

The Precautionary Premium and the Risk-Downside Risk Tradeoff

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Abstract

This paper shows that the precautionary premium embodies a tradeoff between risk and downside risk. It is the size of a mean-preserving spread for which the strength of aversion to risk just offsets the strength of aversion to downside risk. Using this result, decreasing absolute prudence can be interpreted as meaning that the amount of exposure to risk (as measured by a spread) for which aversion to risk just offsets aversion to downside risk decreases as wealth increases. This happens when an increase in wealth causes a smaller percentage change in absolute downside risk aversion than in absolute risk aversion.

Key Words: Precautionary premium, risk, downside risk

1. Introduction

A fundamental concept in the Arrow-Pratt theory of risk aversion is the risk premium π , defined by the equation $u(w-\pi) = Eu(w+\tilde{z})$, where $u(w)$ is a von Neumann-Morgenstern utility function for wealth w and \tilde{z} is an actuarially neutral random variable. A mathematically parallel concept, the precautionary premium γ , was introduced by Kimball (1990), who defined it by the equation $u'(w-\gamma) = Eu'(w+\tilde{z})$. Unlike the risk premium whose preference meaning follows directly from its defining equation, the precautionary premium has been interpreted in the context of a decision model. Specifically, Kimball has shown that γ can be interpreted as the sure reduction in future income that has the same effect on current saving as does the addition of a zero-mean risk on future income. Parallel to the hypothesis of decreasing absolute risk aversion, Kimball formulated the hypothesis that the precautionary premium decreases in wealth, a hypothesis increasingly used in the literature and is equivalent to decreasing absolute prudence (DAP).¹

In this paper, we present an interpretation of the precautionary premium directly based on its defining equation. Our interpretation is in terms of preferences between simple pairs of lotteries. Consider a basic lottery that offers w and $w+\Delta w$ with equal likelihood. For a choice of subtracting s from one of the two states, all risk-averse individuals prefer to subtract s from the good state. For a choice of adding a zero-mean risk \tilde{z} to one of the two states, all downside risk-averse individuals prefer to add \tilde{z} to the good state.² Now consider the choice of subtracting s from one state and adding \tilde{z} to the other state. How will an individual who is both risk averse and downside risk averse choose? For a given \tilde{z} , when s is small, the individual would choose to add \tilde{z} to the good state and subtract s from the bad state. The intuition behind this is that the individual's exposure to risk is small and therefore aversion to risk is dominated by aversion to downside risk. The reverse happens when s is large. Our main result shows that there exists a unique s^* such that aversion to risk just offsets aversion to downside risk. We show that this

¹ See Gollier (2001).

² Menezes et al (1980) showed that $u''' > 0$ is equivalent to aversion to downside risk.

s^* is equal to γ . Thus, the precautionary premium is the size of a mean-preserving spread such that aversion to risk exactly offsets aversion to downside risk.³

In the literature, an individual is said to be prudent if adding a zero-mean risk to his future income raises his optimal saving.⁴ Equivalently, an individual is prudent if and only if u' is convex. Hence, prudence is equivalent to aversion to downside risk. However, decreasing absolute prudence is related to but conceptually distinct from decreasing downside risk aversion. We show that for small risks, u'''/u' is a measure of absolute downside risk aversion. As is well-known, $-u''/u'$ is a measure of absolute risk aversion and $-u'''/u''$ is a measure of absolute prudence. From our main result, we derive a reinterpretation for DAP in terms of aversion to risk and aversion to downside risk. We show that a necessary and sufficient condition for DAP is that an increase in wealth causes a smaller proportional change in absolute downside risk aversion than in absolute risk aversion.

