İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

A NEW APPROACH TO INVERSE KINEMATIC SOLUTIONS OF SERIAL ROBOT ARMS BASED ON QUATERNINONS IN THE SCREW THEORY FRAMEWORK

M.Sc. Thesis by
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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

SERİ ROBOT KOLLARININ TERS KİNEMATİK ÇÖZÜMÜNE SCREW TEORİ VE KUATERNİYON CEBRİ TABANLI YENİ BİR YAKLAŞIM

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FOREWORD

This master thesis where written during the time-period from spring 2008 until spring 2009, under the theaching supervision of Prof. Dr. Hakan Temeltaş, Istanbul Technical University (ITU) Robotic Research Laboratoary. The intent of this thesis is to develop inverse kinematic solution for serial robot manipulators. I believe that this thesis will be very helpful for whom concerned about robot kinematic.

First of all I would like to thank my supervisor, Hakan Temeltaş, for his great help, patience and indispensable advises.

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ABBREVIATIONS

App	: Appendix
D-H	: Denavit - Hartenberg
DOF	: Degree of freedom
E(n)	: N-dimensional Euclidean group
GL	: General linear group
ITU	: Istanbul Technical University
O(n)	: N-dimensional orthogonal group
SE(n)	: N-dimensional special Euclidean group
SO(n)	: N-dimensional special orthogonal group
SU(n)	: N-dimensional special unitary group
U(n)	: N-dimensional unitary group

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A NEW APPROACH TO INVERSE KINEMATIC SOLUTIONS OF SERIAL ARMS BASED ON QUATERNIONS IN THE SCREW THEORY FRAMEWORK

SUMMARY

Screw theory is a way to express displacements, velocities, forces and torques in three dimensional space combining both rotational and translational parts. Any motion along a screw can be decomposed into a rotation about an axis followed by a translation along that axis. Any general displacement of a rigid body can therefore be described by a screw. In general, a three dimensional motion can be defined using a screw with a given direction and pitch. Four parameters are required to fully define a screw motion, the 3 components of a direction vector and the angle rotated about that line. In contrast, the traditional method of characterizing 3-D motion using Euler Angles requires 6 parameters, 3 rotation angles and a 3x1 translation vector. Several application of screw theory has been introduced in robot kinematic. Compared with other methods, screws theory method just establish two coordinates, its geometrical meaning is obvious and it avoids singularities due to the use of the local coordinates. Therefore, screw theory has regained importance and has become an important method in robot kinematic.

The major intents of this thesis are to formulize inverse kinematic problem in a compact closed form and to avoid singularity problem. Non-singular inverse kinematic solutions are obtained by using screw theory. Quaternion algebra is used to formulize kinematic problem in a compact closed form. Quaternions are hyper-complex numbers of rank 4, constituting a four dimensional vector space over the field of real numbers. Any rotation can be represented by unit-quaternion and also any screw motion can be defined by unit dual-quaternion. Screw motion can also be defined by using two quaternions however dual operators are the best way to describe screw motion and also the dual-quaternion is the most compact and efficient dual operator to express screw displacement.

In this thesis, three inverse kinematic solution methods of 6-DOF serial robot manipulator, which are based on screw theory is presented. The first one is exponential mapping method. This method uses matrices as a screw operator. There are 16 parameters to describe screw motion in matrix operators while just 6 parameters are needed. Thus, however this method is singularity avoding, it is not compact closed. And also two new formulations of the inverse kinematic solution of the 6-DOF serial robot manipulator are proposed by using quaternion algebra. In these two new formulation methods, one of them uses quaternions as a screw operator which combines a unit quaternion plus pure quaternion and the other one uses dual-quaternions as a screw operator. These three methods and also the D-H convantion, which is the most common method in robot kinematic are compared with respect to singularity, computation efficiency, design complexity and accuracy.

Simulation results are obtained by using Matlab and animation applications are obtained by using the virtual reality toolbox of MATLAB (VRML). Simulation experiments are made for single and cooperative working of Staubli TXL60 serial robot arm.

SCREW TEORİ ÇERÇEVESİNDE KUATERNİYONLAR KULLANILARAK SERİ ROBOT KOLLARININ TERS KİNEMATİK ÇÖZÜMÜNDE YENİ BİR YAKLAŞIM

ÖZET

Screw teori, üç boyutlu uzayda dönme ve öteleme hareketlerinin birleşimi ile oluşan, genel hareket, hız, kuvvet ve torkların ifade edilmesini sağlayan bir yöntemdir. Genel olarak screw hareketi bir doğru etrafında dönme ve yine aynı doğru boyunca öteleme hareketlerinin bir birleşimidir. Katı cisimlerin tüm hareketleri bu yaklaşımla ifade edilebilir. Genel olarak üç boyutlu uzayda screw hareketi bir doğru ve bir oran (pitch) kullanılarak ifade edilir. (Burada kullanılan oran (pitch), dönme başına meydana gelen öteleme miktarıdır). Genel screw hareketi toplamda dört eleman kullanılarak tanımlanabilir. Bunlardan üç tanesi dönme ve ötelemenin meydana geldiği doğruyu, bir tanesi de doğru etrafında meydana gelen dönme miktarını ifade etmek için kullanılır. Katı cisimlerin hareketinde kullanılan en geleneksel yöntem Euler açılarıdır. Euler açıları bir katı cismin hareketini 6 eleman kullanarak ifade eder. Bunlardan üç tanesi kartezyen koordinatlarda öteleme hareketinin ifadesinde kullanılırken, diğer üç tanesi de bu koordinat sistemlerinde meydana gelen dönmelerin ifadesinde kullanılır. Screw teorinin robot kinematiğinde çeşitli uygulamaları vardır. Diğer yöntemlere kıyasla screw teorinin robot kinematiğinde şu üstünlükleri vardır; yalnız iki koordinat sistemiyle kinematik analiz yapılır, geometrik olarak çok anlaşılırdır ve ters kinematik çözümlemede tekil nokta probleminden etkilenmez. Bu nedenlerden dolayı screw teorinin robot kinematiğinde çok önemli bir yeri vardır.

En genel anlamda bu tezin amaçlarını iki temel başlık altında toplayabiliriz. Bunlardan birincisi seri robotların ters kinematiğinde tekil nokta problemlerinden etkilenmeden çözümlerin elde edilmesidir. Bunun için önerilen yöntemler screw teori tabanlı olarak seçilmiştir. İkinci temel amaç ise kinematik problemin etkin bir cebir kullanılarak ifade edilmesidir. Bunun içinde önerilen yöntemlerde kuaterniyon cebiri kullanılmıştır. Kuaterniyonlar rankı dört olan hiper-kompleks sayılardır. Kuaterniyon cebirinde bu dört eleman kullanılarak bir doğru tanımlanır ve bu doğru etrafında herhangi bir dönme temsil edilebilir. Fakat genel katı cisim hareketi tek bir kuaterniyon ile ifade edilemez. Bunun için ya iki kuaterniyon (bunlardan bir tanesi "birim kuaternion" dönmeyi ifade etmede, diğeri ötelemeyi ifade etmede kullanılır) ya da dual kuaterniyonlar kullanılmalıdır. Dual operatörler screw hareketi ifade etmede kullanılabilecek en iyi operatörlerdir. Aynı zamanda dual operatörlerin içinde de dual kuaterniyon operatörü screw hareketin temsilinde kullanılabilecek en verimli ve en az parametreli dual operatördür.

Bu tezde seri robot kollarının ters kinematik çözümlerine yönelik screw teori tabanlı yöntemler incelenmiştir. Bunlardan ilki ekponensiyel haritalama yöntemidir. Bu yöntemde screw teori ve matris cebiri kullanılır. Bu nedenle tekil nokta problemi olmamasına karşın denklemler çok fazla parametre ile ifade edilmiştir. Bu durumu ortadan kaldırmaya yönelik iki farklı ters kinematik çözümü önerilmiştir. Bunlardan

birincisi birim kuaterniyon (dönme operatörü) ve bir kuaterniyon (öteleme oerpatörü) kullanılarak elde edilmiştir. İkinci çözüm ise dual kuaterniyonlar kullanılarak elde edilmiştir. Bu üç yöntem ve robot kinematiğinde en çok kullanılan yöntem olan D-H yöntemi tekil nokta problemleri, hesaplama verimi, dizayn zorluğu ve çözüm doğruluğu açısından karşılaştırılmışlardır.

Simulasyon çalışmaları Matlab ortamında geçekleştirilmiştir. Animasyon uygulamaları ise Matlabın sanal gerçeklik araç kutusu kullanılarak gerçekleştirilmiştir (VRML). Simulasyon denemelerinde Staubli TXL60 seri robotunun tek ve kooperatif çalışma örnekleri yapılmıştır.

1. INTRODUCTION

The problem of kinematic is to describe the motion of the manipulator without consideration of the forces and torques causing the motion. There are two main kinematic problems. First one is forward kinematic problem, which is to determine the position and orientation of the end effector given the values for the joint variables of the robot. The second one is inverse kinematic problem is to determine the values of the joint variables given the end effector's position and orientation. Inverse kinematic problem is more complicated then forward kinematic problem [1].

Several methods are used in robot kinematic. The most common method is Denavit and Hartenberg notation for definition of special mechanism [2]. This method is based on point transformation approach and it is used 4 x 4 homogeneous transformation matrix which is introduced by Maxwell [3]. Maxwell used homogeneous coordinate systems to represent points and homogeneous transformation matrices to represent the transformation of points. The coordinate systems are described with respect to previous one. For the base point an arbitrary base coordinate system is used. Hence some singularity problems may occur because of this description of the coordinate systems. And also in this method 16 parameters are used to represent the transformation of rigid body while just 6 parameters are needed to describe of rigid body motion.

Another main method in robot kinematic is screw theory which is based on line transformations approach. The elements of screw theory can be traced to the work of Chasles and Poinsot in the early 1800s. Using the theorems of Chasles and Poinsot as a starting point, Robert S. Ball developed a complete theory of screws which he published in 1900 [4]. In screw theory every transformation of a rigid body or a coordinate system with respect to a reference coordinate system can be expressed by a screw displacement, which is a translation by along a λ axis with a rotation by a θ angle about the same axis [4]. This description of transformation is the basis of the screw theory. There are two main advantages of using screw theory for describing rigid body kinematics. The first is that it allows a global description of rigid body motion that does not suffer from singularities due to the use of local coordinates. The second advantage is that the screw theory provides a geometric description of rigid motion which greatly simplifies the analysis of mechanisms [5].

There are many applications of screw theory in kinematic. Yang and Freudenstein were the first to apply line transformation operator mechanism by using the dual-quaternion as the transformation operator [6]. Yang also investigated the kinematics of special five bar linkages dual 3×3 matrices [7]. Pennock and Yang extended this method to robot kinematics [8]. In these methods dual 3×3 matrices are used as a transformation operator to represent position and orientation of robot manipulators. This transformation operator has 18 parameters while just 6 parameters are needed to represent screw motion. Kumar and Kim obtained kinematic equations of 6-DOF robot manipulator by using dual-quaternion and D-H parameters [9]. (Dual-quaternion has 8 parameters. Thus dual-quaternion representation is more compact then matrix representation). Dual–quaternion parameters are obtained from D-H parameters. In inverse kinematic they used geometrical solution approach with D-H parameters. Thus they couldn't avoid singularity problem. M. Murray solved 3-DOF and 6-DOF robot manipulator kinematics by using screw theory with 4x4 matrix operator [10]. Then J.Xie, W.Qiang, B.Liang and C.Li extended this method to 6-DOF space manipulator [11]. In this method non-singular inverse kinematic solutions are obtained using 4x4 matrix operator. This operator needs 16 parameters while just 6 parameters are needed for description of screw motion. J. Funda analyzed transformation operators and he found that dual operators are the best way to describe screw motion and also the dual-quaternion is the most compact and efficient dual operator to express screw displacement [12], [13]. Finally, E. Sariyildiz and H. Temeltaş gave solution of inverse kinematic problem using screw theory and quaternion / dual-quaternion operators [14], [15]. These solutions are singularity avoiding and also just eight parameters are used for description of screw motion.

In this thesis, I gave two new solution methods of inverse kinematic problem of serial robot manipulator. Both of these methods are based on screw theory and quaternion algebra. The first one is solved by using unit-quaternion and the second one is solved by using dual-quaternion. These solutions are given in chapter five. This thesis also include mathematical preliminary of geometry of motion in chapter two, general representations of rigid body motion in chapter three, kinematic analyze of serial robot manipulators using screw theory and exponential mapping in chapter four, simulation and animation results of kinematic solutions of serial robot arms in chapter six and conclusion in chapter seven.

1.1 Purpose of the Thesis

In this thesis I present two new formulation methods to solve kinematic problem of serial robot manipulators. In these methods my major aims are to formulize kinematic problems in a compact closed form and to avoid singularity problems. These formulations are based on screw theory. Because, compared with other methods, screw theory has two main advantages in rigid body kinematic [5]. The first is that it allows a global description of rigid body motion that does not suffer from singularities due to the use of local coordinates. The second advantage is that the screw theory provides a geometric description of rigid motion which greatly simplifies the analysis of mechanisms [5]. Quaternion is used as a screw operator. Because, quaternion has four parameters and any rotational motion can be represented by using this four parameters in quaternion algebra however, nine parameters are needed to represent rotational motion in matrix algebra. And also at last dual-quaternions are used as a screw operator. Dual operators are the best way to describe screw motion and also the dual-quaternion is the most compact and efficient dual operator to express screw displacement [12], [13].

2. GEOMETRY OF A MOTION

2.1 Objectives

A rigid motion of an object is a motion which preserves distance between points. The study of robot kinematics has at its heart the study of the motion of the rigid objects. In this chapter I briefly introduce geometrical preliminary of motion. Firstly, some differential geometry concepts are introduced. Then a brief introduction to the basic of the lie group theory and its connections with the rigid body kinematics are given. I will end this chapter with lie algebra and its transformation to the lie groups (exponential mapping).

2.2 Some Differential Geometry Concepts

2.2.1 Differentiable manifolds and maps

In mathematics, more specifically in differential geometry and topology, a manifold is a mathematical space that on a small enough scale resembles the Euclidean space (see App. A.2 for definition Euclidean space) of a certain dimension, called the dimension of the manifold. Thus a line and a circle are one-dimensional, a plane and the surface of a ball are two-dimensional, and so forth. Infinitely differentiable manifold (C^{∞}), also called a differentiable (smooth) manifold. A smooth manifold is a topological manifold together with its "functional structure" and so differs from a topological manifold because the notion of differentiability exists on it. Every smooth manifold is a topological manifold, but not necessarily vice versa [16].

Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets and $f: U \longrightarrow V$ is a smooth map. f is a smooth map if all partial derivatives of f exist and are continuous. If $m = n$ and f is bijective and both f and f^{-1} are smooth then f is called diffeomorphism and U and V are said to be diffeomorphic. A manifold of dimension n is a set M which is locally homeomorphic to $\mathbb{R}^n$. A manifold can be parameterized using a set of local coordinate charts. A local coordinate chart is a pair (ϕ, U), where ϕ is a function which maps points in the set $U \subset M$ to an open subset of $\mathbb{R}^n$. Let (ϕ, U) and (ψ, V) be two overlapping charts. They are C^∞ related if $\psi^{-1} o \phi$ is a diffeomorphism where it is defined. A collection of such charts with the additional property that U's cover M is called a smooth atlas. A manifold M is a smooth manifold if it admits a smooth atlas. Let $F: M \longrightarrow N$ be a mapping between two smooth manifolds and let (ϕ, U) and (ψ, V) be coordinate charts for M, N respectively. The mapping $F: M \longrightarrow N$ is smooth if $\hat{F}: \phi(U) \longrightarrow \psi(V)$ is smooth for all choices of coordinate charts on M and N [10].

2.2.2 Tangent spaces and tangent maps

Let M be a smooth manifold of dimension n and let p be a point in M. $C^\infty(p)$ is real-valued functions on M for the set of smooth. Its domain of definition includes some open neighborhood of p. A map $X_p: C^\infty(p) \longrightarrow \mathbb{R}$ is called derivation if, for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$, it satisfies:

$$X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g) \qquad \text{(Linearity)} \qquad \textbf{(2.1)}$$

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g) \qquad \text{(Leibniz rule)} \qquad \textbf{(2.2)}$$

The tangent space of M at a point p, denoted $T_p M$, is the set of all derivations $X_p: C^\infty(p) \longrightarrow \mathbb{R}$. Elements of the tangent space are called tangent vectors. Let (ϕ, U) be a coordinate chart on M with local coordinates $(x_1, x_2, \ldots, x_n)$. X_p can be written as:

$$X_p = X_1 \frac{\partial}{\partial x_1} + \cdots + X_n \frac{\partial}{\partial x_n}, \qquad \textbf{(2.3)}$$

where $(X_1, \ldots, X_n) \in \mathbb{R}$ is a local coordinate representation of X_p.

Let $F: M \longrightarrow N$ be a smooth map. The tangent map of F at p is defined as the linear map $F_{*p}: T_p M \longrightarrow T_{F(p)} N$ defined b

$$F_{*p} X_p(f) = X_p(f^\circ F) \qquad \textbf{(2.4)}$$

where $X_p \in T_p M$ and $f \in C^\infty(F(p))$ and $T_p F$ is the tangent map of F at p [10].

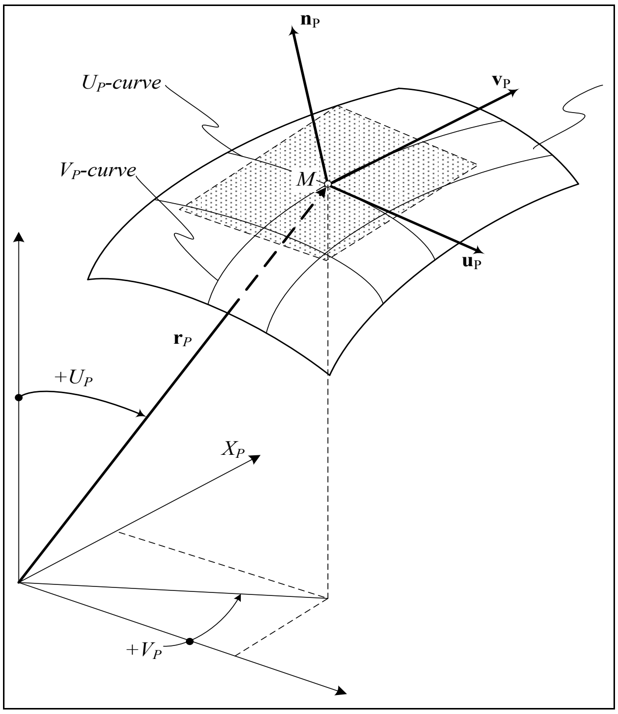

Figure 2.1: Local topology of a surface

We can define a surface uniquely by using two independent variables. A part of surface can be expressed by using its rectangular coordinates. It can be seen in figure 2.1. Rectangular coordinates are X_p, Y_p and Z_p as functions of two Gaussian coordinates U_p and V_p in a certain closed interval:

$$r_p = r_p(U_p, V_p) = \begin{bmatrix} X_p(U_p, V_p) \\ Y_p(U_p, V_p) \\ Z_p(U_p, V_p) \\ 1 \end{bmatrix}; (U_{1.p} \leq U_p \leq U_{2.p};\ V_{1.p} \leq V_p \leq V_{2.p}) \qquad \textbf{(2.5)}$$

where r_p is the position vector of a point of the surface P; U_p and V_p are curvilinear (Gaussian) coordinates of the point on the surface; X_p, Y_p, Z_p are Cartesian coordinates of the point of the surface.

If the parameters U_p and V_p are not independent, the point on the surface is singular. The parameters must be independent which means that the matrix M has rank 2.

$$M = \begin{bmatrix} \frac{\partial X_p}{\partial U_p} & \frac{\partial Y_p}{\partial U_p} & \frac{\partial Z_p}{\partial U_p} \\ \frac{\partial X_p}{\partial V_p} & \frac{\partial Y_p}{\partial V_p} & \frac{\partial Z_p}{\partial V_p} \end{bmatrix} \qquad \textbf{(2.6)}$$

Positions where the rank is 1 or 0 are singular points; when rank at all points is 1 then equation 2.5 represents a curve. The first derivatives of r_p with respect to

Gaussian coordinates U_p and V_p are $\boldsymbol{u}_p$ and $\boldsymbol{v}_p$ tangent vectors of the point M on the surface P. Tangent vectors yield an equation of the tangent plane to the surface P at M [17].

2.3 Groups Lie Groups and Lie Algebra

2.3.1 Groups

The concept of a group was introduced into mathematics by Cayley in the 1860s, generalizing the older notion of substitutions [18]. A group is a set of symmetries operations. However, it is usual to define a group independently from the objects whose symmetry is being considered. Because, the same group may represent the symmetries of several different kind of object. So, a group can be defined, as a set with a binary operation. The binary operation or product is supposed to represent the composition of symmetries that is performing one symmetry followed by another. The binary operation must satisfy the following axioms [19],

Let G be a set and g_1, g_2 be the elements of G.

1. Closure: The product of two group elements is always another group element.
$$g_1 \; and \; g_2 \epsilon \; G \;\; \Rightarrow \;\; g_1 g_2 \; \epsilon \; G$$
2. Associativity: The product of group elements must be associative
$$g_1(g_2 g_3) = (g_1 g_2) g_3$$
3. Identity element: The group must always contain a unique distinguished identity element $e \; \epsilon \; G$ such that
$$g_1 e = e g_1 = g_1 \; \epsilon \; G$$
4. Inverses: For every element in the group there is a unique inverse element.
$$g_1 g_1^{-1} = g_1^{-1} g_1 = e$$

If all group elements satisfy this commutativity, the group is said to be abelian. If there exist group elements for which $g_1 g_2 \neq g_2 g_1$, the group is said to be non-abelian.

The set of integers $\mathbb{Z}$ with addition as the law of composition, form a group. The identity element is 0, the inverse of the integer n is the integer –n, and addition is closed in $\mathbb{Z}$. This group is also commutative. Therefore it is an abelian group.

An example of a non-abelian group is the set of all real nxn matrices with non-vanishing determinant, where the law of composition is matrix multiplication. The condition of nonvanishing determinant ensures that every group element a has an inverse (the usual matrix inverse A^{-1}). However, matrix multiplication is non-commutative, and so in general $AB \neq BA$.

2.3.2 Lie groups

Continuous groups were first studied in great detail by the Norwegian mathematician Sophus Lie (1842-1899), [20]. One of his first examples was the group of isometries of three dimensional space. This could also be called the group of proper rigid motions in $\mathbb{R}^3$. This group is perhaps the most important one for robotics.

A Lie group is a differentiable manifold obeying the group properties and that satisfies the additional condition that the group operations are differentiable. These group operations are as follows:

1. The set of group elements G must form a differentiable manifold.
2. The group product must be differentiable mapping.
3. The map from group element to its inverse must be differentiable mapping.

2.3.2.1 Examples of Lie groups

Example 1: First example is the group of unit modulus complex numbers. Any element of the group can be represented as:

$$z = \cos(\theta) + isin(\theta) \tag{2.7}$$

where θ is between $0 \leq \theta < \pi 2$.

