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An electric charge has no screw sense—a comment on the twist-free formulation of electrodynamics by da Rocha & Rodrigues

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Da Rocha and Rodrigues (RR) claim (i) that in classical electrodynamics in *vector calculus* the distinction between polar and axial vectors and in *exterior calculus* between twisted and untwisted forms is inappropriate and superfluous, and (ii) that they can derive the Lorentz force equation from Maxwell's equations. As to (i), we point out that the distinction of polar/axial and twisted/untwisted derives from the property of the electric charge of being a pure scalar, that is, not carrying any screw sense. Therefore, the mentioned distinctions are necessary ingredients in any fundamental theory of electrodynamics. If one restricted the allowed coordinate transformations to those with positive Jacobian determinants (or prescribed an equivalent constraint), then the RR scheme could be accommodated; however, such a restriction is illegal since electrodynamics is, in fact, also covariant under transformations with negative Jacobians. As to (ii), the "derivation" of the Lorentz force from Maxwell's equations, we point out that RR forgot to give the symbol F (the field strength) in Maxwell's equations an operational meaning in the first place. Thus, their proof is empty. Summing up: the approach of RR does not bring in any new insight into the structure of electrodynamics.

This paper is a reaction to some claims of da Rocha & Rodrigues [1] related to classical electrodynamics. For this purpose we begin with a brief and rough sketch of how the modern premetric form of Maxwell's equations came about and how the premetric framework is based on an appropriate operational interpretation.

1 Maxwell's equations in space and time

1.1 In components

Maxwell [2] formulated his equations in terms of Cartesian components. If we use Cartesian coordinates x^a, with $a, b, .. = 1, 2, 3$, and the time $t = x^0$, then we have the inhomogeneous and the homogeneous Maxwell equations as, respectively,

$$\partial_1 D_1 + \partial_2 D_2 + \partial_3 D_3 = \rho\,, \qquad \partial_2 H_3 - \partial_3 H_2 - \partial_0 D_1 = j_1 \quad \text{and cyclic}\,, \tag{1}$$

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0\,, \qquad \partial_2 E_3 - \partial_3 E_2 - \partial_0 B_1 = 0 \quad \text{and cyclic}\,. \tag{2}$$

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Clearly, in this framework with its so-called Cartesian vectors, we don't need to distinguish between upper and lower indices, nor talk about densities. A screw sense is naturally defined by the sequence x^1, x^2, x^3 of the Cartesian axes. The quantities $D, H; B, E; j$ are all 3-dimensional (3d) vectors, as mathematical objects they are all alike. Accordingly, the concept of a polar and an axial vector does not exist in electrodynamics as long as we restrict ourself to proper rotations $SO(3)$, that is, as long as the 3d Jacobian $J_3 := \det ||\partial x^a/\partial x^{a'}||$ is positive: $J_3 = +1$. However, this restriction is unphysical since we know that Mawell's equations are, in fact, covariant under improper transformations, too.

With hindsight we know that the general 3-dimensional orthogonal group $O(3)$ is a symmetry group of the field equations (1) and (2), and $D, H; B, E; j$ are vector representations of $SO(3)$ and ρ a scalar representation therefrom.

1.2 In vector calculus

Curious as mankind is, one didn't want to restrict oneself to proper transformations, that is, space reflections should be included, and at the same time a transition to curvilinear coordinates was desirable. As a nice formulation, the vector form of the Maxwell equations came up around the turn of the 19th to the 20th century, see Abraham & Föppl [4]. In the form given by Jackson [5], they read

$$\operatorname{div} \mathbf{D} = \rho\,, \qquad \operatorname{curl} \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}\,, \tag{3}$$

$$\operatorname{div} \mathbf{B} = 0\,, \qquad \operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,. \tag{4}$$

An integral part of electrodynamics is the equation that defines $\mathbf{E}$ and $\mathbf{B}$ in the first place, namely the expression of the Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)\,. \tag{5}$$

If we make a space reflection $x^a \longrightarrow -x^a$, we want that (5) stays invariant. If we write (5) in components,

$$F^1 = q(E^1 + v^2 B^3 - v^3 B^2) \quad \text{etc.}, \tag{6}$$

we immediately recognize that in the product $v^2 B^3$ only one vector component can turn around its sign upon reflection. Since the velocity $\mathbf{v}$ is the prototype of a (contravariant) vector, it must be $\mathbf{B}$ that is promoted to an axial vector that remains invariant under reflections.