In recent years a sizable literature has found the need for including downside risk (usually in terms of skewness) as well as risk (variance) in the empirical analysis of a variety of economic phenomena (Bekaert and Harvey, 1997; Bekaert et al, 1998; Pownall and Koedijk, 1999).⁵ Tradeoffs between mean, variance and skewness have been considered in horse race betting behavior (Golec and Tamarkin, 1998), and in the purchase of state lotteries in the U.S. (Garrett and Sobel, 1999). Harvey and Siddique (2000) have found that “Systematic skewness is economically important and commands a risk premium, on average, of 3.60 percent per year.” Despite these studies, there has been no general characterization of the tradeoff between risk and downside risk. In this paper we provide such a characterization and show that the precautionary premium embodies a tradeoff between risk and downside risk.

³ As will be seen from the risk pair (r_1, r_2) defined in the following section, s is a measure of a mean-preserving spread.

⁴ See Gollier (2001, pp. 236-237).

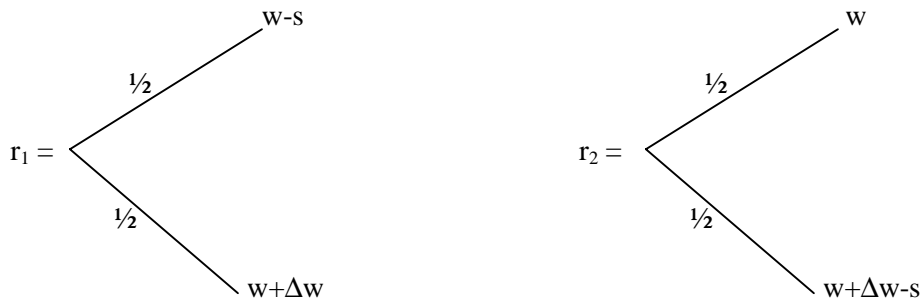
⁵ Pownall and Koedijk state “Using data on Asian equity markets, we observe that during period of financial turmoil, deviations from the mean-variance framework become more severe, resulting in periods with additional downside risk to investor. Current risk management techniques failing to take this additional downside risk into account will underestimate the true Value-at-Risk with greater severity during periods of financial turmoil.” (p. 853)

Section 2 contains our analysis of the risk-downside risk tradeoff and its relationship to the precautionary premium. Section 3 gives a reinterpretation of DAP. Section 4 briefly reviews the literature. In it, we show how our interpretation of the precautionary premium is related to Kimball's analysis of precautionary saving. Section 5 concludes the paper.

2. The Precautionary Premium and the Risk-Downside Risk Tradeoff

In this section we describe three classes of risk pairs. The first class of risk pairs represents an increase in risk. The second class represents an increase in downside risk. The third class has the property that the strength of preference between any pair of risks in this class can be decomposed into the difference between the strength of aversion to risk pairs in the first class and the strength of aversion to downside risk pairs in the second class. We show that for any individual who is both risk averse and downside risk averse, there exists a unique pair of risks in the third class for which the individual is indifferent between the risks in this pair. That is, for this risk pair, aversion to downside risk just offsets aversion to risk. For this risk pair, the change in risk as measured by the size of a mean-preserving spread exactly offsets the change in downside risk. This mean-preserving spread is shown to be the precautionary premium defined in the literature.

Let w and Δw denote wealth and increment in wealth, respectively. Consider the class \mathbf{R} of risk pairs (r_1, r_2) , where $s \geq 0$:



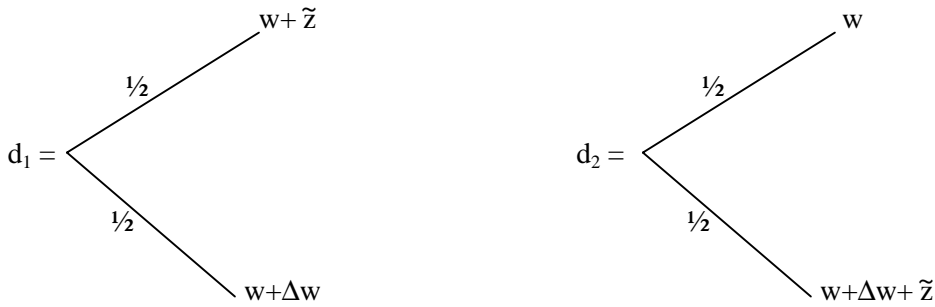
This pair has the characteristic that r_1 can be obtained from r_2 by a mean-preserving spread. Since s is the

