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DISTANCES SETS THAT ARE A SHIFT OF THE INTEGERS AND FOURIER BASIS FOR PLANAR CONVEX SETS

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Abstract. The aim of this paper is to prove that if a planar set A has a difference set $\Delta(A)$ satisfying $\Delta(A)\subset\mathbb{Z}^++s$ for suitable s than A has at most 3 elements. This result is motivated by the conjecture that the disk has not more than 3 orthogonal exponentials.

Further, we prove that if A is a set of exponentials mutually orthogonal with respect to any symmetric convex set K in the plane with a smooth boundary and everywhere non-vanishing curvature, then $\#(A\cap[-q,q]^2)\leq C(K)q$ where $C(K)$ is a constant depending only on K. This extends and clarifies in the plane the result of Iosevich and Rudnev. As a corollary, we obtain the result from [IKP01] and [IKT01] that if K is a centrally symmetric convex body with a smooth boundary and non-vanishing curvature, then $L^2(K)$ does not possess an orthogonal basis of exponentials.

Contents

1. Introduction

The aim of this paper is to investigate size properties of a set A whose distance set

$$\Delta(A):=\{|a-a'|\,:a,a'\in A\}$$

has some prescribed arithmetic properties. For instance, Solymosi ([So04a], *see also* [So04b]) proved the following:

Theorem 1.1 (Solymosi, [So04a])**.**
Let $A\subset\mathbb{R}^2$ such that $\Delta(A)\subset\mathbb{Z}^+$. Then

$$\#(A\cap[-q,q]^2)\lesssim q. \qquad (1)$$

This result is essentially sharp as can be seen in the following way. Let A_N be the subset of the plane consisting of $(n,0)$, where n is a large positive integer, and pairs of the form $(0,m)$ such that m is a positive integer and $n^2+m^2=l^2$ for some positive integer l. By elementary number theory, one can find approximately $\frac{N}{\sqrt{\log(N)}}$ such integers m that are less than N.

Our main result can be seen as a shifted version of this theorem:

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Theorem 1.2.
Let $A \subset \mathbb{R}^2$ be a set such that $\Delta(A) \subset \mathbb{Z}^+ + s$ where s is either a trancendental number or a rational that is not a half-integer nor an integer. Then A has at most 3 elements.

Our original motivation to the above question comes from questions related to the Fuglede conjecture[1] on the existence of orthonormal bases for given sets. To be more precise, if K is a compact set in $\mathbb{R}^d$, we will say that the collection $\{e^{2\pi i\langle x,a\rangle}\}_{a\in A}$ is orthogonal with respect to K if

$$\int_K e^{2\pi i x\cdot(a-a')}dx = 0, \tag{2}$$

whenever $a \neq a' \in A$. In other words, we ask for a set $A \subset \mathbb{R}^d$ such that the difference set $A - A := \{a - a' \,:\, a, a' \in A\}$ is included in in the zero set of the Fourier transform $\widehat{\chi_K}$ of the characteristic function χ_K of K.

In the particular case where K is the unit disc in the plane, $\widehat{\chi_K}$ is a radial function and its zero set is the set of circles with radius the zeroes of the Bessel function J_1. Further, up to re-scaling, it is known that these zeroes are of the form $k + \frac{1}{4} + error$ with $k \in \mathbb{Z}_+$. Dropping the error term, we thus see that Theorem 1.2 brings ground on a conjecture of Fuglede [Fu72] that the disc has at most 3 orthogonal exponentials.

In the case of more general convex sets, many results are known. For instance, assume that K is a compact convex set in $\mathbb{R}^d$, symmetric with respect to the origin, such that the boundary of K is smooth and has everywhere non-vanishing curvature. It was proved in [IKT01] that such sets do not admit an orthonormal *basis* of exponentials. Further, in [IR03], the authors proved that $\{e^{2\pi i\langle x,a\rangle}\}_{a\in A}$ is an infinite set of orthogonal exponentials, then A is contained in a line. Moreover, they proved that if $d \neq 1 \mod (4)$, then A is necessarily finite. When $d = 1 \mod (4)$ the authors produced examples of convex bodies K for which there exists infinite A's satisfying (2). Unfortunately, the proof in [IR03] does not provide a finite upper bound for the size of A leaving open the possibility of having arbitrarily large sets of orthogonal exponentials.

