$

Taking a covariant derivative from the equality (3.10) and using the fact that $\nabla_{[\alpha} l_{\beta]} = 0$ (since $l_\alpha \equiv \nabla_\alpha \varphi$) we obtain that l_α satisfies the geodesic equation

$$l^\beta \nabla_\beta l^\alpha = 0 \; , \tag{3.11}$$

showing that the wave vector is parallel transported along itself. Equations (3.10) and (3.11) combined constitute an important result of the geometrical optics: in the lowest order approximation light rays are null geodesics.

Substituting (3.4) into (3.3) and considering terms of the order $1/\varepsilon$ one can obtain the law of propagation for the amplitude of the electromagnetic tensor

$$D_\lambda a_{\alpha\beta} + \vartheta a_{\alpha\beta} = 0 \; , \tag{3.12}$$

where $D_\lambda \equiv l^\alpha \nabla_\alpha$ and $\vartheta \equiv (1/2) \nabla_\alpha l^\alpha$ is the expansion of the null congruence l^α.

3.2. Equations for Trajectory of a Photon

It was shown in the previous section that in the lowest order approximation of geometrical optics light rays, or trajectories of photons, are represented by null geodesics of the space-time under consideration. In what follows we shall consider propagation of photons in a space-time around an isolated gravitating system. With the metric of the space-time defined by (2.7). We shall assume that the wavelength of the electromagnetic radiation is much smaller than the characteristic wavelength of the gravitational waves emitted by the gravitating system, so that the conditions for validity of geometrical optics approximation are satisfied.

Let us consider propagation of photons subject to the initial-boundary conditions

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \qquad \frac{d\boldsymbol{x}(-\infty)}{dt} = \boldsymbol{k} \; , \tag{3.13}$$

which specify the spacial velocity of photons at the past null infinity by a unit spacial vector $\mathbf{k}$ and the positions of the photons at some initial instant of time t_0.

Taking into account that $l^\alpha = dx^\alpha/d\lambda$, where λ is an affine parameter along the photon's trajectory, we can rewrite the equations of geodesics (3.11) in terms of coordinates of a photon

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \; , \tag{3.14}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}\left(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma} - \partial_\mu g_{\beta\gamma}\right) \tag{3.15}$$

are the Christoffel symbols. Combining the four equations (3.14) we obtain the equations for spacial position of a photon as a function of the coordinate time

$$\ddot{x}^i(t) = -\left(\Gamma^i_{\alpha\beta} - \Gamma^0_{\alpha\beta}\dot{x}^i\right)\dot{x}^\alpha\dot{x}^\beta \; . \tag{3.16}$$

Substituting (3.15) into (3.16), taking into account that $\dot{x}^i = k^i + O(h)$ and keeping only linear in metric perturbations terms we can rewrite the equations (3.16) in the form

$$\begin{aligned} \ddot{x}^i(t) = & \frac{1}{2}h_{00,i} - h_{0i,0} - \frac{1}{2}h_{00,0}k^i - h_{ik,0}k^k - (h_{0i,k} - h_{0k,i})k^k - \\ & h_{00,k}k^kk^i - \left(h_{ik,j} - \frac{1}{2}h_{kj,i}\right)k^kk^j + \left(\frac{1}{2}h_{kj,0} - h_{0k,j}\right)k^kk^jk^i \; . \end{aligned} \tag{3.17}$$

In section 4 we will show how these equations can be analytically integrated in the space-time with metric (2.7) defining gravitational field of an isolated gravitating system.

3.3. Equations for Propagation of Polarization Properties

In this section, following [2] and [3], we introduce the relativistic description of polarized electromagnetic radiation and give the equation (derived in [3]) for gravity-induced rotation of the polarization plane of an electromagnetic wave (the Skrotskii effect).

For description of the polarization properties of electromagnetic radiation it is necessary to introduce local reference frames along the light rays. At each point of space-time we introduce a complex null tetrad $(l^\alpha, n^\alpha, m^\alpha, \overline{m}^\alpha)$ associated with a congruence of light rays [20, 21]. All the vectors of the tetrad are lightlike. The vectors l^α and n^α are real; vectors m^α and $\overline{m}^\alpha$ are complex and complex conjugate with respect to each other. The tetrad is normalized in such a way that the only nonvanishing products among the vectors are $l_\alpha n^\alpha = -1$ and $m_\alpha \overline{m}^\alpha = 1$.

It is obvious that such field of complex null tetrads is not uniquely determined by the congruence of null rays and the normalization conditions. The transformations

$$
\begin{aligned}
l'^\alpha &= A l^\alpha \ , \\
n'^\alpha &= A^{-1}\left(n^\alpha + \overline{B} m^\alpha + B \overline{m}^\alpha + B\overline{B} l^\alpha\right) \ , \\
m'^\alpha &= e^{-i\Theta}\left(m^\alpha + \overline{B} l^\alpha\right) \ , \\
\overline{m}'^\alpha &= e^{i\Theta}\left(\overline{m}^\alpha + B l^\alpha\right) \ ,
\end{aligned}
\tag{3.18}
$$

where A and Θ are real and B is a complex parameter, preserve the direction of the vector l^α and the normalization conditions [21]. The transformations (3.18) form a four-parameter subgroup of the Lorentz group. In addition to the complex null tetrad we introduce at each point in space-time an orthonormal reference tetrad $e^\alpha_{(\beta)}$ defined as follows. Suppose at each point of space-time there is an observer moving with four-velocity u^α. Let two of the vectors of the observer's local reference frame be

$$
e^\alpha_{(0)} = u^\alpha \ , \qquad e^\alpha_{(3)} = (-l_\alpha u^\alpha)^{-1}\left[l^\alpha + (l_\beta u^\beta) u^\alpha\right] \ . \tag{3.19}
$$

With such orientation of the reference frame the observer will see the electromagnetic wave propagating in the $+z$ direction. Spacelike unit vectors $e^\alpha_{(1)}$ and $e^\alpha_{(2)}$ are orthogonal to each other as well as to both $e^\alpha_{(0)}$ and $e^\alpha_{(3)}$ and thus are specified up to a spacial rotation.

The relationship between the vectors of the complex null tetrad and the frame $e^\alpha_{(\beta)}$ is given by

$$
l^\alpha = -(l_\gamma u^\gamma)\left(e^\alpha_{(0)} + e^\alpha_{(3)}\right) \ , \tag{3.20}
$$

$$
n^\alpha = -\frac{1}{2}(l_\gamma u^\gamma)\left(e^\alpha_{(0)} - e^\alpha_{(3)}\right) \ , \tag{3.21}
$$

$$
m^\alpha = \frac{1}{\sqrt{2}}\left(e^\alpha_{(1)} + i e^\alpha_{(2)}\right) \ , \tag{3.22}
$$

$$
\overline{m}^\alpha = \frac{1}{\sqrt{2}}\left(e^\alpha_{(1)} - i e^\alpha_{(2)}\right) \ . \tag{3.23}
$$

Vectors $e^{\alpha}{}_{(1)}$ and $e^{\alpha}{}_{(2)}$ and corresponding to them m^{α} and $\overline{m}^{\alpha}$ play an important role in description of polarization properties of electromagnetic radiation since they form a basis in the polarization plane.