distance by which each point of the support of r_2 is moved outward to obtain r_1 , s serves as a measure of the size of the spread. We call the difference between the distributions of r_1 and r_2 an s-spread. Since r_1 has more risk than r_2 , all risk averters ($u'' < 0$) prefer r_2 to r_1 . Following Friedman and Savage (1948),⁶

$$SR = Eu(r_2) - Eu(r_1) \tag{1}$$

is the disutility attached to the s -spread and represents the strength of aversion to the increase in risk.⁷

Let \tilde{z} be an actuarially neutral random variable (i.e., $E\tilde{z} = 0$) with distribution function $F(z)$ on support $[-a, b]$, $a > 0$ and $b > 0$. Consider the class D of risk pairs (d_1, d_2) :



This pair has the characteristic that d_1 can be obtained from d_2 by a mean-variance-preserving downward transfer of dispersion (actuarially neutral noise \tilde{z} is shifted from $w + \Delta w$ to w). Hence, d_1 has more downside risk than d_2 .⁸ All downside risk averters ($u''' > 0$) prefer d_2 to d_1 . Let⁹

$$SD = Eu(d_2) - Eu(d_1). \tag{2}$$

SD is the disutility attached to the increase in downside risk and represents the strength of downside risk aversion.

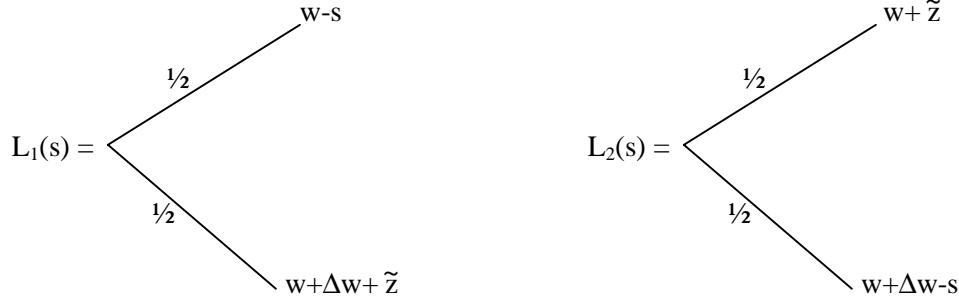
⁶ $SR = 1/2 \int_{-s}^0 \int_0^{\Delta w} [-u''(w + t + x)] dt dx$ is positive for all risk averters ($u'' < 0$).

⁷ $SR = Eu(r_2) - Eu(r_1) = [Eu(r_2) - u(\bar{r})] - [Eu(r_1) - u(\bar{r})]$, where \bar{r} is the mean value of r_1 and r_2 . Following Friedman and Savage (1948), each of the bracketed terms is a utility-based measure of risk aversion since they represent the strength of an individual's preference for the sure option \bar{r} over the risky alternative r_1 or r_2 . It follows that the difference (i.e., R) represents the strength of the individual's preference for r_2 over r_1 .

⁸ Menezes et al (1980) provided the characterization of downside risk in terms of integral conditions.

⁹ $SD = 1/2 \int_{-a}^b \int_0^{\Delta w} \int_0^z (z - y) u'''(w + t + y) dy dt dF(z)$ is positive for downside risk averters ($u''' > 0$).

Consider now the class L of lotteries $(L_1(s), L_2(s))$ defined by ¹⁰



The strength of preference for pairs in L can be decomposed into the difference between the strength of preference for pairs in R and the strength of preferences for pairs in D .¹¹ This follows immediately from

$$Eu(L_2) - Eu(L_1) = [Eu(r_2) - Eu(r_1)] - [Eu(d_2) - Eu(d_1)] = SR - SD. \quad (3)$$

From (3),

$$Eu(L_2) - Eu(L_1) > (<, =) 0 \text{ iff } Eu(r_2) - Eu(r_1) > (<, =) Eu(d_2) - Eu(d_1). \quad (4)$$

For a small s (close to zero), L_1 is preferred to L_2 so that the strength of downside risk aversion exceeds the strength of risk aversion; the reverse happens for a large s . As s increases, SR increases while SD remains unchanged.¹² That is, the strength of risk aversion increases as s increases and the strength of aversion to downside risk remains unchanged as s increases. This leads to the following proposition.