This knowledge applied to $(4)_2$, uncovers the curl operator as axial vector. In $(3)_2$, since $\mathbf{j}$ is polar because of $\mathbf{j} = \rho\mathbf{v}$, with a scalar ρ, the magnetic excitation $\mathbf{H}$ is recognized as axial and the electric excitation $\mathbf{D}$ as polar. Accordingly, in this context, $\mathbf{E}, \mathbf{D}$ are polar and $\mathbf{H}, \mathbf{B}$ axial vectors.

RR claim, see last phrase of their abstract, that "We recall also a formulation of the engineering version of Maxwell equations using electric and magnetic fields as objects of the same nature, i.e., without using polar and axial vectors." We do, too, see equations (1) and (2). However, then, in a Cartesian calculus, they have to require $J_3 > 0$. The more the symmetry group of a physical system is widened, the more refined the description becomes of the quantities involved. In contrast to RR, we believe that the property of a vector being axial or polar is observable, one just has to apply a space reflection in an electrostatic or in a induction experiment, respectively. Accordingly, RR want to widen the transformation group—after all they work in arbitrary coordinates—but they do not want to use the more refined description of the field quantities involved.

In order to understand the procedure of RR better, we made another attempt, compare also Gelman [6] and Brevik [7]. We introduced a constant pseudoscalar field β. Then, the polar fields $\widetilde{\mathbf{H}} := \beta\,\mathbf{H}$ and $\widetilde{\mathbf{B}} := \beta\,\mathbf{B}$ can be introduced and Maxwell's equations rewritten in terms of the polar fields $\mathbf{E}, \mathbf{D}, \widetilde{\mathbf{H}}, \widetilde{\mathbf{B}}$. If one wants to preserve the meaning of the differential operators div and curl, we have to require $\beta^2 = 1$, that is, $\beta = \pm 1$, with the positive sign for a chosen orientation and the negative sign for an opposite orientation.

To achieve consistency, we have to redefine the cross product and the curl operator by multiplying each with β. Formally, this can be done. However, one runs into all sorts of strange behavior. We arrive at a modified determinant with rather curious properties: Its sign changes under a permutation of rows but does not change under a permutation of columns (which is a transformation of coordinates with a negative Jacobian). Moreover, this modified determinant must remain invariant under multiplication of its columns by (-1). Accordingly, most properties of determinants are lost. Even worse, the *mass* of a body will then necessarily become negative in response to the change of the orientation, as it is demonstrated in Sec. 6.1 of [1]. In a domino effect, the elastic stress will become orientation-dependent, too [8]. This also will apply to the classical action. Why should we redefine the vector product, handle strange "determinants" and negative masses in order to be able to follow RR on their adventurous journey to Absurdistan? We prefer to stick with polar and axial vectors and follow the usual rationale of vector calculus.

Whatever the constructions of RR may mean, they certainly do not yield a simpler representation of electrodynamics.

1.3 In tensor calculus

One could ask, why should we turn to tensor calculus, see Schouten [9], if the vector calculus works so well. There are two reasons: (i) the transition to arbitrary coordinates is more smooth, (ii) the transition to spacetime is more smooth; hence we catch two flies at once.

In vector calculus the operators div and curl take a fairly complicated form in curvilinear coordinates. It is desirable to circumvent this complication. The technique is well known: One introduces ρ as scalar *density*, we call it $\hat{\rho}$, and $\mathbf{j}$ as contravariant vector *density* $\hat{j}^a$. As a consequence also the electric excitation $\mathbf{D}$ becomes a density $\mathfrak{D}^a$; densities will be printed in fracture style or with a hat. The divergence operator then translates into div $\mathbf{D} \rightarrow \partial_a \mathfrak{D}^a$ and the curl into curl $\mathbf{E} \rightarrow \partial_a E_b - \partial_b E_a$; the new operators are covariant under arbitrary coordinate transformations. Accordingly, the tensor version of (3) and (4) reads

$$\partial_1 \mathfrak{D}^1 + \partial_2 \mathfrak{D}^2 + \partial_3 \mathfrak{D}^3 = \hat{\rho}\,, \qquad \partial_2 H_3 - \partial_3 H_2 - \partial_0 \mathfrak{D}^1 = \hat{j}^1 \quad \text{and cyclic}\,, \tag{7}$$

$$\partial_1 \mathfrak{B}^1 + \partial_2 \mathfrak{B}^2 + \partial_3 \mathfrak{B}^3 = 0\,, \qquad \partial_2 E_3 - \partial_3 E_2 + \partial_0 \mathfrak{B}^1 = 0 \quad \text{and cyclic}\,. \tag{8}$$

With $\mathfrak{D}^a$ and $\mathfrak{B}^a$ as contravariant vector densities and H_a and E_a as ordinary covectors, this system of equations is generally covariant, in spite of containing only partial (and not covariant) derivatives.

