THE RESONANCE COUNTING FUNCTION FOR SCHRÖDINGER OPERATORS WITH GENERIC POTENTIALS

T. CHRISTIANSEN AND P. D. HISLOP

Abstract. We show that the resonance counting function for a Schrödinger operator has maximal order of growth for generic sets of real-valued, or complex-valued, L^∞-compactly supported potentials.

1. Introduction

The purpose of this note is to show that for a generic set of compactly supported potentials, the resonance counting function for the associated Schrödinger operator has maximal order of growth. We consider odd dimensions $d \geq 1$, and any potential $V \in L^\infty_{\rm comp}(\mathbb{R}^d)$. We define the set of *scattering poles* or *resonances* of the Schrödinger operator $H_V \equiv -\Delta + V$ on $L^2(\mathbb{R}^d)$ through the meromorphic continuation of the resolvent. To make this precise, let χ_V be a smooth, compactly supported function equal to one on the support of V. It is well-known that the operator-valued function $\lambda \to \chi_V(H_V - \lambda^2)^{-1}\chi_V$ admits a meromorphic continuation (denoted by the same symbol) from $\operatorname{Im}\lambda \geq 0$, taken as the physical half-plane, to the entire complex plane. The poles of this continuation (including multiplicities) are independent of the choice of χ_V satisfying these conditions. There are at most a finite number of poles with $\operatorname{Im}\lambda > 0$ corresponding to the finitely-many eigenvalues of H_V. The set of scattering poles of H_V is defined by

$$\mathcal{R}_V = \{\lambda_j \in \mathbb{C} : \chi_V(H_V - \lambda^2)^{-1}\chi_V \text{ has a pole at } \lambda = \lambda_j, \text{ listed with multiplicity}\}. \tag{1}$$

This definition can be made for both real-valued and complex-valued potentials. The *resonance counting function* $N_V(r)$ for H_V on $L^2(\mathbb{R}^d)$, is defined as

$$N_V(r) = \#\{\lambda_j \in \mathcal{R}_V : |\lambda_j| < r\}. \tag{2}$$

The large r properties of $N_V(r)$ have been extensively studied, and we refer the reader to the review article of Zworski [17]. The leading asymptotic behavior is known in one dimension [3, 12, 20], and for certain spherically symmetric potentials for odd $d \geq 3$ [18]. Moreover, the following upper bound on $N_V(r)$ for compactly supported potentials is well-known

$$N_V(r) \leq C_{V,d}(1 + r^d), \tag{3}$$

TC partially supported by NSF grant DMS 0088922 and PDH partially supported by NSF grant DMS 0202656.

see, for example, [4, 5, 14, 18, 19]. In addition, for nontrivial real-valued, compactly supported potentials, it is known that an infinite number of resonances exist [6, 10]. More recently, Sá Barreto [8] proved a lower bound of the form

$$\limsup_{r\to\infty} \frac{N_V(r)}{r} > 0, \tag{4}$$

for nontrivial $V \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$. The situation is different for complex-valued, L^∞ compactly supported potentials. There are nontrivial examples of such potentials *with no resonances* for $d \geq 3$ [1].

The purpose of this note is to prove that the resonance counting function $N_V(r)$, defined in (2), has the maximal order of growth d for a generic family of either real-, or complex-valued, compactly supported potentials. Following B. Simon [11], for a metric space X, we call a dense G_δ set $S \subset X$ *Baire typical*. Our main result is the following theorem.

Theorem 1.1. *Let $d \geq 3$ be odd, let $K \subset \mathbb{R}^d$ be a compact set with nonempty interior, and let $F = \mathbb{R}$ or $F = \mathbb{C}$. Then the set*

$$\mathcal{M} = \{V \in L^\infty(K;F) : \limsup_{r\to\infty} \frac{\log N_V(r)}{\log r} = d\}$$

is Baire typical in $L^\infty(K;F)$.

The term *generic* is often used to designate a property that typically occurs for a given family. If X is a complete metric space, a property is *generic* if it holds for a family $\mathcal{F}$ of dense G_δ sets in X. Such a family is closed under countable intersections and has the property that if $A \in \mathcal{F}$, and X is perfect, then $A \cap B_X$ is uncountable for any open ball $B_X \subset X$. In this sense, our theorem says that the resonance counting function for a generic family of real- or complex-valued, L^∞ compactly supported potentials has the maximum order of growth given by the dimension $d \geq 1$. Since there are nontrivial, complex-valued, L^∞ compactly supported potentials for which $N_V(r)$ has zero order of growth [1], and since $N_0(r)$ for the Laplacian (zero real potential) has zero order of growth, our result is the best possible. We remark that it would be interesting to find nontrivial potentials $V \in L^\infty_{\text{comp}}(\mathbb{R}^d;\mathbb{R})$, $d \geq 3$, for which the order of growth of $N_V(r)$ is *strictly less than d*.