Proposition 1. If $u'' < 0$ and $u''' > 0$ then there exists a unique value of s for which $Eu(L_1) = Eu(L_2)$, or equivalently the strength of risk aversion is equal to the strength of downside risk aversion ($SR = SD$).

¹⁰ For any $s \geq 0$, the lotteries L_1 and L_2 have the same mean, and the variance of L_1 is greater than or equal to that of L_2 since $Var(L_1) - Var(L_2) = (\Delta w)s \geq 0$.

¹¹ Let F^α denote the distribution function for any lottery. It can be shown that

$$F^{L_1} - F^{L_2} = [F^{r_1} - F^{r_2}] + [F^{d_2} - F^{d_1}].$$

It follows that the comparative structure of L_1 and L_2 is a mixture of risk and downside risk.

¹² $\frac{\partial SR}{\partial s} = \frac{1}{2} \int_0^{\Delta w} [-u''(w+t-s)] dt > 0$. Since $\frac{\partial^2 SR}{\partial s^2} = \frac{1}{2} \int_0^{\Delta w} u'''(w+t-s) dt > 0$, SR is convex in s .

Let s^* be the value of s for which $SR = SD$. We now show how s^* is related to the precautionary premium γ , that was introduced by Kimball (1990) and is defined by the equation $u'(w - \gamma) = Eu'(w + \tilde{z})$.

Rearranging the equation $SR = SD$ gives

$$u(w + \Delta w - s^*) - u(w - s^*) = Eu(w + \Delta w + \tilde{z}) - Eu(w + \tilde{z}). \quad (5)$$

For an infinitesimal increase in wealth ($\Delta w \rightarrow 0$), (5) becomes

$$u'(w - s^*) = Eu'(w + \tilde{z}). \quad (6)$$

Hence, s^* is equal to the precautionary premium γ . Proposition 1 provides a new interpretation for the precautionary premium. That is, the precautionary premium is the change in risk as measured by the size of a mean-preserving spread that just offsets a given change in downside risk.

Figure 1 graphically depicts the precautionary premium. In it, the curves SR and SD are drawn from a fixed level of wealth w . The precautionary premium γ is the value of s where the curves SR and SD intersect. For $s < \gamma$, SR lies below SD , indicating that the strength of downside risk aversion exceeds that of risk aversion. The reverse is true for $s > \gamma$.¹³

<Insert Figure 1 about here>

3. A Reinterpretation of DAP

The absolute prudence function $p(w) = -u'''(w)/u''(w)$ was first introduced by Kimball (1990), who advanced the hypothesis that absolute prudence decreases in wealth (DAP). This hypothesis is equivalent to the statement that the precautionary premium γ decreases in wealth and has been generally accepted in the literature. In this section we show how DAP can be reinterpreted in terms of the behavior of risk aversion and downside risk aversion.

We begin by defining the premiums for risk pairs in \mathbf{D} and in \mathbf{R} . Let Π^α denote the Arrow-Pratt risk premium for lottery α and define

¹³ For $s < \gamma$, $u'(w-s) < Eu'(w + \tilde{z})$; for $s > \gamma$, $u'(w-s) > Eu'(w + \tilde{z})$.

$$\pi_r = \Pi^{r_1} - \Pi^{r_2} \text{ and } \pi_d = \Pi^{d_1} - \Pi^{d_2}. \quad (7)$$

π_r is the difference between the cash equivalent of r_2 and the cash equivalent of r_1 and π_d is the difference between the cash equivalent of d_2 and the cash equivalent of d_1 .¹⁴ Since r_1 is obtained from r_2 by a mean-preserving spread, π_r is a dollar measure of aversion to the increase in risk. Similarly, as d_1 is obtained from d_2 by a downside transfer of dispersion, π_d is a dollar measure of aversion to the increased downside risk.¹⁵