The group operation is complex multiplication. Let z_1 and z_2 be two elements of group. Then the multiplication of two elements

$$\begin{aligned} z_1 z_2 &= (\cos(\theta_1) + isin(\theta_1)(\cos(\theta_2) + isin(\theta_2)) \\ &= (\cos(\theta_1)\cos(\theta_2) - sin(\theta_1)sin(\theta_2)) + i\,(\sin(\theta_1)\cos(\theta_2) + cos(\theta_1)sin(\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\,(\sin(\theta_1 + \theta_2) \end{aligned} \tag{2.8}$$

is also continuous since addition and multiplication are continuous.

The group of unit modulus complex numbers is a lie group. The inverse of the unit modulus complex number is its complex conjugate; therefore the identity element is the complex number 1. Complex multiplication is commutative, thus it is an abelian group (A group for which the elements commute (i.e., AB = BA for all elements A and B) is called an *Abelian group).* The manifold underlying this group can be identified with the unit circle in the complex plane.

Example 2: Second example is the unit modulus quaternions. (Quaternions will be investigated deeply in the next chapters. Here I will just focus on the group properties of unit modulus quaternions.) Hamilton's quaternions are numbers of the form

$$q \;= (q_0, \boldsymbol{q_v}) = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_2 \qquad \textbf{(2.9)}$$

where q_0 is a scalar and $\boldsymbol{q_v} = (q_1, q_{2,}q_3)$ is a vector.

Addition and multiplication of quaternions are a new quaternion. They can be represented as:

$$q_a + q_b = (q_{a0} + q_{b0}), (\boldsymbol{q_{av}} + \boldsymbol{q_{bv}}) = q_{ad} \qquad \textbf{(2.10)}$$

$$q_a \otimes q_b = q_{a0}q_{b0} - \boldsymbol{q_{av}} \cdot \boldsymbol{q_{bv}}\, , q_{a0}\boldsymbol{q_{bv}} + q_{b0}\boldsymbol{q_{av}} + \boldsymbol{q_{av}} \times \boldsymbol{q_{bv}} = q_{mult} \qquad \textbf{(2.11)}$$

Notice that while addition is commutative, multiplication is not because of vector product. Note that $\boldsymbol{q_{av}} \times \boldsymbol{q_{bv}} = -\boldsymbol{q_{bv}} \times \boldsymbol{q_{av}}$.

Conjugate and inverse of the quaternion can be expressed as:

$$q^* = (q_0, -\boldsymbol{q_v}) = (q_0, -q_1 -, q_{2,} - q_3) \qquad \textbf{(2.12)}$$

$$q^{-1} = \frac{1}{||q||^2}\, q^* \quad and \quad \big||q|\big| \neq 0 \qquad \textbf{(2.13)}$$

A lie group can be obtained by restricting our attention to quaternions for which $q \otimes q^* = 1$. When $q \otimes q^* = 1$ or $||q||^2 = 1$ we get an unit quaternion. The inverse of the unit quaternion is its conjugate and the identity element of the unit quaternion is the quaternion 1 ($q_{id} = (1,0,0,0,)$). The manifold underlying this group can be identified with the unit sphere in $\mathbb{R}^4$.

2.3.3 Matrix groups

Several matrix groups have been defined. These matrix groups are also known as the classical groups [21]. Since our group attention must be associative and have inverses, we are led to consider groups whose elements are square matrices. We should restrict our attention to nxn matrices that are invertible to obtain Lie groups.

These groups are usually denoted by $GL(n, \mathbb{R})$ which means general linear group, with n the number of rows and columns in the matrices.

If we multiply two matrices with unit determinant, the result is another matrix with determinant 1. $(\det(AB) = \det(A)\det(B))$. If we take all the nxn matrices with determinant 1, we get more examples of Lie groups. These groups are usually called special linear groups ($SL(n, \mathbb{R})$). That is the element of the group lie on a non-singular algebraic variety in$\mathbb{R}^{n^2}$. This variety is the group manifold and its dimension is $n^2 - 1$. In this dimension formulization, n^2 term comes from the dimension on nxn matrix which has n^2 parameters and minus one comes from the non-singular case. Because in non-singular case there is not solution of the equation $det(.) = 0$. Some matrix groups are as follow:

(Note that: In $GL(n, \mathbb{R})$ notation $\mathbb{R}$ refers to the field of scalar. For instance, for complex field $GL(n, \mathbb{C})$ must be used.)

2.3.3.1 Orthogonal and special orthogonal groups

The orthogonal group $O(n)$ is the group of nxn orthogonal matrices. These matrices form a group because they are closed under multiplication and taking inverses. The effect of group element on the vector is given by

$$\boldsymbol{x}' = M\boldsymbol{x} \tag{2.14}$$

where M is nxn orthogonal matrix. The scalar product of two vectors after transformation is same as before transformation.

$$\boldsymbol{x}'\boldsymbol{y}' = \boldsymbol{x}^{\boldsymbol{T}} M^T M \boldsymbol{y} = \boldsymbol{x}^{\boldsymbol{T}} \boldsymbol{y} \tag{2.15}$$

Hence matrices of the orthogonal group must satisfy

$$M^T M = I_n. \tag{2.16}$$

where I_n is the nxn identity matrix.

The product of two orthogonal matrices is again orthogonal. This property can be proved using equation 2.16

$$(M_1 M_2)^T M_1 M_2 = M_2^T M_1^T M_1 M_2 = M_2^T I_n M_2 = I_n \tag{2.17}$$

The determinant of an orthogonal matrix is either 1 or -1, and so the orthogonal group has two components. The component containing the identity is the special orthogonal group $SO(2)$. For example, the group $O(2)$ has group action on the plane that is a rotation:

$$O(2) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad and \quad \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \qquad \textbf{(2.18)}$$

The first of these correspond to anticlockwise rotation by θ about the origin, while the second are reflection in a line through the origin at an angle $\theta/2$ from the first axis.

The special orthogonal group $SO(n)$ is the subgroup of the elements of orthogonal group $O(n)$ with determinant 1. $SO(2)$ and $SO(3)$ are the most important special orthogonal groups, since they are the rigid body rotations about a fixed center in two and three dimensions. In three dimension rotation matrices can be written as:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \; R_y = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{(2.19)}$$

2.3.3.2 Unitary and special unitary groups:

Unitary matrix is a nxn matrix U satisfying the condition

$$UU^* = U^*U = I_n \qquad \textbf{(2.20)}$$

where I_n is the identity matrix and U^* is the conjugare transpose of U. Note this condition says that a matrix U is unitary if and only if it has an inverse which is equal to its conjugate transpose.

$$U^* = U^{-1} \qquad \textbf{(2.21)}$$

The unitary group $U(n)$ is the set of nxn unitary matrices. The special unitary groups consist of unitary matrices with unit determinant. For example $SU(2)$ consists of matrices of the form:

$$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$$

where the real parameters are a, b, c, d satisfy

$$a^2 + b^2 + c^2 + d^2 = 1 \qquad \textbf{(2.22)}$$

Hence $SU(2)$ can be identified with the points of a three-dimensional sphere in $\mathbb{R}^4$.

2.3.3.3 Euclidian and special Euclidian groups:

Define an element of the Euclidean group $E(n)$ to be a pair (R, t), where $R \in O(n)$ and $u \in \mathbb{R}^n$. Any elemen(R, t)t gives a transformation of n-dimensional Euclidean space built from an orthogonal transformation and a translation. This is the group of transformation of the vector space $\mathbb{R}^n$ that preserves the Euclidean metric. Thus it is also known as isometry group.

General rigid transformation on an arbitrary vector can be written as:

$$\boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{t} \tag{2.23}$$

where $R\ and\ t$ denote rotation and translation transformation respectively.

Two successive transformations on a single vector can be written as:

$$\boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{t} \tag{2.24}$$

and then

$$\boldsymbol{x}'' = R_2\boldsymbol{x}' + \boldsymbol{t_2} = R_2R_1\boldsymbol{x} + R_2\boldsymbol{t_1} + \boldsymbol{t_2} \tag{2.25}$$

Thus the product of two transformations is

$$(R_2, \boldsymbol{t_2})(R_1, \boldsymbol{t_1}) = (R_2R_1, R_2t_1 + t_2) \tag{2.26}$$

The group of rigid body motions in $\mathbb{R}^n$ is thus the semi-direct product (see App. B.3 for definition semi-direct product) of the orthogonal group with $\mathbb{R}^n$ itself. Thus Euclidean groups can be denoted as:

$$E(n) = O(n) \rtimes \mathbb{R}^n \tag{2.27}$$

where $\rtimes$ denotes semi-direct product.

Orthogonal matrices determinant 1 $(SO(n), Special\ Orthogonal\ matrices)$ correspond to rotations about the origin in $\mathbb{R}^n$. Orthogonal matrices determinant -1 correspond to reflections. Physical machines can not effect on rigid bodies. Thus we should use special-orthogonal matrices for rigid body transformation. Using special orthogonal matrices, special Euclidean groups can be written as:

$$SE(n) = SO(n) \rtimes \mathbb{R}^n \tag{2.28}$$

There is a convention $n + 1$ dimensional representation of $SE(n)$, that is, an injective homomorphism $SE(n) \longrightarrow GL(n + 1)$. A general rigid body motion can be represented by using $SE(3)$. That is given by

$$(R, t) \longrightarrow \begin{pmatrix} R_{3x3} & \boldsymbol{t}_{3x1} \\ 0_{1x3} & 1 \end{pmatrix}$$

Multiplication and inverse of these matrices are also special-Euclidean group elements. These are can be represented as:

$$\begin{pmatrix} R_2 & \boldsymbol{t}_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R_1 & \boldsymbol{t}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2R_1 & R_2\boldsymbol{t}_1 + \boldsymbol{t}_2 \\ 0 & 1 \end{pmatrix} \tag{2.29}$$

$$\begin{pmatrix} R & \boldsymbol{t} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^T & -R^T\boldsymbol{t} \\ 0 & 1 \end{pmatrix} \tag{2.30}$$

2.3.4 Subgroups

In rigid body motion, generally Lie subgroups are used instead of Lie groups. For instance, special orthogonal groups are used for rotations. Because, while orthogonal matrices with $\det(O(n)) = 1$ (Special Orthogonal matrices) indicate pure rotation, orthogonal matrices with $\det(O(n)) = -1$ indicate improper rotation. Orthogonal matrices with determinant minus one are perfectly good rigid transformation but no machine can perform such an operation. Thus subgroups are very important for rigid body motions.

A subgroup is a subset of elements of the original group that is closed under the group operation. That is, the product of any two elements of the subgroup is again an element of the subgroup. It is possible for subgroup not to be a Lie subgroup but we will consider only Lie subgroups. Some subgroups examples are given above. For instance any matrix group of $GL(n)$ for the appropriate n, $SO(n)$ (subgroup of $O(n)$), $SU(n)$ (subgroup of $U(n)$) and $SE(n)$ (subgroup of $E(n)$).

Euclidean subgroups are very important to represent rigid body motion. Thus I will restrict attention to Euclidean subgroups. Finding all the subgroups of a Lie group is not generally possible; however we can find all the subgroups of Euclidean groups because of semi-direct product property. Let H be a subgroup of $SE(3)$, and let g be any element of $SE(3)$. If we conjugate every element of H by g, we get an isomorphic subgroup to H. The conjugate subgroup has elements of the form ghg^{-1} where $h \in H$, and is usually written, as gHg^{-1}. The conjugate (see App. B.1 for definition conjugate) subgroup is a subgroup since the product of any pair of elements ghg^{-1} and $gh'g^{-1}$ is given by $gh_1h_2g^{-1}$, in other words conjugation gives a homomorphism (see App. B.2 for definition homomorphism) between subgroups. This simplifies the classification of subgroups because we need only consider conjugacy classes of subgroups. Consequently we can find all subgroups of $SE(3)$ a by using conjugation. For example, the subgroup of rotations about the origin is $SO(3)$. In the Euclidean group $SE(3)$, there are many copies of $SO(3)$, since we could consider the subgroup of rotations about any point in space. The important point is that all these subgroups are conjugate to each other; the conjugation is by the translation which moves one centre to the other.

2.3.5 Lie algebra

The Lie algebra is an indispensable tool in studying matrix Lie groups. On the one hand, Lie algebras are simpler than matrix Lie groups, because (as we will see) the Lie algebra is a linear space. On the other hand, the Lie algebra of a matrix Lie group contains much information about that group. Thus many questions about matrix Lie groups can be answered by considering a similar but easier problem for the Lie algebra.

Lie algebra can be defined many different but equivalent ways. It can be thought of as infinitesimal group elements, that is, group elements very near the identity. Let $A \; and \; B$ be any elements of group. We can write these groups elements using identity elements and Lie algebra elements (X, Y) as:

$$A = I + \varepsilon X$$

$$B = I + \varepsilon Y \quad \textbf{(2.31)}$$

and

$$AB = I + \varepsilon(X + Y) + \varepsilon^2 XY \quad \textbf{(2.32)}$$

where ε is infinitesimal. Thus we can ignore ε^2. Hence we can obtain group product as:

$$AB = I + \varepsilon(X + Y) \quad \textbf{(2.33)}$$

We can see that A and B are almost commutative and multiplication of A and B almost correspond to addition of X and Y. And also it can be seen that X and Y as analogous to logarithms of A and B respectively [22].

Lie algebra can also be defined as the tangent space (tangent vectors) to the identity element. Tangent space definition is more useful to understand the following lie algebra applications. Firstly, Lie algebra elements correspond to generalized velocities. Also it can be used to find position and orientation error. And at last it can be used as a linearization of the original group.

To find a tangent vectors we should take the difference between nearby elements and divide by proceeding to the limit of zero difference in parameters value. To find instantaneous velocity, we can take time as parameter. Let's first consider special orthogonal matrices ($SO(3)$), which satisfy $RR^T = I_3$ and $det(R) = 1$. A path will be given by a matrix valued function $R(t)$. Here t is parameter of the path or time if we want to think velocity. For convenience, we will assume that $R(0) = I_n$, this simply means that we agree to measure time from the instant that the path passes through the identity. Taking the derivative of the relation $RR^T = I_3$ gives,

$$\frac{\partial R^T}{\partial t} R + R^T \frac{\partial R}{\partial t} = 0 \qquad \textbf{(2.34)}$$

When $t = 0$ we get

$$\frac{\partial R^T}{\partial t} + \frac{\partial R}{\partial t} = 0 \qquad \textbf{(2.35)}$$

Hence, the tangent space to the identity consists of anti-symmetric matrices. This gives a simple way to find the dimension of the groups, since the dimension of a manifold is the same as the dimension of a tangent space. Anti-symmetric matrices can be found by looking at rotations about the three coordinate axes. Let's first consider rotation about x axis. Derivative of rotation about x axis is:

$$\frac{dR_x}{dx} = \dot{\theta} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(\theta) & -\cos(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) \end{pmatrix} \qquad \textbf{(2.36)}$$

and when $t = 0$ we get,

$$\Omega_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_x \\ 0 & \omega_x & 0 \end{pmatrix} \qquad \textbf{(2.37)}$$

where $\boldsymbol{\omega}_{\mathrm{x}}$ is the angular velocity vector parameter which is on the $\boldsymbol{x}$ axis and Ω_x is $3x3$ anti-symmetric matrix. It can be represented by using same name as the Lie group but in lower case, so Lie algebra of $SO(3)$ is $so(3)$.

For the other axes we have,

$$\Omega_y = \begin{pmatrix} 0 & 0 & \omega_{\mathrm{y}} \\ 0 & 0 & 0 \\ -\omega_{\mathrm{y}} & 0 & 0 \end{pmatrix}, \; \Omega_z = \begin{pmatrix} 0 & -\omega_{\mathrm{z}} & 0 \\ \omega_{\mathrm{z}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.38}$$

A general Lie algebra element for the rotation group therefore has the form,

$$\Omega = \begin{pmatrix} 0 & -\omega_{\mathrm{z}} & \omega_{\mathrm{y}} \\ \omega_{\mathrm{z}} & 0 & -\omega_{\mathrm{x}} \\ -\omega_{\mathrm{y}} & \omega_{\mathrm{x}} & 0 \end{pmatrix} \tag{2.39}$$

where $\boldsymbol{\omega} = (\omega_{\mathrm{x}}, \omega_{\mathrm{y}}, \omega_{\mathrm{z}})$ is angular velocity vector.

We can easily find Lie algebra of the group of rigid body transformation $\mathrm{SE}(3)$ using $\mathrm{so}(3)$. A general $\mathrm{SE}(3)$ can be represented as,

$$(R, t) \longrightarrow \begin{pmatrix} R_{3x3} & \boldsymbol{t}_{3x1} \\ 0_{1x3} & 1 \end{pmatrix}$$

where R is special orthogonal matrix. Taking the derivative and setting $\theta = 0$ gives a typical element of lie algebra ($se(3)$),

$$S = \begin{pmatrix} 0 & -\omega_{\mathrm{z}} & \omega_{\mathrm{y}} & \mathrm{v}_{\mathrm{x}} \\ \omega_{\mathrm{z}} & 0 & -\omega_{\mathrm{x}} & \mathrm{v}_{\mathrm{y}} \\ -\omega_{\mathrm{y}} & \omega_{\mathrm{x}} & 0 & \mathrm{v}_{\mathrm{z}} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \Omega_{3x3} & \boldsymbol{v}_{3x1} \\ 0_{1x3} & 0 \end{pmatrix} \tag{2.40}$$

where $\boldsymbol{v} = (\mathrm{v}_{\mathrm{x}}, \mathrm{v}_{\mathrm{y}}, \mathrm{v}_{\mathrm{z}})$ is a characteristic linear velocity of the motion. If the motion is pure translation then $\Omega = 0$ and $\boldsymbol{v}$ is the velocity of any point in space.

These matrices form a six dimensional vector space, elements of this space are more commonly written as columns vectors,

$$\boldsymbol{s} = \begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix} \tag{2.41}$$

2.4 Exponential Mapping

In the theory of Lie groups the exponential map is a map from the Lie algebra of a Lie group to the group which allows one to recapture the local group structure from the Lie algebra. If we extend exponential matrix of a Lie algebra element into a power series we can obtain first definition of Lie algebra. Let's X be a matrix representing an element of Lie algebra. The power series of the exponential matrix X is,

$$e^X = I + X + \frac{1}{2!}X^2 + \cdots + \frac{1}{n!}X^n + \cdots \quad \textbf{(2.42)}$$

where e^X is a matrix representing an element of the corresponding Lie group.

Lie algebra elements is also left-invariant vector fields (see App. B.4 for definition left invariant vector field) on the group. So far, we have only mentioned vectors at the identity; for a vector field, we need a tangent at every group element. Let X be a matrix representing a tangent vector at the identity. The tangent vector at the point g of the group can be given by gX. Hence, there is a one-to-one correspondence between, tangent vectors at the identity and left-invariant vector fields.

Integral curves of these left-invariant vector fields play an important role in exponential mapping. Integral curve can be defined as,

A smooth curve $\gamma(t), t \in (-\varepsilon, \varepsilon)$ is an integral curve of vector field X, if $\frac{d\gamma(t)}{dt} = X(\gamma(t))$. For a left-invariant vector field such a curve would satisfy the following differential equation:

$$\frac{d\gamma}{dt} = \gamma X \quad \textbf{(2.43)}$$

This equation has analytic solution. The solution which passes through the identity element is

$$\gamma(t) = e^{Xt} \quad \textbf{(2.44)}$$

For exponential matrices we have the following relation:

$$e^X e^Y = e^{X+Y} \quad if\ and\ only\ if \quad XY = YX \quad \textbf{(2.45)}$$

Certainly the elements $t_1 X\ and\ t_2 X$ commute. This means that the elements of the group of the form e^{Xt} form a subgroup

$$e^{t_1 X} e^{t_2 X} = e^{(t_1+t_2)X} \qquad and \quad e^{tX} e^{-tX} = I \quad \textbf{(2.46)}$$

These are the one-dimensional or one parameter subgroup of the group. Each Lie algebra element generates a one-parameter subgroup in this way.

However, owing to the noncommutativity of the Lie group, it is not quite true that $exp(X + Y)$ equals $exp(X)\ exp(Y)$; instead, the correct identity is the *Baker–Campbell–Hausdorff Formula*

$$exp(X)\,exp(Y) = exp(X + Y) + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) \quad \textbf{(2.47)}$$

where the missing terms consist of a moderately complicated infinite series involving the Lie bracket see App. B.5 for definition lie bracket). The exponential map that connects Lie algebras and Lie groups is closely related to the Lie bracket, and because of this it is possible to study and classify Lie groups by first studying and classifying Lie algebras with their Lie bracket operation.

For example, if the Lie group is $SO(2)$, then we can identify it with the unit circle in $\mathbb{C}$. The tangent to this circle at 1 is a vertical line, so we can identify the Lie algebra with the set $i\mathbb{R}$ of purely imaginary numbers. The rotation through an angle θ can then be written as $exp(i\theta)$. Note that this representation is not unique, since $exp(i\theta) = exp(i(\theta + 2\pi))$.

3. RIGID BODY MOTION USING SCREW THEORY

3.1 Objectives

Any way of moving all the points in the plane such that the relative distance between points stays the same and the relative position of the points stays same is called rigid motion. Generally there are two main transformation approaches to define rigid body motion. The first one is point transformation approach. D-H (Denavit & Hartenberg) convention which is the most common method in robot kinematic is based on point transformation approach [1] and [23]. It is generally used 4 x 4 homogeneous transformation matrix which is introduced by Maxwell [3]. Maxwell used homogeneous coordinate systems to represent points and homogeneous transformation matrices to represent the transformation of points. For more detail about this method references [1] and [23] can be viewed. The second one is line transformation approach. Lines are very important in robotics because:

- They model joint axes: a revolute joint makes any connected rigid body rotate about the line of its axis; a prismatic joint makes the connected rigid body translate along its axis line.
- They model edges of the polyhedral objects used in many task planners or sensor processing modules.
- They are needed for shortest distance calculation between robots and obstacles.

In this chapter I will provide a description of rigid body motion using the tools linear algebra and screw theory which is based on line transformation. The elements of screw theory can be traced to the work of Chasles and Poinsot in the early 1800s. Using the theorems of Chasles and Poinsot as a starting point, Robert S. Ball developed a complete theory of screws which he published in 1900 [4]. In screw theory every transformation of a rigid body or a coordinate system with respect to a reference coordinate system can be expressed by a screw displacement, which is a translation by along a λ axis with a rotation by a θ angle about the same axis [4]. This description of transformation is the basis of the screw theory. There are two main advantages of using screw, theory for describing rigid body kinematics. The first one is that it allows a global description of rigid body motion that does not suffer from singularities due to the use of local coordinates. The second one is that the screw theory provides a geometric description of rigid motion which greatly simplifies the analysis of mechanisms [5].

In this chapter I will introduce rigid body motion by using screw theory. Firstly, I will give rigid body transformation properties and its connection with Lie groups. Then exponential coordinates will be given for rotation and rigid motion transformations. I will end this chapter with screw motion.

