İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

A STUDY OF WAVE EQUATIONS IN FIVE DIMENSIONAL SPACETIMES WITH COMPUTATIONAL METHODS

Ph.D. Thesis by

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Department : Physics Engineering

Programme : Physics Engineering

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BİLGİSAYARLI HESAPLAMA YÖNTEMLERİ İLE BEŞ BOYUTLU UZAYZAMANLARDA DALGA DENKLEMLERİNİN İNCELENMESİ

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ABBREVATIONS

APS	:	Atiyah-Patodi-Singer
BC	:	Boundary Conditions
NP	:	Newman-Penrose
5D	:	Five Dimensions
4D	:	Four Dimensions

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LIST OF SYMBOLS

$\partial\mathscr{M}$	:	Boundary of the manifold $\mathscr{M}$
$\mathbf{g}$	:	Determinant of the metric tensor
γ^{μ}	:	Dirac matrices
$\mathbf{H_D}$	:	Double confluent Heun functions
$\mathbf{Se}$	:	Even Mathieu functions
$\mathscr{H}$	:	Laplace operator
$\mathbf{l}, \mathbf{m}$	:	Legs of the complex Newman-Penrose dyad
$\mathbf{So}$	:	Odd Mathieu functions
$\mathbf{R}$	:	Ricci scalar
$\mathbf{R}_{\mu\nu}$	:	Ricci tensor
$\psi_{\mathbf{i}}$	:	Spinor components

A STUDY OF WAVE EQUATIONS IN FIVE DIMENSIONAL SPACETIMES WITH COMPUTATIONAL METHODS

SUMMARY

The interest in higher dimensional wave equations is driven by the usage of higher dimensional metrics in general relativity and string theory. Instanton solutions of general relativity are the counterparts of Yang-Mills instantons which are finite-action solutions of the Yang-Mills equations. They have an important contribution to the path-integral in the quantization of the Yang-Mills fields. The general relativistic instantons are also expected to play a similar role in the path-integral approach to quantum gravity. Weierstrass' general local solution of minimal surfaces yields a general instanton metric and Nutku's helicoid metric is a special case which corresponds to the helicoid minimal surface of this general metric. Dirac and Laplace equations can be solved in terms of Mathieu functions in the four dimensional case. If a time coordinate is added trivially to the metric, the solutions become double confluent Heun functions which are known to arise in higher dimensional solutions in the literature. One can trade the irregular singularity at zero by two regular singularities at plus and minus 1 by a transformation, to reach at the same singularity structure of the Mathieu equation and give the solutions in this form. But the main difference between the two cases comes from the presence of different constants in the radial and the angular parts of the Mathieu functions, modified by the presence of the new term coming from the time-dependence. This fact makes the summation of these functions to form the propagator quite difficult. In four dimensions one can use the summation formula for the product of four Mathieu functions -two of them for the angular and the other two for the radial part- summing them to give us a Bessel type expression. The metric has a curvature singularity at the origin. This point has to be excised. Then, the usage of the appropriate boundary conditions is important. In odd dimensions, using the non-local spectral boundary conditions which are described by Atiyah, Patodi and Singer is obligatory because of topological obstructions. One is free to choose local boundary conditions like Neumann, Dirichlet or Robin in even dimensional spacetimes. However, if one needs to conserve γ^5 and charge conjugation symmetries of the Dirac operator, the non-local spectral boundary conditions should be used. The Atiyah-Patodi-Singer formalism is used to impose spectral boundary conditions in this problem. The equations written on the boundary of the manifold are more singular than the ones in the bulk and this makes the solution more difficult. A qualitative analysis of these equations is studied in order to give information about the soultion. The derivation and analysis of the equations are done by using symbolic packages intensively. A Maple package using the Goldblatt's Newman-Penrose formalism for instanton spaces is developed for the analytical computations needed in the work. The package also supplies a complete Newman-Penrose calculator for instanton metrics.

BİLGİSAYARLI HESAPLAMA YÖNTEMLERİ İLE BEŞ BOYUTLU UZAYZAMANLARDA DALGA DENKLEMLERİNİN İNCELENMESİ

ÖZET

Yüksek boyutlardaki dalga denklemlerine duyulan ilgi, genel görelilik ve sicim kuramlarında yüksek boyutlu metriklerin kullanılmasıyla birlikte artmıştır. Genel görelilikte kullanılan instantonlar, Yang-Mills denklemlerinin sonlu eylem çözümleri olan Yang-Mills instantonlarına karşılık gelen çözümlerdir. Yang-Mills instantonları, Yang-Mills alanlarının kuantizasyonundaki iz integraline önemli bir katkı yaparlar ve genel görelilik kuramındaki karşılıklarının da kuantum kütleçekiminin iz integrali yaklaşımında benzer bir rol oynayacakları tahmin edilmektedir. Weierstarss'ın genel yerel en-küçük yüzeyler çözümü genel bir instanton metriği verir. Nutku helikoit metriği de bu genel metriğin helikoit en-küçük yüzeyine karşılık gelen özel bir durumudur. Dirac ve Laplace denklemleri dört boyutlu durumda Mathieu fonksiyonları cinsinden çözülebilir. Bir zaman koordinatı metriğe doğrudan eklenirse çözümler, literatürde yüksek boyutlu çözümlerde karşılaşılan çift konfluent Heun fonksiyonlarını içerirler. Bir dönüşüm yardımıyla sıfırdaki düzensiz tekillik, artı ve eksi 1'deki iki düzenli tekillikle değiştirilebilir ve Mathieu denkleminin tekillik yapısı elde edilir. Beş boyutlu durum, bu dönüşüm sayesinde Mathieu fonksiyonları cinsinden ifade edilebilir. Zaman koordinatından gelen ek terimle birlikte, radyal ve açısal kısımlar değişik sabitler içerdiğinden, dört boyutlu durumdaki gibi bir ilerletici yazmak için bu fonksiyonların toplanması oldukça zorlaşır. Halbuki, dört boyutlu durumda, (ikisi radyal, ikisi açısal olmak üzere) dört Mathieu fonksiyonunun çarpımı için tanımlanmış olan toplama kuralı yardımıyla Bessel türü bir ifade elde edilir. Metriğin orijinde bir eğrilik tekilliğine sahip olması, bu bölgenin dışarlanmasını gerektirir. Bu da uygun sınır koşullarının kullanımını önemli kılar. Tek boyutlarda, Atiyah, Patodi ve Singer tarafından tanımlanmış olan yerel olmayan spektral sınır koşulları, topolojik engeller sebebiyle zorunludur. Çift boyutlarda Neumann, Dirichlet veya Robin gibi yerel sınır koşulları kullanılabilse de, Dirac operatörünün γ^5 ve yük eşleniği simetrilerinin korunması isteniyorsa yerel olmayan spektral sınır koşulları kullanılmalıdır. Bu problemde sınır koşullarını uygularken Atiyah-Patodi-Singer formalizmi kullanılmıştır. Manifoldun sınırında yazılan denklemler, sınır tanımlanmadan yazılan denklemlerden daha tekildir ve bu da çözümü zorlaştırır. Literatürde ve sembolik paketler kullanılarak yapılan çalışmada denklemlerin analitik çözümleri bulunamamış, çözüm hakkında bilgi vermesi amacıyla denklemlerin nitel bir analizi yapılmıştır. Sembolik paketler, denklemlerin çıkarılması ve analizlerinde yoğun olarak kullanılmıştır. Goldblatt'ın instanton uzayları için geliştirdiği Newman-Penrose formalizmini kullanan bir Maple paketi, çalışmadaki analitik hesapları yapmak için geliştirilmiştir. Paket ayrıca instanton metrikleri için tam bir Newman-Penrose hesaplayıcısı olarak kullanılabilir.

1. INTRODUCTION

The interest in higher dimensional wave equations is driven by the usage of higher dimensional metrics in general relativity and string theory [1–10]. Instanton solutions of general relativity are the counterparts of Yang-Mills instantons which are finite-action solutions of the Yang-Mills equations and they have an important contribution to the path-integral [11]. The general relativistic instantons are expected to play a similar role in the path-integral approach to quantum gravity [12–14]. Weierstrass' general local solution of minimal surfaces yields a general instanton metric and Nutku's helicoid metric is a special case which corresponds to the helicoid minimal surface of this general metric [12, 14]. The metric has a curvature singularity at the origin. Dirac and Laplace equations in this background can be solved in terms of Mathieu functions in the four dimensional case [15–17]. If a time coordinate is added trivially to the metric, the solutions of the Dirac and Laplace equations become double confluent Heun functions (see Appendix A) [18, 19].

Phenomena described by Heun equations are common when one studies problems in atomic physics with certain potentials which combine different inverse powers starting from the first up to fourth or combining the quadratic potential with inverse powers of two, four and six, etc. [20]. They also arise when one studies symmetric double Morse potentials. Slavyanov and Lay describe different physical applications of these equations [21]. Atomic physics problems like separated double wells, Stark effect [22], hydrogen-molecule ion [23] use forms of these equations. Many different problems in solid state physics, like dislocation movement in crystalline materials, quantum diffusion of kinks along dislocations are also solved in terms of these functions [21]. The famous Hill equation [24] for lunar perigee can be cited for an early application in celestial mechanics [21]. In general relativity, while solving wave equations, we also encounter different forms of the Heun equations. Teukolsky [25] studied the perturbations of the Kerr metric and found out that they were described by two coupled singly confluent Heun equations. Quasi-normal modes of rotational gravitational singularities were also studied by solving this system of equations [26]. In recent applications they become indispensable when one studies phenomena in higher dimensions, for example the article by G. Siopsis [27], or phenomena using different geometries. An example of the latter case is seen in the example of wave equations written in the background of these metrics. For instance, in four dimensions, we may write wave equations in the background of four dimensional Euclidean gravity solutions. For the metric written in the Eguchi-Hanson instanton [28] background, the hypergeometric function is sufficient to describe the spinor field solutions [16, 17]. One, however, has to use Mathieu functions to describe even the scalar field in the background of the Nutku helicoid instanton [12, 13] when the separation of variables method is used for the solution. Schmid et al [29] have written a short note describing the occurrence of these equations in general relativity. Their examples are the Dirac equation in the Kerr-Newman metric and static perturbations of the non extremal Reisner-Nordström solution. They encounter the generalized Heun equation [19, 30, 31] while looking for the solutions in these metrics. Here we see that as the metric becomes more complicated, one has to solve equations with larger number of singular points, with no simple recursion relations if one attempts a series type solution. As a particular case of confluent Heun equation, Fiziev studied the exact solutions of the Regge-Wheeler equation [32, 33]. One also sees that if one studies similar phenomena in higher dimensions, unless the metric is a product of simple ones, one has higher chances of encountering Heun type equations as in the references given [34, 35].

In the second chapter, we solved the Dirac equation in four and five dimensional spaces and the scalar equation in five dimensions. The four dimensional scalar equation was studied in reference [12] giving Mathieu functions as the solution as in the case of four dimensional Dirac equation. The five dimensional Dirac and Laplace equations are solved in terms of double confluent Heun equations but one can trade the irregular singularity at zero by two regular singularities at plus and minus one by a transformation, to reach at the same singularity structure of the Mathieu equation and give the solutions in this form. But the main difference between the two cases is that, although both the radial and the angular parts can be written in terms of Mathieu functions, the constants are different, due to the presence of the new term coming from the time-dependence, which makes the summation of these functions to form the propagator quite difficult. In four dimensions one can use the summation formula for the product of four Mathieu functions -two of them for the angular and the other two for the radial part-summing them to give us a Bessel type expression [12].

As the metric has a curvature singularity at the origin [12], this part has to be excised to get a well defined problem. Then, the usage of the appropriate boundary conditions is important. In odd dimensions, the use of the non-local spectral boundary conditions, which are described by Atiyah, Patodi and Singer, is obligatory because of topological obstructions [36–38]. One is free to choose local boundary conditions like Neumann, Dirichlet or Robin in even dimensional spacetimes. However, if one needs to conserve γ^5 and charge conjugation symmetries of the Dirac operator, the non-local spectral boundary conditions should be used [39, 40]. The Atiyah-Patodi-Singer formalism is used to impose spectral boundary conditions in this problem [41]. We can no longer interpret the additional dimension as time because of the causality principle [42].

The equations written on the boundary of the manifold are more singular than the ones in the bulk and this makes the solution more difficult [43]. Moreover, a constant of motion disappears when we restrict the system by setting the radial coordinate x to a constant value. A qualitative analysis of these equations is studied [44–46]. In the third chapter, we studied the Atiyah-Patodi-Singer spectral boundary conditions with a qualitative analysis and singularity analysis of the little Dirac equations in four and five dimensional cases.

In the fourth chapter, we gave the details of the computer package which is developed for the symbolic computations of our study. The interest in symbolic computational study of general relativity is growing rapidly as the capacity of the the computer systems increase. The computer is no longer an apparatus for the numerical relativist, but as the symbolic manipulators get easier to use, the more researchers get into the subject by using these systems. Thanks to the symbolic calculators, lengthy calculations needing time and care are no longer a problem for scientists. The preliminary works on symbolic calculation goes back to 1965, Fletcher's program GRAD ASSISTANT could calculate Ricci tensor for simple metrics [47]. Two years later, in 1967, M. Veltman developed SCHOONSCHIP, a symbolic manipulation program to study renormalizability of gauge theories [48]. At 1968, Macsyma, the first comprehensive symbolic mathematics system, began to be developed by MIT scientists under the leadership of B. Martin [49]. Following the path of SCHOONSCHIP and Macsyma, a group leaded by S. Wolfram began designing SMP which is the ancestor of today's Mathematica [50]. A comprehensive historical review can be found in reference [51]. Recent platforms like Maple [52], Mathematica [53], Matlab [54], Maxima [55] and Reduce [56] offer developed and optimized solutions with their easy to learn interfaces. There are more alternatives and all these platforms have their advantages and disadvantages. Therefore, the user should make a decision before getting deeper into programming. Another important point to consider is the specialized packages which run under these platforms. Using a profession based package reduces the work of the end-user. A package having a nice designed interface can be used even by the people having no idea about the platform. There are such packages for the general relativistic purposes. Some of the best known examples are: GRTensorII [57], Riemann [58], Riegeom [59], Riccir [60] and MathTensor [61]. GRTensorII, being a widespread package led significant progress in the area [62–67]. We developed a Maple package using the Goldblatt's Newman-Penrose formalism for instanton spaces for the analytical computations needed in the work. The package also supplies a complete Newman-Penrose calculator for instanton metrics [68].