For small risk \tilde{z} , Kimball (1990) showed that (6) implies

$$\gamma = (\sigma^2/2) \left[-\frac{u'''(w)}{u''(w)} \right] = (\sigma^2/2)p(w). \quad (8)$$

For small risk \tilde{z} and small increment in wealth Δw , by (7)

$$\pi_r = (\Delta w)(s/2) \left[-\frac{u''(w)}{u'(w)} \right], \quad (9)$$

$$\pi_d = (\Delta w)(\sigma^2/4) \left[\frac{u'''(w)}{u'(w)} \right]. \quad (10)$$

It follows from (8) to (10) that $\pi_r = \pi_d$ at $s = \gamma$.

Denote absolute risk aversion by $a(w) = -u''(w)/u'(w)$ and define absolute downside risk aversion as $d(w) = u'''(w)/u'(w)$. (9) indicates that for small risks the behavior of $d(w)$ controls the behavior of the downside risk premium π_d . By definition,

$$p(w) = d(w)/a(w). \quad (11)$$

From (11),

$$p'(w) < (>, =) 0 \quad \text{iff} \quad d'(w)/d(w) < (>, =) a'(w)/a(w).$$

To highlight this result, we state it as Proposition 2.

¹⁴ The cash equivalent of a lottery is the sure amount of money that makes the individual indifferent between this amount and the lottery. It is equal to the difference between the expected value of a lottery and the Arrow-Pratt risk premium.

¹⁵ The risk premium π_r is in keeping with the literature when considering a pair of non-degenerate risks. Since an increase in downside risk involves two non-degenerate risks, it cannot be defined in terms of indifference between a risk and a surety. Our definition of the downside risk premium π_d is analogous to that of π_r .

Proposition 2. Absolute prudence is decreasing (increasing) in wealth if and only if an increase in wealth causes a smaller (greater) percentage change in absolute downside risk aversion than in absolute risk aversion.

The DAP hypothesis has been generally accepted in the literature. Proposition 2 indicates that the intuition behind this hypothesis is that risk aversion dominates downside risk aversion as wealth increases.

4. Literature

Precautionary saving refers to saving that arises from uncertainty about future income. Irving Fisher (1930) attributed such saving to the need “to lay up for a rainy day”, arguing that “the greater the risk of rainy days in the future, the greater is the impulse to provide for them at the expense of the present.” (pp. 77-78) Fisher’s insight was formalized by contributions by Leland (1968) and Sandmo (1970), who analyzed precautionary saving in a two-period expected utility framework. They showed that precautionary saving is positive (negative) according as the individual is a downside risk averter (preferrer).

Kimball (1990) extended the analysis of precautionary saving by considering the intensity of the precautionary saving motive, as measured by the precautionary premium. He defined this premium as the sure reduction in future income that has the same effect on saving as the addition of a zero-mean risk to future income. We can illustrate precautionary saving and its relationship to the precautionary premium in a simple two-period model. This enables us to show how the risk-downside risk tradeoff underlies the intensity of the precautionary saving motive.

Let k^c be the optimal saving under certainty:

$$\text{Max}_k v(w_1-k)+u(w_2+k), \tag{12}$$

where w_1 and w_2 are income in period 1 and period 2, respectively, v is the period-one utility function.

Let k^u be the optimal saving under income risk when period-two income is $w_2 + \tilde{z}$:

$$\text{Max}_k v(w_1-k)+Eu(w_2 + \tilde{z} + k). \tag{13}$$

Finally, let $k^*(C)$ denote the optimal saving when second period income is $w_2 - C$:

$$\text{Max}_k v(w_1 - k) + u(w_2 - C + k), \quad (14)$$

where C is the reduction in period two income. The precautionary saving is given by $k^u - k^c$. It is positive under downside risk aversion ($u''' > 0$). The precautionary premium γ is the value of C for which $k^* = k^u$. From the first order conditions for (13) and (14), this happens when $u'(w - \gamma) = Eu'(w + \tilde{z})$, where $w = w_2 + k^*$.