The tensor of the electromagnetic field can be represented as

$$F_{\alpha\beta} = Re(\mathcal{F}_{\alpha\beta}) \; , \tag{3.24}$$

where $\mathcal{F}_{\alpha\beta}$ is the complex field. The complex field can be expressed as

$$\mathcal{F}_{\alpha\beta} = \Phi l_{[\alpha} m_{\beta]} + \Psi l_{[\beta} \overline{m}_{\alpha]} \tag{3.25}$$

where Φ and Ψ are complex scalar functions.

The electric and magnetic fields in the rest frame of an observer moving with four-velocity u^{α} are defined as

$$E^{\alpha} = F^{\alpha\beta} u_{\beta} \; , \quad H^{\alpha} = (-1/2)\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} u_{\beta} \; . \tag{3.26}$$

We also define the complex electric field as

$$\mathcal{E}^{\alpha} = \mathcal{F}^{\alpha\beta} u_{\beta} \; . \tag{3.27}$$

Polarization properties of light are completely characterized by the polarization tensor [7] (coherency matrix in [22])

$$J_{\alpha\beta} = \langle \mathcal{E}_{\alpha} \overline{\mathcal{E}}_{\beta} \rangle \; , \tag{3.28}$$

where the angular brackets notify an ensemble average equivalent to averaging over many periods of the wave.

The electromagnetic Stokes parameters are defined with respect to two of the observer's tetrad vectors $e^{\alpha}_{(1)}$ and $e^{\alpha}_{(2)}$ as follows [2]

$$S_0 = J_{\alpha\beta} \left[e^{\alpha}_{(1)} e^{\beta}_{(1)} + e^{\alpha}_{(2)} e^{\beta}_{(2)} \right] \; , \tag{3.29}$$

$$S_1 = J_{\alpha\beta} \left[e^{\alpha}_{(1)} e^{\beta}_{(1)} - e^{\alpha}_{(2)} e^{\beta}_{(2)} \right] \; , \tag{3.30}$$

$$S_2 = J_{\alpha\beta} \left[e^{\alpha}_{(1)} e^{\beta}_{(2)} + e^{\alpha}_{(2)} e^{\beta}_{(1)} \right] \; , \tag{3.31}$$

$$S_3 = i J_{\alpha\beta} \left[e^{\alpha}_{(1)} e^{\beta}_{(2)} - e^{\alpha}_{(2)} e^{\beta}_{(1)} \right] \; . \tag{3.32}$$

It is worth noting that both the Stokes parameters and the components of the polarization tensor of an electromagnetic wave can be determined from simple experiments by measuring the intensities of the wave after it passes through devices that transmit only radiation of certain polarization [22].

Using the definition of the polarization tensor (3.28) we can also rewrite the expressions for the Stokes parameters in terms of the electric field components

$$S_0 =< |\mathcal{E}_{(1)}|^2 + |\mathcal{E}_{(2)}|^2 > , \tag{3.33}$$

$$S_1 =< |\mathcal{E}_{(1)}|^2 - |\mathcal{E}_{(2)}|^2 > , \tag{3.34}$$

$$S_2 =< \mathcal{E}_{(1)}\overline{\mathcal{E}}_{(2)} + \overline{\mathcal{E}}_{(1)}\mathcal{E}_{(2)} > , \tag{3.35}$$

$$S_3 = i < \mathcal{E}_{(1)}\overline{\mathcal{E}}_{(2)} - \overline{\mathcal{E}}_{(1)}\mathcal{E}_{(2)} > , \tag{3.36}$$

where $\mathcal{E}_{(n)} = \mathcal{E}_\alpha e^\alpha_{\ (n)}$ for $n = 1, 2$. The Stokes parameters can also be expressed in terms of the complex functions Φ and Ψ since

$$\mathcal{E}_\alpha = -\frac{l^\alpha u_\alpha}{2} (\Phi \, m_\alpha + \Psi \, \overline{m}_\alpha) \tag{3.37}$$

and therefore

$$\mathcal{E}_{(1)} = -\frac{l^\alpha u_\alpha}{\sqrt{8}} (\Phi + \Psi) \ , \quad \mathcal{E}_{(2)} = -i\frac{l^\alpha u_\alpha}{\sqrt{8}} (\Phi + \Psi) \ . \tag{3.38}$$

To determine how the polarization properties of light vary along the ray we should specify the law of propagation for the tetrad vectors. The vectors of both the null tetrad and the tetrad $e^\alpha_{\ (\beta)}$ are postulated to be parallelly transported along the light rays. We specify the tetrads at the point of observation and thus specify them at every point along the ray. Thus, the equations for propagation of the tetrad vectors read

$$\frac{dm^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} \, l^\beta \, m^\gamma = 0 , \tag{3.39}$$

$$\frac{d\overline{m}^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} \, l^\beta \, \overline{m}^\gamma = 0 , \tag{3.40}$$

$$\frac{de^\alpha_{\ (\mu)}}{d\lambda} + \Gamma^\alpha_{\beta\gamma} \, l^\beta \, e^\gamma_{\ (\mu)} = 0 , \tag{3.41}$$

and the same laws hold for vectors l^α and n^α of the null tetrad.

It follows from (3.12), (3.27) and (3.41) that the complex amplitude of the electric field propagates along the ray according to the law

$$D_\lambda \mathcal{E}_\alpha + \vartheta \mathcal{E}_\alpha = 0 \ . \tag{3.42}$$

For the complex amplitudes Φ and Ψ one has

$$D_\lambda \Phi + \vartheta \Phi = 0 \ , \quad D_\lambda \Psi + \vartheta \Psi = 0 \ . \tag{3.43}$$

Finally, from the equations (3.33) - (3.36) it follows that the Stokes parameters propagate according to the law

$$D_\lambda S_\alpha + 2\vartheta S_\alpha = 0 \ , \tag{3.44}$$

where $\alpha = 0, 1, 2, 3$. We would like to note that despite suggestive notation the Stokes parameters do not form a four-vector, because they do not behave as a vector under coordinate system transformations.