Equation $(7)_2$ can be written more coherently, if we introduce the contravariant bi-vector density $\mathfrak{H}^{ab} := \epsilon^{abc} H_c = -\mathfrak{H}^{ba}$; analogously in $(7)_2$ we take $B_{ab} = \epsilon_{abc}\mathfrak{B}^c = -B_{ba}$; here $\epsilon_{abc} = \pm 1, 0$ is the totally antisymmetric Levi-Civita tensor density. Collecting all terms, we have (summation convention),

$$\partial_a \mathfrak{D}^a = \hat{\rho}\,, \qquad \partial_b \mathfrak{H}^{ab} - \partial_0 \mathfrak{D}^a = \hat{j}^a\,, \tag{9}$$

$$\partial_{[a} B_{bc]} = 0\,, \qquad \partial_{[a} E_{b]} + \frac{1}{2}\partial_0 B_{ab} = 0\,, \tag{10}$$

together with the Lorentz force

$$F_a = q\left(E_a + B_{ab} v^b\right)\,, \qquad F_a v^a = q E_a v^a\,. \tag{11}$$

We have then the electric field strength E_a as covector and the magnetic field strength $B_{ab} = -B_{ba}$ as bi-covector.

The generally covariant formula (11) can be read as defining operationally the electric and magnetic field strengths. Hence in future we treat E_a, B_{ab} as belonging to the primary variables of electrodynamics. The second set, namely the electric and the magnetic excitation $\mathfrak{D}^a, \mathfrak{H}^{ab}$, is important in the context of formulating charge conservation, since $(9)_2$, upon differentiation, and substituting the time derivative of $(9)_1$, yields the charge conservation law in its differential version:

$$\partial_a \hat{j}^a + \partial_0 \hat{\rho} = 0\,. \tag{12}$$

In this way, $\mathfrak{D}^a, \mathfrak{H}^{ab}$ can be understood as potentials of charge and current, respectively, see [10]. We count them also as primary field variables in electrodynamics. Lorentz force (11) and charge conservation (12) are two *interfaces* between the theoretical formalism of Maxwell's equations and experiment. Activating these interfaces makes out of a theoretical construct a physical theory (provided the constitutive relations are additionally specified). It is for this reason that the field variables E_a, B_{ab} and $\mathfrak{D}^a, \mathfrak{H}^{ab}$, together with $\hat{\rho}, \hat{j}^q$, in the mathematical form specified, are measurable quantities.

This is as far as we can go unless we introduce Lorentz and Poincaré transformations. The advantage of the generally covariant system (9), (10) as compared to (3), (4) is that in (9), (10) only *partial* differentiation ∂_a occurs whereas in (3), (4) we have the nabla operator with components $\nabla_a = \partial_a + \Gamma_a$, wherein Γ_a denotes the (abbreviated) Christoffel symbols.

2 Maxwell's equations in spacetime

It is already clear from (9), (10) that we do not need to make a transition to Poincaré covariance. We can go directly to general covariance since the 4d covariance can be read off from (9), (10).

2.1 In 4d tensor calculus

We introduce in the conventional way the 4d excitation $\check{\mathcal{H}}^{ij} = (\mathfrak{H}^{ab}, \mathfrak{D}^a)$, with $i, j, ... = 0, 1, 2, 3$, the current $\check{\mathcal{J}}^i = (\hat{\rho}, \hat{j}^a)$, and the field strength $F_{ij} = (B_{ab}, E_a)$. We find

$$\partial_j \check{\mathcal{H}}^{ij} = \check{\mathcal{J}}^i \, , \qquad \partial_{[i} F_{jk]} = 0 \, . \tag{13}$$