3.2 Rigid Body Motion

All rigid body motion can be defined using translation and rotation transformations [24], [25]. Lets O be an object which is described as a subset of $\mathbb{R}^3$. Using Euclidean space properties we can define any object in Cartesian coordinates. A rigid motion of an object can be represented by a continuous family of mappings $g(t): O \longrightarrow \mathbb{R}^3$. This mapping describes how individual points in the body move as a function of time, relative to some fixed Cartesian coordinate frame. And also we can use vector definition instead of point. Given two points $\boldsymbol{p}, \boldsymbol{q} \in O$, the vector $\boldsymbol{v} \in \mathbb{R}^3$ connecting $\boldsymbol{p}\ to\ \boldsymbol{q}$ is defined to be the directed line segment going from $\boldsymbol{p}\ to\ \boldsymbol{q}$. $(g_*(v) = g(\boldsymbol{p}) - g(\boldsymbol{q}))$. Although both point and vector are defined using similar three components in Cartesian coordinates, they are conceptually quite different.

A mapping $g: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a rigid body transformation if it satisfies the following properties

1. Length preserved: $||g(\boldsymbol{p}) - g(\boldsymbol{q})|| = ||\boldsymbol{p} - \boldsymbol{q}||$ for all points $p, q \in \mathbb{R}^3$
2. The cross product is preserved: $g_*(\boldsymbol{v} \times \boldsymbol{w}) = g_*(\boldsymbol{v}) \times g_*(\boldsymbol{w})$

The first one gives us distance between points on a rigid body are not altered by rigid motions. However this condition is not sufficient since it allows internal reflections, which are not physically realizable. Thus a rigid body transformation must also satisfy second property to preserve orientation. The distance between points and cross product between vectors is fixed. This does not mean that particles in a rigid body can't move relative to each other. However that particles in a rigid body can't translate, they can rotate with respect each other. Thus, to keep track of the motion, of a rigid body, we need to keep track of the motion of any one particle of the rigid, body and the rotation of the body about this point. In order to do this, we represent the configuration of a rigid body by attaching a Cartesian coordinate frame to some point on the rigid body and keeping track of the motion of this body coordinate frame relative to a fixed frame. The motion of the individual particles in the body can then be retrieved from the motion of the body frame and the motion of the point of attachment of the frame to the body [10].

Let's first consider pure rotation. It can be seen in figure 3.1.

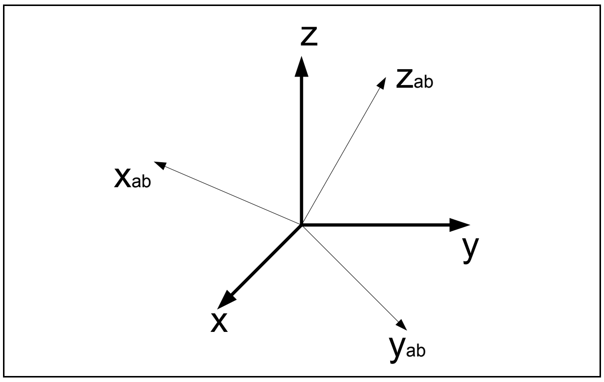

Figure 3.1: Coordinate frames for specifying pure rotational motion

As we mentioned chapter 1, $SO(n)$ matrices are rotational transformation matrices. Any rotation matrices can be represented as:

$$R_{ab} = [\boldsymbol{x_{ab}}, \boldsymbol{y_{ab}}, \boldsymbol{z_{ab}}]_{nxn} \tag{3.1}$$

where R_{ab} denotes rotation frame B relative to frame A and $\boldsymbol{x_{ab}}, \boldsymbol{y_{ab}}, \boldsymbol{z_{ab}} \in \mathbb{R}^3$ denotes the coordinates of the principal axes of B relative to A. When $n = 2$, R denotes planar rotation and $n = 3$, R denotes rigid body rotation. Thus we will generally use $3x3$ rotation matrices ($SO(3)$). Let $\boldsymbol{q_b} = [x_b, y_b, z_b]$ be the coordinates of q relative to frame B. The coordinates of q relative to frame A can be computed given by

$$q_a = [x_{ab}, y_{ab}, z_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = R_{ab}\boldsymbol{q_b} \tag{3.2}$$

In other words R_{ab}, when considered as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$, rotates the coordinates of a point from frame B to frame A.

In general rigid motions consist of rotation and translation. Representation of pure rotation is given above and also representation of pure translation is very simple. To represent any translational motion we should just determine a point on the body and keep track of the coordinates of the point in the body relative to some known frame. However representation of general rigid motion, involving both rotation and translation is more involved. Let's consider a rigid motion which is shown in figure 3.2.

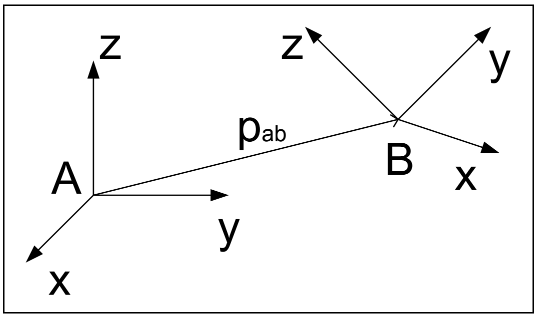

Figure 3.2: Coordinate frames for specifying rigid motion (rotation and translation)

Here A and B are two frames, $\boldsymbol{p_{ab}} \epsilon\ \mathbb{R}^3$ be the position vector of the origin of frame B from the origin of frame A and $R_{ab} \epsilon\ SO(3)$ the orientation of frame B relative to frame A. A configuration of the space of the system consists of the pair $(\boldsymbol{p_{ab}}, R_{ab})$ and the configuration space of the system is the product space of $\mathbb{R}^3$ with $SO(3)$, which shall be denoted as $SE(3)$ [2] (for special Euclidean group):

$$SO(3) = \{(\boldsymbol{p}, R) : \boldsymbol{p} \epsilon\ \mathbb{R}^3, R\ \epsilon\ SO(3)\} = \mathbb{R}^3 \times SO(3) \qquad \textbf{(3.3)}$$

Let $\boldsymbol{q_b} = [x_b, y_b, z_b]$ be the coordinates of q relative to frame B. The coordinates of q relative to frame A can be computed given by

$$\boldsymbol{q_a} = \boldsymbol{p_{ab}} + R_{ab}\boldsymbol{q_b} \qquad \textbf{(3.4)}$$

Using the equation 3.4, we may represent it in linear form by writing it as

$$\begin{bmatrix} \boldsymbol{q_a} \\ 1 \end{bmatrix}_{4x1} = \begin{bmatrix} R_{ab} & \boldsymbol{p_{ab}} \\ 0 & 1 \end{bmatrix}_{4x4} \begin{bmatrix} \boldsymbol{q_b} \\ 1 \end{bmatrix} = T_{ab}\boldsymbol{q_b} \qquad \textbf{(3.5)}$$

The $4x4$ matrix is called homogeneous representation (see App. A.1 for definition homogeneous transformation) of $T_{ab}\ \epsilon\ SE(3)$. $SE(3)$ matrix properties can be seen in chapter 2.

3.3 Exponential Coordinates For Rigid Motion

Firstly I will just consider pure rotational motion to analyze exponential mapping by using exponential coordinates. Then both rotational and translational motion will be analyzed by using same approach as pure rotation. We can define any rotation by using unit vector which specifies the direction of rotation. Let's analyze this motion using a robot which has two rotational joint as shown in figure 3.3. Here, $\boldsymbol{p}$ is any point that is attached to the last link, $\boldsymbol{w_z}, \boldsymbol{w_x} \in \mathbb{R}^3$ are a unit vector which specify the direction of rotation joint1 (J_1) and joint2 (J_2) respectively and $\theta_z, \theta_x \in \mathbb{R}$ is the angle of rotation about $\boldsymbol{w_z}$ and $\boldsymbol{w_x}$ axes respectively.

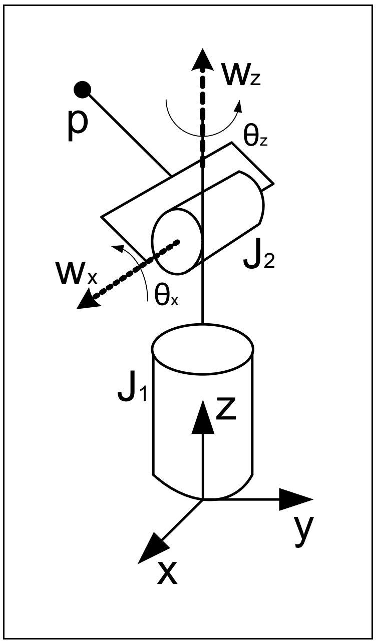

Figure 3.3: Tip point trajectory generated by rotation about the w_z axis

If we rotate the body at constant unit velocity about the axis $\boldsymbol{w_z}$, the velocity of the point can be written as

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{w_z} \times \boldsymbol{p}(t) = \widehat{w}_z \boldsymbol{p}(t) \qquad \textbf{(3.6)}$$

Recall that the vector cross product ($\times$) can be represented as the product of a special skew-symmetric matrix,

$$\widehat{w} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \qquad \textbf{(3.7)}$$

with the vector, i.e.

$$w \times p = \widehat{w}_z p \qquad \textbf{(3.8)}$$

So we have

$$\dot{\boldsymbol{p}}(t) = \hat{w}_z \boldsymbol{p}(t) \tag{3.9}$$

This is a first order differential equation with the solution

$$\boldsymbol{p}(t) = e^{\hat{w}_z t} \boldsymbol{p}(0) \tag{3.10}$$

where $\boldsymbol{p}(0)$ is the initial position of the point $(t = 0)$, $\boldsymbol{p}(t)$ is current position and $e^{\hat{w}_z t}$ is the matrix exponential

$$e^{\hat{w}_z t} = I + \hat{w}_z t + \frac{1}{2!}(\hat{w}_z t)^2 + \frac{1}{3!}(\hat{w}_z t)^3 + \cdots \tag{3.11}$$

If we rotate about the axis w at unit velocity for θ for units of time, then the net rotation is given by

$$R(\boldsymbol{w}, \theta) = e^{\hat{w}\theta} \qquad \text{where} \qquad \theta = |\boldsymbol{w}| t \tag{3.12}$$

Given a skew-symmetric matrix $\hat{w} \in so(3)$ and $\theta \in \mathbb{R}$, $e^{\hat{w}\theta} \in SO(3)$

$so(3)$ matrices hold these relations

- $\hat{a}^2 = \boldsymbol{a}\boldsymbol{a}^{\boldsymbol{T}} - ||\boldsymbol{a}||^2 I$ **(3.13)**
- $\hat{a}^3 = -||\boldsymbol{a}||^2 \hat{a}$ **(3.14)**

Using these relations with $a = w\theta$ exponential of any skew-symmetric matrices $\hat{w}\theta$ can be represented as

$$\begin{aligned} e^{\hat{w}\theta} &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\hat{w} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} \ldots\right)\hat{w}^2 \\ &= I + \hat{w} sin\theta + \hat{w}^2(1 - cos\theta) \end{aligned} \tag{3.15}$$

This formula commonly referred to as Rodrigues formula gives an efficient method for computing exp $(\hat{w}\theta)$.

So exp $(\hat{w}\theta)$ is the rotation matrix which expresses rotation by θ about axis w. We can also find θ and w for a given any exp $(\hat{w}\theta)$ rotation matrices given by,

$$\exp(\hat{w}\theta) = R(\boldsymbol{w}, \theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{3.16}$$

$$\theta = cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \tag{3.17}$$

$$\boldsymbol{w} = \frac{1}{2sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \qquad \textbf{(3.18)}$$

The components of the vector $\boldsymbol{w}\theta \in \mathbb{R}^3$ given by equation 3.17 and 3.18 are called exponential coordinates.

Now we can find exponential mapping of general rigid motion (rotation and translation) by generalizing pure rotation. Let's again analyze this motion using a robot which has four rotational joint as shown in figure 3.4. Here, p is any point that is attached to the last link, $\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}, \boldsymbol{w_4} \in \mathbb{R}^3$ are a unit vector which specify the direction of rotation joint1 (J_1), joint2 (J_2), joint3 (J_3) and joint4 (J_4) respectively, $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R}$ is the angle of rotation about $\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}, \boldsymbol{w_4}$ axes respectively and $\boldsymbol{q}$ is any point on the $\boldsymbol{w_4}$ axis. Assume that the robot has translational motion along the $\boldsymbol{w_4}$ axis with rotational motion about the same axis as shown in figure 3.4.

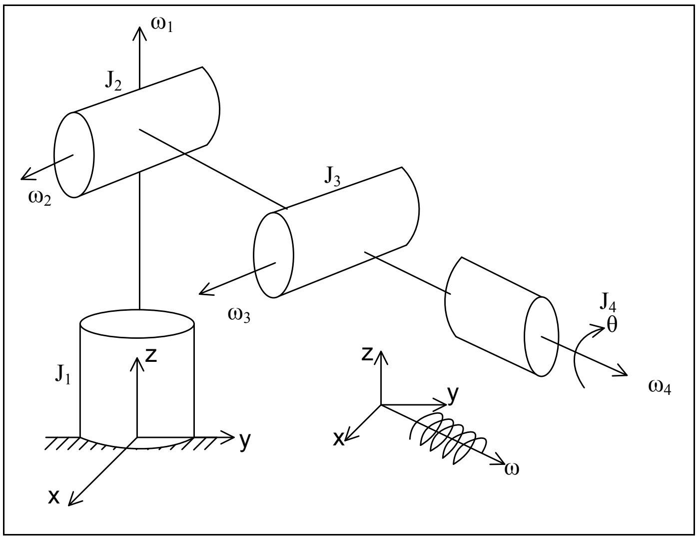

Figure 3.4: General rigid motion (Translation along the w_4 axis plus rotation about the the w_4 axis)

The velocity of the tip point $p(t)$ is then

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{w_4} \times (\boldsymbol{p}(t) - \boldsymbol{q}) \quad or \quad \dot{\boldsymbol{p}}(t) = \boldsymbol{w_4} \times \boldsymbol{p}(t) - \boldsymbol{w_4} \times \boldsymbol{q} \qquad \textbf{(3.19)}$$

Using $\widehat{w}$ as defined above, if we define

$$\hat{\xi} = \begin{bmatrix} \widehat{w} & -\boldsymbol{w} \times \boldsymbol{q} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \widehat{w} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \qquad \textbf{(3.20)}$$

and we express $p(t)$ and its derivative in homogeneous coordinates,

$$\dot{\boldsymbol{p}}(t) = \begin{bmatrix} \widehat{w} & -\boldsymbol{w} \times \boldsymbol{q} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix} = \hat{\xi}\begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix} \Rightarrow \dot{\boldsymbol{p}} = \hat{\xi}\boldsymbol{p} \qquad \textbf{(3.21)}$$

This is a first order differential equation which has the solution:

$$\boldsymbol{p}(t) = e^{\hat{\xi}t}\boldsymbol{p}(0) \tag{3.22}$$

where $e^{\hat{\xi}t}$ is the matrix exponential of $4x4$ matrix $\hat{\xi}t$ defined by

$$e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{1}{2!}(\hat{\xi}t)^2 + \frac{1}{3!}(\hat{\xi}t)^3 + \cdots \tag{3.23}$$

If we rotate about the axis w at unit velocity for θ for units of time, then the net rotation is given by

$$T(\boldsymbol{w}, \theta) = e^{\hat{\xi}\theta} \qquad \text{where} \qquad \theta = |\xi|t \tag{3.24}$$

Here $\hat{\xi}$ is $4x4$ matrix which has $\hat{w} \epsilon so(3)$ and $\boldsymbol{v} \epsilon \mathbb{R}^3$ is referred to as a twist, or a (infinitesimal) generator of the Euclidean group, $\xi \epsilon \mathbb{R}^6$: $(\boldsymbol{v}, \boldsymbol{w})$ is twist coordinates for the twist $\hat{\xi} \epsilon se(3)$. Twists also represent velocity of a body. It contains 6 quantities $(\boldsymbol{v}, \boldsymbol{w})$. Three of them are for linear velocity $(\boldsymbol{v})$ and the other three of them are for angular velocity$(\boldsymbol{w})$.

Given a skew-symmetric matrix $\hat{\xi} \epsilon se(3)$ and $\theta \epsilon \mathbb{R}$,$e^{\hat{\xi}\theta} \epsilon SE(3)$

We can proof this by explicit calculation. In general form, $\hat{\xi} \epsilon se(3)$ can be written as given in equation 3.20. Let's first consider pure translation. In this case $\boldsymbol{w} = \boldsymbol{0}$. If $\boldsymbol{w} = \boldsymbol{0}$ then $\hat{\xi}^2 = \hat{\xi}^3 = \hat{\xi}^4 = \cdots = 0$. Hence

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta = \begin{bmatrix} I & \boldsymbol{v}\theta \\ 0 & 1 \end{bmatrix} \tag{3.25}$$

Secondly assume that $||w|| = 1$. Define a rigid transformation g by

$$g = \begin{bmatrix} I & \boldsymbol{w} \times \boldsymbol{v} \\ 0 & 1 \end{bmatrix} \tag{3.26}$$

Using conjugation we can write another skew symmetric matrices as

$$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -\boldsymbol{w} \times \boldsymbol{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{w} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \boldsymbol{w} \times \boldsymbol{v} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{w} & \boldsymbol{w}\boldsymbol{w}^T\boldsymbol{v} \\ 0 & 0 \end{bmatrix} \tag{3.27}$$

Using $\hat{w}\boldsymbol{w} = \boldsymbol{w} \times \boldsymbol{w} = 0$ that

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\xi}'^3 = \begin{bmatrix} \hat{w}^3 & 0 \\ 0 & 0 \end{bmatrix}, \ldots \tag{3.28}$$

Hence

$$e^{\hat{\xi}'\theta} = \begin{bmatrix} e^{\hat{w}\theta} & h\boldsymbol{w}\theta \\ 0 & 1 \end{bmatrix} \tag{3.29}$$

Using the inverse conjugation, we can find $e^{\hat{\xi}\theta}$ as follow

$$e^{\hat{\xi}\theta} = e^{g(\hat{\xi}'\theta)g^{-1}} = g e^{\hat{\xi}'\theta} g^{-1} = \begin{bmatrix} e^{\hat{w}\theta} & \left(I - e^{\hat{w}\theta}\right)(\boldsymbol{w} \times \boldsymbol{v}) + (\boldsymbol{w}\boldsymbol{w}^{T}\boldsymbol{v}\theta) \\ 0 & 1 \end{bmatrix} \quad \textbf{(3.30)}$$

This transformation can be interpreted as mapping points from their initial coordinates, $\boldsymbol{p}(0) \in \mathbb{R}^3$ to their coordinates after the rigid motion is applied $\boldsymbol{p}(\theta) = e^{\hat{\xi}t}\boldsymbol{p}(0)$. In this equation, both $\boldsymbol{p}(0)$ and $\boldsymbol{p}(\theta)$ are specified with respect to a single reference frame. Thus, the exponential map for a twist gives the relative motion of a rigid body.

3.4 Screw Motion

Screw motion is a specific class of a rigid body motion which is naturally associated with twist. A general screw motion can be defined as a rotation about an axis $\boldsymbol{l}$ with the direction ($\boldsymbol{w}$) in space through an angle of θ radians, followed by translation along the same axis by an amount d as shown in 3.5.

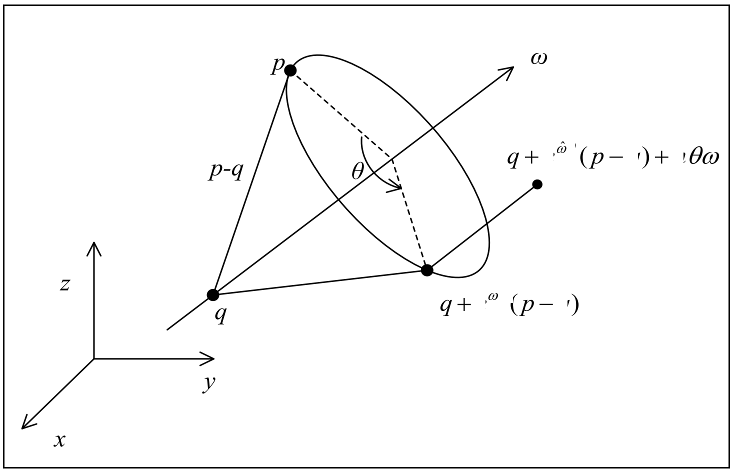

Figure 3.5: General screw motion

where $\boldsymbol{w} \in \mathbb{R}^3$ is the direction vector of line ($\boldsymbol{l}$) and $\boldsymbol{q} \in \mathbb{R}^3$ is any point on the line. Translation can be defined in terms of θ by given, $d := h\theta$. Here $h = {}^{d}/_{\theta}$ that is the ratio of translation to rotation is called pitch. Using figure 3.5 we can define the motion of a point p associated with a screw given by

$$g\boldsymbol{p} = \boldsymbol{q} + e^{\hat{w}\theta}(\boldsymbol{p} - \boldsymbol{q}) + h\boldsymbol{w}\theta \quad \textbf{(3.31)}$$

or in homogenous coordinates

$$g\begin{bmatrix}\boldsymbol{p}\\1\end{bmatrix} = \begin{bmatrix}e^{\widehat{w}\theta} & \left(I - e^{\widehat{w}\theta}\right)\boldsymbol{q} + h\boldsymbol{w}\theta\\0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{p}\\1\end{bmatrix} \quad \textbf{(3.32)}$$

Since the relationship must hold for all $p \in \mathbb{R}^3$, the rigid motion given by the screw is

$$g = \begin{bmatrix}e^{\widehat{w}\theta} & \left(I - e^{\widehat{w}\theta}\right)\boldsymbol{q} + h\boldsymbol{w}\theta\\0 & 1\end{bmatrix} \quad \textbf{(3.33)}$$

Note that the rigid motion given by the screw is same as the exponential of a twist that is given in equation 3.30. Here $\boldsymbol{q} = \boldsymbol{w} \times \boldsymbol{v}$ and $h = \frac{\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{v}}{||w||^2}$.

We can define a twist which realizes the screw motion and has the proper geometric attributes. We can prove that by splitting the proof into usual cases: pure translation and translation plus rotation.

Case 1: $h = (\infty)$. Let $\boldsymbol{l} = \{\boldsymbol{q} + \mu\boldsymbol{v}: \left||\boldsymbol{v}|\right| = 1, \mu \in \mathbb{R}\}$ and define

$$\hat{\xi} = \begin{bmatrix}0 & \boldsymbol{v}\\0 & 0\end{bmatrix} \quad \textbf{(3.34)}$$

The rigid body motion $e^{\hat{\xi}\theta}$ corresponds to pure translation along the screw axis by an amount d.

Case 2: $h = (finite)$. Let $\boldsymbol{l} = \{\boldsymbol{q} + \mu\boldsymbol{w}: \left||\boldsymbol{w}|\right| = 1, \mu \in \mathbb{R}\}$ and define

$$\hat{\xi} = \begin{bmatrix}\widehat{w} & -\boldsymbol{w} \times \boldsymbol{q} + h\boldsymbol{w}\\0 & 0\end{bmatrix} \quad \textbf{(3.35)}$$

The fact that the rigid body motion $e^{\hat{\xi}\theta}$ is the appropriate screw motion is verified by direct calculation.

These special cases of screw motion are very important for robotic. For instance we can define a zero pitch is a screw motion for which $h = 0$, corresponding to a pure rotation about an axis. We can use this definition for revolute joints. And also an infinite pitch is a motion for which $h = \infty$, as previously mentioned. This case corresponds to a pure translation and is the model for the action of a prismatic joint.