The reason for this is reliance on an asymptotic generalization of the following combinatorial principle due to Anning and Erdös ([AE45] *see also* [Er45]):

Theorem 1.3 (Anning and Erdös, [AE45, Er45])**.**
Let $d \geq 2$, and let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^d$. For $A \subset \mathbb{R}^d$, let

$$\Delta(A) = \{|a - a'| : a, a' \in A\}. \tag{3}$$

If $\#A = \infty$ and $\Delta(A) \subset \mathbb{Z}^+$, then A is a subset of a line.

The reason explicit bounds are difficult to extract from any application of this principle is that for any N there exists $A_N \subset \mathbb{R}^d$ not contained in a line such that $\#A_N = N$ and $\Delta(A_N) \subset \mathbb{Z}^+$. This suggests that a different geometric point of view is needed to extract an explicit numerical bound if one exists. Moreover, such a point of view is likely to be dimension specific because, as we mention, above, when $d = 1 \mod (4)$, the set of orthogonal exponentials may be infinite.

Even though an explicit numerical upper bound still eludes us, we have been able to prove the following:

Theorem 1.4.
Let K be a convex planar set, symmetric with respect to the origin. Suppose that the boundary of K is smooth and has everywhere non-vanishing curvature. Then there exists a constant C depending only on K such that, whenever $A \subset \mathbb{R}^d$ is a set such that (2) holds, then

$$\#(A \cap [-q, q]^d) \leq Cq^{d-1}. \tag{4}$$

In particular, K does not have an orthogonal basis of exponentials.

[1] Strictly speaking, this is no longer a conjecture as it has been disproved in high enough dimension by Tao [Ta04], Matolcsi [Ma05] and Kolountzakis-Matolcsi [KM06a], see also [KM06b]

The constant $C(K)$ actually only depends on the inner and outer radius of K as well as on bounds on its curvature. The second part of the theorem is well known and was first proved in [IKT01] ([IKP01, Fu01] when K is the Euclidean ball) and also follows immediately from [IR03]. Our proof is not substantially different from previous proofs in the field. Nevertheless it slightly simplifies the argument in [IKT01] and allows to obtain a quantitative bound.

This paper is organized as follows. We start by proving Theorem 1.2 and some generalizations of it. We then complete the paper by devoting Section 3 to the proof of Theorem 1.4.

2. A shifted Erdös-Solymosi theorem

In the remaining of the paper, we will identify $\mathbb{R}^2$ and $\mathbb{C}$.

2.1. Proof of Theorem 1.2.

In this section, we will prove the following slightly stronger version of Theorem 1.2

Theorem 2.1.
Let $s\in\mathbb{R}$ be such that $8s^8\notin\mathbb{Z}+4s\mathbb{Z}+2s^2\mathbb{Z}+4s^3\mathbb{Z}+s^4\mathbb{Z}+2s^5\mathbb{Z}+2s^6\mathbb{Z}+4s^7\mathbb{Z}$. If $A\subset\mathbb{R}^2$ is such that $\Delta(A)\subset\mathbb{Z}^++s$, then $\#A\leq 3$.

The result is best possible as if $a_0=0$, $a_1=r$, $a_2=re^{i\pi/3}$, then all distances are equal to r.

Proof. Assume that a_0,a_1,a_2,a_3 are four different elements from A. For $j=1,\ldots,3$ let $\alpha_j=a_j-a_0$ and write $\alpha_j=(k_j+s)e^{i\theta_j}$. There is no loss of generality in assuming that α_1 is real, that is $\theta_1=0$.

For $j=2,3$ let $\beta_j=a_j-a_1=\alpha_j-\alpha_1$. Write $|\beta_j|=l_j+s$ with $l_j\in\mathbb{Z}^+$.