Any stationary or time-dependent axisymmetric gravitational field in general causes a relativistic effect of rotation of the polarization plane of an electromagnetic wave [3]. This effect was first discussed by Skrotskii [17] and afterwards by many researchers (see, for example, [18] and references therein). Recently the effect was studied in [3] where authors derived an expression for the angle of rotation of the polarization plane of an electromagnetic wave propagating through a weak gravitational field described by metric perturbations $h_{\alpha\beta}$. Here we give this expression without derivation which can be found in [3] or in [6]. If ϕ is the characterizing the orientation of the polarization ellipse, then one has

$$\frac{d\phi}{dt} = \frac{1}{2} k^\alpha k^j \epsilon_{j\hat{p}\hat{q}} \partial_q h_{\alpha\hat{p}} \ , \tag{3.45}$$

where the hat over the spatial indices denotes the projection onto the plane orthogonal to the propagation of light ray, for instance, $A^{\hat{i}} \equiv P^i_{\ j} A^j$. In the following section we shall integrate this equation analytically to obtain the angle of the rotation of the polarization plane for an electromagnetic signal propagating in the gravitational field of an isolated system.

4. SOLUTION TO THE EQUATIONS OF PROPAGATION OF ELECTROMAGNETIC SIGNALS

4.1. Method of Analytical Integration of the Equations of Propagation of Electromagnetic Signals

In this section we describe the method of analytical integration of equations of light propagation in the field of an isolated gravitating system developed in the series of publications [15], [1] and [6]. In subsequent sections this method will be applied to integrate the equations of geodesics (3.17) and the equation for Skrotskii effect (3.45) in the space-time with the metric given by (2.7) around an isolated gravitating system.

We introduce new variables τ and ξ^i as follows

$$\tau \equiv k_i x^i \ , \qquad \xi^i \equiv P^i_j x^j \ , \tag{4.1}$$

where P^i_j is the operator of projection onto the plane perpendicular to k^i. If one considers unperturbed trajectory of a photon $x^i_N = k^i(t-t_0)+x_0$, it is easy to see that the variable τ characterizing the position of the photon is proportional to time

$$\tau \equiv k_i x^i_N = t - t^*, \tag{4.2}$$

where $t^* \equiv k_i x^i_0 - t_0$ is the time of the closest approach of the electromagnetic signal to the origin of the coordinate system. Since t^* is a constant for a particular ray, one has $d\tau = dt$. This allows to change to the variable τ when calculating integrals along the unperturbed ray. Vector ξ^i is the vector from the origin of the coordinate system to the point of the closest approach. In terms of the new variables the unperturbed trajectory can be written as

$$x^i_N(\tau) = k^i \tau + \xi^i \ . \tag{4.3}$$

Since the vectors k^i and ξ^i are orthogonal to each other, the distance from the point on unperturbed trajectory with coordinates τ and ξ^i to the origin of the coordinate system can be expressed as

$$r_N = \sqrt{\tau^2 + d^2} \ , \tag{4.4}$$

where $d = |\boldsymbol{\xi}|$ is the impact parameter of the unperturbed light-ray trajectory with respect to the origin of the coordinate system.

We introduce the operators of differentiation with respect to τ and ξ^i

$$\hat{\partial}_\tau \equiv \frac{\partial}{\partial \tau} \ , \qquad \hat{\partial}_i \equiv P_i^{\ j} \frac{\partial}{\partial \xi^j} \ . \tag{4.5}$$

Then for any smooth function $F(t, x^i)$ the following relationships are valid between the derivatives in the old and new variables taken on the unperturbed trajectory

$$\left[\left(\frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial t}\right) F(t\,,\,\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}(t-t_0)} = \left(\frac{\partial}{\partial \xi^i} + k_i \frac{\partial}{\partial \tau}\right) F(t^* + \tau\,,\,\boldsymbol{\xi} + \boldsymbol{k}\tau) \ , \tag{4.6}$$

$$\left[\frac{\partial}{\partial t} F(t\,,\,\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}(t-t_0)} = \frac{\partial}{\partial t^*} F(t^* + \tau\,,\,\boldsymbol{\xi} + \boldsymbol{k}\tau). \tag{4.7}$$

In the left-hand sides of the equations (4.6) and (4.7) one has to first calculate the derivatives and only after that substitute the unperturbed trajectory $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{k}(t-t_0)$ while in the right hand sides one has to first substitute the unperturbed trajectory parameterized by the variables τ and ξ^i and then differentiate. Using the equation (4.7) in (4.6) one can obtain the expression for spacial derivatives

$$\left[\frac{\partial F(t,\boldsymbol{x})}{\partial x^i}\right]_{\boldsymbol{x}=\boldsymbol{x}_0+\boldsymbol{k}(t-t_0)} = \left(\frac{\partial}{\partial \xi^i} + k_i\frac{\partial}{\partial \tau} - k_i\frac{\partial}{\partial t^*}\right) F(t^*+\tau\,,\,\boldsymbol{\xi}+\boldsymbol{k}\tau)\;. \tag{4.8}$$

Equations (4.7) and (4.8) can be used for changing over to the variables τ and ξ^i in the equations of light propagation. A useful property of the variables τ and ξ^i is that when one calculates integrals along the light rays the following formulae are valid

$$\int \frac{\partial}{\partial \tau} F(\tau,\boldsymbol{\xi})\,d\tau = F(\tau,\boldsymbol{\xi}) + C(\boldsymbol{\xi})\;, \tag{4.9}$$

$$\int \frac{\partial}{\partial \xi^i} F(\tau,\boldsymbol{\xi})\,d\tau = \frac{\partial}{\partial \xi^i}\int F(\tau,\boldsymbol{\xi})\,d\tau\;, \tag{4.10}$$

where $C(\boldsymbol{\xi})$ is a function of $\boldsymbol{\xi}$. Equation (4.9) shows that the terms represented as partial derivatives with respect to τ can be immediately integrated. Equation (4.10) states that one can change the order of integration and differentiation which crucially simplifies the problem of integration of the equations as we show below.

Let us consider the geodesic equations (3.17). All terms in the right-hand sides of the equations are proportional to the first-order derivatives of the metric perturbations $h_{\alpha\beta}$ with respect to time and spacial coordinates. The metric perturbations, given by the equations (2.7), consist of the canonical and gauge-dependent parts. The canonical part (Eqns. (2.2)-(2.5)) is expressed as a linear combination of derivatives of different orders with respect to spacial variables of functions $[F(t-r)/r]$ (where $F(t-r)$ denotes a mass- or spin-type multipole moment of the system). If one changes to the variables τ and ξ^i in the geodesic equations (3.17), integrates and changes the order of differentiation and integration in the right-hand sides of the equations, then the only two types of integrals that will appear in the solutions (for a moment, leave apart the terms produced by the gauge-dependent components in the metric perturbations) will be the integrals of the type $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$. The gauge-dependent terms, after changing the variables to τ and ξ^i, will appear in the equations of geodesics under the second-order derivative with respect to τ and thus can be immediately integrated (cf. Eq. (4.10)). A similar consideration of the integration procedure can be done for the equations describing the Skrotskii effect.

As it follows from the consideration above, the problem of integration of the equations of light propagation reduces to evaluation of integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$ where $F(t-r)$ denotes multipole moments of different type and order of the gravitating system.