This scheme was known to Einstein [11] in 1916. Among many other texts, a lucid exposition can be found in Schrödinger [12]. Contravariant bi-vector densities can alternatively be written as covariant bi-vectors, that is, $\mathcal{H}_{ij} = \frac{1}{2}\epsilon_{ijkl}\check{\mathcal{H}}^{ab}$, and contravariant vector densities as $\mathcal{J}_{ijk} = \frac{1}{3!}\epsilon_{ijkl}\check{\mathcal{J}}^l = \mathcal{J}_{[ijk]}$; here $\epsilon_{ijkl} = \pm 1, 0$ is the totally antisymmetric Levi-Civita tensor density. Accordingly we find the alternative version

$$\partial_{[i} \mathcal{H}_{jk]} = \mathcal{J}_{ijk} \, , \qquad \partial_{[i} F_{jk]} = 0 \, . \tag{14}$$

This form is particularly suited for passing over to the calculus of exterior forms. But before doing so, we will have to look at the exact properties of the fields $\mathcal{H}_{ij}$, $\mathcal{J}_{ijk}$, and F_{ij}.

In contrast to RR, we do not want that a Clifford algebra formalism dictates us which explicit form of electrodynamics we have to take as the valid one. We refer to experiment in order to support operationally the appropriate form of electrodynamics. First, according to classical mechanics, *force is a covector* (or covariant vector); then, by (11), since the charge q is a scalar and the velocity v^a a (contravariant) vector, the field strength F_{ij} is a conventional bi-covector in 4 dimensions.

What about $\mathcal{H}_{ij}$? Well, we have to be a bit careful here. In (12), $\hat{j}$ and $\hat{\rho}$ are conventional *densities*, that is, they transform with $|J_3|$ (the absolute value of the 3d Jacobian), since charge has no screw sense, see above. As a consequence $\check{\mathcal{J}}^i$ is a 4d density with transformation factor $|J_4|$. If we lower its indices according to $\mathcal{J}_{ijk} = \frac{1}{3!}\epsilon_{ijkl}\check{\mathcal{J}}^l$, we have to take care of the transformation properties of ϵ_{ijkl}. It is a scalar J-density of weight -1, in the (adapted) language of Schouten [9], and as such transforms with the factor $1/J_4$. Accordingly, the factor in the transformation behavior of $\mathcal{J}_{ijk}$ is sign J_4. In other words, $\mathcal{J}_{ijk}$ is a *twisted* tri-covector and, as a consequence, $\mathcal{H}_{ij}$ a *twisted* bi-covector. Therefore, the twisted nature of the excitation and the current in electrodynamics is a natural consequence of the mentioned interface to charge conservation. Of course, if we restrict the considered coordinate transformations in an ad hoc way to those of a positive Jacobian, we don't need to care about it. But if we opt for the most general group under which electrodynamics is invariant, then the electric current and the electromagnetic excitation both are twisted quantities—this is a logical consequence of the fact that a charge carries no screw sense.

Note that our results are consistent in the sense that the charge integral $Q := \int \rho$ turns out to be a scalar, exactly as the charge features in the expression for the Lorentz force (11).

2.2 In 4d exterior differential form calculus

Starting from (14), it is trivial to rewrite Maxwell's equations in exterior form notation,[1]

$$d\mathcal{H} = \mathcal{J}\,, \qquad dF = 0\,. \tag{15}$$

Here the twisted 2-form $\mathcal{H} = \frac{1}{2}\mathcal{H}_{ij}dx^i \wedge dx^j$ etc.; the exterior differential form calculus is presented in Frankel [14], for application to electromagnetism one should compare, for example, Lindell [15].

Hence eventually we found the genuine face of Maxwell's equations. Relying on differential forms, the complete independence of Maxwell's equations of coordinates is now manifest. And the insight about charge conservation and the Lorentz force allowed us to interpret $\mathcal{H}$ as *twisted* and F as *untwisted* differential 2-forms. Where RR made a mistake is apparent, they messed up the transformation behavior of the electric charge density and attributed to the charge a screw sense that cannot be found in nature.