Finally the geometric meaning of a screw can be given by

Chasles Theorem: Every rigid motion can be realized by a rotation about an axis combined with a translational parallel to that axis. [10]

4. KINEMATIC SOLUTION USING SCREW THEORY AND EXPONENTIAL MAPPING

4.1 Objectives

Kinematic is a branch of classical mechanics which describes the motion of objects without consideration of the causes leading to the motion. The kinematics problem has a wide research area in robotics.

D-H notation is the most common method in robot kinematic however it has some disadvantages like singularity and analyzes complexity. Another main method is screw theory. Screw theory is a singularity avoiding method and it provides a geometric description of rigid motion which greatly simplifies the analysis of mechanisms.

In this chapter I will give a description of the kinematics for a general n degree of freedom open-chain robot manipulator using screw theory and exponential mapping. Firstly M.Murray studied this method and proposed to solve the inverse kinematic problem [10], then J. Xie and W.Qiang applied this method to 6-DOF Space manipulator [11]. This method has an advantage that avoids a large amount of matrices inverse multiply operation, establish just two coordinates and the expression is simple that it is convenient for the trajectory planning and simulation.

I will end this chapter by analyzing forward and inverse kinematic problem of 6-DOF serial arm manipulators which is shown in figure 4.1.

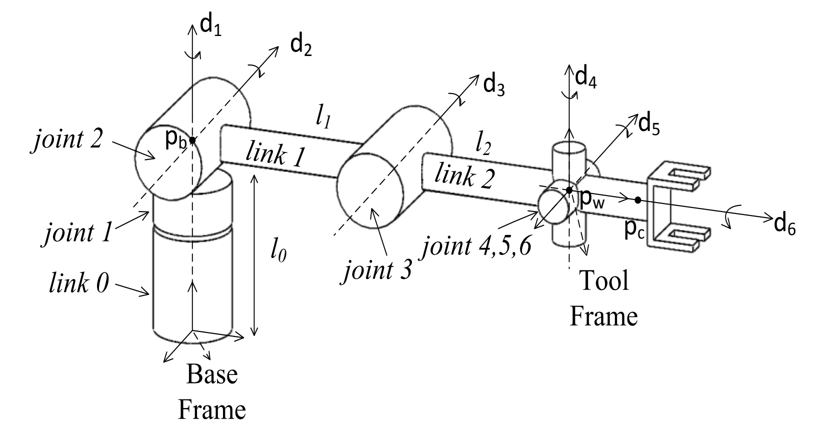

Figure 4.1: 6-dof serial arm robot manipulator in its reference configuration

4.2 Forward Kinematic

The forward kinematic problem is to determine the position and orientation of the end effector given the values for the joint variables of the robot. To find forward kinematic of serial robot manipulator we followed these steps:

4.2.1 Notation

1. Label the joints and the links:

Joints are numbered from number 1 to n, starting at the base, and the links are numbered from number 0 to n. The joints connect link i-1to link i.

2. Configuration of joint spaces:

For revolute joint we describe rotational motion about an axis and we measure all joint angles by using a right-handed coordinate system. For prismatic joint we describe a linear displacement along the direction of the axis.

3. Attaching coordinate frames (Base and Tool Frames):

Two coordinate frames are needed for n degree of freedom open-chain robot manipulator. The base frame can be attached arbitrary but in general it is attached directly to link 0 and the tool frame is attached to the end effector of robot manipulator.

This notation is given for 6-DOF serial robot manipulator in figure 4.1.

4.2.2 Formulization

1. Determining joint axis vector:

First we attach an axis vector which describes the motion of the joint and determine the exponential coordinates. For the i.th revolute joint, the twist has the form

$$\xi_i = \begin{bmatrix} -\boldsymbol{w}_i \times \boldsymbol{q}_i \\ \boldsymbol{w}_i \end{bmatrix} \quad \textbf{(4.1)}$$

where $\boldsymbol{w}_i \in \mathbb{R}^3$ is a unit vector in the direction of the twist axis and $\boldsymbol{q}_i \in \mathbb{R}^3$ is any point on the axis. For the i.th prismatic joints,

$$\xi_i = \begin{bmatrix} \boldsymbol{v}_i \\ 0 \end{bmatrix} \quad \textbf{(4.2)}$$

where $\boldsymbol{v}_i \in \mathbb{R}^3$ is a unit vector pointing in the direction of translation. All vectors and points are specified relative to the base coordinate frame.

2. Obtaining transformation operator:

To obtain transformation operator, firstly exponential coordinates transform to $se(3)$ matrices given by

$$\xi_i = \begin{bmatrix} \boldsymbol{v}_i \\ \boldsymbol{w}_i \end{bmatrix} \quad \Rightarrow \quad \hat{\xi}_i = \begin{bmatrix} \widehat{w}_i & \boldsymbol{v}_i \\ 0 & 0 \end{bmatrix} \quad \textbf{(4.3)}$$

where $\boldsymbol{w}_i \in \mathbb{R}^3$ is a unit vector in the direction of the twist axis, $\boldsymbol{v}_i = -\boldsymbol{w}_i \times \boldsymbol{q}_i: \in \mathbb{R}^3$ is the moment vector (or translational part of twist) of the twist axis and $\widehat{w}_i \in so(3)$ is skew-symmetric matrix corresponding to w_i direction vector.

Then the transformation matrix $T_i \in SE(3)$ of the i.th frame can be obtained using exponential mapping given by

$$T_i = \exp(\hat{\xi}_i) \quad \textbf{(4.4)}$$

3. Formulization of rigid motion:

Combining the individual joint motions, the forward kinematics map, $g_{st}: Q \longrightarrow SE(3)$ is given by

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \ldots e^{\hat{\xi}_n \theta_n} g_{st}(0) \quad \textbf{(4.5)}$$

where $\boldsymbol{Q} = (q_1, q_2, \dots, q_n)$ is the variable vector of joints. For revolute joints $q = \theta$ and for prismatic joints $q = d$. The $\hat{\xi}_i$ must be numbered sequentially starting from the base, but $g_{st}(\theta)$ gives the configuration of the tool frame independently of the order in which the rotations and translations are actually performed. Equation 4.5 is called the product of exponentials formula for the manipulator forward kinematics.

4.3 Inverse Kinematic

The inverse kinematic problem is to determine the values of the joint variables given the end effector's position and orientation. I will use Paden - Kahan subproblems to obtain inverse kinematic solution of serial robot manipulator. There are some Paden-Kahan subproblems and also new extended subproblems [10], [11] and [26]. I will just give three of them which occur frequently in inverse solutions for common manipulator design. To solve the inverse kinematics problem, we reduce the full inverse kinematics problem into appropriate sub-problems. Here are some subproblems.

1. Rotation about a single axis.

2. Rotation about two subsequent axes.

3. Rotation to a given distance

These subproblem solutions are given at App C.5

4.4 6-DOF Serial Robot Manipulator Kinematic Model

In this section we will give an application. 6-DOF serial robot manipulators forward and inverse kinematic problem will be solved. The manipulator is shown in figure 4.1.

4.4.1 Forward kinematic of 6-dof serial robot manipulator

First we must determine the axes for all joints. Then we will find the moment vector for all axes. The axes and the moment vectors can be written as:

$$\boldsymbol{d}_1 = [0\ 0\ 1] \qquad \boldsymbol{d}_2 = [0\ 1\ 0] \qquad \boldsymbol{d}_3 = [0\ 1\ 0]$$

$$\boldsymbol{d}_4 = [0\ 0\ 1] \qquad \boldsymbol{d}_5 = [0\ 1\ 0] \qquad \boldsymbol{d}_6 = [1\ 0\ 0] \qquad \textbf{(4.6)}$$

Any point on these axes can be written as:

$$\boldsymbol{q}_1 = [0\ \ 0\ \ l_0] \qquad \boldsymbol{q}_2 = [0\ \ 0\ \ l_0] \qquad \boldsymbol{q}_3 = [0\ \ l_1\ \ l_0]$$

$$\boldsymbol{q}_4 = [0\ \ l_1 + l_2\ \ l_0] \qquad \boldsymbol{q}_5 = [0\ \ l_1 + l_2\ \ l_0] \qquad \boldsymbol{q}_6 = [0\ \ l_1 + l_2\ \ l_0] \qquad \textbf{(4.7)}$$

Hence the moment vectors of these axes are obtained as:

$$\boldsymbol{v}_1 = -\boldsymbol{d}_1 \times \boldsymbol{q}_1 \qquad \boldsymbol{v}_2 = -\boldsymbol{d}_2 \times \boldsymbol{q}_2 \qquad \boldsymbol{v}_3 = -\boldsymbol{d}_3 \times \boldsymbol{q}_3$$

$$\boldsymbol{v}_4 = -\boldsymbol{d}_4 \times \boldsymbol{q}_4 \qquad \boldsymbol{v}_5 = -\boldsymbol{d}_5 \times \boldsymbol{q}_5 \qquad \boldsymbol{v}_6 = -\boldsymbol{d}_6 \times \boldsymbol{q}_6 \qquad \textbf{(4.8)}$$

We can find twist coordinates and $se(3)$ matrices by using direction and moment vectors. Now we can find forward kinematic by using equation 4.5 given by

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) \qquad \textbf{(4.9)}$$

where $g_{st}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 + l_2 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(4.10)**

is initial position.

4.4.2 Inverse kinematic of 6-dof serial robot manipulator

In the inverse kinematic problem of the serial manipulator, we have rotation and the position of the end effector knowledge as:

$$g_{in} = \begin{bmatrix} R_{in} & \boldsymbol{v_{in}} \\ 0 & 1 \end{bmatrix} \qquad \textbf{(4.11)}$$

where $R_{in} \in SO(3)$ matrix denotes the orientation of robot and $v_{in} \in \mathbb{R}^3$ denotes the position of robot. The equation we wish to solve is

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_{in} \qquad \textbf{(4.12)}$$

Post-multiplying this equation by $g_{st}^{-1}(0)$ isolates the exponential maps:

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = g_{in} g_{st}^{-1}(0) = g_1 \qquad \textbf{(4.13)}$$

Apply both sides of equation 4.13 to a point $\boldsymbol{p}_w \in \mathbb{R}^3$ which is the common point of intersection for the wrist axes. Since $\exp(\hat{\xi}\theta)\boldsymbol{p}_w = \boldsymbol{p}_w$ if $\boldsymbol{p}_w$ is on the axis of $\boldsymbol{\xi}$, this yields

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \boldsymbol{p}_w = g_1 \boldsymbol{p}_w \qquad \textbf{(4.14)}$$

Subtract for both sides of equation 4.14 a point $\boldsymbol{p}_b$ which is at the intersection of axis ξ_1 and ξ_2

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}\boldsymbol{p_w} - \boldsymbol{p_b} = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\left(e^{\hat{\xi}_3\theta_3}\boldsymbol{p_w} - \boldsymbol{p_b}\right) = g_1\boldsymbol{p_w} - \boldsymbol{p_b} \quad \textbf{(4.15)}$$

Using the property that the distance between points is preserved by rigid motions, take the norm of both sides of equation 4.15.

$$||e^{\hat{\xi}_3\theta_3}\boldsymbol{p_w} - \boldsymbol{p_b}|| = ||g_1\boldsymbol{p_w} - \boldsymbol{p_b}|| \quad \textbf{(4.16)}$$

This equation is in the form required for Subproblem 3 with $\boldsymbol{p} = \boldsymbol{p_w}, \boldsymbol{q} = \boldsymbol{p_b}$ and $\delta = ||g_1\boldsymbol{p_w} - \boldsymbol{p_b}||$. Applying subproblem 3, we solve for θ_3.

$$\theta_3 = \theta_0 \pm cos^{-1}\left(\frac{\left\|\boldsymbol{u}'\right\|^2 + \left\|\boldsymbol{v}'\right\|^2 - \delta'^2}{2\left\|\boldsymbol{u}'\right\|\left\|\boldsymbol{v}'\right\|}\right) \quad \textbf{(4.17)}$$

$$\theta_0 = atan2(\boldsymbol{w_3^T}(\boldsymbol{u}' \times \boldsymbol{v}'), \boldsymbol{u}'^{\boldsymbol{T}}\boldsymbol{v}') \quad \textbf{(4.18)}$$

where

$$\boldsymbol{u}' = (\boldsymbol{p_w} - \boldsymbol{r}) - \boldsymbol{w_3}\boldsymbol{w_3^T}(\boldsymbol{p_w} - \boldsymbol{r}) \quad \textbf{(4.19)}$$

$$\boldsymbol{v}' = (\boldsymbol{p_b} - \boldsymbol{r}) - \boldsymbol{w_3}\boldsymbol{w_3^T}(\boldsymbol{p_b} - \boldsymbol{r}) \quad \textbf{(4.20)}$$

$$\delta'^2 = \left(\sqrt[2]{(\boldsymbol{q_{in}^0} - \boldsymbol{pb})(\boldsymbol{q_{in}^0} - \boldsymbol{pb})}\right)^2 - |\boldsymbol{w_3^T}(\boldsymbol{p_w} - \boldsymbol{p_b})|^2 \quad \textbf{(4.21)}$$

$\boldsymbol{r}$ is any point on the axis of $\boldsymbol{w_3}$.

Since $\theta_3$is known equation 4.14 becomes

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}\boldsymbol{p_w}) = g_1\boldsymbol{p_w} \quad \textbf{(4.22)}$$

Applying subproblem 2 with $\boldsymbol{p} = e^{\hat{\xi}_3\theta_3}\boldsymbol{p_w}$ and $\boldsymbol{q} = g_1\boldsymbol{p_w}$ gives the values for θ_1 and θ_2.

$$\theta_2 = atan2(\boldsymbol{w_2^T}(\boldsymbol{u}' \times \boldsymbol{v}')\boldsymbol{u}'^{\boldsymbol{T}}\boldsymbol{v}') \quad \textbf{(4.23)}$$

where

$$\boldsymbol{u}' = (\boldsymbol{p} - \boldsymbol{r}) - \boldsymbol{w_2}\boldsymbol{w_2^T}(\boldsymbol{p} - \boldsymbol{r}) \quad \textbf{(4.24)}$$

$$\boldsymbol{v}' = (\boldsymbol{c} - \boldsymbol{r}) - \boldsymbol{w_2}\boldsymbol{w_2^T}(\boldsymbol{c} - \boldsymbol{r}) \quad \textbf{(4.25)}$$

where

$$c = \alpha\boldsymbol{w_1} + \beta\boldsymbol{w_2} + \gamma(\boldsymbol{w_1} \times \boldsymbol{w_2}) + \boldsymbol{r} \quad \textbf{(4.26)}$$

$$\alpha = \frac{(\boldsymbol{w_1^T}\boldsymbol{w_2})\boldsymbol{w_2^T}(\boldsymbol{p}-\boldsymbol{r}) - \boldsymbol{w_1^T}(\boldsymbol{q}-\boldsymbol{r})}{(\boldsymbol{w_1^T}\boldsymbol{w_2})^2 - 1} \quad \textbf{(4.27)}$$

$$\beta = \frac{(w_1^T w_2) w_1^T (q-r) - w_2^T (p-r)}{(w_1^T w_2)^2 - 1} \quad \textbf{(4.28)}$$

$$\gamma^2 = \frac{||p-r||^2 - \alpha^2 - \beta^2 - 2\alpha\beta w_1^T w_2}{||w_1 \times w_2||^2} \quad \textbf{(4.29)}$$

And θ_1 is

$$\theta_1 = atan2(-\boldsymbol{w}_1^T(\boldsymbol{u}' \times \boldsymbol{v}')\boldsymbol{u}'^T\boldsymbol{v}') \quad \textbf{(4.30)}$$

where

$$\boldsymbol{u}' = (\boldsymbol{q} - \boldsymbol{r}) - \boldsymbol{w}_1 \boldsymbol{w}_1^T (\boldsymbol{q} - \boldsymbol{r}) \quad \textbf{(4.31)}$$

$$\boldsymbol{v}' = (\boldsymbol{c} - \boldsymbol{r}) - \boldsymbol{w}_1 \boldsymbol{w}_1^T (\boldsymbol{c} - \boldsymbol{r}) \quad \textbf{(4.32)}$$

where $\boldsymbol{c}$ is same as equation 4.26 and $\boldsymbol{r}$ is the intersection point of the axis one and axis two.

The remaining kinematics can be written as

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_{in} g_{st}^{-1}(0) = g_2 \quad \textbf{(4.33)}$$

Apply both sides of equation 4.33 to a point $\boldsymbol{p}$ which is on the axis of $\boldsymbol{\xi}_6$ but it is not on the $\boldsymbol{\xi}_4$ and $\boldsymbol{\xi}_5$ axes. This gives

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} \boldsymbol{p} = g_2 \boldsymbol{p} = \boldsymbol{q} \quad \textbf{(4.34)}$$

Apply subproblem 2 to find θ_4 and θ_5.

$$\theta_5 = atan2(\boldsymbol{w}_5^T(\boldsymbol{u}' \times \boldsymbol{v}')\boldsymbol{u}'^T\boldsymbol{v}') \quad \textbf{(4.35)}$$

where

$$\boldsymbol{u}' = (\boldsymbol{p} - \boldsymbol{r}) - \boldsymbol{w}_5 \boldsymbol{w}_5^T (\boldsymbol{p} - \boldsymbol{r}) \quad \textbf{(4.36)}$$

$$\boldsymbol{v}' = (\boldsymbol{c} - \boldsymbol{r}) - \boldsymbol{w}_5 \boldsymbol{w}_5^T (\boldsymbol{c} - \boldsymbol{r}) \quad \textbf{(4.37)}$$

where

$$\boldsymbol{c} = \alpha \boldsymbol{w}_4 + \beta \boldsymbol{w}_5 + \gamma(\boldsymbol{w}_4 \times \boldsymbol{w}_5) + \boldsymbol{r} \quad \textbf{(4.38)}$$

$$\alpha = \frac{(w_4^T w_5) w_5^T (p-r) - w_4^T (q-r)}{(w_4^T w_5)^2 - 1} \quad \textbf{(4.39)}$$

$$\beta = \frac{(w_4^T w_5) w_4^T (q-r) - w_5^T (p-r)}{(w_4^T w_5)^2 - 1} \quad \textbf{(4.40)}$$

$$\gamma^2 = \frac{||\boldsymbol{p}-\boldsymbol{r}||^2-\alpha^2-\beta^2-2\alpha\beta \boldsymbol{w}_4^T \boldsymbol{w}_5}{||\boldsymbol{w}_4 \times \boldsymbol{w}_5||^2} \quad \textbf{(4.41)}$$

And θ_4 is

$$\theta_4 = atan2(-\boldsymbol{w}_4^T(\boldsymbol{u}' \times \boldsymbol{v}')\boldsymbol{u}'^T\boldsymbol{v}') \quad \textbf{(4.42)}$$

where

$$\boldsymbol{u}' = (\boldsymbol{q}-\boldsymbol{r}) - \boldsymbol{w}_4\boldsymbol{w}_4^T(\boldsymbol{q}-\boldsymbol{r}) \quad \textbf{(4.43)}$$

$$\boldsymbol{v}' = (\boldsymbol{c}-\boldsymbol{r}) - \boldsymbol{w}_4\boldsymbol{w}_4^T(\boldsymbol{c}-\boldsymbol{r}) \quad \textbf{(4.44)}$$

The only remaining unknown is θ_6. Rearranging the kinematics equation and applying both sides to any point $\boldsymbol{p}$ which is not on the axis of $\boldsymbol{\xi}_6$,

$$e^{\hat{\xi}_6\theta_6}\boldsymbol{p} = e^{-\hat{\xi}_5\theta_5}e^{-\hat{\xi}_4\theta_4}e^{-\hat{\xi}_3\theta_3}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}g_{in}g_{st}^{-1}(0)\boldsymbol{p} = \boldsymbol{q} \quad \textbf{(4.45)}$$

Apply subproblem 1 to find θ_6.

$$\theta_6 = atan2(\boldsymbol{w}_6^T(\boldsymbol{u}' \times \boldsymbol{v}')\boldsymbol{u}'^T\boldsymbol{v}') \quad \textbf{(4.46)}$$

where

$$\boldsymbol{u}' = (\boldsymbol{p}-\boldsymbol{r}) - \boldsymbol{w}_6\boldsymbol{w}_6^T(\boldsymbol{p}-\boldsymbol{r}) \quad \textbf{(4.47)}$$

$$\boldsymbol{v}' = (\boldsymbol{q}-\boldsymbol{r}) - \boldsymbol{w}_6\boldsymbol{w}_6^T(\boldsymbol{q}-\boldsymbol{r}) \quad \textbf{(4.48)}$$

where $\boldsymbol{p}$ is a new point which is not on the sixth joint axis and $\boldsymbol{r}$ is the intersection point of the wrist axes.

5. KINEMATIC SOLUTION USING SCREW THEORY AND QUATERNION ALGEBRA

5.1 Objectives

In 1843, the Irish mathematician W. R. Hamilton invented quaternions in order to extend three-dimensional vector algebra for inclusion of multiplications and divisions [27], [28]. However, quaternions have had a revival in the late 20th century, primarily due to their utility in describing spatial rotations. Representations of rotations by quaternions are more compact and faster to compute than representations by matrices [12] [13]. Several operators can be used in screw theory. However, quaternion is the best operator to describe screw motion.

In this chapter, first I will briefly introduce quaternion algebra. Then line transformation by using dual-quaternions will be introduced. I will end this chapter by giving two different solution methods for 6-DOF serial robot manipulator which is shown in figure 4.1.

5.2 Quaternion

Quaternions are hyper-complex numbers of rank 4, constituting a four dimensional vector space over the field of real numbers [27]. A quaternion can be represented as:

$$q = (q_0, \boldsymbol{q_v}) \tag{5.1}$$

where q_0 is a scalar and $\boldsymbol{q_v} = (q_1, q_{2,} q_3)$ is a vector. A quaternion with $\mathbf{q_v} = 0$, is called as a real quaternion, and a quaternion with $\mathrm{q}_0 = 0$, is called as a pure quaternion (or vector quaternion). Addition of two quaternions is simpler and it can be expressed as:

$$q_a + q_b = (q_{a0} + q_{b0}), (\boldsymbol{q_{av}} + \boldsymbol{q_{bv}}) \tag{5.2}$$

Multiplication of two quaternions is harder than addition and it can be expressed as:

$$q_a \otimes q_b = q_{a0} q_{b0} - \boldsymbol{q_{av}} \cdot \boldsymbol{q_{bv}} \, , q_{a0} \boldsymbol{q_{bv}} + q_{b0} \boldsymbol{q_{av}} + \boldsymbol{q_{av}} \times \boldsymbol{q_{bv}} \tag{5.3}$$

where "⊗", ".", "×" denotes quaternion product, dot product and cross product respectively. The quaternion addition is associative and commutative. The quaternion multiplication is associative, and distributes over addition but it is not commutative.

Conjugate of the quaternion can be expressed as:

$$q^* = (q_0, -\boldsymbol{q_v}) = (q_0, -q_1 -, q_{2,} - q_3) \qquad \textbf{(5.4)}$$

The above equations allow us to define the quaternion norm ||q|| as:

$$||q||^2 = q \otimes q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2 \qquad \textbf{(5.5)}$$

When $||q|| = 1$, we get a unit quaternion. Any quaternion q can be normalized by dividing by its norm, to obtain a unit quaternion.