2. CALCULATIONS FOR THE NUTKU HELICOID METRIC WITH A TIME COORDINATE

2.1 The Metric Tensor

The Nutku helicoid metric is given by,

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{1+\frac{a^2}{r^2}}}[dr^2+(r^2+a^2)d\theta^2+\left(1+\frac{a^2}{r^2}\sin^2\theta\right)dy^2 \\ &\quad -\frac{a^2}{r^2}\sin 2\theta dydz+\left(1+\frac{a^2}{r^2}\cos^2\theta\right)dz^2], \end{aligned} \tag{2.1}$$

where $0 < r < \infty$, $0 \leq \theta \leq 2\pi$, y and z are along the Killing directions and will be taken to be periodic coordinates on a torus. This is a special form [12] which corresponds to the helicoid minimal surface of a general instanton metric [13,14]. A helicoid in three dimensions is shown in Figure 2.1.

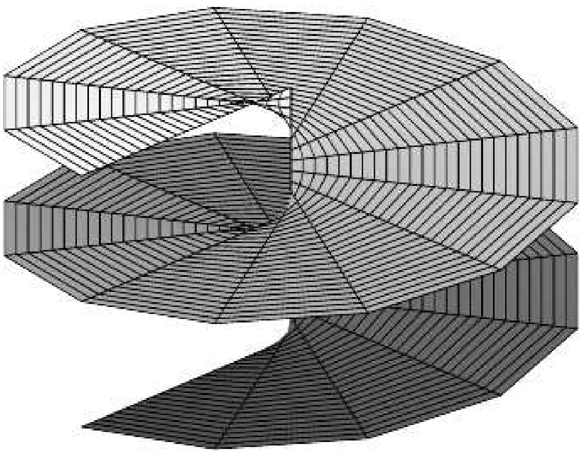

Figure 2.1: Helicoid in Three Dimensions

When the Einstein's tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.2}$$

is written, it is seen that all of the components are zero. Therefore, this is a vacuum solution. This metric reduces to the flat metric if we take $a = 0$:

$$ds^2 = dr^2 + r^2d\theta^2 + dy^2 + dz^2. \tag{2.3}$$

If we make the following transformation

$$r = a \sinh x, \tag{2.4}$$

the metric becomes

$$\begin{aligned} ds^2 &= \frac{a^2}{2}\sinh 2x(dx^2 + d\theta^2) \\ &+ \frac{2}{\sinh 2x}[(\sinh^2 x + \sin^2\theta)dy^2 \\ &- \sin 2\theta dydz + (\sinh^2 x + \cos^2\theta)dz^2]. \end{aligned} \tag{2.5}$$

We use the Newman-Penrose (NP) formalism [69,70] in four Euclidean dimensions [71–73]. To write the Dirac equation in this formalism, we need to choose the base vectors and the γ matrices in curved space accordingly.

We choose

$$l^\mu = \frac{1}{a\sqrt{\sinh 2x}}(1, i, 0, 0), \tag{2.6}$$

$$m^\mu = \frac{1}{\sqrt{\sinh 2x}}(0, 0, \cosh(x - i\theta), i\sinh(x - i\theta)), \tag{2.7}$$

as the legs of the NP dyad (see Appendix B).

2.2 Dirac Operator

The relativistic mass-shell condition is given as $p_\mu p^\mu = E^2 - \mathbf{p}^2 = M^2$ where p_μ is the 4-momentum, E is energy, M is mass and Planck units ($\hbar = c = 1$) are used. Dirac started with the square root of this expression to write an equation which is linear in time (Schrödinger-like) for the relativistic electron [74–76]:

$$i\frac{\partial\psi}{\partial t} = (-i\alpha_i\nabla^i + \beta M)\psi \tag{2.8}$$

where, α_i and β are constant matrices. The square of the operator which is acting on ψ field should give the mass-shell condition,

$$-\frac{\partial^2\psi}{\partial t^2} = (-i\alpha.\nabla + \beta M)^2\psi = (-\nabla^2 + M^2)\psi \tag{2.9}$$

Thus, the matrices should satisfy,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{2.10}$$

$$\{\alpha_i, \beta\} = 0 \tag{2.11}$$

$$\alpha_i^2 = \beta^2 = 1. \tag{2.12}$$

These matrices are called as Dirac matrices or γ matrices and are denoted as $\beta = \gamma^0$, $\beta\alpha^i = \gamma^i$. Consequently, the Dirac equation is given as,

$$(i\gamma^\mu \partial_\mu - M)\psi = 0 \tag{2.13}$$

and the condition that the γ-matrices should satisfy is given as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{2.14}$$

The massive Dirac equation in curved spacetimes reads

$$i\gamma^\mu \nabla_\mu \psi = M\psi, \tag{2.15}$$

where

$$\nabla_\mu = \partial_\mu - \Gamma_\mu, \tag{2.16}$$

using the spin connection Γ_μ which is the representative of the curvature of the spacetime. The γ matrices can be written in terms of the base vectors as (see Appendix B),

$$\gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & 0 & l^\mu & m^\mu \\ 0 & 0 & -\bar{m}^\mu & \bar{l}^\mu \\ \bar{l}^\mu & -m^\mu & 0 & 0 \\ \bar{m}^\mu & l^\mu & 0 & 0 \end{pmatrix}. \tag{2.17}$$

The spin connection is written as

$$\Gamma_\mu = \frac{1}{4}\gamma^\nu_{;\mu}\gamma_\nu \tag{2.18}$$

using the γ matrices given above.

2.2.1 Solutions in Four Dimensions

For the Nutku helicoid metric background, Dirac equations read,

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x + i\partial_\theta)\psi_3 + a[\cos(\theta + ix)\partial_y + \sin(\theta + ix)\partial_z]\psi_4 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\psi_1\} = 0, \tag{2.19}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x - i\partial_\theta)\psi_4 - a[\cos(\theta - ix)\partial_y + \sin(\theta - ix)\partial_z]\psi_3 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\psi_2\} = 0, \tag{2.20}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x - i\partial_\theta + \coth 2x)\psi_1$$
$$-a[\cos(\theta+ix)\partial_y + \sin(\theta+ix)\partial_z]\psi_2 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\psi_3\} = 0, \quad (2.21)$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x + i\partial_\theta + \coth 2x)\psi_2$$
$$+a[\cos(\theta-ix)\partial_y + \sin(\theta-ix)\partial_z]\psi_1 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\psi_4\} = 0. \quad (2.22)$$

To simplify the calculations, we will study the massless ($M=0$) case. Then we see that only $\{\psi_1, \psi_2\}$ and $\{\psi_3, \psi_4\}$ components are coupled to each other. Using (2.21) we can write,

$$\psi_2 = \frac{(\partial_x - i\partial_\theta + \coth 2x)\psi_1}{iak[\cos(\theta+ix)\cos\phi + \sin(\theta+ix)\sin\phi]} \quad (2.23)$$

or by using (2.22) we get

$$\psi_1 = \frac{-(\partial_x + i\partial_\theta + \coth 2x)\psi_2}{iak[\cos(\theta-ix)\cos\phi + \sin(\theta-ix)\sin\phi]} \quad (2.24)$$

and similarly, by using (2.19) and (2.20) we get

$$\psi_4 = \frac{-(\partial_x + i\partial_\theta)\psi_3}{iak[\cos(\theta+ix)\cos\phi + \sin(\theta+ix)\sin\phi]} \quad (2.25)$$

and

$$\psi_3 = \frac{(\partial_x - i\partial_\theta)\psi_4}{iak[\cos(\theta-ix)\cos\phi + \sin(\theta-ix)\sin\phi]} \quad (2.26)$$

after making the transformations

$$k_y = k\cos\phi, k_z = k\sin\phi. \quad (2.27)$$

We are free to take

$$\psi_i = e^{i(k_y y + k_z z)}\Psi_i(x,\theta) \quad (2.28)$$

because of the properties of the metric. We take

$$\Psi_1 = \frac{\sinh[x - i(\theta-\phi)]}{\sqrt{\sinh 2x}}\tilde{\Psi}_1 \quad (2.29)$$

and

$$\Psi_2 = \frac{\sinh[x + i(\theta-\phi)]}{\sqrt{\sinh 2x}}\tilde{\Psi}_2. \quad (2.30)$$

Now we have to solve

$$\frac{-2}{2ak\sqrt{\sinh 2x}}\left\{\partial_{xx}+\partial_{\theta\theta}+\frac{a^2k^2}{2}\left\{\cos[2(\theta+\phi)]-\cosh 2x\right\}\right\}\tilde{\Psi}_{1,2}=0. \qquad (2.31)$$

When similar transformations are done for the other components we get:

$$\frac{\cosh[x-i(\theta-\phi)]}{ak}\left\{\partial_{xx}+\partial_{\theta\theta}+\frac{a^2k^2}{2}\left\{\cos[2(\theta+\phi)]-\cosh 2x\right\}\right\}\Psi_3=0, \qquad (2.32)$$

$$\frac{\cosh[x+i(\theta-\phi)]}{ak}\left\{\partial_{xx}+\partial_{\theta\theta}+\frac{a^2k^2}{2}\left\{\cos[2(\theta+\phi)]-\cosh 2x\right\}\right\}\Psi_4=0. \qquad (2.33)$$

We can seek a solution in the product form,

$$\Psi_i(x,\theta)=R_i(x)S_i(\theta) \qquad (2.34)$$

and we end up with the equations

$$\frac{d^2R_{1,2,3,4}}{dx^2}-[\frac{a^2k^2}{2}\cosh 2x+\zeta_{1,2,3,4}]R_{1,2,3,4}=0, \qquad (2.35)$$

$$\frac{d^2S_{1,2,3,4}}{d\theta^2}+\frac{a^2k^2}{2}\{\cos[2(\theta+\phi)]+\zeta_{1,2,3,4}\}S_{1,2,3,4}=0 \qquad (2.36)$$

where the subscripts (1, 2, 3, 4) relate the solutions with the spinor components. The solutions of these equations can be expressed in terms of Mathieu functions:

$$\begin{aligned}\psi_1 = e^{ik(z\sin\phi+y\cos\phi)}\frac{\sinh[x-i(\theta-\phi)]}{\sqrt{\sinh 2x}}\times\\ \{\left[C_1Se(-\zeta_1,-\frac{a^2k^2}{4},-ix)+C_2So(-\zeta_1,-\frac{a^2k^2}{4},-ix)\right]\\ \times\left[C_3Se(-\zeta_1,-\frac{a^2k^2}{4},\theta+\phi)+C_4So(-\zeta_1,-\frac{a^2k^2}{4},\theta+\phi)\right]\},\end{aligned} \qquad (2.37)$$

$$\begin{aligned}\psi_2 &= e^{ik(z\sin\phi+y\cos\phi)}\frac{\sinh[x+i(\theta-\phi)]}{\sqrt{\sinh 2x}}\times\\ &\{\left[C_5Se(-\zeta_2,-\frac{a^2k^2}{4},-ix)+C_6So(-\zeta_2,-\frac{a^2k^2}{4},-ix)\right]\\ &\times\left[C_7Se(-\zeta_2,-\frac{a^2k^2}{4},\theta+\phi)+C_8So(-\zeta_2,-\frac{a^2k^2}{4},\theta+\phi)\right]\},\end{aligned} \qquad (2.38)$$

$$\begin{aligned}\psi_3 &= e^{ik(z\sin\phi+y\cos\phi)}\{\left[C_9Se(-\zeta_3,-\frac{a^2k^2}{4},-ix)+C_{10}So(-\zeta_3,-\frac{a^2k^2}{4},-ix)\right]\\ &\times\left[C_{11}Se(-\zeta_3,-\frac{a^2k^2}{4},\theta+\phi)+C_{12}So(-\zeta_3,-\frac{a^2k^2}{4},\theta+\phi)\right]\},\end{aligned} \qquad (2.39)$$

$$\psi_4 = e^{ik(z\sin\phi + y\cos\phi)}\{\left[C_{13}Se(-\zeta_4, -\frac{a^2k^2}{4}, -ix) + C_{14}So(-\zeta_4, -\frac{a^2k^2}{4}, -ix)\right] \times \left[C_{15}Se(-\zeta_4, -\frac{a^2k^2}{4}, \theta+\phi) + C_{16}So(-\zeta_4, -\frac{a^2k^2}{4}, \theta+\phi)\right]\}. \tag{2.40}$$

Here ζ_i are the separation constants and C's are arbitrary constant. The sketches of these solutions are shown in Figures 2.2, 2.3 and 2.4 for $\phi = 0.3$, $a = 1$, $k = 1$, $\zeta_i = 2$ and $C_i = 1$.