Let us put this into a bit broader perspective: In formulating electrodynamics, the basic difference between the approach of RR and the one of us is that they take a prescribed spacetime with orientation, metric g_{ij}, etc. and press Maxwell's equations into that straightjacket. In contrast, we are much more careful. We may want to put charge on a (non-orientable) Möbius band or a Klein bottle, for example, and we are aware that the metric represents the gravitational potential in Einstein's theory of gravitation. Therefore, we want to expel as many orientational and gravitational structures as possible from the fundamental laws of electrodynamics. That is, we subscribe to the *premetric* approach of electrodynamics in the tradition of Murnaghan [16], Kottler [17], Cartan [18], and van Dantzig [19], see also [10, 20, 21, 22, 23, 24, 25, 26]. The premetric Maxwell equations (15) incorporate *topological* information, that is, whether certain forms are closed or exact. In particular, they are independent of metric and connection. We want to learn about the *genuine face* of Maxwell's equations, not about the illusive "... Many Faces of Maxwell ... Equations ..." that at least one of the authors of [1] is searching for [27].

In the corresponding axiomatic scheme [23]—for an elementary introduction see [28]—we make minimal assumptions about spacetime, just a 4-dimensional manifold that we decompose into 1+3 by means of an arbitrary normalized 4d vector n. One coordinate, longitudinal to n, is related to the physical dimension of time and 3 coordinates, transversal to n, related to the dimension of length. Then, postulating electric charge conservation, the form of the Lorentz force density, and magnetic flux conservation, we arrive at what we think is the genuine coordinate-free representation of Maxwell's equations in a 4-dimensional (4d) version, see (15). Both conservation laws are based on counting procedures, the Lorentz force law on force measurements known from mechanics—no measurements of time intervals or length are involved, that is, no metric needed: This axiomatic system is premetric.

We are even able, assuming for the vacuum a local and linear constitutive law between electromagnetic excitation $\mathcal{H}$ and electromagnetic field strength F to *derive* the light cone—and thus the metric, including its signature, up to a factor—in the geometric optics limit [29], provided birefringence is forbidden [30]. We also find a relation between the Lenz rule, the sign of the energy density, and the signature of the metric [23, 31, 32]. Whereas RR a priori put in the light cone into their spacetime picture, we get it out from local and linear electrodynamics—giving the light cone its proper place in a theory of electromagnetism, and not presupposing it as an intrinsic structure of spacetime.

2.3 In 4d Clifford calculus

RR motivated their negative and biased attitude towards twisted forms by their wish to reformulate electrodynamics in the Clifford bundle language. They write in the introduction to their paper: "...if the charge argument is indeed correct, it seems to imply that the Clifford bundle cannot be used to describe electromagnetism or any other physical theory." In our view, this claim is unsubstantiated, see also the work of Demers [33] on the relation of twisted forms with Clifford algebra.

[1] This is to be compared with Minkowski's symbolic representation of 1907 of Maxwell's equations $\mathrm{lor} f = s$, $\mathrm{lor} F^\star = 0$, with the metric dependent differential operator lor; for details see the discussion in [13].

We will not go into the details of constructing the complete Clifford-based formulation of electrodynamics; however, we would like to demonstrate that Maxwell's equations in vacuum can be straightforwardly recast into the Clifford formalism without eliminating the twisted forms. In contrast to RR, we will use the 4-dimensional covariant language throughout.

Given the 2-form of the electromagnetic field strength, $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$, the corresponding Clifford field (that is, the section of the Clifford bundle over the spacetime manifold) reads $\mathcal{F} = \frac{1}{2}F_{ij}\gamma^{[i}\gamma^{j]}$. Let us now apply the Dirac operator $\mathcal{D} = \gamma^i\nabla_i$ to $\mathcal{F}$. With the well-known identity of Clifford algebra,

$$\gamma^i\gamma^{[j}\gamma^{k]} \equiv g^{ij}\gamma^k - g^{ik}\gamma^j + \eta^{ijkl}\gamma_5\gamma_l\,, \quad \text{with} \quad \eta^{ijkl} = \epsilon^{ijkl}/\sqrt{-g}\,, \tag{16}$$

we immediately find

$$\mathcal{D}\mathcal{F} = \gamma_j\nabla_iF^{ij} + \frac{1}{2}\gamma_5\gamma_l\eta^{ijkl}\nabla_iF_{jk}. \tag{17}$$

For the electric current $J^i = (\rho, \mathbf{j})$ we define in the usual way the Clifford field $\mathcal{J} = \gamma_iJ^i$. Then we can verify that the Clifford-algebra equation of the Dirac type

$$\mathcal{D}\mathcal{F} = -\mathcal{J} \tag{18}$$

is completely equivalent to Maxwell's inhomogeneous and homogeneous equations in vacuum:

$$\partial_j\check{\mathcal{H}}^{ij} = \check{\mathcal{J}}^i\,, \qquad \partial_iF_{jk} + \partial_jF_{ki} + \partial_kF_{ij} = 0\,. \tag{19}$$

Here $\check{\mathcal{J}}^i$ is the current density and the vacuum constitutive relation is assumed to be $\check{\mathcal{H}}^{ij} = \sqrt{-g}g^{ik}g^{jl}F_{kl}$.