Figure 2 illustrates precautionary saving and the precautionary premium in terms of the relationship between the solutions to the problems (12)-(14). S^c , S^u and S^* are the optimal points for (12), (13) and (14), respectively, where S^* corresponds to $C = \gamma$.¹⁶ In the diagram, precautionary saving is the horizontal distance between S^c and S^u while the precautionary premium γ is the vertical distance between S^u and S^* .

<Insert Figure 2 about here>

We now show the sense in which the risk-downside risk tradeoff underlies the intensity of the precautionary saving motive. In Figure 2, the curve connecting S^c and S^* is the income-consumption curve (ICC) under certainty resulting from changes in period-two income. Suppose $C < \gamma$. Then the optimal solution to (14) is at some point on ICC that is above S^* . Hence, saving under income risk (k^u) is greater than saving (k^*) under certainty with second period income $w_2 - C$. We next show that this is equivalent to the individual's preference between two lotteries that involve a tradeoff between risk and downside risk and for which the individual's aversion to risk is dominated by aversion to downside risk.

By the first-order conditions for (13) and (14) and that $k^u > k^*$,

$$-v'(w_1 - k^*) + Eu'(w_2 + k^* + \tilde{z}) > 0,$$

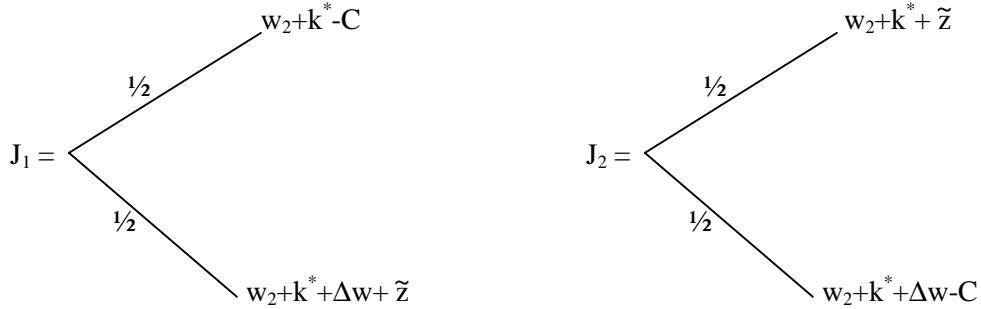
$$-v'(w_1 - k^*) + u'(w_2 - C + k^*) = 0.$$

Hence,

¹⁶ The introduction of a zero-mean risk \tilde{z} causes indifference curves to shift upward due to risk aversion and to rotate counter-clockwise due to downside risk aversion. Hence, S^u lies to the left of S^c on the budget line: $c_1 + c_2 = w_1 + w_2$. See Menezes and Auten (1978).

$$Eu'(w_2 + k^* + \tilde{z}) > u'(w_2 - C + k^*).$$

This holds if and only if J_1 is preferred to J_2 for small $\Delta w > 0$, where



Matching w with w_2+k^* and s with C , J_1 and J_2 respectively become L_1 and L_2 defined in Section 2. Hence, from the analysis in Section 2, aversion to downside risk dominates aversion to risk. The intuition underlying this is that the exposure to risk as determined by the size of the spread (C) is lower than the exposure to risk when the size of the spread is γ . Since downside risk is unaffected by the size of the spread, exposure to downside risk dominates that of risk. The reverse happens when $C > \gamma$. Finally, when $C = \gamma$, aversion to risk just offsets aversion to downside risk.

5. Concluding Remarks

This paper shows that the precautionary premium involves a tradeoff between risk and downside risk. We show that u'''/u' is a measure of absolute downside risk aversion, and decreasing absolute downside risk aversion is related to but distinct from DAP. DAP can be interpreted as saying that the amount of exposure to risk (as measured by a spread) for which aversion to risk just offsets aversion to downside risk decreases as wealth increases. This happens when an increase in wealth causes a smaller percentage change in absolute downside risk aversion than in absolute risk aversion.