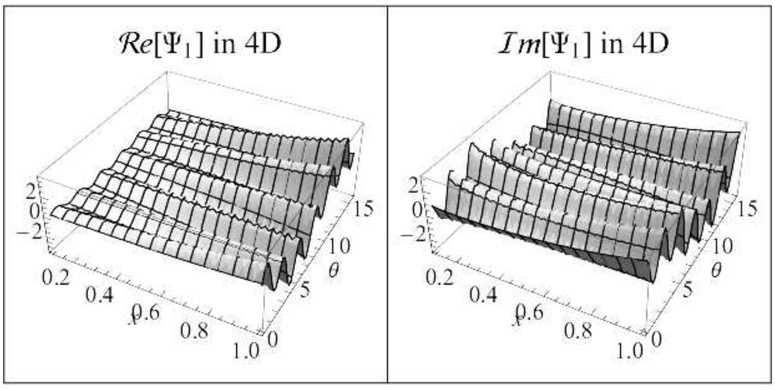

Figure 2.2: Ψ_1 Solution in Four Dimensions

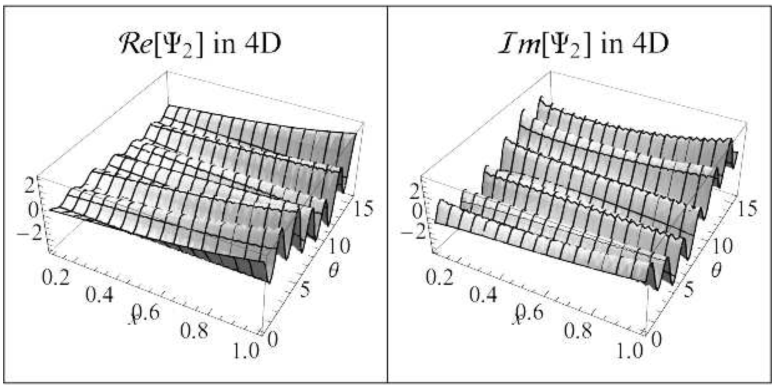

Figure 2.3: Ψ_2 Solution in Four Dimensions

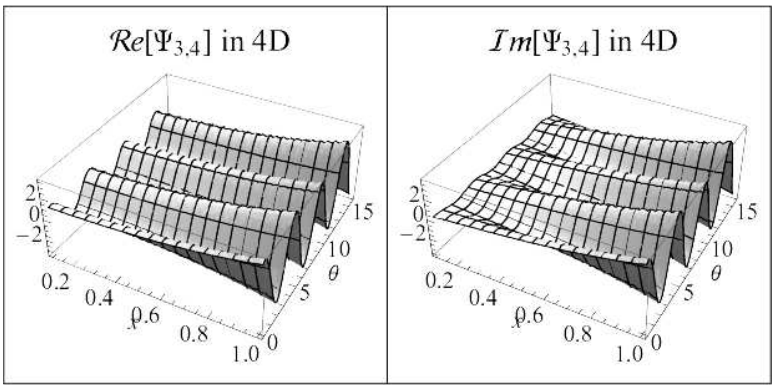

Figure 2.4: Ψ_{34} Solution in Four Dimensions

At this point also note that we can get solutions in the plane wave form, which are given as $e^{ka[sin(\theta-\phi+ix)+sin(\theta-\phi-ix)]}$, similar to the ones given by Sucu and Ünal [16].

But the former form turns out to be more useful when we try to compare the solution with the solution in the five dimensional case.

2.2.2 Solutions in Five Dimensions

The addition of the time component trivially to the previous metric gives:

$$ds^2 = -dt^2 + ds_4^2, \tag{2.41}$$

resulting in the massless Dirac equation as:

$$(\gamma^\mu \partial_\mu + \gamma^t \partial_t - \gamma^\mu \Gamma_\mu - \gamma^t \Gamma_t)\psi = 0, \quad (\mu = x, \theta, y, z). \tag{2.42}$$

Here, the spin connection corresponding to the time coordinate

$$\Gamma_t = 0. \tag{2.43}$$

The Dirac matrix corresponding to the time component

$$\gamma^t = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \tag{2.44}$$

can be taken according to the anti-commutation relation of the Dirac matrices, to give the set of equations,

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x + i\partial_\theta)\psi_3 + a[\cos(\theta + ix)\partial_y + \sin(\theta + ix)\partial_z]\psi_4 + i\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_1\} = 0, \tag{2.45}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x - i\partial_\theta)\psi_4 - a[\cos(\theta - ix)\partial_y + \sin(\theta - ix)\partial_z]\psi_3 + i\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_2\} = 0, \tag{2.46}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x - i\partial_\theta + \coth 2x)\psi_1 - a[\cos(\theta + ix)\partial_y + \sin(\theta + ix)\partial_z]\psi_2 - i\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_3\} = 0, \tag{2.47}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x + i\partial_\theta + \coth 2x)\psi_2 + a[\cos(\theta - ix)\partial_y + \sin(\theta - ix)\partial_z]\psi_1 - i\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_4\} = 0. \tag{2.48}$$

If we solve for ψ_1 and ψ_2 and replace them in the latter equations, we get two equations which involve only ψ_3 and ψ_4 . If we take

$$\psi_i = e^{i(k_t t + k_y y + k_z z)} \Psi_i(x,\theta), \tag{2.49}$$

and

$$\begin{aligned} \Psi_1 &= \frac{\sqrt{2}}{ak_t\sqrt{\sinh 2x}}\{(\partial_x + i\partial_\theta)\Psi_3 \\ &\quad + iak[\cos(\theta+ix)\cos\phi + \sin(\theta+ix)\sin\phi]\Psi_4]\}, \end{aligned} \tag{2.50}$$

$$\begin{aligned} \Psi_2 &= \frac{\sqrt{2}}{ak_t\sqrt{\sinh 2x}}\{(\partial_x - i\partial_\theta)\Psi_4 \\ &\quad - iak[\cos(\theta-ix)\cos\phi + \sin(\theta-ix)\sin\phi]\Psi_3]\} \end{aligned} \tag{2.51}$$

The resulting equations read:

$$\left\{\partial_{xx} + \partial_{\theta\theta} + \frac{a^2k^2}{2}\{\cos[2(\theta+\phi)] - \cosh 2x\} + \frac{a^2k_t^2}{2}\sinh 2x\right\}\Psi_{3,4} = 0. \tag{2.52}$$

If we assume that the result is expressed in the product form $\Psi_{3,4} = T_1(x)T_2(\theta)$, the angular part is again expressible in terms of Mathieu functions.

$$\begin{aligned} T_2(\theta) &= C_1 Se\left[\eta, -\frac{a^2k^2}{4}, \arccos(\sqrt{\frac{1+\cos(\theta+\phi)}{2}})\right] \\ &\quad + C_2 So\left[\eta, -\frac{a^2k^2}{4}, \arccos(\sqrt{\frac{1+\cos(\theta+\phi)}{2}})\right]. \end{aligned} \tag{2.53}$$

Here C_i's are arbitrary constants. η is the separation constant and the periodicity of the solution makes it equal to the square of an integer.

The equation for T_1 reads:

$$\left\{\frac{d^2}{dx^2} - \frac{a^2k^2}{2}\cosh 2x + \frac{a^2k_t^2}{2}\sinh 2x - \eta\right\}T_1 = 0. \tag{2.54}$$

The solution of this equation is expressed in terms of double confluent Heun (H_D) functions [19]:

$$\begin{aligned} T_1(x) &= C_3 H_D\left[0, \frac{a^2k^2}{2} - \eta, -a^2k_t^2, \frac{a^2k^2}{2} + \eta, \tanh x\right] \\ &\quad + C_4 H_D\left[0, \frac{a^2k^2}{2} - \eta, -a^2k_t^2, \frac{a^2k^2}{2} + \eta, \tanh x\right] \\ &\quad \times \int \frac{-dx}{H_D\left[0, \frac{a^2k^2}{2} - \eta, -a^2k_t^2, \frac{a^2k^2}{2} + \eta, \tanh x\right]^2}. \end{aligned} \tag{2.55}$$

One can see that as x goes to infinity, the function given above diverges. The function is finite at $x = 0$ though. In order to get well defined functions, we study the region where $x \leq$ F, where F is a finite value. We will give a way to determine F below.

Just to show the differences with the four dimensional solution, we attempt to write this expression in terms of Mathieu functions. This can be done after few transformations. We define

$$A = \frac{a^2 k_t^2}{2}, \tag{2.56}$$

$$B = -\eta, \tag{2.57}$$

$$C = -\frac{a^2 k^2}{2}, \tag{2.58}$$

and use the transformation

$$z = e^{-2x}. \tag{2.59}$$

Then the differential operator is expressed as

$$O = 4z^2 \frac{d^2}{dz^2} + 4z \frac{d}{dz} + A' z + B + C' \frac{1}{z}, \tag{2.60}$$

where d_z denotes derivative with respect to z. Here

$$A' = \frac{C - A}{2} \tag{2.61}$$

$$C' = \frac{C + A}{2} \tag{2.62}$$

If we take

$$\sqrt{\frac{A'}{C'}} u = z, \tag{2.63}$$

and

$$w = \frac{1}{2}(u + \frac{1}{u}) \tag{2.64}$$

and set $E = \sqrt{A' C'}$ we get,

$$O = (w^2 - 1) \frac{d^2}{dw^2} + w \frac{d}{dw} + \frac{E}{2} w + \frac{B}{4}. \tag{2.65}$$

The solution of the equation obtained by acting this operator on a function, is also expressible in terms of Mathieu functions given as:

$$R(z) = C_3' Se(-B, -E, \arccos \sqrt{\frac{w+1}{2}}) + C_4' So(-B, -E, \arccos \sqrt{\frac{w+1}{2}}). \tag{2.66}$$

To satisfy the regularity conditions of the Mathieu functions we set the coefficient of the second solution in equation (2.55) equal to zero. The Maple commands and the output for this transformation is given in Figure 2.5.

The sketch of this solution is shown in Figure 2.6 and 2.7 for $\phi = 0.3$, $a = 1$, $k = 1$, $\eta = 2$ and two different values of k_t.

At this point we see a natural limitation in the values that can be taken by our radial variable since the argument of the function arccos can not exceed unity. The fact that $\sqrt{\frac{w+1}{2}}$ can not exceed unity, limits the values our initial variable x can take, thus determining F which imposed on our solution in equation (2.55). The calculation gives $\mathrm{F}{=}\frac{1}{4}\ln(\frac{k^2-k_t^2}{k^2+k_t^2})$ as the limiting value. As $\sqrt{\frac{w+1}{2}}$ becomes infinity at $k_t = -k$ and at $k_t = k$, these k_t values should be excluded.

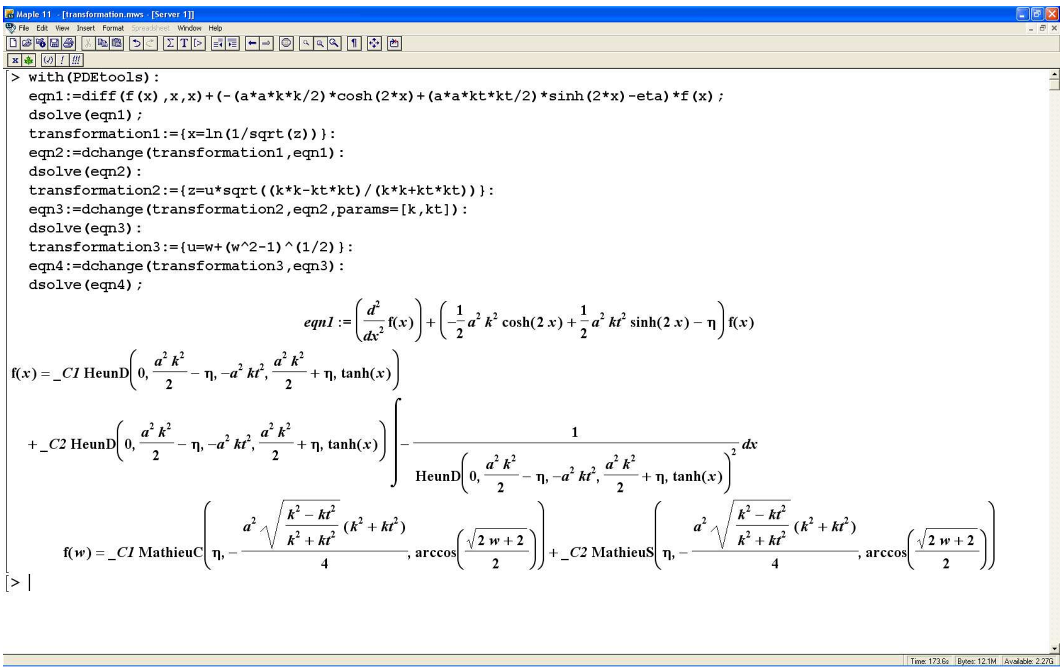

Figure 2.5: Transformation for the Dirac Equation

Here we also see a difference from the four dimensional case. Although both the radial and the angular part can be written in terms of Mathieu functions, the constants are different, modified by the presence of the new $\frac{a^2k_t^2}{2}$ term, which makes the summation of these functions to form the propagator quite difficult.