2. Proof of Theorem 1.1

We shall denote the scattering matrix for $H_V = -\Delta + V$ by $S_V(\lambda)$. The operator $S_V(\lambda)$ acts on $L^2(S^{d-1})$ and if V is real-valued, then it is a unitary operator for $\lambda \in \mathbb{R}$. The S-matrix is given explicitly by

$$S_V(\lambda) = I + c_d \lambda^{d-2} \pi_\lambda (V - V R_V(\lambda) V) \pi_{-\lambda}^t \equiv I + T_\lambda, \tag{5}$$

where $R_V(\lambda) = (H_V - \lambda^2)^{-1}$ and $(\pi_\lambda f)(\omega) = \int e^{-i\lambda x\cdot\omega} f(x)dx$ [15]. Under the assumption that $V \in L^\infty_{\text{comp}}(\mathbb{R}^d;F)$, the operator $T_\lambda : L^2(S^{d-1}) \to L^2(S^{d-1})$ is trace class. The S-matrix has a meromorphic continuation to the entire complex plane with finitely many poles in $\operatorname{Im}\lambda > 0$, corresponding to eigenvalues of H_V, and resonances in $\operatorname{Im}\lambda < 0$. We recall that if $\operatorname{Im}\lambda_0 \geq 2\|V\|_{L^\infty}+1$, the multiplicities of λ_0, as a zero of $\det S_V(\lambda)$, and $-\lambda_0$, as a pole of $(H_V-\lambda^2)^{-1}$, coincide, cf. Section 3 of [16]. We will work with the function $\det S_V(\lambda)$. For $N,\ M,\ q>0,\ j>2N+1$, let

$$A(N,M,q,j) = \{V \in L^\infty(K;F) : \|V\|_{L^\infty} \leq N,\ \log|\det(S_V(\lambda))| \leq M|\lambda|^q \text{ for } \operatorname{Im}\lambda \geq 2N+1 \text{ and } |\lambda| \leq j\}.$$

We remark that $\det S_V(\lambda)$ is holomorphic in this region.

Lemma 2.1. *The set $A(N,M,q,j) \subset L^\infty(\mathbb{R}^d)$ is closed.*

Proof. Let $V_k \in A(N,M,q,j)$, such that $V_k \to V$ in the L^∞ norm. Then clearly $\|V\|_{L^\infty} \leq N$. We shall use (5) and the bound

$$|\det(I+A) - \det(I+B)| \leq \|A-B\|_1 e^{\|A\|_1+\|B\|_1+1}, \tag{6}$$

cf. [13]. We let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the trace and Hilbert-Schmidt norms, respectively, on $L^2(S^{d-1})$. We wish to show that $\|S_{V_k}(\lambda)-S_V(\lambda)\|_1 \to 0$ as $k\to\infty$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a function that is equal to one on K. Using (5), we have

$$\begin{aligned} &\|S_{V_k}(\lambda) - S_V(\lambda)\|_1 \\ &\quad\leq\ |c_d||\lambda|^{d-2}\|\pi_\lambda\chi\|_2(\|V_k-V\|_{L^\infty} + \|V_kR_{V_k}V_k - VR_VV\|_{L^2\to L^2})\|\chi\pi^t_{-\lambda}\|_2. \end{aligned}$$

As in Lemma 3.3 of [4], using the explicit Schwartz kernel of π_λ, one can see that if $|\lambda| \leq j$ there is a constant C_j such that $\|\pi_\lambda\chi\|_2 \leq C_j$ and $\|\chi\pi^t_{-\lambda}\|_2 \leq C_j$. We need only show that $\|V_kR_{V_k}V_k - VR_VV\|_{L^2\to L^2} \to 0$ as $k\to\infty$. But since $\operatorname{Im}\lambda \geq 2N+1 \geq 2\max(\|V_k\|_{L^\infty}, \|V\|_{L^\infty})+1$, the operators $R_{V_k}(\lambda)$ and $R_V(\lambda)$ are holomorphic functions of λ, with norms that are uniformly bounded in this region. Since

$$R_{V_k}(\lambda) - R_V(\lambda) = R_{V_k}(\lambda)(V-V_k)R_V(\lambda),$$

$\|R_{V_k}(\lambda) - R_V(\lambda)\| \to 0$ as $k\to\infty$. Thus $\|V_kR_{V_k}V_k - VR_VV\|_{L^2\to L^2} \to 0$ as $k\to\infty$.