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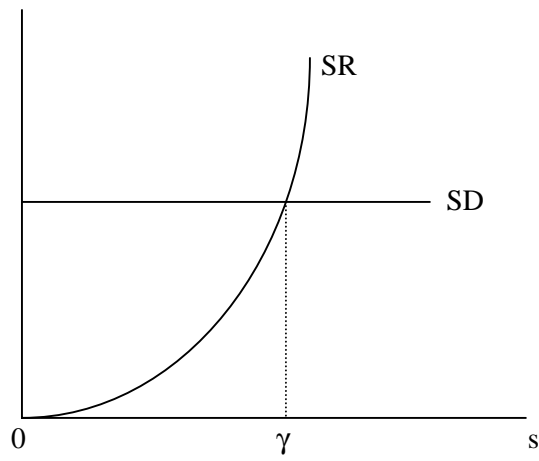


Figure 1

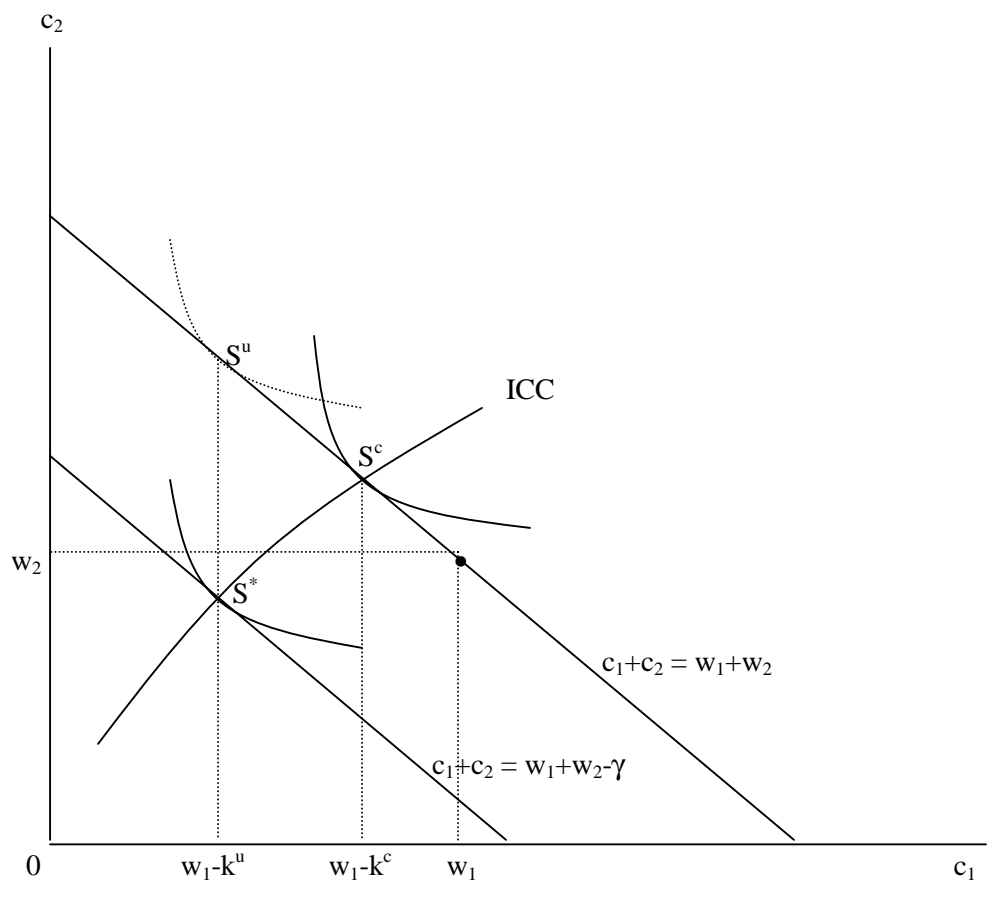


Figure 2