Note that the Maxwell equations (19) are in their standard form, and the electromagnetic excitation $\check{\mathcal{H}}^{ij}$ as well as the electric current density $\check{\mathcal{J}}^i$ are both *twisted*. This fact presents absolutely no difficulty to the *equivalent* Clifford-algebra formulation specified in equation (18).

As a final remark we feel it necessary to stress that although we admit that there is a certain beauty in the Clifford-algebra approach, the latter seems to be strongly confined to *vacuum* electrodynamics, and the equation (18) cannot be satisfactorily extended to more general constitutive laws, except for the case of a moving isotropic medium, see Jancewicz [34].

3 Discussion

In our quasi-historical description we wanted to show in a simple manner how the concepts developed over time for the description of electrodynamics: from 3-dimensional Cartesian vector calculus to the differential form presentation in (15). There are alternative, and perhaps still more convincing approaches by starting from discrete electrodynamics and using the theory of chains and cochains, see Bossavit [22], Tonti [21, 26], Zirnbauer [35], and others. In the end, in a continuum limit, these authors found the same structures as in (15), in particular, they also found unequivocally twisted forms. Classical electrodynamics is a closely knit structure and one cannot introduce changes at one structure without affecting badly other structures.

As we argued, RR did not recognize the importance of the interfaces between mathematical theory and experiment. This can also be seen in another way. In the Abstract RR claimed that "... we derive directly from Maxwell equation the density of [the Lorentz] force ..." How come? RR started from Maxwell's equations (see their Sec. 6) in which there occurs a mathematical symbol F without physical meaning; at least they did not specify an operational definition of F, that is, how one has to measure it. Then RR manipulated the Maxwell's equations in the conventional way and came up with the energy-momentum tensor of the electromagnetic field in vacuum, which they assumed to be known—besides Maxwell's equations. For the divergence of the energy-momentum tensor they found [if we translate it into the notation of our equations in (15)] the force density $f_\alpha = (e_\alpha \rfloor F) \wedge \mathcal{J}$. In this formula, the force density f_α is linked to the unidentified freely floating object F and the electric current $\mathcal{J}$. *Provided F is identified with the electromagnetic field strength*—and this is what RR did—this formula is the expression for the Lorentz force density. Consequently, RR identified their F as field strength by means of the Lorentz formula.

Now, they claimed that two of us [23] took the Lorentz force density as an axiom, but they *derived* it. This is an empty claim because their F was an unidentified object and nothing else. RR found out eventually that F can be be identified as the electromagnetic field strength via the Lorentz formula, whereas we took it as an axiom. RR did not recognize that *logically* they did the same as we did; but we, for reasons of transparency and straightforwardness, formulated our assumptions at the beginning within our axiomatic scheme whereas RR made the same assumption in a hidden way at the end of their calculations.

The experience one had won from the classical electrostatic experiments of the 18th and 19th centuries was that electric charge $Q := \int_{\Omega_3} \rho$ is a 3d scalar. In modern elementary particle physics, the conservation of electric charge is a well-tested law. Consider a scattering process of an electron with a neutron (or a proton). One ascribes to the charge of each particle a number and adds up these numbers before scattering and afterwards: The electron carries a negative elementary charge -1, the neutron or the proton carry a 0 or a $+1$, respectively. There is no screw, chirality, or handedness involved in testing charge conservation. *Scalar numbers* and adding up them is all what is required.

This confirms the conclusion that the charge also in classical electrodynamics is a pure scalar. Any other attribute to the charge would not be reflected in nature, would be superfluous and redundant. This conclusion is consistent with Schouten's verdict: "... an electric charge has no screw sense." See [9], p. 132.

But doesn't carry an electron, the carrier of an elementary charge, a spin? Isn't then a screw attached to it? Yes, indeed, but this screw is exclusively related to spin angular momentum of the electron, but not to its charge. Take a negatively charged pion π^-. It carries no spin, but an electric charge—and in this case there is no screw related to the π^-.