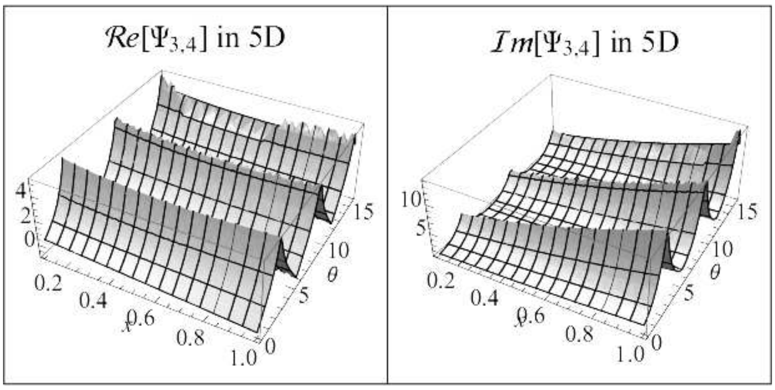

Figure 2.6: Ψ_{34} Solutions in Five Dimensions for $k_t = 0.01$

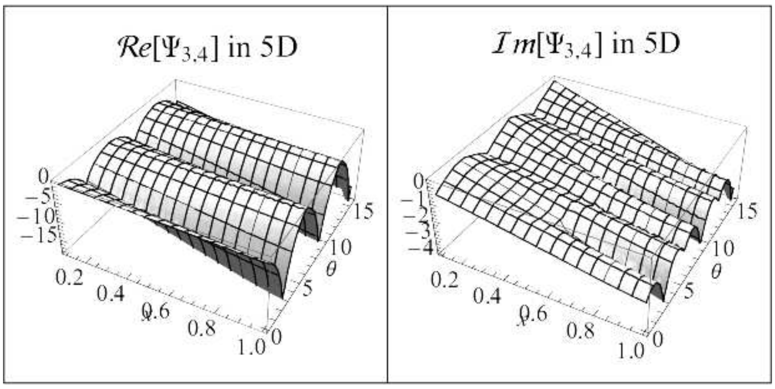

Figure 2.7: Ψ_{34} Solutions in Five Dimensions for $k_t = 1.2$

In four dimensions we can use the summation formula [12, 77] for the product of four Mathieu functions, two of them for the angular and the other two for the radial part, summing them to give us a Bessel type expression [12].

2.3 Laplace Operator

In this section we will discuss the Laplace operator written in the five dimensional background explained above. It is used for the calculation of the field equation for the scalar particle, similar to the case studied in reference [12]. The definition of the Laplace operator is [74],

$$\mathscr{H} \equiv \frac{1}{\sqrt{-g}} \partial_\nu \sqrt{-g} g^{\mu\nu} \partial_\mu. \tag{2.67}$$

For the five dimensional metric, the operator becomes,

$$\begin{aligned} \mathscr{H} \quad \equiv \quad & \frac{1}{\sqrt{-g}} \partial_\nu \sqrt{-g} g^{\mu\nu} \partial_\mu = -\partial_{tt} + \partial_{xx} + \partial_{\theta\theta} + a^2 \sinh^2 2x (\partial_{yy} + \partial_{zz}) \\ & + a^2 (\cos\theta \partial_y + \sin\theta \partial_z)^2 - a^2 \sinh x \cosh x \partial_{tt}. \end{aligned} \tag{2.68}$$

There are three Killing vectors and one quadratic Killing tensor with eigenvalues given below.

From the Killing tensor we can construct a second order tensor operator [12],

$$K \equiv -\partial_{\theta\theta} - a^2(\cos\theta\partial_y + \sin\theta\partial_z)^2. \tag{2.69}$$

The eigenvalues can be defined by the equations,

$$K\Phi = \lambda\Phi, \tag{2.70}$$

$$\partial_t\Phi = k_t\Phi, \tag{2.71}$$

$$\partial_y\Phi = k_y\Phi, \tag{2.72}$$

$$\partial_z\Phi = k_z\Phi. \tag{2.73}$$

We have

$$\Phi = e^{i(k_t t + k_y y + k_z z)} R(x) S(\theta) \tag{2.74}$$

where S obeys the equation

$$\frac{d^2\tilde{S}(\Theta)}{d\Theta^2} + (\lambda - a^2k^2\cos^2\Theta)\tilde{S}(\Theta) = 0. \tag{2.75}$$

Here

$$\Theta = \theta - \phi \tag{2.76}$$

and the solution reads

$$\begin{aligned} S(\theta) &= C_1 Se(\frac{-a^2k^2}{2} + \lambda, \frac{a^2k^2}{4}, \theta - \phi) \\ &\quad + C_2 So(\frac{-a^2k^2}{2} + \lambda, \frac{a^2k^2}{4}, \theta - \phi). \end{aligned} \tag{2.77}$$

The radial part obeys

$$\frac{d^2R}{dx^2} + a^2(\sinh x \cosh x k_t^2 - k^2\sinh^2 x - \frac{\lambda}{a^2})R = 0. \tag{2.78}$$

The solution to this equation, the radial solution, can be written in terms of double confluent Heun functions,

$$\begin{aligned} R(x) &= C_3 H_D\left[0, a^2k^2 - \lambda, -a^2k_t^2, \lambda, \tanh x\right] \\ &\quad + C_4 H_D\left[0, a^2k^2 - \lambda, -a^2k_t^2, \lambda, \tanh x\right] \\ &\quad \times \int \frac{-dx}{H_D\left[0, a^2k^2 - \lambda, -a^2k_t^2, \lambda, \tanh x\right]^2}. \end{aligned} \tag{2.79}$$

The radial equation can be reduced to the modified Mathieu function after performing a transformation similar to the spinor case treated above. Taking

$A = \frac{a^2 k_t^2}{2}$, $B = \frac{a^2 k^2}{2} - \lambda$ and $C = \frac{-a^2 k^2}{2}$, and using $z = e^{-2x}$ transformation we get a result similar to equation (2.66). To satisfy the regularity conditions of the Mathieu functions we set the coefficient of the second solution equal to zero. The Maple commands and the output for this transformation is given in Figure 2.8. The sketch of this solution is shown in Figure 2.9 for $\phi = 0.3$, $a = 1$, $k = 1$, $k_t = 0.01$, $\lambda = 2$.

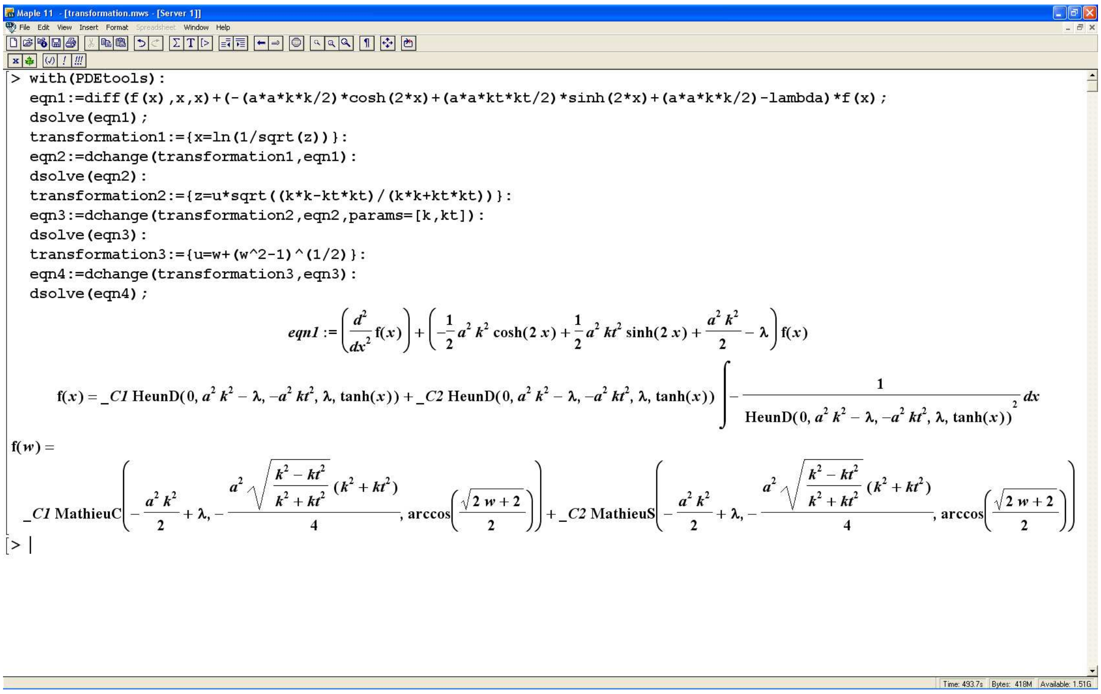

Figure 2.8: Transformation for the Scalar Equation

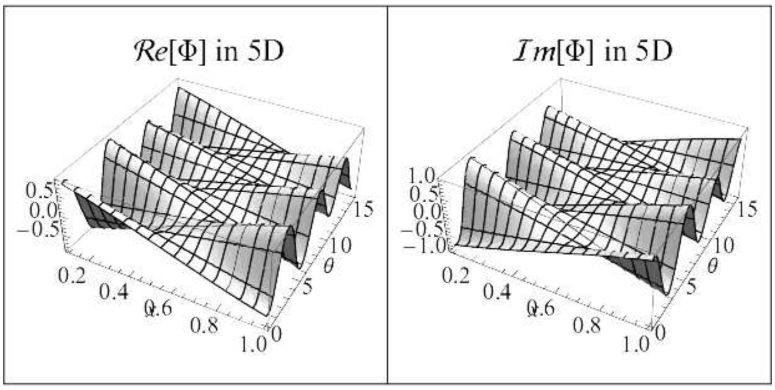

Figure 2.9: Scalar Solution in Five Dimensions

3. ATIYAH-PATODI-SINGER SPECTRAL BOUNDARY CONDITIONS

Non-local spectral boundary conditions were first explained by Atiyah et al in the paper where they study the Hirzebruch signature of the manifolds with boundary [11, 78, 79]. Local boundary conditions like Neumann or Dirichlet can be used for the solutions of the Dirac equation on a bounded even dimensional manifold. However, if one needs to conserve γ^5 and charge conjugation symmetries of the Dirac operator, the non-local spectral boundary conditions should be used [39,40]. In the odd dimensional cases, one is obliged to use the non-local spectral boundary conditions because of the topological obstructions which are obtained by Atiyah et al [36–38, 80].

Dirac equation,

$$i\gamma^{\mu}\nabla_{\mu}\psi_{\Lambda} = i\gamma^{\mu}\nabla_{\mu}\begin{pmatrix} u_{\Lambda} \\ \upsilon_{\Lambda} \end{pmatrix} = \Lambda\begin{pmatrix} u_{\Lambda} \\ \upsilon_{\Lambda} \end{pmatrix} = \Lambda\psi_{\Lambda} \tag{3.1}$$

and the "little" Dirac equation (Dirac equation on the boundary) can be written as,

$$i\gamma^{\mu}\hat{\nabla}_{\mu}e_{\lambda} = \lambda e_{\lambda} \tag{3.2}$$

Operator $\hat{\nabla}$ is Hermitian and λ's are real [42]. The functions e_{λ} construct a basis on the boundary of the manifold.

Near the boundary, u_{Λ} and υ_{Λ} can be expanded in terms of e_{λ} [42]:

$$u_{\Lambda} = \sum_{\lambda} f^{\lambda}_{\Lambda}(\xi)e_{\lambda}(q) \quad , \; f^{\lambda}_{\Lambda}(\xi) = \int_{\partial\mathscr{M}} e^{\dagger}{}_{\lambda}(q)u_{\Lambda}(\xi,q)\sqrt{|g|}d\tau, \tag{3.3}$$

$$\upsilon_{\Lambda} = \sum_{\lambda} g^{\lambda}_{\Lambda}(\xi)e_{\lambda}(q) \quad , \; g^{\lambda}_{\Lambda}(\xi) = \int_{\partial\mathscr{M}} e^{\dagger}{}_{\lambda}(q)\upsilon_{\Lambda}(\xi,q)\sqrt{|g|}d\tau, \tag{3.4}$$

where $d\tau$ is the volume element and $\sqrt{|g|}$ is the square root of the determinant of the metric.

On the boundary, spectral boundary conditions restricts the λ values for upper and lower spinor components:

$$f^{\lambda}_{\Lambda}|_{\partial\mathscr{M}} = 0, \quad \lambda > 0 \tag{3.5}$$

and

$$g^{\lambda}_{\Lambda}|_{\partial\mathscr{M}} = 0, \quad \lambda < 0 \tag{3.6}$$

can be written. Projectors P^+ and P^- can be constructed for the boundary modes with positive and negative eigenvalues [42]:

$$P^+(q,q') = \sum_{\lambda>0} e_\lambda(q) {e^\dagger}_\lambda(q'), \tag{3.7}$$

$$P^-(q,q') = \sum_{\lambda<0} e_\lambda(q) {e^\dagger}_\lambda(q') \tag{3.8}$$

for which I being the unity operator for the function space spanned by e_λ,

$$P^+ + P^- = I \tag{3.9}$$

is satisfied. If we write the two dimensional P^+ and P^- operators as a 4x4 matrix P, the spectral boundary condition for the 4-spinor Ψ becomes,

$$P\Psi|_{\partial\mathscr{M}} = \begin{pmatrix} P^+ & 0 \\ 0 & P^- \end{pmatrix} \begin{pmatrix} u \\ \upsilon \end{pmatrix} |_{\partial\mathscr{M}} = 0. \tag{3.10}$$

The matrix γ^5 is given by

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

and the projector P commutes with γ^5 matrix:

$$[P, \gamma^5] = 0. \tag{3.11}$$

Therefore, spectral boundary conditions conserve chirality [42]. The spinor on the boundary can be written as:

$$\Psi|_{\partial\mathscr{M}} = \begin{pmatrix} \sum_{\lambda<0} f_\lambda e_\lambda \\ \sum_{\lambda>0} g_\lambda e_\lambda \end{pmatrix}. \tag{3.12}$$

The charge conjugation property is defined as,

$$\Psi^C(x, A_\mu) = C\Psi^*(x, -A_\mu) \tag{3.13}$$

where C is the charge conjugation operator which satisfies $\gamma^5 C = -C\gamma^5$:

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.14}$$

and A_μ is the gauge (or curvature) related term in Dirac equation ($i\gamma^\mu(\partial_\mu + A_\mu)\psi = \Lambda\psi$).