From $|\beta_j|^2=|\alpha_j-\alpha_1|^2=|\alpha_1|^2+|\alpha_j|^2-2|\alpha_1||\alpha_j|\cos\theta_j$, we deduce that

$$2|\alpha_1||\alpha_j|\cos\theta_j=|\alpha_1|^2+|\alpha_j|^2-|\beta_j|^2. \tag{5}$$

From this, we get that

$$(2|\alpha_1||\alpha_j|\sin\theta_j)^2=4|\alpha_1|^2|\alpha_j|^2-(|\alpha_1|^2+|\alpha_j|^2-|\beta_j|^2)^2 \tag{6}$$

On the other hand $\alpha_3-\alpha_2=a_3-a_2$ thus $|\alpha_3-\alpha_2|=m+s$ for some $m\in\mathbb{Z}^+$ and

$$2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2)=|\alpha_2|^2+|\alpha_3|^2-|\alpha_3-\alpha_2|^2, \tag{7}$$

from which we get that

$$\begin{aligned}
4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2) &= 2|\alpha_1|^2(|\alpha_2|^2+|\alpha_3|^2-|\alpha_3-\alpha_2|^2)\\
&= 2(k_1+s)^2\big((k_2+s)^2+(k_3+s)^2-(m+s)^2\big)\\
&= 2k_1^2(k_2^2+k_3^2-m^2)+4s\big(k_1^2(k_2+k_3-m)+k_1(k_2^2+k_3^2-m^2)\big)\\
&\quad +s^2\big(k_1^2+k_2^2+k_3^2-m^2+4k_1(k_2+k_3-m)\big)\\
&\quad +4s^3(k_1+k_2+k_3-m)+2s^4\\
&\in 2\mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+2s^4.
\end{aligned} \tag{8}$$

On the other hand,

$$4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2)=4|\alpha_1|^2|\alpha_2||\alpha_3|(\cos\theta_2\cos\theta_3+\sin\theta_2\sin\theta_3).$$

But, with (5),

$$\begin{aligned}
4|\alpha_1|^2|\alpha_2||\alpha_3|\cos\theta_2\cos\theta_3 &= \big((k_1+s)^2+(k_2+s)^2-(l_2+s)^2\big)\\
&\quad\times\big((k_1+s)^2+(k_3+s)^2-(l_3+s)^2\big)\\
&\in \mathbb{Z}+s2\mathbb{Z}+s^2\mathbb{Z}+s^32\mathbb{Z}+s^4,
\end{aligned} \tag{9}$$

as can be seen by expanding the expression in the first line. It follows that

$$4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3=4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2)-4|\alpha_1|^2|\alpha_2||\alpha_3|\cos\theta_2\cos\theta_3$$

has to be in $\mathbb{Z}+s2\mathbb{Z}+s^2\mathbb{Z}+s^32\mathbb{Z}+s^4$, thus

$$(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3)^2\in\mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+s^4\mathbb{Z}+s^54\mathbb{Z}+s^62\mathbb{Z}+s^74\mathbb{Z}+s^8.$$

Now, from (6), we get that

$$(2|\alpha_1||\alpha_j|\sin\theta_j)^2 = 4(k_1+s)^2(k_j+s)^2-\bigl((k_1+s)^2+(k_j+s)^2-(l_j+s)^2\bigr)^2 \in \mathbb{Z}+s4\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+3s^4 \qquad (10)$$

as previously, thus

$$(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3)^2\in\mathbb{Z}+4s\mathbb{Z}+s^22\mathbb{Z}+s^34\mathbb{Z}+s^4\mathbb{Z}+s^54\mathbb{Z}+s^62\mathbb{Z}+s^712\mathbb{Z}+9s^8.$$

We thus want that

$$8s^8\in\mathbb{Z}+4s\mathbb{Z}+2s^2\mathbb{Z}+4s^3\mathbb{Z}+s^4\mathbb{Z}+2s^5\mathbb{Z}+2s^6\mathbb{Z}+4s^7\mathbb{Z}$$

which contradicts our assumption on s. □

Remark: The assumption on s is quite mild as it is satisfied by all transcendental numbers, all algebraic numbers of order at least 9 and also by all rational numbers that are not integers nor half-integers. For this last fact, if $s=\frac{p}{q}$ with p,q mutually prime and $q\neq 1$, then the assumption reads