A similar argument shows that $\|I-S_{V_k}(\lambda)\|_1$ and $\|I-S_V(\lambda)\|_1$ are bounded uniformly for $\operatorname{Im}\lambda \geq 2N+1$, $|\lambda| \leq j$. Using (6), then, we see that $\det S_{V_k}(\lambda) \to \det S_V(\lambda)$ and thus

$$\log|\det S_V(\lambda)| \leq M|\lambda|^q \text{ if } \operatorname{Im}\lambda \geq 2N+1 \text{ and } |\lambda| \leq j.$$

□

In the next step, we characterize those $V \in L^\infty_{\text{comp}}(K;F)$ for which the order of growth of the resonance counting function is strictly less than the dimension d. For $N,\ M,\ q>0$, let

$$B(N,M,q) = \bigcap_{j\geq 2N+1} A(N,M,q,j).$$

Note that $B(N,M,q)$ is closed by Lemma 2.1.

Lemma 2.2. *Let $V \in L^\infty(K;F)$, with*

$$\limsup_{r\to\infty} \frac{\log N_V(r)}{\log r} < d.$$

Then there exist $N,\ M,\ l \in \mathbb{N}$ such that $V \in B(N, M, d-1/l)$.

Proof. By [2, Lemma 4.2], there is a $p < d$ such that

$$\limsup_{r\to\infty} \frac{\log \max_{0<\theta<\pi} \log |\det S_V(2\|V\|_{L^\infty} + 1 + re^{i\theta})|}{\log r} = p.$$

In fact, the continuity of $\det S_V(\lambda)$ in this region implies that this bound is true for $0 \leq \theta \leq \pi$. It follows that there is a $p' \geq p$, $p' < d$, and an $M \in \mathbb{N}$ such that

$$\log |\det S_V(\lambda)| \leq M|\lambda|^{p'}$$

when $\operatorname{Im}\lambda \geq 2\|V\|_\infty + 1$. Choose $l \in \mathbb{N}$ so that $p' \leq d - 1/l$ and $N \in \mathbb{N}$ so that $N \geq \|V\|_\infty$, and then $V \in B(N, M, d-1/l)$ as desired. □

Lemma 2.3. *The set*

$$\mathcal{M} = \{V \in L^\infty_{\text{comp}}(K;F) : \limsup_{r\to\infty} \frac{\log N_V(r)}{\log r} = d\}$$

is a G_δ set.

Proof. By Lemma 2.2, the complement of $\mathcal{M}$ is contained in

$$\bigcup_{(N,M,l)\in\mathbb{N}^3} B(N, M, d-1/l),$$

which is an F_σ set since it is a countable union of closed sets. By [2, Lemma 4.2], if $V \in \mathcal{M}$, then $V \notin B(N, M, d-1/l)$ for any $N,\ M,\ l \in \mathbb{N}$. Thus $\mathcal{M}$ is the complement of an F_σ set. □

We can now prove our theorem.

Proof of Theorem 1.1. Since Lemma 2.3 shows that $\mathcal{M}$ is a G_δ set, we need only show that $\mathcal{M}$ is dense in $L^\infty(K;F)$. To do this, we use a slight modification of the proof of [2, Corollary 1.3]. We give the proof here for the convenience of the reader. Let $V_0 \in L^\infty(K;F)$ and let $\epsilon > 0$. By [18, Theorem 2], we may choose a spherically symmetric $V_1 \in L^\infty(K;\mathbb{R})$ so that $V_1 \in \mathcal{M}$ and $\|V_1\|_{L^\infty} < \epsilon/2$. We now consider the function $V(z) \equiv V(z,x) = zV_1(x) + (1-z)V_0(x)$. This potential satisfies the conditions of [2, Theorem 1.1], and $V(0) = V_0$. Thus, by [2, Theorem 1.1], for some pluripolar set $E \subset \mathbb{C}$, we have

$$\limsup_{r\to\infty} \frac{\log N_{V(z)}(r)}{\log r} = d,$$

for $z \in \mathbb{C} \setminus E$. In particular, since $E \restriction \mathbb{R} \subset \mathbb{R}$ has Lebesgue measure 0 (e.g. [7, Section 12.2]), we may choose a point $z_0 \in \mathbb{R}$, $z_0 \notin E$, with $|z_0| < \epsilon/2(1 + \|V_0\|_{L^\infty})$. Then $V(z_0) \in \mathcal{M}$ and $\|V(z_0) - V_0\|_{L^\infty} < \epsilon$. Note that if V_0 is real-valued (respectively, complex-valued) then so is $V(z_0)$. □

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Department of Mathematics
University of Missouri
Columbia, Missouri 65211
e-mail:tjc@math.missouri.edu

Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506–0027
e-mail: hislop@ms.uky.edu