According to these definitions,

$$\psi^C|_{\partial\mathscr{M}} = \begin{pmatrix} -\sum\limits_{\lambda>0} g^*_\lambda (e^*_\lambda)_{A\to -A} \\ \sum\limits_{\lambda<0} f^*_\lambda (e^*_\lambda)_{A\to -A} \end{pmatrix}. \tag{3.15}$$

Using the little Dirac equations, it can be written that,

$$(e^*_\lambda)_{A\to -A} = e^*_{-\lambda}, \tag{3.16}$$

Therefore,

$$\psi^C|_{\partial\mathscr{M}} = \begin{pmatrix} -\sum\limits_{\lambda<0} g^*_{-\lambda} e_\lambda \\ \sum\limits_{\lambda>0} f^*_{-\lambda} e_\lambda \end{pmatrix}, \tag{3.17}$$

is obtained. Using the spectral boundary conditions, it is determined that the upper spinor components stay in the subspace spanned by eigenvectors corresponding to negative eigenvalues and lower components stay in the subspace spanned by eigenvectors corresponding to positive eigenvalues and, hence, the charge conjugation symmetry is conserved [39, 42].

3.1 Solutions in Five Dimensions

We write the system in the form $L\psi = \Lambda\psi$ and try to obtain the solutions for the different components. The aim is to write the upper components (ψ_1, ψ_2) in terms of the lower components (ψ_3, ψ_4). We will impose the boundary conditions on the upper components in terms of derivatives of the lower components below.

The equations read

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x + i\partial_\theta)\psi_3 \\ +a[\cos(\theta+ix)\partial_y + \sin(\theta+ix)\partial_z]\psi_4 - \frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_1\} = \Lambda\psi_1, \tag{3.18}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x - i\partial_\theta)\psi_4 \\ -a[\cos(\theta-ix)\partial_y + \sin(\theta-ix)\partial_z]\psi_3 - \frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_2\} = \Lambda\psi_2, \tag{3.19}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x - i\partial_\theta + \coth 2x)\psi_1 \\ -a[\cos(\theta+ix)\partial_y + \sin(\theta+ix)\partial_z]\psi_2 + \frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_3\} = \Lambda\psi_3, \tag{3.20}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\{(\partial_x + i\partial_\theta + \coth 2x)\psi_2$$
$$+a[\cos(\theta - ix)\partial_y + \sin(\theta - ix)\partial_z]\psi_1 + \frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t\psi_4\} = \Lambda\psi_4. \quad (3.21)$$

These are the coupled equations involving three different components. The usual method to obtain solutions at this stage is the separation of variables method. Three of the variables, t, y, z define the Killing directions. This is exploited by writing the solution as a product of exponentials in y, z, t times a function of x and θ.

$$\psi_i = e^{i(k_t t + k_y y + k_z z)}\Psi_i(x, \theta). \quad (3.22)$$

We take $k_y = k\cos\phi, k_z = k\sin\phi$, since then we can absorb the variable ϕ in the functions of the remaining variables x and θ.

We note that these four equations are not similar in form. The transformation $\Psi_{1,2} = \frac{1}{\sqrt{\sinh 2x}} f_{1,2}$ can be used for the upper components. This transformation eliminates the $\coth 2x$ terms in the third and the fourth equations. With this transformation, the equations read,

$$(\partial_x + i\partial_\theta)\Psi_3 \ + iak[\cos(\theta - \phi + ix)]\Psi_4 - iak_t f_1 = \Lambda\frac{a}{\sqrt{2}} f_1, \quad (3.23)$$

$$(\partial_x - i\partial_\theta)\Psi_4 \ - iak[\cos(\theta - \phi - ix)]\Psi_3 - iak_t f_2 = \Lambda\frac{a}{\sqrt{2}} f_2, \quad (3.24)$$

$$(-\partial_x + i\partial_\theta) f_1 \ + iak[\cos(\theta - \phi + ix)] f_2 + iak_t\Psi_3 = -\Lambda\frac{a\sinh 2x}{\sqrt{2}}\Psi_3, \quad (3.25)$$

$$(-\partial_x - i\partial_\theta) f_2 \ - iak[\cos(\theta - \phi - ix)] f_1 + iak_t\Psi_4 = -\Lambda\frac{a\sinh 2x}{\sqrt{2}}\Psi_4. \quad (3.26)$$

We solve $f_{1,2}$ from the last two equations and substitute in the first two equations given above. This substitution gives us second order, but uncoupled equations for the lower components.

$$[\partial_{xx} + \partial_{\theta\theta} + \frac{a^2}{2}\{k^2[-\cos[2(\theta - \phi)] - \cosh 2x] - (k_t^2 + \Lambda^2)\sinh 2x\}]\Psi_{3,4} = 0. \quad (3.27)$$

We can separate this equation into two ordinary differential equations by the ansatz $\Psi_{3,4} = R(x)S(\theta - \phi)$. Separation of equation (3.27) gives us two ordinary differential equations. The equation for S reads

$$\frac{d^2 S(\Theta)}{d\Theta^2} - \left(\frac{a^2}{2}k^2\cos(2\Theta) - n\right)S(\Theta) = 0, \quad (3.28)$$

where $(\theta-\phi)=\Theta$. This equation is of the Mathieu type and the solution can be written immediately.

$$S(\theta)=C_1 Se(n,\frac{a^2k^2}{4},\theta-\phi)+C_2 So(n,\frac{a^2k^2}{4},\theta-\phi), \tag{3.29}$$

where C_1 and C_2 are arbitrary constants. The solutions should be periodic in the angular variable Θ. This fact forces n, the separation constant, to take discrete values. It is known that the angular Mathieu functions satisfy an orthogonality relation such that functions with different n values are perpendicular to each other according to the McLachlan normalization [81].

The equation for $R(x)$ reads

$$\left\{\frac{d^2}{dx^2}-[\frac{a^2}{2}(k^2\cosh 2x+(k_t^2+\Lambda^2)\sinh 2x]+n\right\}R(x)=0, \tag{3.30}$$

whose solution can be reduced to the form

$$R(x)=D_1 Se(n,A_6,i(x+b))+D_2 So(n,A_6,i(x+b)), \tag{3.31}$$

where C_1,C_2,D_1,D_2 are arbitrary constants. The other constants (A_6 and b) will be defined below as we explain how one can reduce our initial equation to give these solutions. The solutions for the lower components Ψ_3,Ψ_4 are given in terms of sums over n and integrals over k,ϕ,k_t we used in the separation ansatz. To get this solutions one has to go through several steps.

In reference books the solution of equation (3.30) is not listed as a Mathieu function. In fact, we find that the solution of this equation is expressed in terms of double confluent Heun functions [19]. We denote these functions as H_D in our expressions.

$$\begin{aligned} R(x)=C_1 H_D&\left(0,\frac{a^2k^2}{2}-n,a^2(\Lambda^2+k_t^2),\frac{a^2k^2}{2}+n,\tanh x\right)\\ +C_2 H_D&\left(0,\frac{a^2k^2}{2}-n,a^2(\Lambda^2+k_t^2),\frac{a^2k^2}{2}+n,\tanh x\right)\\ \times\int&\frac{-dx}{H_D\left(0,\frac{a^2k^2}{2}-n,a^2(\Lambda^2+k_t^2),\frac{a^2k^2}{2}+n,\tanh x\right)^2}. \end{aligned} \tag{3.32}$$

Normally one takes the first function and discards the second solution as explained in the previous chapter.

We can show that we can express this result in terms of Mathieu functions by using a simpler independent variable than the one given in (3.32) after performing few transformations. We define

$$A_1 = \frac{-a^2(k_t^2 + \Lambda^2)}{2}, \tag{3.33}$$

$$A_2 = -n, \tag{3.34}$$

$$A_3 = -\frac{a^2k^2}{2}, \tag{3.35}$$

and use the transformation

$$z = e^{-2x}. \tag{3.36}$$

Then the differential operator in (3.30) is expressed as

$$\left(4z^2\frac{d^2}{dz^2} + 4z\frac{d}{dz} + (\frac{A_3 - A_1}{2})z + A_5 + (\frac{A_3 + A_1}{2})\frac{1}{z}\right) f = 0. \tag{3.37}$$

This equation is still of the double confluent Heun form, since it still has two irregular singularities at zero and infinity. We define $A_4 = (\frac{A_3 - A_1}{2})$ and $A_5 = (\frac{A_3 + A_1}{2})$. If we take

$$\sqrt{\frac{A_4}{A_5}}u = z, \tag{3.38}$$

and

$$w = \frac{1}{2}(u + \frac{1}{u}), \tag{3.39}$$

and set $A_6 = \sqrt{A_4 A_5}$ we get,

$$\left((w^2 - 1)\frac{d^2}{dw^2} + w\frac{d}{dw} + (\frac{A_6}{2}w + \frac{A_2}{4})\right) f = 0. \tag{3.40}$$

With the new transformations, we have traded the irregular singularity at zero by two regular singularities at plus and minus one. This is the same singularity structure of the Mathieu equation.

The solution of this equation is indeed expressible in terms of Mathieu functions. It is given as:

$$R(w) = C_1' Se\left(n, A_6, \arccos\sqrt{\frac{w+1}{2}}\right) + C_2' So\left(n, A_6, \arccos\sqrt{\frac{w+1}{2}}\right), \tag{3.41}$$

where $A_6 = \frac{a^2}{4}[k^4 - (k_t^2 + \Lambda^2)^2]^{1/2}$.

Going back through the transformations we made, it is not hard to express $\arccos\sqrt{\frac{w+1}{2}}$ in terms of our original variable x up to a constant, $i(x+b)$. We simply write

$$\frac{w+1}{2} = \frac{e^{-2x}\left(\frac{e^{2x}+\sqrt{k^2-k_t^2-\Lambda^2}}{k^2+k_t^2+\Lambda^2}\right)^2}{4\sqrt{\frac{k^2-k_t^2-\Lambda^2}{k^2+k_t^2+\Lambda^2}}} = \cosh^2(x+b). \tag{3.42}$$

Here

$$e^{-2b} = \sqrt{\frac{k^2-k_t^2-\Lambda^2}{k^2+k_t^2+\Lambda^2}}. \tag{3.43}$$

Taking the arccos of this expression gives the result given in equation (3.31).

After these transformations, we see that the solution of the lower components of the Dirac equation can be expressed in terms of functions that are regular at zero. We can not say this for the upper components, though. They are expressed in terms of these solutions and their derivatives divided by a function, which blows up at zero. Even if we take the so called odd Mathieu functions, which can be expanded in terms of hyperbolic sine functions, their derivatives will be hyperbolic cosine functions. To obtain the upper components we have to divide them by $\sqrt{\sinh 2x}$. Then these functions will still diverge at the origin.

A finite scalar product can be defined around the origin for these solutions in the form

$$\int \Psi_i^*\Psi_i\sqrt{|g|}d\tau, \qquad (i=1,..,4), \tag{3.44}$$

in a finite domain, including the origin. Here $d\tau$ is the volume element in our five dimensional space. Repeated indices are not summed over. $\sqrt{|g|}=\frac{a^2}{2}\sinh 2x$ is the square root of the determinant of the metric, necessary to get an invariant volume element. The zero of the invariant measure cancels the singularity of the wave functions at the origin.

We find, however, that the solutions are not normalizable as x goes to infinity. Furthermore, our metric has curvature singularities at the origin. Although, using our new measure ($\sqrt{|g|}d\tau$), we can make all four components normalizable at the origin, we still define our solutions for the domain $0<x<$F. We integrate x variable in the domain up to the point $x\leq$F, where F is the point where the function starts to diverge. Since our radial solution is multiplied by the angular solution and the exponential function to make up the total solution of the Dirac equation, the orthogonality of the angular solutions for different values of the discrete n makes our solutions orthogonal to each other. By dividing by the appropriate factors, we can normalize them.

We note that the domain where the solutions are normalizable is restricted. These differential equations do not have a meaning unless we define the boundary conditions the solutions obey at the boundary of our domain. Since our system exists in an odd dimensional manifold with a boundary, we have to study this equation using spectral boundary conditions [36–38].

The method used in applying these boundary conditions requires first studying the little Dirac equation, the tangential operator of the Dirac operator restricted to the boundary, where the variable x takes a fixed value x_0.

For this purpose we write the equations (3.23-3.26) for a fixed $x = x_0$ value.

$$\frac{\sqrt{2}}{a}\{i\partial_\theta \Psi_3 \; + iak\cos(\theta - \phi + ix_0)\Psi_4 - \frac{iak_t}{\sqrt{2}} f_1\} = \lambda f_1, \tag{3.45}$$

$$\frac{\sqrt{2}}{a}\{-i\partial_\theta \Psi_4 \; - iak\cos(\theta - \phi - ix_0)\Psi_3 - \frac{iak_t}{\sqrt{2}} f_2\} = \lambda f_2, \tag{3.46}$$

$$\frac{\sqrt{2}}{a}\{-i\partial_\theta f_1 \; - iak\cos(\theta - \phi + ix_0) f_2 + \frac{iak_t}{\sqrt{2}} \Psi_3\} = \lambda \Psi_3, \tag{3.47}$$

$$\frac{\sqrt{2}}{a}\{i\partial_\theta f_2 \; + iak\cos(\theta - \phi - ix_0) f_1 + \frac{iak_t}{\sqrt{2}} \Psi_4\} = \lambda \Psi_4, \tag{3.48}$$

where λ is the eigenvalue of the little Dirac equation.

After studying the system with symbolic packages and referring to literature [82] we could not find analytical solutions of these equations in terms of known functions. We could not even write uncoupled equations in the second order. One needs to go to fourth order in derivatives to be able to write equations that involves a single unknown function.

At this point we follow closely our references [39, 40, 42, 83]. We formally expand our solutions at the boundary, fixed x_0 in terms of eigenfunctions of the little Dirac equations with both positive and negative eigenvalues λ. For the lower components we have

$$\Psi^{\Lambda}_{3,4}(\Theta, x_0) = \sum_{\lambda} h_{\lambda}(\Theta, x_0), \tag{3.49}$$

for fixed values of k_t, k_y, k_z. We set

$$\Psi^{\Lambda}_{3,4}(\Theta, x_0)|_{\partial\mathcal{M}} = \sum_{\lambda>0} h_{\lambda}(\Theta, x_0). \tag{3.50}$$

The negative λ eigenvectors are all set to be zero at the boundary to satisfy the spectral boundary conditions.