The inverse of a quaternion can be expressed as:

$$q^{-1} = \frac{1}{||q||^2} q^* \quad and \quad ||q|| \neq 0 \qquad \textbf{(5.6)}$$

that satisfies the relation $q^{-1} \otimes q = q \otimes q^{-1} = 1$ **(5.7)**

For a unit-quaternion we have

$$q^{-1} = q^* \qquad \textbf{(5.8)}$$

Unit quaternion can be defined as a rotation operator [29], [30]. Rotation about a unit axis $\boldsymbol{n}$ with angle θ is expressed as:

$$q = (\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\boldsymbol{n}) \qquad \textbf{(5.9)}$$

Further information about rotation representation of quaternion can be found in App D.5.

5.3 Dual-Quaternion

A dual-quaternion can be defined as:

$$\hat{q} = (\hat{q}_0, \hat{q}_v) \quad \text{or} \quad \hat{q} = q + \epsilon q^0 \qquad \textbf{(5.10)}$$

where $\hat{q}_0 = q_0 + \varepsilon q_0^0$, is a dual scalar, $\hat{q}_v = \boldsymbol{q_v} + \varepsilon \boldsymbol{q_v^0}$, is a dual vector, q and $q^0$are both quaternions, $\epsilon : \epsilon^2 = 0$ is the dual factor.

Addition of two dual-quaternions is simple and it is very similar as quaternion addition .It can be expressed as:

$$\hat{q}_a + \hat{q}_b = (q_a + q_b) + \epsilon(q_a^0 + q_b^0) \quad \textbf{(5.11)}$$

Multiplication of two dual-quaternions is also harder than addition and it can be expressed as:

$$\hat{q}_a \odot \hat{q}_b = (q_a \otimes q_b) + \epsilon(q_b \otimes q_a^0 + q_a \otimes q_b^0) \quad \textbf{(5.12)}$$

where "⊗"denotes quaternion and "⊙" denotes dual-quaternion product. The dual-quaternion addition is associative and commutative. The dual-quaternion multiplication is associative, and distributes over addition but it is not commutative.

Conjugate, norm and inverse of the dual-quaternion is similar with quaternion. They can be expressed as:

$$\hat{q}^* = q^* + \varepsilon(q^0)^* \quad \textbf{(5.13)}$$

$$||\hat{q}||^2 = \hat{q} \otimes \hat{q}^* \quad \textbf{(5.14)}$$

$$\hat{q}^{-1} = \frac{1}{||\hat{q}||^2}\hat{q}^* \quad \textbf{(5.15)}$$

When $||q|| = 1$, we get a unit dual-quaternion. For unit dual-quaternion these equations can be written as

$$\left||\hat{q}|\right|^2 = \hat{q} \otimes \hat{q}^* = 1 \quad \text{and} \quad q \otimes q^* = 1 \quad \textbf{(5.16)}$$

$$q^* \otimes q^0 + (q^0)^* \otimes q = 0 \quad \textbf{(5.17)}$$

Unit dual-quaternion is also rigid motion transformation operator. [31], [32]

5.4 Line Transformation by Using Dual-Quaternions

A general rigid transformation has 6 DOF. 3 DOF is for orientation and 3 DOF is for translation. Hence we need a transformation operator which has at least six parameters. A unit-quaternion can be used as a rotation operator. A point p_b can be transformed to a point p_a by using unit quaternions as follow:

$$p_a = q \otimes p_b \otimes q^* \quad \textbf{(5.18)}$$

where q is unit-quaternion. Unit-quaternions can be used for transformation of a point but general rigid transformation can't be implemented by using unit-quaternions. A general rigid transformation has 6 DOF. Hence we need a transformation operator which has at least six parameters. We can use dual-quaternion for general rigid transformation [31]. Although it has eight parameters and it is not minimal, it is the most compact and efficient dual operator [12], [13]. Now, we will explain how dual-quaternion allows a rigid-transformation. This transformation is very similar with pure rotation; however, not for a point but for a line.

A line in plücker coordinates ($L_p(\boldsymbol{m}, \boldsymbol{d})$) can be expressed by using dual quaternios as:

$$\hat{l}_a = \boldsymbol{l_a} + \varepsilon \boldsymbol{m_a} \tag{5.19}$$

After transformation of $\hat{l}_a$ (R: rotation and t: translation) we obtain a transformed line $\hat{l}_b$ ($\hat{l}_b = \boldsymbol{l_b} + \varepsilon \boldsymbol{m_b}$). Transformation of line can be expressed as:

$$\boldsymbol{l_b} = R\boldsymbol{l_a} \tag{5.20}$$

$$\boldsymbol{m_b} = \boldsymbol{p_b} \times \boldsymbol{l_b} = (R\boldsymbol{p_a} + \boldsymbol{t}) \times R\boldsymbol{l_a} = R(\boldsymbol{p_a} \times \boldsymbol{l_a}) + \boldsymbol{t} \times R\boldsymbol{l_a} = R\boldsymbol{m_a} + \boldsymbol{t} \times R\boldsymbol{l_a} \tag{5.21}$$

We change vector notations with quaternion notations by using pure quaternions as:

$$l_a = (0\,, \boldsymbol{l_a}) \tag{5.22}$$

And the cross product can be written as:

$$(0, \boldsymbol{t})xq = \frac{1}{2}(q \otimes (0, \boldsymbol{t})^* + (0, \boldsymbol{t}) \otimes q) \tag{5.23}$$

where $\mathbf{t}$ is the translation and t=(0,$\mathbf{t}$) is the quaternion notation of translation. Using equation 5.20 and 5.21 we obtain

$$l_b = q \otimes l_a \otimes q^* \tag{5.24}$$

$$m_b = q \otimes m_a \otimes q^* + \frac{1}{2}(q \otimes l_a \otimes q^* \otimes t^* + t \otimes q \otimes l_a \otimes q^*) \tag{5.25}$$

where m_a, m_b, l_a, l_b and t are pure quaternion notation of vectors.

If we define a new quaternion $q' = \frac{1}{2} t \otimes q$ and dual quaternion $\hat{q} = q + \varepsilon q'$, equation 5.22 is equivalent to

$$l_b + \varepsilon m_b = (q + \varepsilon q') \odot (l_a + \varepsilon m_a) \odot (q^* + \varepsilon q'^*) \tag{5.26}$$

If we denote the lines by dual quaternions we obtain a very similar formulization with pure rotation transformation:

$$\hat{l}_b = \hat{q} \odot \hat{l}_a \odot \hat{q}^* \quad \textbf{(5.27)}$$

Note that:

$$||\hat{q}||^2 = \hat{q} \otimes \hat{q}^* = q \otimes q^* + \varepsilon(q \otimes q'^* + q' \otimes q^*)$$

$$= q \otimes q^* + \frac{1}{2}\varepsilon(q \otimes q^* \otimes t^* + t \otimes q \otimes q^*) = 1 \quad \textbf{(5.28)}$$

hence $\hat{q}$ is a unit dual-quaternion.

5.5 Screw Motion with Quaternion

In equation 3.30, screw motion is expressed by using 4x4 transformation matrices. It uses sixteen parameters while just 6 parameters are needed. We can express screw motion more compact form than transformation matrices by using quaternion. If we separate general screw motion as a rotation and translation, we have

Rotation: $R(\theta, \boldsymbol{d})$

Translation: $\boldsymbol{t} = \frac{\theta}{2\pi} p\boldsymbol{d} + (I_{3x3} - R(\theta, \boldsymbol{d}))\boldsymbol{p}$ **(5.29)**

It can be expressed by using quaternion as follow:

Rotation: $q = (\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\boldsymbol{t})$

Translation: $t = \acute{q} + p - q \otimes p \otimes q^*$ **(5.30)**

where $\boldsymbol{\acute{q}}$ is the amount of translation and $\boldsymbol{p}$ is the position vector of some point on the line in pure quaternion form.

5.6 Screw Motion with Dual Quaternion

If we separate screw motion as a rotation and translation rotation will be equal to $R(\theta, \boldsymbol{d})$ and translation will be equal to $\boldsymbol{t} = \frac{\theta}{2\pi} p\boldsymbol{d} + (I_{3x3} - R)\boldsymbol{p}$. Assume that $c = \frac{\theta}{2\pi} p$ then $k = \boldsymbol{d}.\boldsymbol{t}$ and $\boldsymbol{p}.\boldsymbol{d} = 0$. Using the Rodrigues formula (see App. D.4 for Rodrigues formula),

$$R\boldsymbol{p} = \boldsymbol{p} + sin(\theta)\boldsymbol{d} \times \boldsymbol{p} + (1 - \cos(\theta)\, \boldsymbol{d} \times (\boldsymbol{d} \times \boldsymbol{p})) \quad \textbf{(5.31)}$$

$$\boldsymbol{p} = \frac{1}{2}(\boldsymbol{t} - (\boldsymbol{t}.\boldsymbol{d})\boldsymbol{d} + \cot\left(\frac{\theta}{2}\right) \boldsymbol{d} \times \boldsymbol{t}) \quad \textbf{(5.32)}$$

The point $\boldsymbol{p}$ can not be defined if the angle θ is either 0^o or 180^o. Otherwise the moment vector can be written as:

$$\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{d} = \frac{1}{2}\left(\boldsymbol{t}\, x\, \boldsymbol{d} + \boldsymbol{d} \times (\boldsymbol{t} \times \boldsymbol{d}) \cot\left(\frac{\theta}{2}\right)\right) \quad \textbf{(5.33)}$$

If we use the unit quaternion notation which is given in equation 5.9 we derived the moment equation as:

$$\sin\left(\frac{\theta}{2}\right)\boldsymbol{m} = \frac{1}{2}\left(\boldsymbol{t} \times \boldsymbol{q_v} + q_0\boldsymbol{t} - \cos\left(\frac{\theta}{2}\right)(\boldsymbol{d}.\boldsymbol{t})\boldsymbol{d}\right) \quad \textbf{(5.34)}$$

If we use $k = \boldsymbol{d}.\boldsymbol{t}$ equation and rewrite equation 5.34 we obtain:

$$\sin\left(\frac{\theta}{2}\right)\boldsymbol{m} + \frac{k}{2}\cos\left(\frac{\theta}{2}\right)\boldsymbol{d} = \frac{1}{2}(\boldsymbol{t} \times \boldsymbol{q_v} + q_0\boldsymbol{t}) \quad \textbf{(5.35)}$$

Equation 5.37 gives the vector part of the dual part of the dual quaternion. Using equation 5.9 as a rotation operator for the real part of the dual quaternion and $q^{'} = \frac{\mathbf{1}}{\mathbf{2}} t \otimes q$ equation as a translation operator for the dual part of the dual quaternion we obtain a new dual quaternion as a rigid motion operator. A rigid motion operator can be written as:

$$\begin{aligned}\hat{q} &= \begin{pmatrix} q_0 \\ \boldsymbol{q_v} \end{pmatrix} + \boldsymbol{\varepsilon}\begin{pmatrix} -\frac{1}{2}(\boldsymbol{q_v}.\boldsymbol{t}) \\ \frac{1}{2}(q_0\boldsymbol{t} + \boldsymbol{t} \times \boldsymbol{q_v}) \end{pmatrix} \\ &= \begin{pmatrix} cos\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right)\boldsymbol{d} \end{pmatrix} + \boldsymbol{\varepsilon}\begin{pmatrix} -\frac{k}{2} sin\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right)\boldsymbol{m} + \frac{k}{2} cos\left(\frac{\theta}{2}\right)\boldsymbol{d} \end{pmatrix}\end{aligned} \quad \textbf{(5.36)}$$

Using equation A.37 and we obtain a new representation as:

$$\hat{q} = \begin{pmatrix} cos\left(\frac{\theta + \varepsilon k}{2}\right) \\ sin\left(\frac{\theta + \varepsilon k}{2}\right)(\boldsymbol{d} + \varepsilon\boldsymbol{m}) \end{pmatrix} \quad \textbf{(5.37)}$$

This representation is very compact and also it uses algebraically separates the angle (θ) and pitch (k) information. Hence it is very powerful [5]. Moreover if we write $\hat{\theta} = \theta + \varepsilon k$ and $\hat{d} = \boldsymbol{d} + \varepsilon\boldsymbol{m}$ equation 5.39 becomes [33]:

$$\hat{q} = cos\left(\frac{\hat{\theta}}{2}\right) + sin\left(\frac{\hat{\theta}}{2}\right)\hat{d} \quad \textbf{(5.38)}$$

Let's define a dual-quaternion for the i.th screw motion such that

$$\hat{q}_i = q_i + \epsilon q_i^0 \tag{5.39}$$

Then real part and dual part of the i.th screw motion's dual-quaternion can be represented given by

$$q_i = \cos\left(\frac{\theta_i}{2}\right) + \sin\left(\frac{\theta_i}{2}\right)\boldsymbol{d_i}$$

$$q_i^0 = \frac{1}{2}(p_i - q_i \otimes p_i \otimes q_i) \otimes q_i \quad \text{or} \quad q_i^0 = [0, sin\left(\frac{\theta_i}{2}\right)\boldsymbol{m_i}] \tag{5.40}$$

where i=1,2,…,m

5.7 6-DOF Serial Robot Manipulator Kinematic Model Using Quaternion

In this section, I will give two new methods to solve forward and inverse kinematics problem of serial robot arm which is shown in figure 4.1. First, forward and inverse kinematic problem will be solved by using quaternions. Then the same problem will be solved by using dual-quaternions and plücker coordinates (see App. C.2 for definition plücker coordinates). I will use same notation as exponential mapping method which is given in chapter 4. Thus, anyone who wants to see notation should look at chapter 4. Also formulization approaches of these two new methods are quite similar as exponential mapping method but formulas are based on quaternion algebra.

5.7.1 Forward kinematic of 6-dof serial robot manipulator

First we must determine the axes for all joints. Axes can be chosen as follow:

$$\boldsymbol{d}_1 = [0\ 0\ 1] \qquad \boldsymbol{d}_2 = [0\ 1\ 0] \qquad \boldsymbol{d}_3 = [0\ 1\ 0]$$

$$\boldsymbol{d}_4 = [0\ 0\ 1] \qquad \boldsymbol{d}_5 = [0\ 1\ 0] \qquad \boldsymbol{d}_6 = [1\ 0\ 0] \tag{5.41}$$

Any point on these axes can be written as:

$$\boldsymbol{p}_1 = [0\ 0\ l_0] \qquad \boldsymbol{p}_2 = [0\ 0\ l_0] \qquad \boldsymbol{p}_3 = [\ l_1\ 0\ l_0]$$

$$\boldsymbol{p}_4 = [\ l_1 + l_2\ 0\ l_0] \qquad \boldsymbol{p}_5 = [\ l_1 + l_2\ 0\ l_0] \qquad \boldsymbol{p}_6 = [\ l_1 + l_2\ 0\ l_0] \tag{5.42}$$

Thus we can write quaternions by using axes and point vectors. Quaternions can be obtained from equation 5.29 and 5.30 where n = 6. To obtain translation, equation 5.30 must be calculated six times. Our general forward kinematic equations are:

$$q_{16} = q_1 \otimes q_2 \otimes q_3 \otimes q_4 \otimes q_5 \otimes q_6 \tag{5.43}$$

$$q_i^o = q_{i-1} \otimes p_i \otimes q_{i-1}^* - q_i \otimes p_i \otimes q_i^* + q_{i-1}^o \quad \textbf{(5.44)}$$

where i = 1, 2,….6 and $q_1^o = p_1 - q_1 \otimes p_1 \otimes q_1^*$. And the position of the end effector is

$$q_{ep} = q_6 \otimes p_{ep} \otimes q_6^* + q_6^o \quad \textbf{(5.45)}$$

5.7.2 Inverse kinematic of 6-dof serial robot manipulator

In the inverse kinematic problem of the serial manipulator, we have orientation and position knowledge of the end effector. These are two quaternions and we will calculate all joint angles by using these quaternions. The first one gives us the orientation knowledge of the robot manipulator (q_{in}) and the second one gives us the position knowledge of the end effector (q_i^o). To find all joint angles complete inverse kinematic problem must be converted into the appropriate subproblems. First we put two points at the intersection of the axes. First one is $p_w$which is at the intersection of the wrist axes and the second one is p_b which is at the intersection of the first two axes. The last three joints angles do not affect the point p_w. Hence we can say the position of point p_w is free from the wrist angles. If we take the end effector position q_{in}^0 =($q_0^0, q_1^0, q_2^0, q_3^0$) we get $p_w = q_{in}^0$. Thus we can write

$$q_3 \otimes p_w \otimes q_3^* + q_3^o - p_b = q_{in}^0 \quad \textbf{(5.46)}$$

Using the property that distance between points is preserved by rigid motions; take the magnitude of both sides of equation 5.46

$$||q_3 \otimes p_w \otimes q_3^* + q_3^o - p_b|| = ||q_{in}^0|| \quad \textbf{(5.47)}$$

We obtain subproblem3. $\theta_3$can be found by using subproblem 3 that is given at App. C.5.

If we translate $p_w$by using known $\theta_3$we obtain a new point p. We get subproblem 2 using the point p as the initial position of the subproblem 2 motion and the point q as the final position of the subproblem 2 motion. The points p and q can be formulized as:

$$p = q_3 \otimes p_w \otimes q_3^* + p_3 - q_3 \otimes p_3 \otimes q_3^* \text{ and } q = q_{in}^0 \quad \textbf{(5.48)}$$

θ_1 and θ_2 can be found by using subproblem 2 that is given at App. C.5.

To find wrist angles we put a point p_c which is on the d_6 axis and it does not intersect with d_4 *and* d_5 axes. Thus the point p_c is not affected from the last joint angle. Fourth and fifth joints angles determine the position of the point p_c. For known θ_1, θ_2 *and* $\theta_3$we can write

$$q_{45} \otimes p_c \otimes q_{45}^* + q_3^* \otimes q_{46}^o \otimes q_3 = q_3^* \otimes q_{in}^o \otimes q_3 - q_3^* \otimes q_3^o \otimes q_3 \quad \textbf{(5.49)}$$

Equation 5.49 gives us subproblem 2. To obtain subproblem parameters we should find p and q points which are given at App. C.5. The point p is equal to p_c. We should just find q. The point q can be found given by

$$q_m = (q_1 \otimes q_2 \otimes q_3)^* \otimes q_{in} \quad \textbf{(5.50)}$$

$$q_{pnt} = q_m^* \otimes p_c \otimes q_m + q_m^* \otimes p_0 \otimes q_m + q_m^* \otimes p_1 \otimes q_m - q_{t1} - q_{t2} \quad \textbf{(5.51)}$$

where

$$q_{t1} = q_3^* \otimes p_3 \otimes q_3 - p_3$$

$$q_{t2} = q_3^* \otimes p_2 \otimes q_3 - (q_2 \otimes q_3)^* \otimes p_3 \otimes (q_2 \otimes q_3) + (q_2 \otimes q_3)^* \otimes p_1 \otimes (q_2 \otimes q_3) \quad \textbf{(5.52)}$$

The parameters of subproblem 2 are

$$p = p_c = pw + d_6 * 5 \quad \text{and} \quad q = q_{pnt} \quad \textbf{(5.53)}$$

Thus first five joints angles are obtained. Only the last joint angle is unknown. The last joint angle can be found from orientation part of input. We can write,

$$q_{16} = q_1 \otimes q_2 \otimes q_3 \otimes q_4 \otimes q_5 \otimes q_6 = q_{in} \quad \textbf{(5.54)}$$

$$q_6 = (q_1 \otimes q_2 \otimes q_3 \otimes q_4 \otimes q_5)^* q_{in} \quad \textbf{(5.55)}$$

We can find the last joint angle from equation 5.55.

5.8 6-DOF Serial Robot Manipulator Kinematic Model Using Dual-Quaternions

5.8.1 Forward kinematic of 6-dof serial robot manipulator

First we must determine the axes for all joints. Then we will find the moment vector for all axes. The axes and the moment vectors can be written as:

$$\boldsymbol{d}_1 = [0\ 0\ 1] \qquad \boldsymbol{d}_2 = [0\ 1\ 0] \qquad \boldsymbol{d}_3 = [0\ 1\ 0]$$

$$\boldsymbol{d}_4 = [0\ 0\ 1] \qquad \boldsymbol{d}_5 = [0\ 1\ 0] \qquad \boldsymbol{d}_6 = [1\ 0\ 0] \quad \textbf{(5.56)}$$

Any point on these axes can be written as:

$\boldsymbol{p}_1 = [0\ 0\ l_0]$ $\boldsymbol{p}_2 = [0\ 0\ l_0]$ $\boldsymbol{p}_3 = [\ l_1\ 0\ l_0]$

$\boldsymbol{p}_4 = [\ l_1 + l_2\ 0\ l_0]$ $\boldsymbol{p}_5 = [\ l_1 + l_2\ 0\ l_0]$ $\boldsymbol{p}_6 = [\ l_1 + l_2\ 0\ l_0]$ **(5.57)**

Hence the moment vectors of these axes are obtained as:

$\boldsymbol{m}_1 = \boldsymbol{p}_1 x \boldsymbol{d}_1$ $\boldsymbol{m}_2 = \boldsymbol{p}_2 x \boldsymbol{d}_2$ $\boldsymbol{m}_3 = \boldsymbol{p}_3 x \boldsymbol{d}_3$

$\boldsymbol{m}_4 = \boldsymbol{p}_4 x \boldsymbol{d}_4$ $\boldsymbol{m}_5 = \boldsymbol{p}_5 x \boldsymbol{d}_5$ $\boldsymbol{m}_6 = \boldsymbol{p}_6 x \boldsymbol{d}_6$ **(5.58)**

Now we can write dual-quaternion by using axes and moment vectors. Dual-quaternion can be obtained from equation 5.39 and 5.40 where i = 1,2….6. Finally forward kinematic equation of serial robot manipulator can be found by using dual-quaternion product. The forward kinematic equation of serial robot manipulator is

$$\hat{q} = \hat{q}_1 \odot \hat{q}_2 \dots \odot \hat{q}_n$$

where $\hat{q} = q + \varepsilon q^0$ and n=6. Here q gives us orientation of the robot manipulator. The position of the robot manipulator can be found using equation A.35.

$$\boldsymbol{t} = (\boldsymbol{q}_{vn} \times \boldsymbol{q}_{vn}^0) + (\boldsymbol{q}_{vn-1} \times \boldsymbol{q}_{vn-1}^0) \cdot \boldsymbol{q}_{vn} * \boldsymbol{q}_{vn}$$

where $\boldsymbol{q}_{vn}$ and $\boldsymbol{q}_{vn}^0$ are real vector parts and dual vector part of dual quaternion product respectively.