Let us consider first the contributions to the relativistic effects in propagation of light due to the mass-monopole and spin-dipole terms. We neglect the loss of energy and angular momentum due to gravitational radiation so that the total mass and angular momentum of the system are considered to be conserved. Then it can be shown that the contributions under consideration are expressed in terms of integrals $[1/r]^{[-1]}$, $\left[\hat{\partial}_i(1/r)\right]^{[-1]}$, $\left[\hat{\partial}_{ia}(1/r)\right]^{[-1]}$, $\left[\hat{\partial}_i(1/r)\right]^{[-2]}$ and $\left[\hat{\partial}_{ia}(1/r)\right]^{[-2]}$ which can be evaluated.

The relativistic corrections to the light ray path and the relativistic effect of rotation of the polarization plane due to higher order multipole moments (starting with the mass- and spin-quadrupole) are expressed in terms of integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$ along the unperturbed light ray, where $F(t-r)$ denotes a multipole moment. To evaluate these integrals we introduce the new variable [1]

$$y \equiv s - t^* = \tau - r(\tau) = \tau - \sqrt{d^2+\tau^2}\ . \tag{4.11}$$

Then the following relationships hold

$$\tau = \frac{y^2-d^2}{2y}\ , \qquad \sqrt{d^2+\tau^2} = -\frac{1}{2}\frac{d^2+y^2}{y}\ , \qquad d\tau = \frac{1}{2}\frac{d^2+y^2}{y^2}dy. \tag{4.12}$$

Making use of the new variable y and the equations (4.12) we can express the integrals under discussion as follows

$$\left[\frac{F(t-r)}{r}\right]^{[-1]} = -\int\limits_{-\infty}^{y} \frac{F(t^*+\zeta)}{\zeta}d\zeta\ , \tag{4.13}$$

$$\left[\frac{F(t-r)}{r}\right]^{[-2]} = -\frac{1}{2}\int\limits_{-\infty}^{y}\int\limits_{-\infty}^{\eta} \frac{F(t^*+\zeta)}{\zeta}d\zeta d\eta - \frac{d^2}{2}\int\limits_{-\infty}^{y}\frac{1}{\eta^2}\int\limits_{-\infty}^{\eta} \frac{F(t^*+\zeta)}{\zeta}d\zeta d\eta\ , \tag{4.14}$$

where ζ, η are new variables of integration, and t^* is the time of the closest approach of photon to the origin of the coordinate system. It is worth to note that the time t^* depends on the choice of coordinate system and therefore in general has no physical meaning.

An important property of the integrals (4.13) and (4.14), expressed in terms of the variable y, is that they depend on τ and ξ^i only through the upper limits of integration (since $y \equiv \tau - \sqrt{d^2+\tau^2}$) and the square of the impact parameter d^2 in the prefactor of the second integral in (4.14).

Differentiating (4.13) with respect to ξ^i and τ yields

$$\hat{\partial}_i \left\{ \left[\frac{F(t-r)}{r}\right]^{[-1]} \right\} = -\frac{F(t^*+y)}{y}\,\hat{\partial}_i y = -F(t^*+y)\,\hat{\partial}_i \ln(-y) = \frac{\xi^i}{yr}\,F(t-r)\;, \tag{4.15}$$

$$\hat{\partial}_\tau \left\{ \left[\frac{F(t-r)}{r}\right]^{[-1]} \right\} = -\frac{F(t^*+y)}{y}\,\hat{\partial}_\tau y = -\frac{F(t^*+y)}{y}\left(1-\frac{\tau}{r}\right) = \frac{F(t-r)}{r}\;. \tag{4.16}$$

The last result also follows from the equation (4.10). Thus derivatives of the integral $[F(t-r)/r]^{[-1]}$ with respect to either ξ^i or τ can be expressed in terms of the integrand.

Differentiating the (4.14) with respect to ξ^i multiple times yields

$$\hat{\partial}_k \left\{ \left[\frac{F(t-r)}{r}\right]^{[-2]} \right\} = \xi^k \left\{ \frac{1}{y}\left[\frac{F(t-r)}{r}\right]^{[-1]} - \int\limits_{-\infty}^{y} \frac{1}{\eta^2} \int\limits_{-\infty}^{\eta} \frac{F(t^*+\zeta)}{\zeta}\,d\zeta d\eta \right\}\;, \tag{4.17}$$

$$\begin{aligned}\hat{\partial}_{jk} \left\{ \left[\frac{F(t-r)}{r}\right]^{[-2]} \right\} =& \frac{\xi^k\xi^j}{y^2}\,\frac{F(t-r)}{r} + \\ & P^{jk} \left\{ \frac{1}{y}\left[\frac{F(t-r)}{r}\right]^{[-1]} - \int\limits_{-\infty}^{y} \frac{1}{\eta^2} \int\limits_{-\infty}^{\eta} \frac{F(t^*+\zeta)}{\zeta}\,d\zeta d\eta \right\}\;,\end{aligned} \tag{4.18}$$

$$\hat{\partial}_{ijk} \left\{ \left[\frac{F(t-r)}{r}\right]^{[-2]} \right\} = \frac{1}{y}\left\{ \left(P^{ij} + \frac{\xi^{ij}}{yr}\right)\hat{\partial}_k + P^{jk}\hat{\partial}_i + \xi^j\hat{\partial}_{ik} \right\} \left[\frac{F(t-r)}{r}\right]^{[-1]}\;, \tag{4.19}$$

In the last expression all terms contain at least a first order derivative of the integral $[F(t-r)/r]^{[-1]}$ with respect to ξ^i. Above it was shown (Eq. (4.15)) that in this case the integration is eliminated. Therefore the third and higher order derivatives of $[F(t-r)/r]^{[-2]}$ with respect to ξ^i can be expressed in terms of $F(t-r)$ and one does not have to perform integration. Explicitly evaluating the derivatives in (4.19) using (4.15) one gets

$$\begin{aligned}\hat{\partial}_{ijk} \left\{ \left[\frac{F(t-r)}{r}\right]^{[-2]} \right\} =& \frac{P^{ij}\xi^k + P^{jk}\xi^i + P^{ik}\xi^j}{y^2 r}\,F(t-r) \\ & + \frac{\xi^i\xi^j\xi^k}{y^2 r^2}\left[\left(\frac{2}{y}-\frac{1}{r}\right)F(t-r) - \dot{F}(t-r)\right]\;.\end{aligned} \tag{4.20}$$

In the equations for trajectory of a photon and gravity-induced rotation of the polarization plane the integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$ are differentiated with respect to τ and ξ^i and it can be shown that in all terms differentiation eliminates integration. In the subsequent sections we write out solutions to the equations formally in terms of integrals $[F(t-r)/r]^{[-1]}$ and $[F(t-r)/r]^{[-2]}$, but one should bear in mind that performing differentiations using the formulae (4.9), (4.15) and (4.19) the integrals can be expressed in terms of integrands.