Then, we solve $f_{1,2}$ in terms of $\Psi_{3,4}$ using the equations (3.23-3.24) and fix the x values to x_0 on the boundary. Note that since $x = x_0$, the derivative with respect to x is evaluated at this point as well as the terms without the x derivative. We can, in general, use the expansion given for $\Psi_{3,4}$ on the boundary, in terms of its eigenfunctions.

$$\Psi^{\Lambda}_{3,4}(\Theta, x_0) = \sum_{\lambda} h_{\lambda,3,4}(\Theta, x_0). \tag{3.51}$$

This sum is over all values of λ. In fixing the values of $f_{1,2}$ in terms of $\Psi_{3,4}$ on the boundary, we use only part of the expansion of $\Psi^{\Lambda}_{3,4}(\Theta, x)$ where

$$\Psi^{\Lambda}_{3,4}(\Theta, x)|_{\partial\mathcal{M}} = \sum_{\lambda<0} h_{\lambda,3,4}(x_0, \Theta). \tag{3.52}$$

In other words, we write $\Psi^{\Lambda}_{1,2}(\Theta, x)|_{\partial\mathcal{M}}$ using the expressions obtained from $f_{1,2}$ in terms of the negative λ values of $\Psi^{\Lambda}_{3,4}(\Theta, x)|_{\partial\mathcal{M}}$. These boundary conditions are non-local, but are shown to be the only consistent ones for odd dimensional Euclidean spaces by Atiyah-Patodi-Singer [36–38].

3.1.1 Qualitative Analysis for the Little Dirac Equations in Five Dimensions

The little Dirac equation is a system of linear differential equations with periodic coefficients. Then we can write the system as [44, 45]:

$$\partial_\theta \psi = P(\theta)\psi, \tag{3.53}$$

where $P(\theta)$ is the coefficient matrix and ψ is the solution vector. Here $P(\theta + \tau) = P(\theta)$ and τ is the period of the coefficients ($\tau \neq 0$). According to Bellman [44], if $\psi(0) = I$, we can write

$$\psi(\theta) = Q(\theta)e^{B\theta}. \tag{3.54}$$

The matrix $Q(\theta)$ is also periodic with the period τ. Then we have

$$\begin{aligned}
\psi(\theta+\tau) &= Q(\theta+\tau)e^{B\theta}e^{B\tau} \\
&= Q(\theta)e^{B\theta}e^{B\tau} \\
&= \psi(\theta)e^{B\tau}.
\end{aligned} \tag{3.55}$$

We can define $e^{B\tau} \equiv C$ and use

$$C = \psi(\theta)^{-1}\psi(\theta+\tau) \tag{3.56}$$

to obtain C. The Jordan normal form of C, T being the transformation matrix,

$$C = T \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_r \end{pmatrix} T^{-1} \tag{3.57}$$

gives us the B matrix:

$$B = \frac{1}{\tau}\ln L. \tag{3.58}$$

The eigenvalues of B give us the characteristic roots. We will use these characteristic roots with different parameters in our stability analysis.

The periodicity of the defined Q can be checked using numerical means.

We use

$$\psi(\theta+\tau) = Q(\theta+\tau)e^{B\theta}C \tag{3.59}$$

and for $\theta = 0$, $\psi(\theta) = Q(\theta)e^{B\theta}$ to give,

$$\psi(\tau)C^{-1} = Q(\tau) = Q(0) = \psi(0) = I. \tag{3.60}$$

We check numerically that this equation is satisfied;

$$\psi(\tau)C^{-1} = \begin{pmatrix} 1.0 - \tilde{}10^{-8}i & \tilde{}10^{-8} & \tilde{}10^{-8} & \tilde{}10^{-8} \\ \tilde{}10^{-8} & 1.0 - \tilde{}10^{-9}i & \tilde{}10^{-9} & \tilde{}10^{-9} \\ \tilde{}10^{-8} & \tilde{}10^{-9} & 1.0 - \tilde{}10^{-8}i & \tilde{}10^{-8} \\ \tilde{}10^{-9} & \tilde{}10^{-8} & \tilde{}10^{-8} & 0.99 + \tilde{}10^{-8}i \end{pmatrix}. \tag{3.61}$$

Hence, Q is periodic (where "~" denotes the magnitude of the numerical value).

The values and the signs of the characteristic roots of the spinor components are given in Table 3.1. In the table, "+ + - -" means that the first two components have positive roots and the last two components have negative characteristic roots. The root value for a set of parameters $(a, k, k_t, x_0, \lambda)$ is the same for the all four spinor components. Our calculations indicate that f_1 and f_2 solutions are not stable (positive characteristic root) while, Ψ_3 and Ψ_4 solutions are stable (negative characteristic root). As it is seen in Table 3.1, when we keep all the other parameters constant and vary only a, the value of the roots are influenced most, whereas the effect of the variation in the value of λ changes the value of the roots least. However, the dependency to the change of parameters is not significant for all cases. We also find that when these parameters exceed unity in absolute value, we encounter instabilities as the separation between consecutive roots increase and some negative roots go to positive values for larger values than unity. If we keep the values of the parameters in the range $[-1,1]$, we seem to have no such problems.

Table 3.1: The Change in the Characteristic Roots With Respect to Parameters in 5D

a	k	k_t	x_0	λ	real parts of the characteristic roots ($\times 2\pi$)	signs of the characteristic roots
1	1	1	1	1	10.33600	$++--$
0.5	”	”	”	”	4.75880	”
0.8	”	”	”	”	8.13868	”
1.1	”	”	”	”	11.42970	”
1.5	”	”	”	”	15.76100	”
1	0.5	1	1	1	7.42689	”
”	0.8	”	”	”	9.03146	”
”	1.1	”	”	”	11.03930	”
”	1.5	”	”	”	11.03830	”
1	1	0.5	1	1	9.53770	”
”	”	0.8	”	”	9.96445	”
”	”	1.1	”	”	10.54930	”
”	”	1.5	”	”	11.52540	”
1	1	1	0.5	1	8.06106	”
”	”	”	0.8	”	9.18092	”
”	”	”	1.1	”	11.06420	”
”	”	”	1.5	”	15.15810	”
1	1	1	1	0.5	9.53770	”
”	”	”	”	0.8	9.96445	”
”	”	”	”	1.1	10.54930	”
”	”	”	”	1.5	11.52540	”

3.2 Solutions in Four Dimensions

Here we repeat the calculations given for the five dimensional case after setting k_t to zero. Our solutions exist only in a finite domain for the variable x, hence we have to use the appropriate boundary conditions at this point. At this point, although we can use local boundary conditions in this case, we choose to use the spectral boundary conditions of Atiyah-Patodi-Singer to conserve chirality and charge conjugation.

We write the system in the form $L\psi = \Lambda\psi$ where ψ is a four component spinor, and try to obtain the solutions for the different components. Then our equations are similar to the ones given in the equations (3.18-3.21). The only difference is taking k_t equal to zero but three components are still coupled in our equations.

The method of solution is exactly like the one used in the five dimensional case. To get our solutions we use the separation of variables method. We write the solution as a product of exponentials in y, z times a function of x and θ.

$$\psi_i = e^{i(k_y y + k_z z)}\Psi_i(x,\theta). \tag{3.62}$$

The same transformations are used as those in the five dimensional case to reduce $\Psi_{1,2}$ to $f_{1,2}$. We solve our equations in terms of $f_{1,2}$ and substitute these expressions in equations, given above. We end up with second order, but uncoupled equations for the lower components.

$$\left(\partial_{xx} + \partial_{\theta\theta} - \frac{a^2}{2}[k^2(\cos 2(\theta-\phi) + \cosh 2x) + \Lambda^2 \sinh 2x]\right)\Psi_{3,4} = 0. \tag{3.63}$$

We can separate this equation into two ordinary differential equations by the ansatz $\Psi_{3,4} = R(x)S(\theta)$. For $S(\theta)$ we get an equation of the Mathieu type and the solution can be written immediately.

$$S(\theta) = C_1 Se[n, \frac{a^2k^2}{4}, \theta-\phi] + C_2 So(n, \frac{a^2k^2}{4}, \theta-\phi). \tag{3.64}$$

Solution for $R(x)$ can be reduced to

$$R(x) = D_1 So(n, B, i(x+b')) + D_2 Se(n, B, i(x+b')). \tag{3.65}$$

Here C_1, C_2, D_1, D_2 are arbitrary constants. $B = \frac{\sqrt{k^4-\Lambda^4}}{2}$. b' is defined as in equation (3.43) with $k_t = 0$. Just note that n is the separation constant which has to take discrete values to get a periodic solutions for the angular Mathieu equation $S(\theta)$. The solutions for the lower components Ψ_3, Ψ_4 are given in terms of sums over n and integrals over k, ϕ we used in the separation ansatz.

We find that we have the upper solutions, $\Psi_{1,2}$ are divergent at the origin, whereas the lower ones are finite. Both of our solutions diverge at infinity. For the same reasons as given for the five dimensional case, we have to limit the domain of our solutions.

To impose these boundary conditions we need to write the little Dirac equation, the Dirac equation restricted to the boundary, where the variable x takes a fixed value x_0. We choose to write the equations in the form,

$$\frac{\sqrt{2}}{a}\{i\partial_\theta\Psi_3 \ + iak\cos(\theta-\phi+ix_0)\Psi_4\} = \lambda f_1, \tag{3.66}$$

$$\frac{\sqrt{2}}{a}\{-i\partial_\theta\Psi_4 \ - iak\cos(\theta-\phi-ix_0)\Psi_3\} = \lambda f_2, \tag{3.67}$$

$$\frac{\sqrt{2}}{a}\{-i\partial_\theta f_1 \ - iak\cos(\theta-\phi+ix_0) f_2\} = \lambda\Psi_3, \tag{3.68}$$

$$\frac{\sqrt{2}}{a}\{i\partial_\theta f_2 \ + iak\cos(\theta-\phi-ix_0) f_1\} = \lambda\Psi_4. \tag{3.69}$$

Here λ is the eigenvalue of the little Dirac equation.

We could not obtain analytical solutions of these equations in terms of known functions. This is the same result as in the five dimensional case. For us it seems very curious being able to solve similar system of partial differential equations, but not even being able to decouple them when this system reduces to ordinary differential equations on the boundary. One possible explanation is that $\theta-\phi\pm ix$ act as z and $\bar{z}$ of complex variables. Sometimes it is easier to find functions of this pair as solutions than a function of a single real variable. Actually for the full Dirac equation, Sucu and Ünal find solutions in a closed form [16]. Same technique, however, does not seem to work when $x = x_0$.

We expand our solutions at the boundary, fixed by two values of x_0 in terms of eigenfunctions of the little Dirac equations with both positive and negative eigenvalues λ.

$$\Psi_i^\Lambda(\Theta, x_0) = \sum_\lambda g_{i,\lambda}(\Theta, x_0) \tag{3.70}$$

for fixed values of k_y, k_z. We set

$$\Psi_{3,4}^\Lambda(\Theta, x)|_{\partial\mathscr{M}} = \sum_{\lambda>0} g_{\lambda,3,4}(\Theta, x_0). \tag{3.71}$$

The negative λ eigenvectors are all set to be zero at the boundary.

The boundary conditions on the upper components are imposed exactly in the same manner as explained in the five dimensional case, namely we solve for $f_{1,2}$ in terms of $\Psi_{3,4}$ using the equations

$$\frac{\sqrt{2}}{a}\{(\partial_x + i\partial_\theta)\Psi_3 \ + a[\cos(\theta - \phi + ix)]\Psi_4\} = \Lambda f_1, \qquad (3.72)$$

$$\frac{\sqrt{2}}{a}\{(\partial_x - i\partial_\theta)\Psi_4 \ - a[\cos(\theta - \phi - ix)]\Psi_3\} = \Lambda f_2, \qquad (3.73)$$

and fix the x values to x_0 on the boundary.

We can use the expansion given for $\Psi_{3,4}$ on the boundary, in terms of its eigenfunctions, equation (3.70). This sum is over all values of λ. In fixing the values of $f_{1,2}$ in terms of $\Psi_{3,4}$ on the boundary, we use only the part where $\lambda < 0$. These boundary conditions are non local, but they respect self adjointness and conserve γ^5 and charge conjugation symmetry.

3.2.1 Qualitative Analysis for the Little Dirac Equations in Four Dimensions

The same analysis explained above can be used for the four dimensional case, giving the characteristic roots of the four dimensional little Dirac equations. The results can be summarized as in the Table 3.2. The arrangement of Table 3.2 is the same as Table 3.1. The behavior of the roots are the same as in the five dimensional case, with different numerical values of the characteristic roots. Therefore, the stability structure of the five and four dimensional cases are the same.

Table 3.2: The Change in the Characteristic Roots With Respect to Parameters in 4D

a	k	x_0	λ	real parts of the characteristic roots ($\times 2\pi$)	signs of the characteristic roots
1	1	1	1	9.25029	$++--$
0.5	”	”	”	4.03926	”
0.8	”	”	”	7.21630	”
1.1	”	”	”	10.25420	”
1.5	”	”	”	14.22150	”
1	0.5	1	1	5.83270	”
”	0.8	”	”	7.75421	”
”	1.1	”	”	10.03190	”
”	1.5	”	”	13.27450	”
1	1	0.5	1	6.56206	”
”	”	0.8	”	7.91586	”
”	”	1.1	”	10.06320	”
”	”	1.5	”	14.46480	”
1	1	1	0.5	8.30164	”
”	”	”	0.8	8.81342	”
”	”	”	1.1	9.49249	”
”	”	”	1.5	10.58880	”

3.3 Singularity Analysis for the Little Dirac Equations

The Dirac equation written in the background of the Nutku helicoid metric was written as

$$(\partial_x + i\partial_\theta)\Psi_3 \; + iak[\cos(\theta - \phi + ix)]\Psi_4 = 0, \tag{3.74}$$

$$(\partial_x - i\partial_\theta)\Psi_4 \; - iak[\cos(\theta - \phi - ix)]\Psi_3 = 0, \tag{3.75}$$

$$(-\partial_x + i\partial_\theta) f_1 \; + iak[\cos(\theta - \phi + ix)] f_2 = 0, \tag{3.76}$$

$$(-\partial_x - i\partial_\theta) f_2 \; - iak[\cos(\theta - \phi - ix)] f_1 = 0. \tag{3.77}$$

These equations have simple solutions [16] which can also be expanded in terms of products of radial and angular Mathieu functions [15,84]. Problem arises when these solutions are restricted to boundary [41].