5.8.2 Inverse kinematic of 6-dof serial robot manipulator

In the inverse kinematic problem of the serial manipulator, we have rotation and the position of the end effector knowledge as:

$$\hat{q}_{in} = (q_{in}, q_{in}^0) \quad \textbf{(5.59)}$$

where $q_{in} = (q_0, q_1, q_2, q_3)$ is the real part (rotation knowledge), and q_{in}^0 =$(q_0^0, q_1^0, q_2^0, q_3^0)$ is the dual part (position knowledge) of the input knowledge. And we will calculate all joint angles by using rotation and the position of the end effector knowledge. Our general forward kinematic equation is:

$$\hat{q} = \hat{q}_{16} \odot (d_6 + \varepsilon m_6) \odot \hat{q}_{16}^* \quad \textbf{(5.60)}$$

where $\hat{q}_{16} = \hat{q}_1 \odot \hat{q}_2 \odot \hat{q}_3 \odot \hat{q}_4 \odot \hat{q}_5 \odot \hat{q}_6$ **(5.61)**

We must convert the complete inverse kinematic problem into the appropriate subproblems. First we put two points at the intersection of the axes. First one is $p_w$which is at the intersection of the wrist axes and the second one is $\boldsymbol{p_b}$ which is at the intersection of the first two axes. The last three joints angles do not affect the point $\boldsymbol{p_w}$. Therefore, we can say that the position of point $\boldsymbol{p_w}$ is free from the wrist angles and we can formulize it as:

$$(\boldsymbol{q}_{v6} \times \boldsymbol{q}_{v6}^0) + (\boldsymbol{q}_{v5} \times \boldsymbol{q}_{v5}^0).\boldsymbol{q}_{v6} * \boldsymbol{q}_{v6} = (\boldsymbol{q}_{v3} \times \boldsymbol{q}_{v3}^0) + (\boldsymbol{q}_{v2} \times \boldsymbol{q}_{v2}^0).\boldsymbol{q}_{v3} * \boldsymbol{q}_{v3} = \boldsymbol{p_w} \quad \textbf{(5.62)}$$

We can write same equation for the point $\boldsymbol{p_b}$.

$$(\boldsymbol{q}_{v2} \times \boldsymbol{q}_{v2}^0) + (\boldsymbol{q}_{v1} \times \boldsymbol{q}_{v1}^0).\boldsymbol{q}_{v2} * \boldsymbol{q}_{v2} = \boldsymbol{p_b} \quad \textbf{(5.63)}$$

The position of the point $\boldsymbol{p_b}$ is free from the angles first two joints. If we subtract equation 5.63 from both of side of the equation 5.62 we obtain

$$(\boldsymbol{q}_{v6} \times \boldsymbol{q}_{v6}^0) + (\boldsymbol{q}_{v5} \times \boldsymbol{q}_{v5}^0).\boldsymbol{q}_{v6} * \boldsymbol{q}_{v6} - \boldsymbol{p_b} = (\boldsymbol{q}_{v3} \times \boldsymbol{q}_{v3}^0) + (\boldsymbol{q}_{v2} \times \boldsymbol{q}_{v2}^0).\boldsymbol{q}_{v3} * \vec{\boldsymbol{q}}_{v3} - \boldsymbol{p_b} \quad \textbf{(5.64)}$$

If we take the end effector position $q_{in}^0 = (q_0^0, q_1^0, q_2^0, q_3^0)$ at the intersection of the wrist axes we have $\boldsymbol{p_w} = q_{in}^0$. Hence we can write

$$(\boldsymbol{q}_{v3} \times \boldsymbol{q}_{v3}^0) + (\boldsymbol{q}_{v2} \times \boldsymbol{q}_{v2}^0).\boldsymbol{q}_{v3} * \boldsymbol{q}_{v3} - (\boldsymbol{q}_{v2} \times \boldsymbol{q}_{v2}^0) + (\boldsymbol{q}_{v1} \times \boldsymbol{q}_{v1}^0).\boldsymbol{q}_{v2} * \boldsymbol{q}_{v2} = \boldsymbol{q}_{in}^0 - \boldsymbol{pb} \quad \textbf{(5.65)}$$

Using the property that distance between points is preserved by rigid motions, take the magnitude of both sides of equation 5.65 we obtain subproblem 3. θ_3 can be written by using subproblem 3 that is given at App. C.5.

If we translate $\boldsymbol{p_w}$ by using known $\theta_3$we obtain a new point $\boldsymbol{p}$ and subproblem 2 that is given at App. C.5. The parameters of subproblem 2 ($\boldsymbol{p}$ and $\boldsymbol{q}$) are as follow:

$$\boldsymbol{q} = \boldsymbol{q}_{in}^0 \text{ and } \boldsymbol{p} = \boldsymbol{d_3} \times \boldsymbol{p_{dual}} \quad \textbf{(5.66)}$$

where $p_{dual} = q_3 \otimes m_w \otimes q_3^* + q_3 \otimes d_3 \otimes q_3^{o*} + q_3^o \otimes d_3 \otimes q_3^*$ **(5.67)**

To find wrist angles we put a point $\boldsymbol{p_i}$ (initial point) which is on the $\boldsymbol{d_6}$ axis and it does not intersect with $\boldsymbol{d_4}$ and $\boldsymbol{d_5}$ axes. Two imaginer axes are used to find $\boldsymbol{p_e}$ (end point), that is, the position of the point $\boldsymbol{p_i}$ after rotation by θ_4 *and* θ_5. It can be found by using the intersection of two lines formula that is given at App. C.3. These two imaginer axes intersect on $\boldsymbol{d_6}$ and the point $\boldsymbol{p_i}$ is the intersection point of these imaginer axes. Its mathematical formulization can be written as follow:

$\boldsymbol{d}_7 = [0\ 1\ 0]$, $\boldsymbol{d}_8 = [0\ 0\ 1]$

$$\boldsymbol{p}_7 = [\ a_{23} + a_{34} + 5\ \ 0\ d_0]\ ,\ \ \boldsymbol{p}_8 = [\ a_{23} + a_{34} + 5\ 0\ d_0] \tag{5.68}$$

$$\boldsymbol{m}_7 = \boldsymbol{p}_7 x \boldsymbol{d}_7\ ,\ \boldsymbol{m}_8 = \boldsymbol{p}_8 x \boldsymbol{d}_8 \tag{5.69}$$

$$\hat{q}_7 = cos(0) + sin(0)\hat{d}_7 \quad \hat{q}_8 = cos(0) + sin(0)\hat{d}_8 \tag{5.70}$$

After making up two imaginer axes we can calculate the point p_e.

$$\hat{q}_{17} = \hat{q}_{in} \odot \hat{q}_7 \qquad and \qquad \hat{q}_{18} = \hat{q}_{in} \odot \hat{q}_7 \odot \hat{q}_8 \tag{5.71}$$

$$\boldsymbol{p_e} = (\vec{\boldsymbol{q}}_{v8} x \vec{\boldsymbol{q}}_{v8}^0) + (\vec{\boldsymbol{q}}_{v7} x \vec{\boldsymbol{q}}_{v7}^0).\vec{\boldsymbol{q}}_{v8} * \vec{\boldsymbol{q}}_{v8} \tag{5.72}$$

Hence we obtained the point $\boldsymbol{p_e}$. The point $\boldsymbol{p_i}$ is on the $\boldsymbol{d_6}$ axis. Hence it is not affected from the last joint angle. Fourth and fifth joints angles determine the position of the point $\boldsymbol{p_i}$. This gives us a subproblem 2. θ_4 and $\theta_5$can be solved by using subproblem 2 that is given at App. C.5. The parameters of subproblem 2 are

$$\boldsymbol{p} = \boldsymbol{p_i} = \boldsymbol{p_w} + \boldsymbol{d}_6 * 5 \ \text{ and } \ \boldsymbol{q} = \boldsymbol{p_e} \tag{5.73}$$

Hence first five joints angles are obtained. Only the last joint angle is unknown. To find last joint angle we need a point which is not on the last joint axis. We call it $\boldsymbol{p_d}$. The position of the point $\boldsymbol{p_d}$ after rotation by θ_6 can be found by using equations 5.70 and 5.73. This gives us a subproblem1. θ_6 can be solved as follow:

$$\theta_6 = arctan2(\boldsymbol{d}_6^T(\boldsymbol{u}' \times \boldsymbol{v}'), \boldsymbol{u}'^T \boldsymbol{v}') \tag{5.74}$$

where $\boldsymbol{u}' = (\boldsymbol{p}_d' - \boldsymbol{r}) - \boldsymbol{d}_6 \boldsymbol{d}_6^T (\boldsymbol{p}_d' - \boldsymbol{r})$ and $\boldsymbol{v}' = (\boldsymbol{p_d} - \boldsymbol{r}) - \boldsymbol{d}_6 \boldsymbol{d}_6^T (\boldsymbol{p_d} - \boldsymbol{r})$

where $\boldsymbol{p}_d'$ is a new point which is the position of the point $\boldsymbol{p_d}$ after rotation by θ_6 and **r** is the intersection point of the wrist axes.

6. SIMULATION RESULTS

6.1 Introduction

Simulation experiments are made for Staubli TX60 serial robot manipulator which is shown in figure 6.1. This series robots feature an articulated arm with 6 degrees of freedom for high flexibility. It spreads a wide area in industrial robot applications. Also it is very similar as the serial robot manipulator which is analyzed in chapter four and chapter five. On this account, this robot has been chosen for simulation experiments. The only different part between Staubli TX60 robot manipulator and the serial robot manipulator that is given in chapter four is that Staubli TX60 serial robot manipulator has offset to avoid singularity.

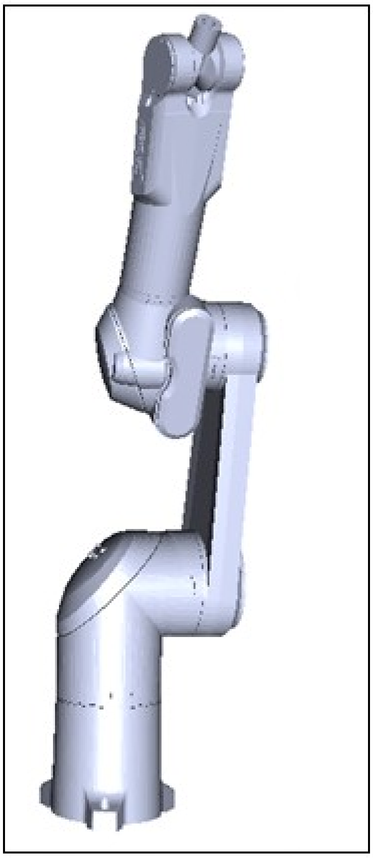

Figure 6.1: Staubli TX60 L serial robot manipulator

General specification of Staubli TX60 as follow:

Table 6.1: Staubli TX 60 Specification

	TX60	TX 60 L
Numbers of DOF	6	6
Nominal Load Capacity	3.5	2
Maximum Load Capacity	9	5
Reach at Wrist	670	920
Repeatability	±0.02 mm	±0.03 mm
Protection Class (*Wrist)	IP 65 (*IP67)	IP 65 (*IP67)

This specification can be found from Staubli's web page [Url-1].

Matlab is chosen for simulations of forward and inverse kinematic of serial robot manipulators because animation applications can be easily made by using virtual reality toolbox of Matlab. Also Staubli TX60 iges file which can be freely obtained from Staubli's web page is used for animation application [Url-2]. Three different simulation experiments are made which are shown in figure 6.2, figure 6.3 and figure 6.4.

The first one is single working of serial robot manipulator. In this case single robot arm carries a box from its initial position to the target position as shown in figure 6.2. To implement this case, first a path is determined for the box. Then inverse kinematic of serial arm is solved by using this path.

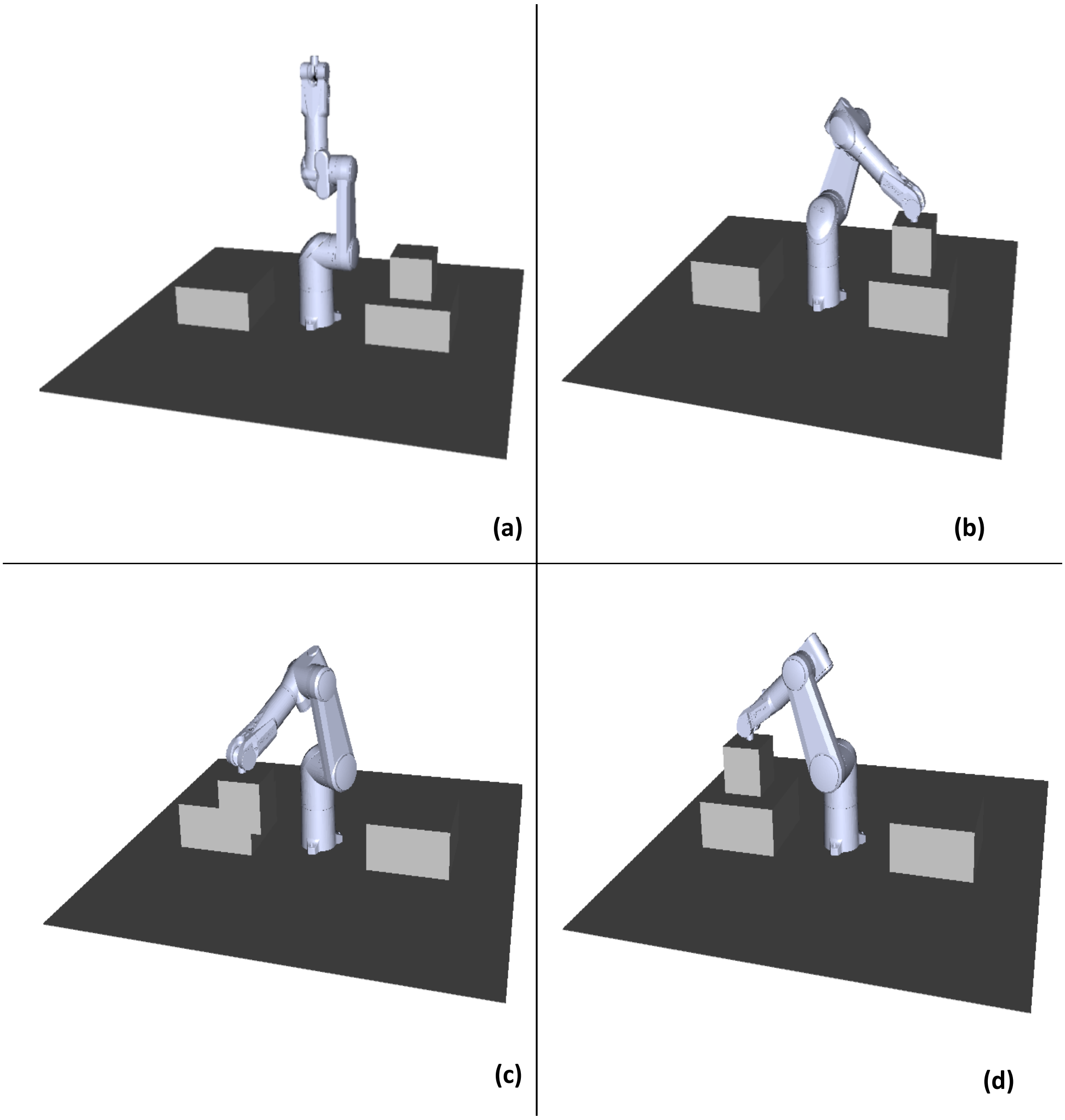

Figure 6.2: Single working (carrying box experiments)

The second simulation experiment is cooperative working of serial robot arms. In this case two robot arms work together and they carry a ball from its initial position to the target position as shown in figure 6.3. To implement this case, first a path is determined for the ball. Then the inverse kinematic of serial robot arm is solved by using this path for both of robot arms. Orientations of robot arms are chosen adversely to each other.

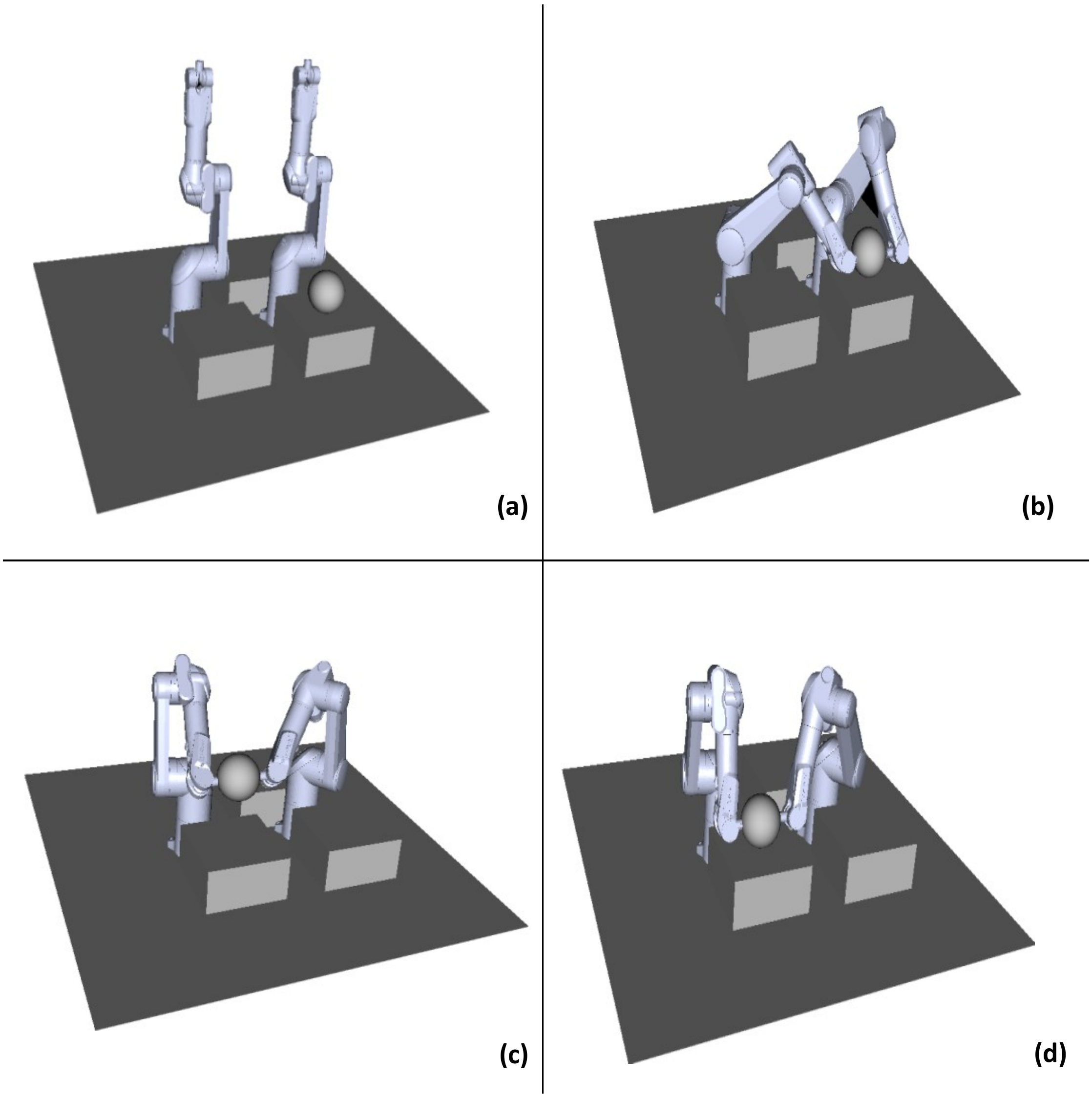

Figure 6.3: Cooperative working (carrying ball experiments)

The third simulation experiment is also cooperative working of serial robot arms. In this case, there is a master-slave mode working. The first robot arm which has a ball at the end effector moves by a given path and the second robot arm follows the tip point of the first robot arm as shown in figure 6.4. To implement this case, first a path is determined for the first robot arm. Then the orientation and the position data of the first robot arm is sent to the second robot arm and inverse kinematic of second robot arm is solved by using these data. The first robot arm which sends its position and orientation data works as a master and the second robot arm which follows the tip point of the first robot arm works as a slave. Similarly, orientations of robots are chosen adversely to each other.

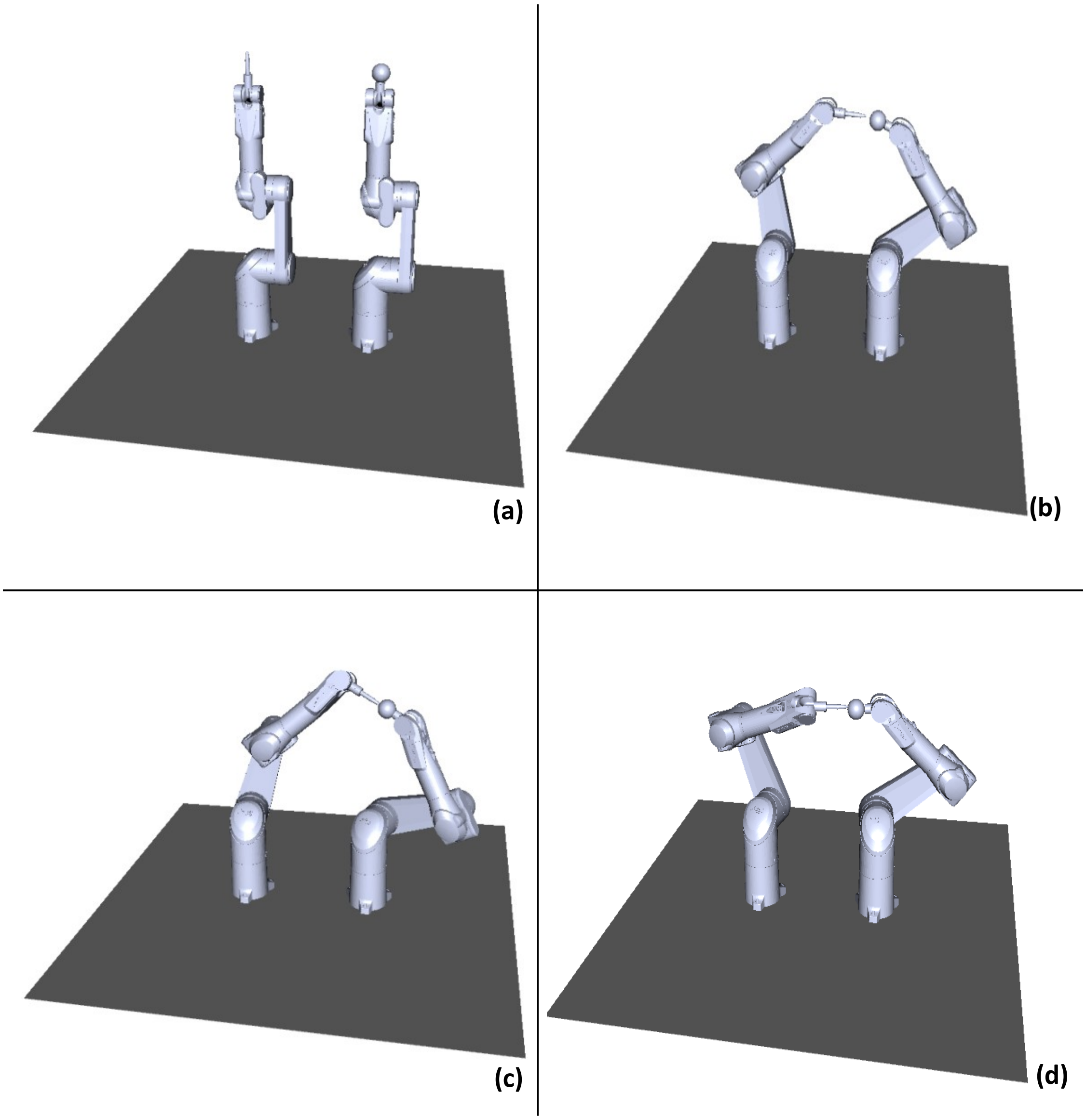

Figure 6.4: Cooperative working (master slave mode working)

6.2 Computational Cost and Comparative Study

Here, a comparative study of the presented methods is worked out. And also D-H method is included in this comparison because it is currently the most common method in robot kinematic. Forward and inverse kinematic solutions of 6-dof robot arm can be found in B. Siciliano and L. Sciavicco book [40].