4.2. Solution to the Equations for Trajectory of a Photon

Using the formulae (4.7) and (4.8) and rewriting the equation (3.17) in new variables one obtains

$$\frac{d^2x^i(\tau)}{d\tau^2} = \frac{1}{2}k^\alpha k^\beta\,\hat{\partial}_i h^{\rm can.}_{\alpha\beta} - \hat{\partial}_\tau\left(k^\alpha h^{\rm can.}_{i\alpha} - \frac{1}{2}\,k^i k^j k^p q^{\rm can.}_{jp}\right) - \hat{\partial}_{\tau\tau}(w^i - k^i w^0)\;, \tag{4.21}$$

where all functions in the right-hand side are taken on the unperturbed light-ray trajectory (before performing differentiation). The gauge functions w^α have not yet been specified which means that equations (4.21) are gauge-invariant. Substituting into (4.21) the metric perturbations expressed in terms of τ and ξ^i (Eqns. (A.1)-(A.9) in Appendix A) and integrating once with respect to τ one obtains [6]

$$\dot{x}^i(\tau) = k^i + \dot{\Xi}^i(\tau,\boldsymbol{\xi})\;, \tag{4.22}$$

where

$$\dot{\Xi}^i(\tau,\boldsymbol{\xi}) = \underset{(G)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi}) + \underset{(M)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi}) + \underset{(S)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi})\;, \tag{4.23}$$

$$\underset{(G)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi}) = \hat{\partial}_\tau\left[(\varphi^i - k^i\varphi^0) + (w^i - k^i w^0)\right]\;, \tag{4.24}$$

$$\begin{aligned}\underset{(M)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi}) =& 2\mathcal{M}\left[\hat{\partial}_i\frac{1}{r}\right]^{[-1]} - k^i\frac{2\mathcal{M}}{r} + \\ & 2\hat{\partial}_i\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)H(2-q)\times \\ & \left(1-\frac{p-q}{l}\right)\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right]^{[-1]} - \\ & 2\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\left(1-\frac{p}{l}\right)\times \\ & \left\{\left(1+\frac{p}{l-1}\right)k_{i<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\left[\frac{\mathcal{I}^{(p)}_{A_l}(t-r)}{r}\right] - \right. \\ & \left.\frac{2p}{l-1}k_{<a_1\ldots a_{p-1}}\hat{\partial}_{a_p\ldots a_{l-1}>}\left[\frac{\mathcal{I}^{(p)}_{iA_{l-1}}(t-r)}{r}\right]\right\}\;,\end{aligned} \tag{4.25}$$

$$\underset{(S)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi}) = 2k_j\epsilon_{jba}\mathcal{S}_b\left[\hat{\partial}_{ia}\frac{1}{r}\right]^{[-1]} - 2\hat{\partial}_a\frac{\epsilon_{iba}\mathcal{S}_b}{r} - \tag{4.26}$$
$$4k_j\hat{\partial}_{ia}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)H(2-q)\times$$
$$\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^q_\tau\left[\frac{\epsilon_{jba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]} +$$
$$4\left(\hat{\partial}_a - k_a\hat{\partial}_{t^*}\right)\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\times$$
$$\left(1-\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{iba}\mathcal{S}^{(p)}_{bA_{l-1}}(t-r)}{r}\right] +$$
$$4k_j\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{jba_{l-1}}\mathcal{S}^{(p+1)}_{\hat{i}bA_{l-2}}(t-r)}{r}\right] .$$

Here $\underset{(G)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi})$ represents the gauge-dependent perturbations, $\underset{(M)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi})$ and $\underset{(S)}{\dot{\Xi}^i}(\tau,\boldsymbol{\xi})$ are the perturbations due to the mass and spin multipole moments, correspondingly. The Heaviside function $H(p-q)$ is defined by the expression (1.18) and $C_l(l-p,p-q,q)$ are the polynomial coefficients (1.17). The gauge functions φ^α were introduced as follows: we collected in the equations (4.21) all terms with second and higher order derivatives with respect to τ and equated them to $k^i\varphi^0-\varphi^i$. By introducing the functions φ^α we single out the terms that can be immediately integrated. We do not separate $k^i\varphi^0-\varphi^i$ into φ^0 and φ^i, such separation is not unique while the functions $k^i\varphi^0-\varphi^i$ are uniquely defined (Eqns. (B.4)-(B.6) in the Appendix B).

The gauge functions w^α can be chosen arbitrarily. For the reasons discussed in Section 2 we choose w^α which make our coordinate system ADM-harmonic, that is satisfying both ADM and harmonic gauge conditions. These functions are given by the expressions (B.2) and (B.3).

The second integration of (4.21) yields the expressions for trajectory of the photon

$$x^i(\tau) = x^i_N + \Delta\underset{(G)}{\Xi^i} + \Delta\underset{(M)}{\Xi^i} + \Delta\underset{(S)}{\Xi^i}\ , \tag{4.27}$$

where

$$\Delta \underset{(G)}{\Xi^i} \equiv \underset{(G)}{\Xi^i} (\tau, \boldsymbol{\xi}) - \underset{(G)}{\Xi^i} (\tau_0, \boldsymbol{\xi}) \ , \tag{4.28}$$

$$\Delta \underset{(M)}{\Xi^i} \equiv \underset{(M)}{\Xi^i} (\tau, \boldsymbol{\xi}) - \underset{(M)}{\Xi^i} (\tau_0, \boldsymbol{\xi}) \ ,$$

$$\Delta \underset{(S)}{\Xi^i} \equiv \underset{(S)}{\Xi^i} (\tau, \boldsymbol{\xi}) - \underset{(S)}{\Xi^i} (\tau_0, \boldsymbol{\xi}) \ .$$

The term

$$\underset{(G)}{\Xi^i} (\tau, \boldsymbol{\xi}) = (\varphi^i - k^i \varphi^0) + (w^i - k^i w^0) \ , \tag{4.29}$$

represents the gauge-dependent part of the trajectory's perturbation, and the physically meaningful perturbations due to the mass and spin multipoles are given by

$$\begin{aligned}
\underset{(M)}{\Xi^i} (\tau, \boldsymbol{\xi}) = & 2\mathcal{M} \left[\hat{\partial}_i \frac{1}{r}\right]^{[-2]} - 2\mathcal{M} k^i \left[\frac{1}{r}\right]^{[-1]} + \\
& 2\hat{\partial}_i \sum_{l=2}^{\infty} \sum_{p=0}^{l} \sum_{q=0}^{p} \frac{(-1)^{l+p-q}}{l!} C_l(l-p, p-q, q) H(2-q) \times \\
& \left(1 - \frac{p-q}{l}\right) \left(1 - \frac{p-q}{l-1}\right) k_{<a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l>} \hat{\partial}_\tau^q \left[\frac{\mathcal{I}_{A_l}^{(p-q)}(t-r)}{r}\right]^{[-2]} - \\
& 2 \sum_{l=2}^{\infty} \sum_{p=0}^{l-1} \frac{(-1)^{l+p}}{l!} C_l(l-p, p) \left(1 - \frac{p}{l}\right) \times \\
& \left\{ \left(1 + \frac{p}{l-1}\right) k_{i<a_1 \dots a_p} \hat{\partial}_{a_{p+1} \dots a_l>} \left[\frac{\mathcal{I}_{A_l}^{(p)}(t-r)}{r}\right]^{[-1]} - \right. \\
& \left. \frac{2p}{l-1} k_{<a_1 \dots a_{p-1}} \hat{\partial}_{a_p \dots a_{l-1}>} \left[\frac{\mathcal{I}_{iA_{l-1}}^{(p)}(t-r)}{r}\right]^{[-1]} \right\} ,
\end{aligned} \tag{4.30}$$