To impose these boundary conditions we need to write the little Dirac equation, the Dirac equation restricted to the boundary, where the variable x takes a fixed value x_0. We choose to write the equations in the form,

$$\frac{\sqrt{2}}{a}\{i\frac{d}{d\theta}\Psi_3 \; + iak\cos(\theta - \phi + ix_0)\Psi_4\} = \lambda f_1, \tag{3.78}$$

$$\frac{\sqrt{2}}{a}\{-i\frac{d}{d\theta}\Psi_4 \ -iak\cos(\theta-\phi-ix_0)\Psi_3\} = \lambda f_2, \tag{3.79}$$

$$\frac{\sqrt{2}}{a}\{-i\frac{d}{d\theta}f_1 \ -iak\cos(\theta-\phi+ix_0)f_2\} = \lambda\Psi_3, \tag{3.80}$$

$$\frac{\sqrt{2}}{a}\{i\frac{d}{d\theta}f_2 \ +iak\cos(\theta-\phi-ix_0)f_1\} = \lambda\Psi_4. \tag{3.81}$$

Here λ is the eigenvalue of the little Dirac equation. We take $\lambda = 0$ as the simplest case. The transformation

$$\tilde{\Theta} = \theta - \phi - ix_0 \tag{3.82}$$

can be used. Then we solve f_1 in the latter two equations in terms of f_2:

$$-\frac{d^2}{d\tilde{\Theta}^2}f_2 - \tan\tilde{\Theta}\frac{d}{d\tilde{\Theta}}f_2 + \frac{(ak)^2}{2}[\cos(2\tilde{\Theta})\cosh(2x_0) \\ -i\sin(2\tilde{\Theta})\sinh(2x_0) + \cosh(2x_0)]f_2 = 0 \tag{3.83}$$

When we make the transformation

$$u = e^{2i\tilde{\Theta}}, \tag{3.84}$$

the equation reads,

$$\left\{4(u+1)u\left[u\frac{d^2}{du^2}+\frac{d}{du}\right] - 2iu(u-1)\frac{d}{du} + \\ \frac{(ak)^2}{2}(u+1)\left[ue^{-2x_0}+\frac{1}{u}e^{2x_0}+\cosh(2x_0)\right]\right\}f_2 = 0. \tag{3.85}$$

This equation has irregular singularities at $u = 0$ and ∞ and a regular singularity at $u = -1$. If we try a solution in the form

$$\sum_{n=-\infty}^{\infty} a_n u^n \tag{3.86}$$

around the irregular singularity $u = 0$ we end up with a four-term recursion relation as

$$a_{n-1}[4(n^2-2n+1) - 2i(n-1) + \frac{(ak)^2}{2}(\frac{3}{2}e^{-2x_0}+\frac{1}{2}e^{2x_0})] \\ +a_n[4n^2+2in+\frac{(ak)^2}{2}(\frac{3}{2}e^{-2x_0}+\frac{1}{2}e^{2x_0})]+ \\ a_{n-2}[\frac{(ak)^2}{2}e^{-2x_0}] + a_{n+1}[\frac{(ak)^2}{2}e^{2x_0}] = 0 \tag{3.87}$$

As it is known, in the Heun equation case, this kind of series solution gives a three-term relation [19].

If we search for a solution of the Thomé type we may try a solution of the form

$$f_2 = e^{\frac{A}{\sqrt{u}}} g(u). \tag{3.88}$$

This form does not allow us to get a Taylor series expansion around the irregular point $u = 0$ [22, 85].

If we try a series solution around the regular singularity at $u = -1$ as

$$\sum_{n=0}^{\infty} a_n (u+1)^{n+\alpha} \tag{3.89}$$

we find a relation between five consecutive coefficients for the solution. Therefore, we may conclude that the solution of this equation cannot be written in terms of Heun functions or simplier special functions.

We note that the Heun equation has four regular singularities. Some of these regular singularities coalesce to form irregular singularities in the confluent forms of this equation. The confluent Heun equation has two regular and one irregular singularities, the double confluent Heun equation has two irregular singularities, the biconfluent Heun equation has a regular and a higher rank irregular singularity and the triconfluent Heun equation has a single irregular singularity of even higher rank [19]. The little Dirac equation we studied has two irregular singularities and one regular singularity which does not fit any of these types.

To check this further, we first set the coefficient of $\frac{1}{u}$ term in equation (3.85) equal to zero to change our irregular singularity at zero to a regular one. Then we keep this term and discard the ue^{-2x_0} term to reduce the singularity structure of infinity. In both cases one can check that the solution can be expressed in terms of confluent Heun functions. This shows that reducing one of the singularities yields a Heun function.

Thus, we conclude that the full equation (3.85) has more singularities then the better known solutions in the literature, which are included in the computer packages like Maple or cited in the book by Ince [43].

3.4 The Integral of Motion

The full metric in four dimensions admits two evident Killing vectors ∂_y and ∂_z (There is an additional one in five dimensional case, ∂_t). Another constant of motion is the square of the 4-momentum $g^{\mu\nu} p_\mu p_\nu = \mu^2$, and an independent integral of motion,

$$K = -{p_\theta}^2 - a^2 \left(\cos\theta\, p_y + \sin\theta\, p_z\right)^2, \qquad (3.90)$$

exists [12]. The existence of this Killing tensor leads to a separation of variables in the Hamilton-Jacobi equation for geodesics. The separability ansatz,

$$S = p_y y + p_z z + S_x(x) + S_\theta(\theta) \qquad (3.91)$$

leads to decoupled ordinary differential equations in the Hamilton-Jacobi equation,

$$\begin{aligned} \left(\frac{dS_x}{dx}\right)^2 + a^2 \left(p_y^2 + p_z^2\right) \sinh^2 x - \frac{\mu^2 a^2}{2} \sinh 2x &= K \\ \left(\frac{dS_\theta}{d\theta}\right)^2 + a^2 \left(\cos\theta\, p_y + \sin\theta\, p_z\right)^2 &= -K \end{aligned} \qquad (3.92)$$

where the separation constant K coincides with the integral of motion [12].

When one restricts the solution to a fixed value of the radial coordinate, the value of K becomes dependent to this fixed value and the system on the boundary lacks the symmetry of the full spacetime.

4. NPInstanton COMPUTER PACKAGE

The package, NPInstanton, consists of procedures to calculate some physical and mathematical quantities for instanton metrics using Newman-Penrose formalism. It is coded under Maple 11 and GRTensorII package. The procedures calculate massless scalar equation, massless Dirac equation, source-free Maxwell equations, covariant and contravariant Dirac matrices, coframe $l(= l_\mu dx^\mu)$ and $m(= m_\mu dx^\mu)$, Ricci rotation coefficients, Weyl scalars, trace-free Ricci scalars, spinor equivalent of the connection 1-forms, basis 2-forms, curvature 2-forms, integrands of the Euler number and the Hirzebruch signature curvature part integrals and Petrov class of the spacetime.

As one can see, some of the objects could be calculated by standard means and without using a signature-dependent package. But for the sake of completeness, we added these features to the program. By these, the program becomes a complete symbolic calculator for an instanton metric. The NP calculator of GRTensorII is not designed for the Euclidean signature and could give unexpected results. Therefore, a complete NP based calculator, combining the power of GRTensorII and NP formalism for these special metrics is useful.

4.1 System Requirements

Any system that can run Maple 11 and GRTensorII is able to run NPInstanton.

The program uses,

1. GRTensorII Package: The famous and widely used non-commercial general relativity package. It has two versions, the one which NPInstanton uses works under Maple and a limited version of the package for Mathematica is also available. The developers of GRTensorII are Peter Musgrave, Denis Pollney and Kayll Lake [57]. Several objects (Ricci scalar, covariant Weyl tensor, etc.) are calculated by this package in our program. The package has a powerful

NP calculator but it is not designed for the metrics with Euclidean signature. No specific knowledge of GRTensorII is needed for using NPInstanton. Our program creates the metric file needed for the calculation itself using the NP legs given in the input file and writes it to the metric directory of GRTensorII as "npinstanton.mpl". This metric file then can be used in GRTensorII independently.

2. DifferentialGeometry Package of Maple 11: One of the most important new features of Maple 11 is the new package: "DifferentialGeometry". It is convenient to use this package with linear algebraic quantities. The rather old "difforms" package would be another solution but it has some conflicts with linear algebraic quantities which are the major elements of our program. The DifferentialGeometry package is based upon the Vessiot package developed by I. M. Anderson, Florin Catrina, Cinnamon Hillyard, Jeff Humphries, Jamie Jorgersen, and Charles Miller at Utah State University. The redesign and expansion of Vessiot to DifferentialGeometry for Maple 11 was done by I. M. Anderson and E. S. Cheb-Terrab [86]. The definition of the wedge product is supplied by this package in our program.

3. linalg package of Maple: This rather old internal package is used for an eigenvalue calculation in Petrov classification section.

The program is set for a computer having a 512 Mb of RAM (Average value for today's personal computers). If the system has less memory, the user must change the line

```
kernelopts(gcfreq=10^7):
```

of the npinstanton.mws file to a lower value (10^6 is the standard value of Maple). For a computer having larger than 1 Gb of memory, the user may change the gcfreq value as 10^8. "gc" is the abbreviation of "garbage collection" and it is the Maple's internal routine which cleans the memory after an amount of memory is allocated. For a computer having a large amount of memory, one can increase the frequency of this process. The larger the gc frequency value results in more memory to be wasted but for a system having a large amount of memory it increases the performance for some calculations.

The programming style is procedure-based. Each command is a Maple procedure which can call other calculation procedures and inherit their outputs. For example, when the dirac() command is given for the calculation of massless Dirac equation, the program calls dirac() procedure. The calculation of Dirac equations needs the information of γ matrices. Therefore, dirac() procedure calls gammamatricespless() procedure for this calculation and its output is sent to dirac() procedure. The suffix "-pless" (printless) means that this procedure is a "secondary" one and it does not print its output to the screen. When dirac() procedure finishes execution, it removes the information that was inherited from gammamatricespless() procedure and sends the user only its permitted output as dirac vector.

The "secondary" procedures do the massive works of the main procedures and the main procedure generally contains the key directives. Therefore, this usage makes the program easier to track. In addition to this, if a user wants to add another command to the program, a procedure of that command can be coded using these "secondary" procedures, or the user can add his/her own "secondary" procedures, as well.

Throughout the program, only the output variables of the procedures are the global ones and only these variables can be reached by the user at the end of the execution of the command. Therefore, the user should be aware of the output names of the procedures that are being used. This global variables are shown at the end of the execution of each command. For example, the output variable of scalaroperator() command will be shown as scalarop and it can be called in the session when needed.

The outputs of the auxiliary calculations are cleared from the memory by setting them as local variables, as these auxiliary calculations may occupy too much memory. For example, γ matrices information is not necessary if the user needs to know only the Dirac equation. Therefore, dirac() procedure clears the information about γ matrices after finishing the execution. If the user needs to know the γ matrices, gammamatrices() procedure is run by the user. As another example, in conn1form() procedure, only output values are connection 1-forms and the outputs from coframepless() and riccirotcoeffpless() procedures, as well as internal calculations are dismissed to gain memory for further calculations.

A few commands for simplification are added to the code which can be evaluated easily by an average personal computer (i.e. having a 512 Mb of memory), but it is always more convenient for the user to choose the right simplification technique for the problem after the calculation. The output must be regarded as a "raw material" for a simplification routine that is to be chosen by the user. The user having a computer with insufficient memory can extract the simplification routines from the program, simply by modifying it in an editor but it is not recommended as it may lead to miscalculation of some properties such as Petrov type or (anti-)self-duality.

Linux and Windows versions are available and for those who have older Maple versions, a limited version of NPInstanton (NPInstantonjr) which does not contain the basis2form(), curv2form() and topologicalnumbers() parts (parts that use DifferentialGeometry Package) is also supplied. The details on how to run the program is given in the README file which comes with the program files.

4.2 Running the Program

The program consists of several procedures for calculations. The user runs the procedures by calling some specific commands and the output will be available for further calculations after the procedure is finished.

The program comes with four files:

i. NPInstanton program (npinstanton.mws)

ii. Sample input files (eguchihanson and nutku)

iii. README file containing short definitions of the commands (README.txt)

Sample input file should be in "C:/npinstanton" directory. If the user wishes to change this location permanently, in the NPInstanton program, in Windows,

currentdir("C:/npinstanton"):

or in Linux,

currentdir("/usr/local/npinstanton"):

line should be changed in the appropriate way. Or the user may choose to give the path of the file after running the program. A permission change of the directory may be needed in Linux systems and it can be completed by

chmod -R 777 /usr/local/npinstanton/

or in Ubuntu Linux,

sudo chmod -R 777 /usr/local/npinstanton/

for the original directory after logging in as the superuser or root.

Changes to the npinstanton.mws file should be made using an editor such as notepad. The changes should not be made within Maple as, on saving the file, the Maple editor adds a "signature" which causes errors when executing the worksheet.