In D-H method, there is a need for storing the transformation matrix or the orientation vector of every coordinate system with respect to its previous one from the beginning. In screw theory method, the storage cost is minimum because it is not necessary to store all the transformation from the beginning, as they are not needed to be known. And also the dual-quaternion requires eight memory locations, while the homogeneous transformation matrix 16. The storage requirement affects the computational time because the cost of fetching an operand from memory exceeds the cost of performing a basic arithmetic operation.

Table 6.2: Performance comparison of rotation operations

Method	Storage	Multiplies	Add/Subtracts	Total
Rotation matrix	9	27	18	45
Quaternion	4	16	12	28

Table 6.3: Performance comparison of rigid transformation operations

Method	Storage	Multiplies	Add/Subtracts	Total
Homogenous matrix	16	64	48	112
Dual-Quaternion	8	48	34	82

In order to determinate the position and the orientation of the end effector for n link serial arm robot manipulator:

- $64\,x\,(n-1)$ multiply and $48\,x\,(n-1)$ addition must be done if D-H method is used.
- $64\,x\,n$ multiply and $48\,x\,n$ addition must be done if screw theory with exponential mapping method is used.
- $32\,x\,(2n-1) + 16\,x\,(n-1)$ multiply and $24\,x\,(2n-1) + 12\,x\,(n-1) + 11$ addition must be done if screw theory with quaternion method is used.
- $48\,x\,(n-1) + 96$ multiply and $34\,x\,(n-1) + 68$ addition must be done if screw theory with dual-quaternion method is used.

If we take n=6 we get 320 multiply and 240 addition for D-H method, 384 multiply and 288 addition for screw theory with exponential mapping method, 432 multiply and 335 addition for screw theory with quaternion method, 336 multiply and 238 addition for screw theory with dual-quaternion method.

If we optimize our transformation algorithms we get for n link

- $48\,x\,(n-1)$ multiply and $36\,x\,(n-1)$ addition must be done if D-H method is used.
- $48\,x\,n$ multiply and $36\,x\,n$ addition must be done if screw theory with exponential mapping method is used.
- $16\,x\,(n-1) + 24\,x\,(2n-n-1)$ multiply and $18\,x\,(2n-n-1) + 12\,x\,(n-1) + 6$ addition must be done if screw theory with quaternion method is used.
- $40\,x\,(n-1) + 80$ multiply and $30\,x\,(n-1) + 18$ addition must be done if screw theory with dual-quaternion method is used.

If we take n=6 we get 240 multiply and 180 addition for D-H method, 288 multiply and 216 addition for screw theory with exponential mapping method, 200 multiply and 156 addition for screw theory with quaternion method, 280 multiply and 168 addition for screw theory with dual-quaternion method.

Simulation results of these methods are as follow:

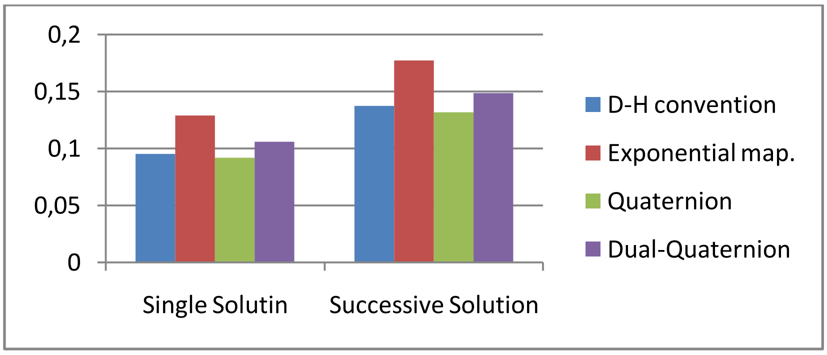

Figure 6.5: Simulation times (second) of the forward kinematic solutions

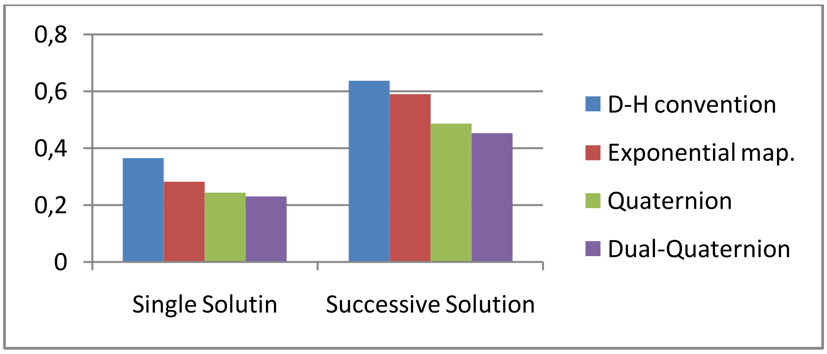

Figure 6.6: Simulation times (second) of the inverse kinematic solutions

Running environment is as table 6.4.

Table 6.4: Running environment

Cpu	Cpu Memory	Operating System	Simulation Software
Intel Core 2 Duo 2.2 GHz	2 GB	Windows XP	Matlab 7

Inverse kinematic solutions of D-H convention and screw theory can be analyzed from table 6.4 and table 6.5. As it can be seen from table 6.5 screw theory solutions are more accurate then D-H convention solution. And also as it can be seen from table 6.6 screw theory is a singularity avoiding method while D-H convention suffers from singularity.

Table 6.5: Inverse kinematic solutions in nonsingular case

Real Angle	Screw Solutions	Screw Error	D-H Solution	D-H Error
θ1=0.6283	θ1=0.6283	0	θ1=0.6283	0
θ2=0.5236	θ2=0.5236	0	θ2=0.5236	0
θ3=0.4488	θ3=0.4488	0	θ3=0.4488	0
θ4=0.5236	θ4=0.5236	0	θ4=0.5255	0.0019
θ5=0.2856	θ5=0.2856	0	θ5=0.2855	0.0001
θ6=1.0472	θ6=1.0471	0.0001	θ6=1.0474	0.0001

Table 6.6: Inverse kinematic solutions in singular case

Real Angle	Screw Solutions	D-H Solution
θ1=0.6283	θ1=0.6283	Unreal
θ2=0.5236	θ2=0.5236	Unreal
θ3=1.5708	θ3=0	Unreal
θ4=0.5236	θ4= 0.6434	Unreal
θ5=0.2856	θ5= 1.5555	Unreal
θ6=1.0472	θ6= 1.1669	Unreal

Note that in singular case some solutions are not same as real angle, because there are infinite solutions in singular case and one solution is found from infinite solutions.

7. CONCLUSION

In this thesis, three methods for the formulation of the kinematic equations of robot manipulators have been presented. Two of them are new formulation methods which based on screw theory and quaternion algebra. In these two new formulation methods, one of them uses quaternions as a screw operator which combines a unit quaternion plus pure quaternion and the other one uses dual-quaternions as a screw operator. The other method is also based on screw theory however it uses 4x4 matrices as a screw operator. Using screw theory in robot kinematics has several advantages. Screw theory based on line transformation. It is an effective way to establish a global description of rigid body and avoids singularities due to the use of the local coordinates. The main advantage of this method lies in its geometrical representation of link and joint axes of a manipulator, giving better understanding of its configuration in the workspace and avoiding singularities due to the use of the local coordinates. Two coordinate frames are needed for n degree of freedom open-chain robot arms. These coordinate systems are established at the base frame which is arbitrary frame and at the end effectors frame. The other joints are represented by using single axes. All joint axes and end effectors coordinate frame are represented with respect to base coordinate frame. This representation simplifies description of mechanism and avoids singularities due to the use of the local coordinates.

These three methods and also the D-H convention are compared with respect to singularity, computation efficiency and accuracy. The D-H convention is added in this comparison, because it is the most common method in robot kinematic. The D-H convention uses homogenous transformation matrices as a transformation operator and it based on point transformation. In this method, coordinate systems are described with respect to previous one. For the base point an arbitrary base coordinate frame can be used. Hence some singularity problems may occur because of this description of the coordinate frames. And also in the D-H representation n coordinate frames are needed for n link robot manipulator. This representation gives rise to complexity. Hence, we can say that screw theory methods are singularity avoiding methods and their geometrical descriptions are quite simple however the D-H convention suffers from singularities and its geometrical description is complex. Comparison with respect to computation efficiency is based on two main factors. The first one is the used theory and the second one is the selected transformation operator. However the D-H convention is more computationally efficient than screw theory methods in the forward kinematics, screw theory methods are more computationally efficient than the D-H convention in the inverse kinematics. Screw theories with quaternion and dual-quaternion results are very close to the D-H convention in the forward kinematics. If we consider only screw theory methods we can understand the affect of the transformation operators on the computation efficiency easier. Screw theories with quaternion and dual-quaternion methods are more computationally efficient than exponential mapping method in both of the forward and inverse kinematics. If we compare screw theory with quaternion and dual-quaternion methods we will get similar results for low degrees of freedom of robot arms however dual-quaternion is more computationally efficient when the degrees of freedom of robot arms grow up. And also in screw theory more accurate solutions are obtained in the inverse kinematics.

On this account, the wider use of the screw theory methods into the robotics community has to be considered. Nevertheless homogenous transformations with the D-H convention applications are more common than screw theory. Because point transformation can be understood easier than line transformation, mathematical substructure of the DH convention is simpler than screw theory and also the D-H convention is well defined method.

Screw theories with quaternion or dual-quaternion describe robot kinematics by using eight parameters without suffering singularities. I believe that these properties are more useful for hyper-degrees of freedom systems like humanoid, quadruped vs. than lower-degrees of freedom like robot arms. Since, there are much more singularity points in hyper-dof systems and these singularity points control is more difficult because of increasing of singularity point numbers. And also lower parameters representation of hyper-degrees of freedom systems is more useful for computational efficient. Thus, in the future works screw theory with quaternion methods should apply to hyper-degrees of freedom systems. In addition the trajectory generation, velocity and dynamic analysis based on screw theory with quaternion should be studied.

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APPENDICES

APPENDIX A.1

Suppose we have a point (x, y) in the Euclidean plane. To represent this same point in the projective plane, we simply add a third coordinate of 1 at the end: $(x, y, 1)$. Overall scaling is unimportant, so the point $(x,y,1)$ is the same as the point (ax, ay, a), for any nonzero a. In other words,

$$(x, y, w) = (ax, ay, aw) \tag{A.1}$$

for any $a \neq 0$. (Thus the point (0,0,0) is disallowed). Because scaling is unimportant, the coordinates (x, y, w) are called the homogeneous coordinates of the point.

To transform a point in the projective plane (X, Y, W) back into Euclidean coordinates, we simply divide by the third coordinate: $(x, y) = (X/W, Y/W)$. Immediately we see that the projective plane contains more points than the Euclidean plane, that is, points whose third coordinate is zero. These points are called ideal points, or points at infinity.

A general rigid body transformation (rotation and translation) is defined by using homogeneous transformation matrices. Generally it is represented using $4x4$ matrices as follow

$$T = \begin{pmatrix} R_{3x3} & \boldsymbol{t_{3x1}} \\ 0_{1x3} & 1 \end{pmatrix} \tag{A.2}$$

Homogeneous matrices have the following advantages [25]:

- Simple explicit expressions exist for many familiar transformations including rotation
- These expressions are n-dimensional
- There is no need for auxiliary transformations, as in vector methods for rotation
- More general transformations can be represented (e.g. projections, translations)
- Directions (ideal points) can be used as parameters of the transformation, or as inputs
- If nonsingular matrix T transforms point P by PT, then hyperplane h is transformed by $T^{-1}h$
- The columns of T (as hyperplanes) generate the null space of T by intersections
- Many homogeneous transformation matrices display the duality between invariant axes and centers.

APPENDIX A.2

Euclidean $n-space$, sometimes called Cartesian space or simply $n-space$, is the space of all $n-tuples$ of real numbers, $(x_1, x_2, \dots, x_n)$. It is commonly denoted by $\mathbb{R}^n$. Elements of $\mathbb{R}^n$ are called $n-vectors$. Euclidean n-space is the most elementary example of an n dimensional manifold. $\mathbb{R}^1 = \mathbb{R}$ is the set of real numbers (i.e., the real line), and $\mathbb{R}^2$ is called the Euclidean plane. In Euclidean space, covariant and contravariant quantities are equivalent so $\vec{\boldsymbol{e}}^j = \vec{\boldsymbol{e}}_j$. [34]

Euclidean Space can also be defined by Euclid's postulates. These postulates are:

1. A straight line may be drawn from any one point to any other point (any 2 points determine a unique line).
2. A finite straight line may be produced to any length in a straight line.
3. A circle may be described with any centre at any distance from that centre.
4. All right angles are equal.
5. If a straight line meets two other straight lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles. [34]

Euclidean vector space

An ordered triple $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})$ can be interpreted geometrically as a point or a vector. In Euclidean space vectors have these properties: Let $\boldsymbol{u}, \boldsymbol{v}$ **and** $\boldsymbol{w}$ be vectors and k be scalar**.**

1) $\boldsymbol{u} \,.\, \boldsymbol{v} = \boldsymbol{v} \,.\, \boldsymbol{u}$
2) $(\boldsymbol{u} + \boldsymbol{v}).\boldsymbol{w} = \boldsymbol{u} \,.\, \boldsymbol{w} + \boldsymbol{v} \,.\, \boldsymbol{w}$
3) $(\boldsymbol{ku}) \,.\, \boldsymbol{v} = \boldsymbol{k}(\boldsymbol{u} \,.\, \boldsymbol{v})$
4) $\boldsymbol{v} \,.\, \boldsymbol{v} = \boldsymbol{0}$ if and only if $\boldsymbol{v} = \boldsymbol{0}$

Norm and distance in Euclidean n-space

Euclidean norm: $$||\boldsymbol{u}|| = \sqrt[2]{\boldsymbol{u} \,.\, \boldsymbol{u}} = \sqrt[2]{\boldsymbol{u}_1^2 + \boldsymbol{u}_2^2 + \cdots + \boldsymbol{u}_n^2} \qquad \textbf{(A.4)}$$

Euclidean length:
$$(\boldsymbol{u}\,, \boldsymbol{v}) = ||\boldsymbol{u} - \boldsymbol{v}|| = \sqrt[2]{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \qquad \textbf{(A.5)}$$

Properties of Euclidean space

- There is no preferred origin in Euclidean space. Any point would be as good as any other as a choice for the origin.

- There is no preferred direction in Euclidean space.
- There is no specific way to define a point at infinity.
- The 'metric' for Euclidean space. That is a function, for a given space, that defines the distance between points. For Euclidean space, if $\boldsymbol{p}$ and $\boldsymbol{q}$ are two points then:

  $$\left\|\boldsymbol{p} - \boldsymbol{q}\right\|^2 = (\boldsymbol{p} - \boldsymbol{q})(\boldsymbol{p} - \boldsymbol{q}) \qquad \textbf{(A.6)}$$
- Euclidean space is flat, linear and continuous (differentiable) [9]

APPENDIX B.1

Let $g \epsilon GL(n)$ be a general linear group element and it acts on M which is the space of all $n\ x\ n$ matrices. This space is in fact a vector space and it satisfied vector space axioms. Similarity transformation can be expressed as:

$$S(g, M) = gMg^{-1} \tag{A.7}$$

This is an action since it satisfies three axioms which are given in semi direct product. We can use the same representation for any subgroup of $GL(n)$ by restricting the maps to the subgroup. In group theory the operation which sends a group element m to gmg^{-1} is called conjugation, where g is another group element.

There is also another action which is called congruence. It can be represented as:

$$C(g, M) = gMg^{T} \tag{A.8}$$

If we take M as symmetric matrix, that is, if it satisfies $M = M^T$ then the transformed matrix will also be symmetric. Hence we will restrict our attention on symmetric matrices for congruence action. Using arguments analogous to the ones above, we can easily show that this action satisfies our three axioms and thus gives another representation of $GL(n)$.

Note that the orthogonal matrices are defined as the set of matrices that preserve die identity matrix under congruence transformations. Orthogonal groups are the set of matrices which satisfy $gg^T = g^Tg = I$. However, for symmetric matrices, it doesn't matter whether we define the congruence as gMg^T or g^TMg [18], [19].

APPENDIX B.2

A group homomorphism is a map $f: G \longrightarrow H$ between two groups such that the group operation is preserved: $f(g_1 g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$, where the product on the left-hand side is in G and on the right-hand side in H. For lie groups this mapping must be differentiable.

A simple consequence of this definition is that the identity element of a group is always mapped to the identity element in the other group by a homomorphism. To see this notice that:

$$f(eg) = f(g) = f(e)f(g) \text{ for all } g \in G \tag{A.9}$$

The inverse element of an element g is always mapped to the inverse of the image of g. That is $f(g^{-1}) = (f(g))^{-1}$. To see this consider:

$$e = f(e) = f(gg^{-1}) = f(g)f(g^{-1}) \tag{A.10}$$

For instance, the group of rotation $S(3)$ can be mapped into the general rigid motion group $SE(3)$ by sending each rotation R to a rotation about the origin $R \longrightarrow \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$. Also rotation can be mapped to rotation about any other point in space. [10], [19]

APPENDIX B.3

Given two groups G and *H,* elements of their direct product, written $G\,X\,H$ are pairs of elements (g_1, h_1), where $g \in G$ and $h \ \in H$. The group operation is:

$$(g_1, h_1)(g_1, h_1) = (g_1 g_2, h_1 h_2) \qquad \textbf{(A.11)}$$

Many of the groups we have defined above have been described as groups of symmetry operations on certain spaces. Consider a group G and a manifold X. G acts on X if there is a differentiable map

$$a : G \times X \longrightarrow X$$

that satisfies

$$a(e, x) = x \quad \text{for all } x \in X \quad \text{(axiom 1)} \qquad \textbf{(A.12)}$$

and

$$a\big(g_1, a(g_2, x)\big) = a(g_1 g_2, x) \quad \text{for all } x \in X \text{ and } g_1, g_2 \in G \text{ (axiom 2)} \qquad \textbf{(A.13)}$$

Take X to be die group manifold of G itself, then G acts on itself by left multiplication: $l(g_1, g) = g_1 g$. Right action of G on itself, given by $r(g_1, g) = g g_1^{-1}$. Suppose X is a vector space, $\mathbb{R}^n$ say. The matrix groups now act on this space by matrix multiplication:

$$a(M, x) = Mx \qquad \textbf{(A.14)}$$

where $M \in GL(n)$ and $x \in \mathbb{R}^n$. Linear actions of groups on vector spaces are called representations, that is, actions which satisfy

$$g(h_1 + h_2) = g(h_1) + g(h_2) \quad \text{for all } g \in G \text{ and all } h_1, h_2 \in H \quad \text{(axiom 3)} \qquad \textbf{(A.15)}$$

Now suppose we have a group G and a commutative group H together with a linear action of G on H. That is, a map $G\,x\,H \longrightarrow H$ given by $g(h)$ satisfying all three axioms given above. The semi-direct product of G and H, written $G\,x\,H$, has the same elements as the direct product, that is, pairs of the form (g, h), where $g \in G$ and $h \in H$ [5]. However, the product of two elements is defined as

$$(g_1, h_1)(g_2, h_2) = g_1 g_2, h_1 + g_1(h_2) \qquad \textbf{(A.16)}$$

APPENDIX B.4

Let left translation be $L_a: G \longrightarrow G, x \longrightarrow a.x$ (Left translation by $a \in G$)

Clearly, La is diffeomorphisms of G onto itself. The corresponding differentials will be denoted by dL_a:

$$dL_a: TG \longrightarrow TG \qquad \textbf{(A.17)}$$

We shall use the same notation for the differential at a fixed point $x \in G$:

$$dL_a: T_xG \longrightarrow T_xG \qquad \textbf{(A.18)}$$

A vector field ξ is called left invariant, if it is preserved under left translations. For instance, for any $\in G$:

$$dL_a(\xi) = \xi \qquad \textbf{(A.19)}$$

In other words, if we consider the values $\xi(x)$ and $\xi(y)$ of our vector field at two distinct points $x \in G$ and $y = ax \in G$, then they must be related by the linear operator [35]

$$dL_a: T_xG \longrightarrow T_yG, \text{i.e. } dL_a(\xi(x)) = \xi(ax) = \xi(y) \qquad \textbf{(A.20)}$$

Properties of left invariant vector fields

- The space of left invariant vector fields is naturally isomorphic (as a vector space) to the tangent space T_eG at the identity.
- Left invariant vector fields are complete
- Integral curves of left invariant vector fields through the identity $e \in G$ aree exactly one-parameter subgroups
- The space of left invariant vector fields is closed under the Lie bracket

APPENDIX B.5

Besides the left and right actions of G on itself, there is the conjugation action

$$c(g) : h \longrightarrow ghg^{-1}$$

Unlike the left and right actions which are transitive, this action has fixed points, including the identity.

Adjoint Representation: The differential of the conjugation action, evaluated at the identity, is called the adjoint action

$$Ad(g) = c_*(g)(e) : T_eG \longrightarrow T_eG \quad \textbf{(A.21)}$$

Identifying g with T_eG and invoking the chain rule to show that

$$Ad(g_1)Ad(g_2) = Ad(g_1g_2) \quad \textbf{(A.22)}$$

this gives a homomorphism

$Ad(g)$: $G \longrightarrow GL(g)$ is called the adjoint representation.

For the matrix group case, the adjoint representation is just the conjugation action on matrices

$$Ad(g)(y) = gYg^{-1} \quad \textbf{(A.23)}$$

since one can think of the Lie algebra in terms of matrices infinitesimally close to the unit matrix and carry over the conjugation action to them.

To find the product we look at the derivative of the adjoint representation.We look at group elements near the identity so that they can be approximated by $g \approx I + tS_1$ and its inverse by $g^{-1} \approx I - tS_1$. This is used to adjoint representation of S_2

$$(I + tS_1)S_2(I - tS_1) = S_2 + t(S_1S_2 - S_2S_1) \quad \textbf{(A.24)}$$

differentiating and setting $t = 0$ gives the Lie algebra element $(S_1S_2 - S_2S_1)$. This is Lie bracket or commutator of pair of elements, it is usually written using square bracket [19],

$$[S_1, S_2] = S_1S_2 - S_2S_1 \quad \textbf{(A.25)}$$

APPENDIX C.1

A line can be completely defined by the ordered set of two vectors as shown in figure A.1. First one is point vector (***p***) which indicates the position of an arbitrary point on line, and the other vector is free direction vector (***d***) which gives the line direction. A line can be expressed as:

$$L(\boldsymbol{p}, \boldsymbol{d})$$

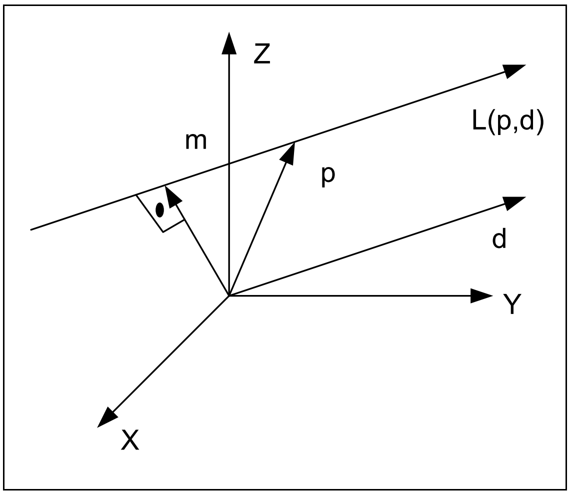


Figure A.1: A line in Cartesian coordinate-system

The representation $L(\boldsymbol{p}, \boldsymbol{d})$ is not minimal, because it uses six parameters for only four degrees of freedom. With respect to a world reference frame, the line's coordinates are given by a six-vector.