$$
\begin{aligned}
\underset{(S)}{\Xi^i}\,(\tau,\boldsymbol{\xi}) =& 2k_j\epsilon_{jba}\mathcal{S}_b\left[\hat{\partial}_{ia}\frac{1}{r}\right]^{[-2]} - 2\epsilon_{iba}\mathcal{S}_b\left[\hat{\partial}_a\frac{1}{r}\right]^{[-1]} - \\
& 4k_j\hat{\partial}_{ia}\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)H(2-q)\times \\
& \left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^q_\tau\left[\frac{\epsilon_{jba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right]^{[-2]} + \\
& 4\left(\hat{\partial}_a - k_a\hat{\partial}_{t^*}\right)\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\times \\
& \left(1-\frac{p}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{iba}\mathcal{S}^{(p)}_{bA_{l-1}}(t-r)}{r}\right]^{[-1]} + \\
& 4k_j\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\frac{(-1)^{l+p}l}{(l+1)!}C_{l-1}\left(l-p-1,p\right)\times \\
& k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\left[\frac{\epsilon_{jba_{l-1}}\mathcal{S}^{(p+1)}_{\hat{i}bA_{l-2}}(t-r)}{r}\right]^{[-1]} \quad .
\end{aligned}
\tag{4.31}
$$

4.3. Solution to the Equations for Rotation of the Polarization Plane

We rewrite the equation (3.45) in terms of the variables τ and ξ^i using the relationship (4.8) and substitute the metric (2.7) expressed in terms of new variables (Eqns. (A.1)-(A.9)):

$$
\frac{d\phi}{d\tau} = \frac{1}{2}k^\alpha k^j\epsilon_{j\hat{p}\hat{q}}\hat{\partial}_q h^{can.}_{\alpha\hat{p}} + \frac{1}{2}k^j\epsilon_{j\hat{p}\hat{q}}\hat{\partial}_{q\tau}w^{\hat{p}} \ . \tag{4.32}
$$

Integrating with respect to τ and changing the order of differentiation and integration yields

$$
\phi = \phi_{(G)} + \phi_{(M)} + \phi_{(S)} + \phi_0 \ , \tag{4.33}
$$

where ϕ_0 is the constant angle characterizing the initial orientation of the polarization ellipse in the plane formed by the vectors $\boldsymbol{e}_{(1)}$ and $\boldsymbol{e}_{(2)}$. The terms $\phi_{(G)}$, $\phi_{(M)}$ and $\phi_{(S)}$ describe the contributions due to gauge-dependent terms, terms depending on mass- and spin-type moments correspondingly.

The gauge-dependent part of the Skrotskii effect can be obtained immediately, since the gauge dependent terms appear in the equation (4.32) under the derivative with respect to τ (cf. Eq. (4.9)). The integration yields

$$
\phi_{(G)} = \frac{1}{2}k^j\epsilon_{j\hat{p}\hat{q}}\hat{\partial}_q\left(w^{\hat{p}} + \chi^{\hat{p}}\right) \ , \tag{4.34}
$$

where the gauge functions w^i and χ^i are given by the equations (2.15) and (B.7). The gauge functions χ^i were introduced by collecting in the equation (4.32) all terms that can be eliminated by a gauge transformation.

The Skrotskii effect due to the mass and spin multipoles of the isolated system is given by

$$\phi_{(M)}(\tau) = 2\sum_{l=2}^{\infty}\sum_{p=0}^{l}\frac{(-1)^{l+p}}{l!}C_l(l-p,p)\frac{l-p}{l-1} \qquad (4.35)$$
$$k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>j}\left[\frac{\epsilon_{j\hat{b}a_l}\mathcal{I}^{(p)}_{\hat{b}A_{l-1}}(t-r)}{r}\right]^{[-1]} ,$$

and

$$\phi_{(S)}(\tau) = 2\sum_{l=1}^{\infty}\sum_{p=0}^{l}\frac{(-1)^{l+p}l}{(l+1)!}C_l(l-p,p)\left(1-\frac{p}{l}\right) \qquad (4.36)$$
$$\left[1+H(l-1)\left(1-\frac{2p}{l-1}\right)\right]k_{<a_1\dots a_p}\hat{\partial}_{a_{p+1}\dots a_l>}\left[\frac{\mathcal{S}^{(p+1)}_{A_l}(t-r)}{r}\right]^{[-1]} .$$

Just like in the case with the equations for light-ray path (4.25)-(4.26) and (4.30)-(4.31) differentiation eliminates integrals in the right-hand sides of the equations (4.35)-(4.36) as it was shown in Section 4 4.1.

5. OBSERVABLE RELATIVISTIC EFFECTS

In this section we give the expressions for the observable relativistic effects of time delay and gravitational deflection of light. The expressions for another observable effect – rotation of the polarization plane were essentially given in Section 4 4.3.

5.1. Time Delay

In the paper [1] the following expression was obtained for the time of propagation of the electromagnetic signals through the gravitational field of an isolated source from the emitter to the observer

$$t - t_0 = |\boldsymbol{x}-\boldsymbol{x}_0| + \Delta(\tau,\tau_0) , \qquad (5.1)$$

$$\Delta(\tau,\tau_0) = \underset{(G)}{\Delta}(\tau,\tau_0) + \underset{(M)}{\Delta}(\tau,\tau_0) + \underset{(S)}{\Delta}(\tau,\tau_0) , \qquad (5.2)$$

where $x_0^\alpha = (t_0, \boldsymbol{x}_0)$ and $x^\alpha = (t, \boldsymbol{x})$ are the coordinates of the points of emission and observation of the signal, $\underset{(G)}{\Delta}(\tau, \tau_0)$, $\underset{(M)}{\Delta}(\tau, \tau_0)$ and $\underset{(S)}{\Delta}(\tau, \tau_0)$ are functions describing the delay of the electromagnetic signal due to the gauge-dependent terms, mass and spin multipoles of the gravitational field of the isolated system correspondingly. These functions are expressed in terms of the perturbations of the signal's trajectory as

$$\underset{(G)}{\Delta}(\tau, \tau_0) = -\, k_i \left[\underset{(G)}{\Xi^i}(\tau, \boldsymbol{\xi}) - \underset{(G)}{\Xi^i}(\tau_0, \boldsymbol{\xi}) \right], \tag{5.3}$$

$$\underset{(M)}{\Delta}(\tau, \tau_0) = -\, k_i \left[\underset{(M)}{\Xi^i}(\tau, \boldsymbol{\xi}) - \underset{(M)}{\Xi^i}(\tau_0, \boldsymbol{\xi}) \right], \tag{5.4}$$

$$\underset{(S)}{\Delta}(\tau, \tau_0) = -\, k_i \left[\underset{(S)}{\Xi^i}(\tau, \boldsymbol{\xi}) - \underset{(S)}{\Xi^i}(\tau_0, \boldsymbol{\xi}) \right]. \tag{5.5}$$