Accordingly, experiments show convincingly that the electric charge has no screw attached to it. Therefore, in formalizing charge conservation, the 4-dimensional electric current $\mathcal{J}$ has to be a 3-form *with* twist. Only then the charge $Q = \int_{\Omega_3} \mathcal{J}$ is really a 4-dimensional scalar, totally independent of any orientation of spacetime. As Perlick [36] remarked so aptly: "... one *must* understand the excitation as a form with twist if one wants that the charge contained in a volume always has the same sign, independent of the orientation chosen." See also Bossavit [22] and Tonti [21, 26] in this context. This is in marked contrast to the RR-formalism: therein the charge Q switches its sign upon turning around the orientation. They try to fix it by additional *ad hoc* assumptions, but it is evident: Their *surrogate charge* carries an additional attribute that has no image in nature. Their 'charge' is over-freighted with a redundant structure.

Note added in proof: In a 'note added in proof' in [1], da Rocha and Rodrigues tried to answer our above formulated objections to their article [1], see also their paper [39]. We will discuss shortly their main points in the sequence chosen by RR:

1. Twistfree electrodynamics and "observed phenomena"

   We agree with RR that one can formulate a twistfree electrodynamics, that is, using only untwisted forms. All what we tried to point out is that this amounts to an amputation: essential properties of electrodynamics are cut-off (see the example of the Möbius strip below). The quoted lemma of de Rhams is a purely geometrical statement, notions such as integrals over forms, like an action, are not discussed, nor observed parity violations like the one in the electrodynamics of the antiferromagnet Cr_2O_3 [37]. That charge carries no screw is—in contrast to RR's statement to the opposite—an experimentally esablished fact, we mentioned the *scattering experiments* in Sec.3 above.

2. Charge on a Möbius strip and twisted differential forms

   Recall that a twisted form is a special mathematical construct that gives a *well-defined* value to an integral over a certain domain. This value is independent of the orientation and even of the orientability of the domain. In particular, twisted forms provide positive values for length, volume, and mass-energy.

Also the total electric charge turns out to have a *well-defined* (positive or negative) value. RR [1] claimed: "Now had our critics read our Remark 13 they could be recalled of the fact that being $\mathcal{J}$ a pair *or* an impair 2-form we cannot define its integral over the Möbius strip. So, we conclude that only in fiction can someone think in putting a real physical charge distribution (made of elementary charge carriers) on a Möbius strip ... and leaving this physical impossiblity aside we cannot see any necessity for the use of impair [twisted] forms." We disagree with this inverse logic strongly. One cannot *conclude* anything about real physics from their specific mathematical constructions. The only possibility for them is *to claim* some physics behavior based on their "good mathematics" and to compare it to real physics.

The facts are: An electrically conducting Möbius strip can be constructed and electrically charged, see Stewart [38]. Its one sidedness can be observed by electrostatic means even in a simple experiment in a school laboratory [38]. Accordingly, who is talking about 'fiction' and a 'physical impossibility'? Perhaps it is safer to adhere to twisted forms in order to be able, unlike RR, to integrate the charge over a Möbius strip.

3. Clifford bundle is consistent with twisted forms

   By their remark concerning the Clifford bundle approach, RR introduced nothing but confusion. Contrary to the original claim (see their introduction) that the twisted forms and axial vectors are incompatible with the Clifford approach, now they seem to agree with the opposite, namely, that it is still possible to keep working with the twisted forms along with the Clifford structures. Fine! This is consistent with what we said. As soon as it is perfectly possible to live happily with the axial vectors and twisted forms within the Clifford bundle framework, one should be strongly advised to keep using them. It is thus satisfactory to see that RR agree with our conclusion that their approach, in which the well-defined charge, mass, and volume are replaced with the orientation-dependent surrogates, is unwarranted and redundant.

4. Lorentz force used for an operational definition of the field strength

   We quote RR: "... we proved that the coupling of F with J must be given by the Lorentz force law, which must then be used in the *operational* way in which those objects must be used when one is doing Physics."

   In other words, the Lorentz force formula must be used operationally in order to define the meaning of F. This is exactly what we claimed. And this operational interpretation is a coditio sine qua non. Whether one does it at the beginning or at the end of an electrodynamic theory doesn't make a logical difference.

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