APPENDIX C.2

An alternative line representation was introduced by A. Cayley and J. Plücker. Finally this representation named after Plücker [20]. Plücker coordinates can be expressed as

$$L_p(\boldsymbol{m}, \boldsymbol{d}) \quad \text{where} \quad \boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{d} \tag{A.26}$$

Both $\boldsymbol{d}$ and $\boldsymbol{m}$ are free vectors: $\boldsymbol{d}$ and $\boldsymbol{p}$ have the same meaning as before (they represent the direction of the line and the position of an arbitrary point on the line respectively) and $\boldsymbol{m}$ is the moment of $\boldsymbol{d}$ about the chosen reference origin. Note that $\boldsymbol{m}$ is independent of which point $\boldsymbol{p}$ on the line is chosen:

$$\boldsymbol{p} \times \boldsymbol{d} = (\boldsymbol{p} + t\boldsymbol{d}) \times \boldsymbol{d} \tag{A.27}$$

The two three-vectors $\boldsymbol{d}$ and $\boldsymbol{m}$ are always orthogonal, $\boldsymbol{d}.\boldsymbol{m} = \boldsymbol{0}$ **(A.28)**

A line has still four degrees of freedom while it is represented by plücker coordinates. Hence plücker coordinates representation is not minimal. The advantage of plücker coordinate representation is that it is homogeneous: $L_p(\boldsymbol{m}, \boldsymbol{d})$ represents same line as $L_p(k\boldsymbol{m}, \mathrm{k}\boldsymbol{d})$, where $k \in \mathbb{R}$

APPENDIX C.3

Two intersect lines are shown in figure A.2.

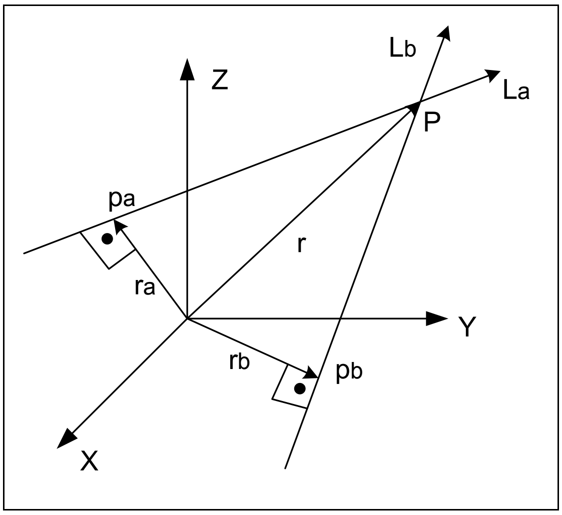


Figure A.2: Two intersection lines in Cartesian coordinate-system

$La = (\boldsymbol{ma}, \boldsymbol{da}) \quad Lb = (\boldsymbol{mb}, \boldsymbol{db})$ **(A.29)**

We can write these equations for two intersect lines as

$\boldsymbol{ra} = \boldsymbol{da} \times \boldsymbol{ma}$ and $\boldsymbol{rb} = \boldsymbol{db} \times \boldsymbol{mb}$ **(A.30)**

We can write

$\boldsymbol{r} = \boldsymbol{ra} + \alpha\boldsymbol{da} = \boldsymbol{rb} + \beta\boldsymbol{db}$ where $\alpha, \beta \in \mathbb{R}$ **(A.31)**

$\alpha\boldsymbol{da} = \boldsymbol{ra} - \boldsymbol{rb} + \beta\boldsymbol{db} \;\; or \;\; \beta\boldsymbol{db} = \boldsymbol{rb} - \boldsymbol{ra} + \alpha\boldsymbol{da}$ **(A.32)**

If we multiply first and second equations in A.32 by $\boldsymbol{pa}$ and $\boldsymbol{pb}$ respectively we obtain:

$\alpha = (\boldsymbol{ra} - \boldsymbol{rb}).\boldsymbol{db} + \beta.\boldsymbol{da}.\boldsymbol{db} = \boldsymbol{ra}.\boldsymbol{db}$ **(A.33)**

$\beta = (\boldsymbol{rb} - \boldsymbol{ra}).\boldsymbol{da} + \alpha.\boldsymbol{da}.\boldsymbol{db} = \boldsymbol{rb}.\boldsymbol{da}$ **(A.34)**

Note that: $\boldsymbol{ra}.\boldsymbol{da} = \boldsymbol{rb}.\boldsymbol{db} = \boldsymbol{0}$ **(A.35)**

Hence we can write:

$\boldsymbol{r} = \boldsymbol{db} \times \boldsymbol{mb} + (\boldsymbol{da} \times \boldsymbol{ma}.\boldsymbol{db}).\boldsymbol{db}$ or $\boldsymbol{r} = \boldsymbol{da} \times \boldsymbol{ma} + (\boldsymbol{db} \times \boldsymbol{mb}.\boldsymbol{da}).\boldsymbol{da}$

APPENDIX C.4

The dual number was originally introduced by Clifford in 1873 [36], [37]. In analogy with a complex number a dual number can be defined as:

$$\hat{u} = u + \epsilon u^0 \quad \textbf{(A.36)}$$

where u and u^0 are real numbers, $\epsilon^2 = 0$

Every function f of the dual numbers obeys the rule

$$f(a + \varepsilon b) = f(a) + \varepsilon b\dot{f}(\mathrm{a}) \quad \textbf{(A.37)}$$

Hence

$$\cos\left(\frac{\theta+\varepsilon d}{2}\right) = cos\frac{\theta}{2} - \varepsilon\frac{d}{2}sin\frac{\theta}{2} \quad \text{and} \quad \sin\left(\frac{\theta+\varepsilon d}{2}\right) = sin\frac{\theta}{2} + \varepsilon\frac{d}{2}cos\frac{\theta}{2} \quad \textbf{(A.38)}$$

We can use dual numbers to express plücker coordinates. We can write orientation vector ($\boldsymbol{u}$) and moment vector ($\boldsymbol{u}^0$) as:

$$\boldsymbol{u} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \quad and \quad \boldsymbol{u}^0 = \boldsymbol{p} \times \boldsymbol{u} \quad \textbf{(A.39)}$$

Hence a line can be expressed in plücker coordinates by using dual number as:

$$\hat{u} = \boldsymbol{u} + \epsilon\boldsymbol{u}^0 \quad \textbf{(A.40)}$$

APPENDIX C.5

Subproblem 1. Rotation about a single axis:

A subproblem 1 is a point rotation formulization [38]. It can be easily explained by using figure A.3.

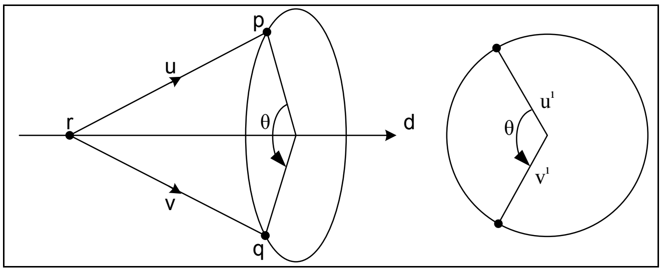


Figure A.3: Rotate p about the axis of d until it is coincident with q

A point p rotates about the axis of d until it is coincident with point $q$as shown in figure 5. Let r be a point on the axis of d, $u = p - r$ and $v = q - r$ are two vectors. We can write general rotation formula for this motion as:

$v = R(\theta, d)u$ **(A.41)**

and

$\theta = atan2(d^T(u'xv'), u'.v')$ **(A.42)**

where $u' = u - dd^Tu$ and $v' = v - dd^Tv$ **(A.43)**

If $u = 0$ then there are infinite number of solutions since in this case $p = q$ and both points lie on the axis of rotation.

Subproblem2. Rotation about two subsequent axes

This problem correspond to rotating a point p first about the axis of d_2 by $\theta_2$and then about the axis of d_1 by θ_2, hence the final location of p is coincident with the point q. It can be easily explained by using figure A.4.

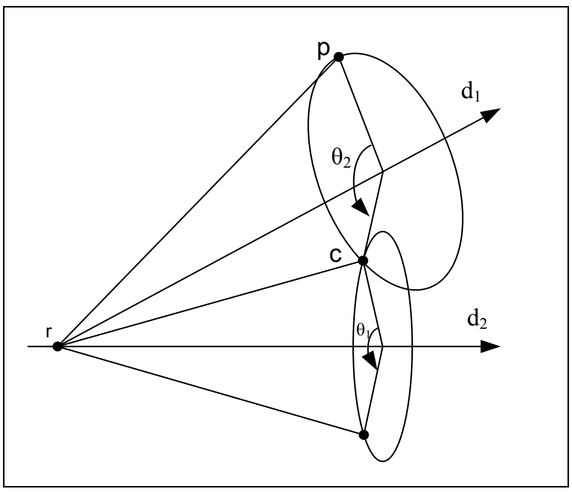


Figure A.4: Rotate p about the axis of d1 followed by a rotation around the axis of d2 until it is coincident with q

Two axes must be intersected. If two axes are coincide, this problem reduce to subproblem 1. If two axes are not parallel , i.e. $d_1xd_2 \neq 0$, then let c be a point such that

$$R(\theta_2, d_2)p = c = R(-\theta_1, d_1) \quad \textbf{(A.44)}$$

In other words c represents the point to which p is rotated about the axis of d_2 by θ_2. Let r be the point of intersection of the two axes so that

$$R(\theta_2, d_2)(p-r) = c - r = R(-\theta_1, d_1)(q-r) \quad \textbf{(A.45)}$$

As before, define vectors $= p - r$, $v = q - r$ and z= $c - r$. Substituting these expressions into equation A.45 gives

$$R(\theta_2, d_2)u = z = R(-\theta_1, d_1)v \quad \textbf{(A.46)}$$

Since d_1 , d_2 and d_1xd_2 are linearly independent, we can write

$$z = \alpha d_1 + \beta d_2 + \gamma(d_1xd_2) \quad \textbf{(A.47)}$$

where

$$\alpha = \frac{(d_1^T d_2)d_2^T u - d_1^T v}{(d_1^T d_2)^2 - 1}, \quad \beta = \frac{(d_1^T d_2)d_1^T v - d_2^T u}{(d_1^T d_2)^2 - 1} \quad \textbf{(A.48)}$$

and

$$\gamma^2 = \frac{||u||^2 - \alpha^2 - \beta^2 - 2\alpha\beta d_1^T d_2}{||d_1xd_2||^2} \quad \textbf{(A.49)}$$

In the case that a solution exists, we can find z- and hence c- given α, β and γ. To find $\theta_1$and θ_2, we solve

$$R(\theta_2, d_2)p = c \quad \text{and} \quad R(-\theta_1, d_1)q = c \quad \textbf{(A.50)}$$

using subproblem 1 [38].

Subproblem 3. Rotation to a given distance

This problem correspond the rotating a point p about the axis d until the point is a distance δ from q as shown in figure A.5.

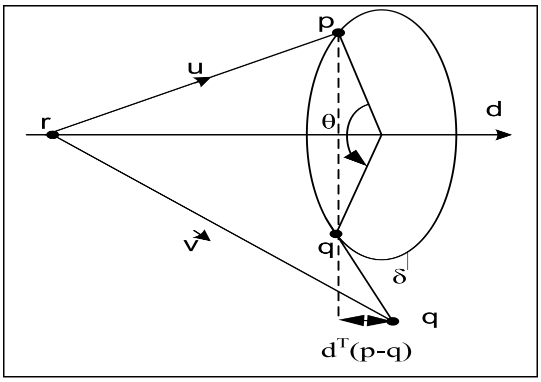


Figure A.5: Rotate p about the axis of d until it is a distance δ from q

To find the explicit solution, we again consider the projection of all points onto the plane perpendicular to d, the direction of the axis of d. Let r be a point on the axis of d and define $u = p - r$ and $v = q - r$. The projections of u, v and δ are

$$u' = u - dd^T u \quad \text{and} \quad v' = v - dd^T v$$

$$\delta'^2 = \delta^2 - |d^T(p-q)|^2, \quad \textbf{(A.51)}$$

If let θ_0 be the angle between the vectors u' and v' we have

$$\theta_0 = atan2(d^T(u'xv'), u'^T v') \quad \textbf{(A.52)}$$

Hence

$$\theta = \theta_0 \pm cos^{-1}\left(\frac{\left||u'|\right|^2 + \left||v'|\right|^2 - \delta'^2}{2||u'|||v'||}\right) \quad \textbf{(A.53)}$$

Equation A.53 has either zero, one or two solutions, depending on the number of points in which the circle of radius $\left||u'|\right|$ intersects the circle of radius δ' [38].

APPENDIX D.1

There are two types of rotation transformation that we want to consider:

1. Finite rotations that is a change from one angular orientation to another.

2. Continuous and infinitesimal rotations, such as when an object is continuously rotating.

When we first think of these types of rotation we might guess that one of these would be the rate of change of the other and that they would obey similar rules.

However it turns out that continuous and infinitesimal rotations are easily combined using vector addition. However finite rotations are more complicated and require other types of algebra.

APPENDIX D.2

The idea behind Euler angles is to split the complete rotation of the coordinate system into three simpler constitutive rotations, in such a way that the complete rotation matrix is the product of three simpler matrices. Euler angles presentation is not unique and in the literature there are many different conventions are used. (There are $3x3x3 = 27$ possible definitions for Euler Angle rotations, but not all of them are real.) These conventions depend on the axes about which the rotations are carried out, and their sequence (since rotations are not commutative). For instance, the engineering and robotics communities typically use $Z - X - Z$ Euler angles [Url-3].

Advantages / disadvantages of Euler representation

Advantages:

Three-angle orientation representations have two advantages:

- They use the minimal number of parameters.
- One can choose a set of three angles that, by design, feels natural or intuitive for a given application.

Disadvantages

- No set of three angles can globally represent all orientations without singularity,
- A small rotation about the Y-axis requires large rotations of the joints in a $Z - X - Z$ wrist.

APPENDIX D.3

From Euler's rotation theorem we know that any rotation can be expressed as a single rotation about some axis. The axis is the unit vector (unique except for sign) which remains unchanged by the rotation. The magnitude of the angle is also unique, with its sign being determined by the sign of the rotation axis. The axis can be represented as a three-dimensional unit vector $\hat{\boldsymbol{e}} = [e_x, e_y, e_z]$, and the angle by a scalar θ. Combining two successive rotations, each represented by an Euler axis and angle, is not straightforward. It is usual to convert to direction cosine matrix or quaternion notation, calculate the product, and then convert back to Euler axis and angle. Euler angle representation can be analyzed using figure A.6.

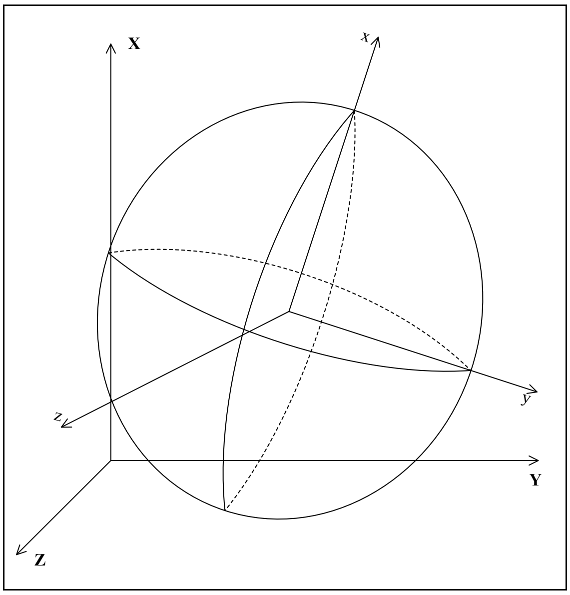


Figure A.6: Euler angles.(XYZ fixed coordinate-system, xyz rotated frame coordinate-system)

APPENDIX D.4

Rodrigues' rotation formula gives an efficient method for computing the rotation matrix $R \epsilon SO(3)$ corresponding to a rotation by an angle $\theta \epsilon \mathbb{R}$ about a fixed axis specified by the unit vector $\boldsymbol{w} = (w_x, w_y, w_z)$. Then R is given by

$$e^{\widehat{w}\theta} = I + \widehat{w} sin\theta + \widehat{w}^2(1 - cos\theta)$$

$$= \begin{bmatrix} cos\theta + w_x^2(1 - cos\theta) & w_x w_y(1 - cos\theta) - w_z sin\theta & w_y sin\theta + w_x w_z(1 - cos\theta) \\ w_z sin\theta + w_x w_y(1 - cos\theta) & cos\theta + w_y^2(1 - cos\theta) & -w_x sin\theta + w_y w_z(1 - cos\theta) \\ -w_y sin\theta + w_x w_z(1 - cos\theta) & w_x sin\theta + w_y w_z(1 - cos\theta) & cos\theta + w_z^2(1 - cos\theta) \end{bmatrix}$$

(A.54)

where I is 3×3 identity matrix and $\widehat{w}$ denotes the skew-symmetric matrix with entries

$$\widehat{w} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \quad \textbf{(A.55)}$$

Note that $\widehat{w}w = 0$, so applying the rotation matrix given by Rodrigues' formula to any point on the rotation axis returns the same point [Url-4].

APPENDIX D.5

It is quite difficult to give a physical meaning to a quaternion, it is just a quantity which represents a rotation. If you need a physical meaning then this is probably the best way to think of it:

$$q = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\vec{\mathbf{n}}\right) \qquad \textbf{(A.56)}$$

where θ angle of rotation and $\vec{\mathbf{n}}$ vector representing axis of rotation. So it is closely related to the axis angle representation of rotations. But the physical meaning of quaternion is some more complicated then axis angle representation [39].

Any rotation in three dimensions is a rotation by some angle about some axis. When the angle is zero the axis does not matter, so rotation by zero degrees is a single point in the space of rotations (the identity rotation). For a tiny but nonzero angle, the set of possible rotations is like a small sphere surrounding the identity rotation, where each point on the sphere represents an axis pointing in a particular direction (compare the celestial sphere). Rotations through increasingly large angles are increasingly far from the identity rotation, and we can think of them as concentric spheres of increasing radius. Thus, near the identity rotation, the abstract space of rotations looks similar to ordinary three-dimensional space (which can also be seen as a central point surrounded by concentric spheres of every radius). However, as the rotation angle increases past 180°, rotations about different axes stop diverging and become more similar to each other, becoming identical (and equal to the identity rotation) when the angle reaches 360°.

We can see similar behavior on the surface of a sphere [Url-5]. If we start at the north pole and draw straight lines (that is, lines of longitude) in many directions, they will diverge but eventually converge again at the south pole. Concentric circles of increasing radius drawn around the north pole (lines of latitude) will eventually collapse to a point at the south pole once the radius reaches the distance between the poles. If we think of different directions from the pole (that is, different longitudes) as different rotation axes, and different distances from the pole (that is, different latitudes) as different rotation angles, we have an analogy to the space of rotations. Since the sphere's surface is two dimensional while the space of rotations is three dimensional, we must actually model the space of rotations as a hypersphere; however, we can think of the ordinary sphere as a slice through the full hypersphere (just as a circle is a slice through a sphere). We can take the slice to represent, for example, just the rotations about axes in the x,y plane. Note that the angle of rotation is twice the latitude difference from the north pole: points on the equator represent rotations of 180°, not 90°, and the south pole represents a rotation of 360°, not 180°.

The north pole and the south pole represent the same rotation, and in fact this is true of any two antipodal points: if one is a rotation by α around the axis $\mathbf{v}$, the other is a rotation by $360° - \alpha$ around the axis $-\mathbf{v}$. In fact, then, the space of rotations is not the (hyper)sphere itself but the (hyper)sphere with antipodal points identified. But for many purposes we can think of rotations as points on the sphere, even though they are twofold redundant (a so-called double cover).

We can parameterize the surface of a sphere with two coordinates, such as latitude and longitude. But latitude and longitude are ill-behaved (degenerate) at the north and south poles, though the poles are not intrinsically different from any other points on the sphere. It can be shown that no two-parameter coordinate system can avoid such degeneracy (the hairy ball theorem). We can avoid such problems by embedding the sphere in three-dimensional space and parameterizing it with three Cartesian coordinates (here w, x, y), placing the north pole at $(w, x, y) = (1,0,0)$, the south pole at $(w, x, y) = (-1,0,0)$, and the equator at $w = 0, x2 + y2 = 1$. Points on the sphere satisfy the constraint $w^2 + x^2 + y^2 = 1$, so we still have just two degrees of freedom though there are three coordinates. A point (w,x,y) on the sphere represents a rotation around the $(x, y, 0)$ axis by an angle $\alpha = 2\cos^{-1} w = 2\sin^{-1}\sqrt{x^2 + w^2}$.

In the same way the hyperspherical space of 3D rotations can be parameterized by three angles (Euler angles), but any such parameterization is degenerate at some points on the hypersphere, leading to the problem of gimbal lock. We can avoid this by using four Euclidean coordinates w, x, y, z, with $w2 + x2 + y2 + z2 = 1$. The point (w, x, y, z) represents a rotation around the (x, y, z) axis by an angle $\alpha = 2\cos^{-1} w$ or $\alpha = 2\sin^{-1}\sqrt{x^2 + w^2 + z^2}$. Quaternion representation can be analyzed using figure A.7.

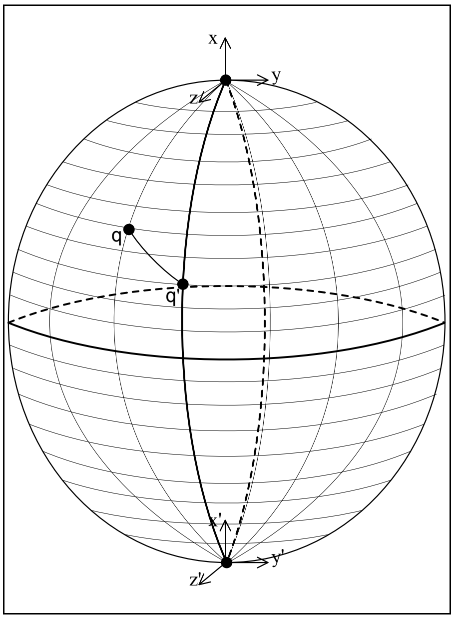
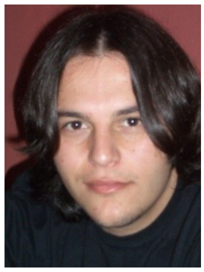


Figure A.7: Rotation representation using complex numbers and two dimensional sphere surface



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- **E. Sariyildiz, H. Temeltaş ,** "Solution of Inverse Kinematic Problem for Serial Robot Using Dual-Quaterninons and Plücker Coordinates", Advanced Intelligent Mechatronics, June 2009