We note that the expressions 5.3 give the effect with respect to the coordinate time which has to be converted to the time measured by the observer. It is shown in [1] that if the observer's velocity is negligible with respect to the global ADM-harmonic coordinate system the relationship between the coordinate time and proper time T of the observer is given by

$$T = \left(1 - \frac{\mathcal{M}}{r}\right)(t - t_i) , \tag{5.6}$$

where $t_{\rm i}$ is the initial epoch of observation. In most cases the distance r is much grater than $\mathcal{M}$ and coordinate time coincide with the proper time of the observer. If observer is moving with respect to the global coordinate system, an additional Lorentz transformation to the rest frame of the observer has to be performed. This case was considered in [6].

5.2. Deflection of Light

The expression for observable vector towards the source of light calculated with respect to the observer's reference frame was derived in [1] and reads

$$s^i(\tau, \boldsymbol{\xi}) = K^i + \alpha^i(\tau, \boldsymbol{\xi}) + \beta^i(\tau, \boldsymbol{\xi}) - \beta^i(\tau_0, \boldsymbol{\xi}) + \gamma^i(\tau, \boldsymbol{\xi}), \tag{5.7}$$

where

$$K^i = \frac{x^i - x_0^i}{|\mathbf{x} - \mathbf{x}_0|} \tag{5.8}$$

is the unit vector in the direction "observer – source of light", defined in the global coordinate system;

$$\alpha^i(\tau, \boldsymbol{\xi}) = -\, P^i_j\, \dot{\Xi}^j \tag{5.9}$$

is the vector describing the angle of light deflection;

$$\beta^i(\tau,\boldsymbol{\xi}) = \frac{P^i_j\left[\underset{(M)}{\Xi^j}(\tau,\boldsymbol{\xi}) + \underset{(S)}{\Xi^j}(\tau,\boldsymbol{\xi})\right]}{|\mathbf{x}-\mathbf{x}_0|} \tag{5.10}$$

is relativistic corrections, introduced by the relationship

$$k^i = -K^i - \beta^i(\tau,\boldsymbol{\xi}) + \beta^i(\tau_0,\boldsymbol{\xi})\,. \tag{5.11}$$

The term

$$\gamma^i(\tau,\boldsymbol{\xi}) = -\frac{1}{2}P^{ij}k^q h^{TT}_{jq}(t,\mathbf{x}) \tag{5.12}$$

appears as a result of transformation from the the global ADM-harmonic system to the local frame of the observer and describes the perturbations of the observer's coordinate system due to the gravitational waves emitted by the isolated system. It was assumed that the observer is at rest with respect to the global ADM-harmonic system. In the case when the observer is moving with respect to the global system one has to perform an additional Lorentz transformation to the rest frame of the observer.

APPENDIX A: THE METRIC TENSOR IN TERMS OF VARIABLES $\boldsymbol{\xi}$ AND τ

The expressions for perturbations of the metric tensor (2.2) – (2.5) in variables $\boldsymbol{\xi}$ and τ have the form as follows

$$h^{\rm can.}_{00} = \frac{2\mathcal{M}}{r} + 2\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(t^*,\tau,\boldsymbol{\xi})\;, \tag{A.1}$$

$$h^{\rm can.}_{0i} = -\frac{2\epsilon_{iba}\mathcal{S}^b N^a}{r^2} + 4\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\left[\underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) + \underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi})\right]\;, \tag{A.2}$$

$$h^{\rm can.}_{ij} = \delta_{ij}h^{\rm can.}_{00} + q^{\rm can.}_{ij}\;, \tag{A.3}$$

$$q^{\rm can.}_{ij} = 4\sum_{l=2}^{\infty}\sum_{p=0}^{l-2}\sum_{q=0}^{p}\underset{(M)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi}) - 8\sum_{l=2}^{\infty}\sum_{p=0}^{l-2}\sum_{q=0}^{p}\underset{(S)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi})\;, \tag{A.4}$$

where

$$\underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}}{l!} C_l(l-p,p-q,q) \tag{A.5}$$
$$k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_l>}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right] ,$$

$$\underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}}{l!} C_{l-1}(l-p-1,p-q,q) \tag{A.6}$$
$$k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}^{(p-q+1)}_{iA_{l-1}}(t-r)}{r}\right] ,$$

$$\underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}\, l}{(l+1)!} C_{l-1}(l-p-1,p-q,q) \tag{A.7}$$
$$(\hat{\partial}_a + k_a\hat{\partial}_\tau - k_a\hat{\partial}_{t^*})k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}^q_\tau\left[\frac{\epsilon_{iab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right] ,$$

$$\underset{(M)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q}}{l!} C_{l-2}(l-p-2,p-q,q) \tag{A.8}$$
$$k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-2}>}\hat{\partial}^q_\tau\left[\frac{\mathcal{I}^{(p-q+2)}_{ijA_{l-2}}(t-r)}{r}\right] ,$$

$$\underset{(S)}{\mathrm{q}}{}^{lpq}_{ij}(t^*,\tau,\boldsymbol{\xi}) = \frac{(-1)^{l+p-q} l}{(l+1)!} C_{l-2}(l-p-2,p-q,q) \tag{A.9}$$
$$(\hat{\partial}_a + k_a\hat{\partial}_\tau - k_a\hat{\partial}_{t^*})k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-2}>}\hat{\partial}^q_\tau\left[\frac{\epsilon_{ba(i}\mathcal{S}^{(p-q+1)}_{j)bA_{l-2}}(t-r)}{r}\right] .$$

All quantities in the right side of expressions (A.5)–(A.9), which are explicitly shown as functions of x^i, $r = |\boldsymbol{x}|$ and t, must be understood as taken on the unperturbed light-ray trajectory and expressed in terms of ξ^i, $d = |\boldsymbol{\xi}|$, τ and t^* in accordance with the equations (4.2), (4.4). For example, the ratio $\mathcal{I}^{(p-q)}_{A_l}(t-r)/r$ in equation (A.5) must be understood as

$$\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r} \equiv \frac{\mathcal{I}^{(p-q)}_{A_l}(t^*+\tau-\sqrt{\tau^2+d^2})}{\sqrt{\tau^2+d^2}} , \tag{A.10}$$

and the same replacement rule is applied to the other equations.