$$8p^8\notin q^8\mathbb{Z}+4pq^7\mathbb{Z}+2p^2q^6\mathbb{Z}+4p^3q^5\mathbb{Z}+p^4q^4\mathbb{Z}+2p^5q^3\mathbb{Z}+2p^6q^2\mathbb{Z}+4p^7q\mathbb{Z}\subset q\mathbb{Z}$$

so that q divides 8. But, then writing $q=2r$ with $r=1,2$ or 3, the assumption reads

$$8p^8\notin 2^8r^8\mathbb{Z}+2^9pr^7\mathbb{Z}+2^7p^2r^6\mathbb{Z}+2^6p^3r^5\mathbb{Z}+2^4p^4r^4\mathbb{Z}+2^4p^5r^3\mathbb{Z}+2^3p^6r^2\mathbb{Z}+2^3p^7r\mathbb{Z}\subset 8r\mathbb{Z}$$

so that $r=1$ and $s=\frac{p}{2}$.

Also, note that one may scale the assumption. For example, if we assume that $\Delta(A)\subset\frac12\mathbb{Z}^++\frac18$ then $\#A\leq 3$, since $B=2A=\{2a,\ a\in A\}$ satisifes $\Delta(B)\subset\mathbb{Z}+\frac14$, which establishes the link with zeroes of the Bessel function J_1 (*see* next section).

Of course, we may scale both ways, and we then for instance get that if $\Delta(A)\subset 4\mathbb{Z}^++1$ or if $\Delta(A)\subset 4\mathbb{Z}^++3$ (that is $s=1/4$ and $s=3/4$ respectively) then $\#A\leq 3$. This result is false when $s=1/2$ since it is not hard to construct sets for which $\Delta(A)\subset 2\mathbb{Z}^++1$. Indeed, let $k,l\in\mathbb{Z}^+$ and assume that $2l+1\leq 2(2k+1)$ and let $\theta=\arccos\frac{2l+1}{2(2k+1)}$. Finally, let $a_0=0$, $a_1=1$, $a_2=(2k+1)e^{i\theta}$ and $a_3=-(2k+1)e^{-i\theta}$. It is then clear that a_i-a_0, $i=1,2,3$ all have odd integer modulus. Further $a_3-a_2=-2(2k+1)\cos\theta=-2l-1$ is an odd integer. Finally

$$\begin{aligned}|a_2-a_1| &= \bigl((2k+1)\cos\theta-1\bigr)^2+(2k+1)^2\sin^2\theta\\ &= (2k+1)^2+1-2(2k+1)\cos\theta=4k^2+2(2k-l)+1\end{aligned}$$

while

$$\begin{aligned}|a_3-a_1| &= \bigl(-(2k+1)\cos\theta-1\bigr)^2+(2k+1)^2\sin^2\theta\\ &= (2k+1)^2+1+2(2k+1)\cos\theta=4k^2+2(2k+l)+3\end{aligned}$$

are also an odd integers. It seems nevertheless clear from the previous proof that some sparcity should happen in this case.

2.2. **A perturbation of Theorem 1.2.**

Recall that our original motivation in proving Theorem 1.2 was to bound the number of exponentials that are orthogonal for the disc in the plane. Recall the well known fact that $\widehat{\chi_B}(\xi)=\frac{1}{|\xi|}J_1(2\pi|\xi|)$ where J_1 is the Bessel function of order 1. It immediately follows that $\{e^{2i\pi\langle a,x\rangle}\}_{a\in A}$ is an orthogonal set of exponentials for the disc if and only if the distance set $\Delta(A)$ of A satisfies $\Delta(A)\subset\mathcal{Z}_{J_1}$ where $\mathcal{Z}_{J_1}$ is the set of zeroes of J_1.