APPENDIX B: GAUGE FUNCTIONS

Gauge functions w^α, generating the coordinate transformation from the canonical harmonic coordinate system to the ADM-harmonic, are given by equations (2.14), (2.15). They transform the metric tensor as follows

$$h^{\rm can.}_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha w_\beta - \partial_\beta w_\alpha \;, \tag{B.1}$$

where $h^{\rm can.}_{\alpha\beta}$ is the canonical form of the metric tensor in harmonic coordinates given by equations (2.2)–(2.5) and $h_{\alpha\beta}$ is the metric tensor given in the ADM-harmonic coordinates by equations (2.16)–(2.19).

The gauge functions taken on the light-ray trajectory and expressed in terms of the variables ξ^i and τ can be written down in the form

$$w^0 \;=\; \sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\int_{-\infty}^{\tau+t^*} du \; \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi}), \tag{B.2}$$

$$\begin{aligned} w^i \;=\; & (\hat\partial_i + k_i\hat\partial_\tau - k_i\hat\partial_{t^*})\sum_{l=2}^{\infty}\sum_{p=0}^{l}\sum_{q=0}^{p}\int_{-\infty}^{\tau+t^*} dv \int_{-\infty}^{v} du \; \underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi}) \\ & -4\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\int_{-\infty}^{\tau+t^*} du \left[\underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi}) + \underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi})\right] \;, \end{aligned} \tag{B.3}$$

where $\underset{(M)}{\mathrm{h}}{}^{lpq}_{00}(u,\tau,\boldsymbol{\xi})$, $\underset{(M)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi})$ and $\underset{(S)}{\mathrm{h}}{}^{lpq}_{0i}(u,\tau,\boldsymbol{\xi})$ are defined by the Eqns. (A.5), (A.6) and (A.7) after making use of the substitution $t^* \to u$.

Linear combination $k^i\varphi^0 - \varphi^i$ of the gauge-dependent functions φ^α introduced in equation (4.24) is given by the expressions

$$k^i\varphi^0 - \varphi^i = (k^i\varphi^0_{(M)} - \varphi^i_{(M)}) + (k^i\varphi^0_{(S)} - \varphi^i_{(S)}) \;, \tag{B.4}$$

$$\begin{aligned} k^i\varphi^0_{(M)} - \varphi^i_{(M)} = {} & 2\hat\partial_i \sum_{l=2}^{\infty}\sum_{p=2}^{l}\sum_{q=2}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)\times \\ & \left(1-\frac{p-q}{l}\right)\left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat\partial_{a_{p+1}\ldots a_l>}\hat\partial_\tau^{q-2}\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right] + \\ & 2\sum_{l=2}^{\infty}\sum_{p=1}^{l}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}}{l!}C_l(l-p,p-q,q)\times \\ & \left(1-\frac{p-q}{l}\right)\left\{\left(1+\frac{p-q}{l-1}\right)k_{i<a_1\ldots a_p}\hat\partial_{a_{p+1}\ldots a_l>}\hat\partial_\tau^{q-1}\left[\frac{\mathcal{I}^{(p-q)}_{A_l}(t-r)}{r}\right] - \right. \\ & \left. 2\frac{p-q}{l-1}\,k_{<a_1\ldots a_{p-1}}\hat\partial_{a_p\ldots a_{l-1}>}\hat\partial_\tau^{q-1}\left[\frac{\mathcal{I}^{(p-q)}_{iA_{l-1}}(t-r)}{r}\right]\right\}, \end{aligned} \tag{B.5}$$

$$
\begin{aligned}
k^i\varphi^0_{(S)} - \varphi^i_{(S)} =& 2\frac{\epsilon_{iab}k^a\mathcal{S}^b}{r} + \\
& 4k_j\hat{\partial}_{ia}\sum_{l=3}^{\infty}\sum_{p=2}^{l-1}\sum_{q=2}^{p}\frac{(-1)^{l+p-q}\,l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times \\
& \left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}_{\tau}^{q-2}\left[\frac{\epsilon_{jab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right] - \\
& 4\left(\hat{\partial}_a - k_a\hat{\partial}_{t^*}\right)\sum_{l=2}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times \\
& \left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}_{\tau}^{q-1}\left[\frac{\epsilon_{iab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right] - \\
& 4k_a\sum_{l=2}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times \\
& \left(1-\frac{p-q}{l-1}\right)k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}_{\tau}^{q}\left[\frac{\epsilon_{iab}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right] + \\
& 4k_j\sum_{l=2}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}\left(l-p-1,p-q,q\right)\times \\
& k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}_{\tau}^{q-1}\left[\frac{\epsilon_{jba_{l-1}}\mathcal{S}^{(p-q+1)}_{\hat{i}bA_{l-2}}(t-r)}{r}\right]
\end{aligned}
\tag{B.6}
$$

Gauge-dependent term generated by equation (4.34) for the rotation of the plane of polarization of electromagnetic wave is a pure spatial vector χ^i that can be decomposed in two linear parts corresponding to the mass and spin multipoles:

$$
\chi^i = \chi^i_{(M)} + \chi^i_{(S)}\ , \tag{B.7}
$$

where

$$\chi^i_{(M)} = 4\sum_{l=2}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}}{l!}C_{l-1}(l-p-1,p-q,q)\left(1-\frac{p-q}{l-1}\right) \times k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}_\tau^{q-1}\left[\frac{\mathcal{I}^{(p-q+1)}_{iA_{l-1}}(t-r)}{r}\right] , \tag{B.8}$$

$$\begin{aligned}\chi^i_{(S)} = & -4\sum_{l=1}^{\infty}\sum_{p=0}^{l-1}\sum_{q=0}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}(l-p-1,p-q,q)\left[1-\frac{p-q}{l-1}\,H(l-1)\right] \\ & \times\left[H(q)(\hat{\partial}_a-k_a\hat{\partial}_{t^*})+k_a\hat{\partial}_\tau\right]k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>}\hat{\partial}_\tau^{q-1}\left[\frac{\epsilon_{iba}\mathcal{S}^{(p-q)}_{bA_{l-1}}(t-r)}{r}\right] \\ & +4\sum_{l=3}^{\infty}\sum_{p=1}^{l-1}\sum_{q=1}^{p}\frac{(-1)^{l+p-q}l}{(l+1)!}C_{l-1}(l-p-1,p-q,q)\left(1-\frac{p}{l}\right)\left(1-\frac{q}{p}\right) \\ & \times k_{<a_1\ldots a_p}\hat{\partial}_{a_{p+1}\ldots a_{l-1}>a}\hat{\partial}_\tau^{q-1}\left[\frac{\epsilon_{baa_{l-1}}\mathcal{S}^{(p-q)}_{ibA_{l-2}}(t-r)}{r}\right] .\end{aligned} \tag{B.9}$$

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