Further, J_1 has the following asymptotic expansion when $r\to+\infty$ (*see e.g.* [St93, VIII 5.2, page 356-357]):

$$J_1(r)\sim-\left(\frac{2}{\pi r}\right)^{1/2}\left[\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma\left(\frac32+2j\right)}{2^{2j}(2j)!\Gamma\left(\frac32-2j\right)}\frac{\sin\left(r-\frac{\pi}{4}\right)}{r^{2j}}-\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma\left(\frac52+2j\right)}{2^{2j+1}(2j+1)!\Gamma\left(\frac12-2j\right)}\frac{\cos\left(r-\frac{\pi}{4}\right)}{r^{2j+1}}\right].$$

From this, it is not hard to see that if $\widehat{\chi_B}(\xi)=0$ and if ξ is big enough, then

$$4|\xi|^2=(k+1/4)^2+\frac{3}{4\pi^2}+O(k^{-2}). \tag{11}$$

We will now show that we can still perturbate Theorem 1.2 so as that the difference set $\Delta(A)$ consists of a small perturbation (and a harmless re-scaling) of the zeroes of the Bessel function in the following sense:

Proposition 2.2.
Let $s\in\mathbb{Q}\setminus\frac{1}{2}\mathbb{Z}$ and let η be either algebraic of order at least 5 or trancendental. Let $A\subset\mathbb{R}^2$ be such that every element $\alpha\in\Delta(A)$ has the property that $|\alpha|^2=(k+s)^2+\eta$, then A has at most 3 elements.

Proof. We use the same notation and assumptions as in the proof of Theorem 1.2. Identity (8) then becomes

$$\begin{aligned}4|\alpha_1|^2|\alpha_2||\alpha_3|\cos(\theta_3-\theta_2) &= 2\big((k_1+s)^2+\eta\big)\big((k_2+s)^2+(k_3+s)^2-(m+s)^2+\eta\big)\\ &= 2k_1^2(k_2^2+k_3^2-m^2)+4s\big(k_1^2(k_2+k_3-m)+k_1(k_2^2+k_3^2-m^2)\big)\\ &\quad+s^2\big(k_1^2+k_2^2+k_3^2-m^2+4k_1(k_2+k_3-m)\big)\\ &\quad+4s^3(k_1+k_2+k_3-m)+2s^4\\ &\quad+2\eta\big((k_1+s)^2+(k_2+s)^2+(k_3+s)^2-(m+s)^2\big)+2\eta^2\\ &\in \mathbb{Q}+\eta\mathbb{Q}+2\eta^2.\end{aligned}$$

Identity (9) becomes

$$\begin{aligned}4|\alpha_1|^2|\alpha_2||\alpha_3|\cos\theta_2\cos\theta_3 &= \big((k_1+s)^2+(k_2+s)^2-(l_2+s)^2+\eta\big)\\ &\quad\times\big((k_1+s)^2+(k_3+s)^2-(l_3+s)^2+\eta\big)\\ &\in \mathbb{Q}+\eta\mathbb{Q}+\eta^2.\end{aligned}$$

It follows that $4|\alpha_1|^2|\alpha_2||\alpha_3|\big(\cos(\theta_3-\theta_2)-\cos\theta_2\cos\theta_3\big)$ belongs to

$$\mathbb{Q}+\eta\mathbb{Q}+\eta^2.$$

Squaring, we get that $\big(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3\big)^2$ is in

$$\mathbb{Q}+\eta\mathbb{Q}+\eta^2\mathbb{Q}+\eta^3\mathbb{Q}+\eta^4.$$

On the other hand, Identity (10) becomes

$$\begin{aligned}(2|\alpha_1||\alpha_j|\sin\theta_j)^2 &= 4\big((k_1+s)^2+\eta\big)\big((k_j+s)^2+\eta\big)\\ &\quad-\big((k_1+s)^2+(k_j+s)^2-(l_j+s)^2+\eta\big)^2\\ &\in \mathbb{Q}+\eta\mathbb{Q}+3\eta^2.\end{aligned}$$

It follows that $\big(4|\alpha_1|^2|\alpha_2||\alpha_3|\sin\theta_2\sin\theta_3\big)^2$ is in $\mathbb{Q}+\eta\mathbb{Q}+\eta^2\mathbb{Q}+\eta^3\mathbb{Q}+9\eta^4$. As we have assumed that $\eta\mathbb{Q}+\eta^2\mathbb{Q}+\eta^3\mathbb{Q}+\eta^4\mathbb{Q}\subset\mathbb{R}\setminus\mathbb{Q}$, we have obtained the contradiction $8s^8\in\mathbb{Q}+\mathbb{R}\setminus\mathbb{Q}\subset\mathbb{R}\setminus\mathbb{Q}$. □

Remark: We have used that the perturbation by η is fixed only in a mild way in order to simplify computations. Actually, it is not hard to see that if we assume that if each $\alpha\in\Delta(A)$ has the property that $|\alpha|^2=(k+s)^2+P_k(\eta)$ where k is an integer, η is a fixed *trancendental* number and P_k is a polynomial with rational coefficients, then the above proof still gives the same result. If moreover the degrees of the $P_k's$ are bounded by M, then the proof stille works provided η is algebraic of order at least $4M+1$.

In particular, let us recall that the asymptotic expansion (11) of the large zeroes ξ of $\widehat{\chi_B}$ can be pushed further to obtain:

$$|\xi|^2=\frac{1}{4}(k+1/4)^2\left(1+\frac{3}{4\pi^2(k+1/4)^2}+\sum_{j=2}^{N}\frac{c_j}{\pi^{2j}(k+1/4)^{2j}}+O\left(\frac{1}{k^{2N+2}}\right)\right). \tag{12}$$

where the c_j's are rational constants. Let us now truncate this formula. More precisely, let us assume that the set A is such that each $\alpha\in\Delta(A)$ has the property that $|\alpha|^2$ is of the form

$$\frac{1}{4}(k+1/4)^2\left(1+\sum_{j=1}^{N}\frac{c_j}{\pi^{2j}(k+1/4)^{2j}}\right).$$

Then A has at most 3 elements.

An even more careful examination of the proof shows that, if each $\alpha\in\Delta(A)$ is of the form $|\alpha|^2=(k+s)^2+\eta_k$ then only 6 η_k's intervene in the proof (corresponding to $k_1,\ldots,k_3$, l_2,l_3 and m). Moreover they are raised to the power at most 4, so A has at most 3 elements as soon as there exists no rational polynomial relation of degree at most 4 between any 6 η_k's *i.e* if, for any $j_1,\ldots,j_6$, the only polynomial of degree ≤ 4 of 6 variables with coefficients in $\mathbb{Q}$ such that $P(\eta_{j_1},\ldots,\eta_{j_6})=0$ is $P=0$. Such relations are highly unlikely between zeroes of the Bessel function J_1, it is thus natural to conjecture, following Fuglede [Fu72] that the disk has no more than 3 orthogonal exponentials.

3. Orthogonal exponentials for planar convex sets

For sake of simplicity, we will concentrate on the proof of Theorem 1.4 in the case of dimension 2.

3.1. Preliminaries.

We will need the following well known facts about convex sets.

Notation: For a convex set K, we call ρ_K its Minkowski function of K, so that $K=\{x:\rho(x)\leq 1\}$, and ρ_K^* its support function given by

$$\rho_K^*(\xi)=\sup_{x\in K}\langle x,\xi\rangle. \tag{13}$$

By the method of stationary phase ([He62], *see e.g.* [St93, Chapter 3]),

$$\widehat{\chi_K}(\xi)=C_1|\xi|^{-\frac{3}{2}}\sin\left(2\pi\left(\rho_K^*(\xi)-\frac{1}{8}\right)\right)+E(\xi), \tag{14}$$

with

$$|E(\xi)|\leq C_2|\xi|^{-\frac{5}{2}}, \tag{15}$$

where C_1 and C_2 are some constants depending only on K.

It should also be noted that if $\{e^{2i\pi ax}\}_{a\in A}$ is orthogonal with respect to $L^2(K)$ then $\widehat{\chi_K}(a-a')=0$ for $a,a'\in A$. But $\widehat{\chi_K}$ is continuous and $\widehat{\chi_K}\neq 0$ so there exists η_0 such that $|a-a'|\geq\eta_0$, that is, the set A is separated with separation depending only on K.

3.2. Proof of Theorem 1.4.

An immediate consequence of (14) and (15) is that, if A is as in the statement of Theorem 1.4, there exists a constant C_3 such that, whenever $a,a'\in A$, then

$$\left|\rho_K^*(a-a')-\frac{k}{2}-\frac{1}{8}\right|\leq\frac{C_3}{k+1} \tag{16}$$

for some integer k. We may now cut A into a finite number of pieces, such that in each piece, any two elements a,a' are separated enough to have $k\geq 100C_3$ in (16).

Now, if $a,a',a''\in A$ are in a q by α rectangle $\mathcal{R}_\alpha$ then $a-a',a-a'',a'-a''$ are all in an angular sector with direction some vector e that depends only on $\mathcal{R}_\alpha$. More precisely, the angle $\theta=\theta(e,a-a')$ between e and $(a-a')$ is at most θ with $\sin\theta=\alpha/|a-a'|$. In particular, $\theta\leq\theta_m$ where $\sin\theta_m=\alpha/L$ and L is the minimal distance between two elements of A. Further, from the curvature assumption on K, for u in such a sector,

$$\big|\rho_K^*(u)-c|u|\big|\leq c'\theta(u,e)^2|u|,$$

where $c=c(e)$ and $c'=c'(e)$ are two consants that depend on e, provided $\theta(u,e)$ is small enough (that is α is taken to be small enought). In particular, for $u=a-a'$,

$$\big|\rho_K^*(a-a')-c|a-a'|\big|\leq\frac{C_4}{|a-a'|}.$$

It follows from this and (16), that there exists an integer k such that

$$\left| c|a-a'| - \frac{k}{2} - \frac{1}{8} \right| \le \frac{C_5}{k+1}. \tag{17}$$

There is no loss of generality to assume that elements in A are sufficiently separated to have $C_5/(k+1) < 1/100$. Similarely, there also exist integers l, m such that

$$\left| c|a-a''| - \frac{l}{2} - \frac{1}{8} \right| \le \frac{1}{100} \quad \text{and} \quad \left| c|a'-a''| - \frac{m}{2} - \frac{1}{8} \right| \le \frac{1}{100}. \tag{18}$$

Now, since a, a', a'' are in a box of size q by α, if $|a-a''| \ge |a-a'|, |a'-a''|$ then

$$\begin{aligned} |a-a''| &\ge (|a-a'|^2-\alpha^2)^{1/2} + (|a'-a''|^2-\alpha^2)^{1/2} \\ &= |a-a'| + |a'-a''| - \frac{\alpha^2}{(|a-a'|^2-\alpha^2)^{1/2}+|a-a'|} - \frac{\alpha^2}{(|a'-a''|^2-\alpha^2)^{1/2}+|a'-a''|} \\ &\ge |a-a'| + |a'-a''| - \frac{1}{100c} \end{aligned}$$

where c is the constant in (17), provided we have taken α small enough. It follows that

$$\big| c|a-a''| - c|a-a'| - c|a'-a''| \big| \le \frac{1}{100}.$$

But the, from (17) and (18),

$$\left| \frac{l-m-k}{2} - \frac{1}{8} \right| \le \frac{1}{25}$$

a clear contradiction. Thus every q by α rectangle contains at most 2 elements of a. It follows that A has at most $2q/\alpha$ elements in a $q \times q$ square.

Remark: The same proof in dimension $d \ne 1 \mod (4)$ works provided we use $q \times \alpha \times \cdots \times \alpha$ tubes. We would then obtain that A has at most $\lesssim q^{d-1}$ elements in any cube of side d.

3.3. Orthogonal exponential bases.

The following result is proved in [IKP01] in the case of the ball, and in [IKT01] in the general case. We shall give a completely self-contained and transparent proof below.

Theorem 3.1 (Iosevich, Katz, Pedersen, Tao, [IKP01, IKT01])**.**
Let K be a symmetric convex set in $\mathbb{R}^d$ with a smooth boundary and everywhere non-vanishing curvature. Then $L^2(K)$ does not possess an orthogonal basis of exponentials.

Proof. To prove Theorem 3.1, assume that $L^2(K)$ does possess an orthogonal basis of exponentials $\{e^{2\pi i x\cdot a}\}_{a\in A}$. From Theorem 1.4, $\#(A\cap[-q,q]^d) \lesssim q^{d-1}$.

But, as is well known [Be66, La67, GR96, IKP01], if $\{e^{2\pi i x\cdot a}\}_{a\in A}$ is an orthonormal basis of exponentials of $L^2(K)$, then $\limsup \frac{\#(A\cap[-q,q]^d)}{q^d} > 0$, a contradiction. □

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