To run NPInstanton:

1. Open a Classic worksheet in Maple. Standard worksheet can also be used technically but GRTensorII suggests Classic worksheet.

2. Run NPInstanton from its location by read command.

For example,

read"C:/npinstanton/npinstanton.mws";

or in Linux systems,

read "/usr/local/npinstanton/npinstanton.mws";

runs the program from "C:/npinstanton/" or "/usr/local/npinstanton/" directory.

3. Enter the input file as requested. No ";" or ":" is required after the file name. Now, the program will load GRTensorII package, DifferentialGeometry package, information from the input file and check the Newman-Penrose legs. After the program checks the Newman-Penrose legs, the user will be prompted as

npinstanton>

and the session is ready for calculations. Most of the calculations are finished in seconds (even less than a second) but some spacetimes may have properties that may need more time to finish the calculation.

4.3 The Input File

The input file contains the necessary definitions for the space and wave equations. General definitions (constants, etc.) can also be given in this file. The definitions should be given using Maple's syntax. The input file of our program plays the role of the metric file of GRTensorII.

The user first sets the coordinate names in the first section as,

firstcoordinate:= x:

secondcoordinate:= theta:

etc. as given in sample input file. Then the components of the covariant NP legs are entered. l_covar[1] being the first component of the covariant l_μ leg and l_bar_covar[1] being the complex conjugate of l_covar[1], etc.. Maple does not do the complex simplifications because they need assumptions that may cause wrong calculations. Therefore, the most appropriate way to define the legs is to give the both by hand. The choice of NP legs are not unique for a metric. For the Eguchi-Hanson NP legs we can have [11]:

$$l = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{1-\frac{a^4}{r^4}}}dr - \frac{ir}{2}\sqrt{1-\frac{a^4}{r^4}}(d\xi + \cos\theta d\phi)\right], \tag{4.1}$$

$$m = \frac{re^{-i\xi}}{2\sqrt{2}}(d\theta + i\sin\theta d\phi). \tag{4.2}$$

Then the input forms are

l_covar[1]:=1/(sqrt(2)*A):

l_bar_covar[1]:=1/(sqrt(2)*A):

etc. as given in sample input file. The A value being $\sqrt{1-\frac{a^4}{r^4}}$ was given in the general definitions section.

Spinor components can be given if the user will be calculating the Dirac equation. They can be chosen as [16],

$$\psi_1 = e^{i(m+\frac{1}{2})\xi}\Psi_1(r,\theta,\phi), \tag{4.3}$$

$$\psi_2 = e^{i(m+\frac{1}{2})\xi}\Psi_2(r,\theta,\phi), \tag{4.4}$$

$$\psi_3 = e^{i(m-\frac{1}{2})\xi}\Psi_3(r,\theta,\phi), \tag{4.5}$$

$$\psi_4 = e^{i(m-\frac{1}{2})\xi}\Psi_4(r,\theta,\phi), \tag{4.6}$$

and the input form is

spinorcomponent1:=exp(I*(m+(1/2))*xi)*psi1[r,theta,phi]:

etc. as given in sample input file.

The scalar function can be given for the calculation of the scalar equation. This definition can be skipped if the used does not need to calculate this object. It can be chosen as

$$\varphi = e^{im\xi}\Phi(r,\theta,\phi) \tag{4.7}$$

for the Eguchi-Hanson space, φ being the scalar function and it is set as

scalarfunction:=exp(I*m*xi)*Phi(r,theta,phi):

in the input file.

4.4 Command Definitions

The list of commands can be given as, scalaroperator(), dirac(), maxwell(), gammamatrices(), coframe(), spinrotcoeff(), weylscalar(), tfricciscalar(), conn1form(), basis2form(), curv2form() and topologicalnumbers().

The definitions of these commands will be given in Appendix C through an example.

5. RESULTS AND DISCUSSION

The wave equations in the background of the Nutku helicoid solution can be expressed in terms of Mathieu functions. The radial and angular equations are found to be appropriate to form a propagator by using summation formulae for Mathieu functions in the four dimensional metric [12]. A time dimension is added trivially to the metric to see a possible higher structure. In the time added Nutku helicoid case, the wave equations are written in terms of the double confluent Heun equation which is a special family of the Heun equation [19]. This solution can be also expressed in terms of the Mathieu function by using the $z = e^{-2x}$ transformation, which maps infinity to zero followed by a rescaling and a further transformation where, aside from the scaling, we are taking the hyperbolic cosine of the original radial variable and this procedure brings a natural limit on the radial variable. By applying this transformation, one trades the irregular singularity at zero by two regular singularities at plus and minus one to end up with the same singularity structure with the Mathieu equation. Although both the radial and the angular part can be written in terms of Mathieu functions, the constants are different, modified by the presence of the new $\frac{a^2k_t^2}{2}$ term coming from the additional coordinate, which makes the summation of these functions to form the propagator quite difficult.

Nutku helicoid solution has a curvature singularity at the origin [12]. Therefore, in order to have a precise result we tried to apply the Atiyah-Patodi-Singer spectral boundary conditions which was necessitated by the dimension of the spacetime. One is free to choose the local boundary conditions in even dimensions. However, we applied the same type of boundary conditions to conserve γ^5 and charge conjugation symmetries of the Dirac operator in the four dimensional case. The application of the spectral boundary conditions involves the solution of the so called the little Dirac equation, the Dirac equation written on the boundary of the manifold. The analytical solutions of the little Dirac equation could not be reached, but a qualitative analysis is done to show the behavior of the solutions. We concluded that two elements of the solution vector ($\mathbf{\Psi}_3$ and $\mathbf{\Psi}_4$) which correspond to negative characteristic roots are stable, and the other two are unstable, given by the positive characteristic roots. The singularity structure of the little Dirac equation can reveal why we could not obtain an analytical solution. The singularity analysis shows that the equation has irregular singularities at zero and infinity and a regular singularity at minus one. We see that this equation has one more singularity than the Heun functions; thus, our solution is not one of the better known solutions in the literature. Moreover, a constant of motion is disappeared when we restrict the system by putting the radial coordinate x to a constant value.

The computer programs are involved intensively in the derivation and analysis of the equations. A Maple package using the Goldblatt's Newman-Penrose formalism (see Appendix B) for instanton spaces is developed for the analytical computations needed in the work. The package also supplies a complete Newman-Penrose calculator for instanton metrics. The numerical study for the qualitative analysis section is done by Mathematica [53] as it is faster when dealing with large data sets but Maple [52] is widely used in all other parts.

The numerical methods, being still important for the analytically non-solvable systems such as N-body dynamics, analytical methods are getting more and more crucial in the applicable cases as the capacity of the computer systems and the symbolic manipulation packages increase.

It is a strong estimate that we will encounter higher equations -in the sense of the singularity structure- than the hypergeometric equations gradually more in the literature, as the phenomena in higher dimensions or in different geometries are studied. Having different confluent types and special cases which reduce to less singular equations, Heun's equation has a special importance to bring together the known literature with the physics of the future.

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A. HEUN'S EQUATION

Heun's equation is the general second-order linear Fuchsian equation with four singularities. The extra singularity brings difficulties when analyzing the equation. The powerful methods used for investigating the hypergeometric equation no longer work for the Heun's equation case. For example, the power series solution has a three term recursion relation while for the hypergeometric equation one ends up with a recursion relation between two successive coefficients. In the case of two way recursion relations, it is easy to study the convergence of the solution series. However, in three way relations this type of study is not as clear as the former case.

The reference of this brief introduction is the book edited by Ronveaux [19], which is the most comprehensive source in the field.

A.1 Heun's General Equation

The canonical form of the Heun's general equation can be given as ($a \neq 0,1$),

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0, \tag{A.1}$$

where the parameters are linked by $\gamma + \delta + \varepsilon = \alpha + \beta + 1$. This equation has four regular singularities at $z = 0, 1, a, \infty$.

This equation reduces to the hypergeometric equation if $a = 1$ and $q = \alpha\beta$ or $a = q = 0$ or $\varepsilon = 0$ and $q = a\alpha\beta$. If $\gamma = \delta = \varepsilon = \frac{1}{2}$ then the equation becomes Lamé's equation.

A.2 The Confluent Heun Equation

This form is obtained when the singularity at $z = a$ is merged with that at $z = \infty$, resulting in an equation having regular singularities at $z = 0, 1$ and an irregular singularity of rank 1 at ∞.

The natural general form can be given as,

$$\frac{d^2y}{dz^2} + \left(\sum_{i=1}^{2}\frac{A_i}{z-z_i} + E_0\right)\frac{dy}{dz} + \left(\sum_{i=1}^{2}\frac{C_i}{z-z_i} + \sum_{i=1}^{2}\frac{B_i}{(z-z_i)^2} + D_0\right)y = 0, \tag{A.2}$$

where A_i, B_i, C_i, D_0, E_0 are arbitrary parameters. This equation has Whittaker-Hill and Mathieu equations, which are the simplest modelling equations with periodical potentials, as special cases.

A.3 The Double Confluent Heun Equation

This form is obtained when the singularity at $z = 1$ is moved to the point $z = b$; then simultaneously let $a \to \infty$ and $b \to 0$. The equation has two irregular singularities at 0 and ∞ of rank 1.

In its canonical form,

$$D^2 y + \left(\alpha_1 z + \frac{\alpha_{-1}}{z}\right) Dy + \left[\left(B_1 + \frac{\alpha_1}{2}\right) z + \left(B_0 + \frac{\alpha_1 \alpha_{-1}}{2}\right) + \left(B_{-1} - \frac{\alpha_{-1}}{2}\right) \frac{1}{z}\right] y = 0, \quad \text{(A.3)}$$

where $D = z\frac{d}{dz}$. $\alpha_1, \alpha_{-1}, B_{-1}, B_0, B_1$ are arbitrary complex parameters. This equation has special cases as confluent hypergeometric, Bessel, Euler and Mathieu equations. $\alpha_i = 0$ degenerate case gives the Mathieu equation.

A.4 The Biconfluent Heun Equation

This form is obtained when the singularity at $z = 1$ is moved to the point $z = b$; then simultaneously let $a \to \infty$ and $b \to \infty$. The singularity at ∞ becomes irregular of rank 2. The singularity at $z = 0$ remains regular.

In its normal form,

$$\frac{d^2 y}{dz^2} + \left(Az^2 + Bz + C + \frac{D}{z} + \frac{E}{z^2}\right) y = 0. \quad \text{(A.4)}$$

It has the Kummer equation as a particular case.

A.5 The Triconfluent Heun Equation

This form cannot be derived directly by confluence from Heun's equation in its standard form. One has to go back to less specialized equation which has singularities at a_1, a_2, a_3 and ∞ and allow $a_i \to \infty$. The result has an irregular singularity at ∞ of rank 3.

It can be written as,

$$\frac{d^2 y}{dz^2} + \left(Az^4 + Bz^3 + Cz^2 + Dz + E\right) y = 0 \quad \text{(A.5)}$$

in its normal form.

B. NEWMAN-PENROSE FORMALISM IN EUCLIDEAN SPACES

B.1 General Definitions

Newman-Penrose (NP) formalism is a powerful technique for investigating the physical properties of the exact solutions of Einstein's field equations [69, 70]. Goldblatt has developed NP formalism for gravitational instantons based on the $SU(2)\times SU(2)$ spin structure of positive definite metrics [71, 72]. In the gravitational instanton case, the gravitational field decomposes into its self-dual and anti-self-dual parts and this decomposition is natural in the spinor approach which necessitates two independent spin frames for the spinor structure of 4-dimensional Riemann manifolds with Euclidean signature [73]. Aliev and Nutku used differential forms to make it easier to treat [73]. The purpose of this appendix is to give a brief outlook to the subject. The full descriptions can be found in references cited in the text.

In Euclidean NP formalism, a complex dyad (complex vectors of l^μ and m^μ) and a tetrad (e_a^μ) is defined:

$$e_a^\mu = \{l^\mu, \bar{l}^\mu, m^\mu, \bar{m}^\mu\} \tag{B.1}$$

The inverse of this basis is given by,

$$e_\nu^a = \{\bar{l}_\nu, l_\nu, \bar{m}_\nu, m_\nu\} \tag{B.2}$$

and the co-frame 1-forms,

$$e^a = e_\nu^a dx^\nu = \{\bar{l}, l, \bar{m}, m\} \tag{B.3}$$

and the metric

$$ds^2 = l\otimes\bar{l} + \bar{l}\otimes l + m\otimes\bar{m} + \bar{m}\otimes m \tag{B.4}$$

is written respectively. The legs of the complex dyad satisfies these normalization relations:

$$\begin{aligned} l_\mu \bar{l}^\mu &= 1, \quad m_\mu \bar{m}^\mu = 1 \\ l_\mu l^\mu &= 0, \quad m_\mu m^\mu = 0 \\ l_\mu m^\mu &= 0, \quad m_\mu \bar{m}^\mu = 0 \end{aligned} \tag{B.5}$$

Completeness relation can be written as:

$$\delta_\nu^\mu = l^\mu \bar{l}_\nu + \bar{l}^\mu l_\nu + m^\mu \bar{m}_\nu + \bar{m}^\mu m_\nu \tag{B.6}$$

The directional derivatives along the legs of the dyad,

$$D = l^\mu \frac{\partial}{\partial x^\mu}, \quad \delta = m^\mu \frac{\partial}{\partial x^\mu} \tag{B.7}$$

and the exterior derivative,

$$d = \bar{l}D + l\bar{D} + m\bar{\delta} + \bar{m}\delta \tag{B.8}$$

can be found [73].

B.2 Spin Frames

There is a rotational degree of freedom in the definition of the tetrad legs [73]. This degree of freedom is given by the group $SO(4)$ and it can be decomposed into a product of two independent $SU(2)$ degrees of freedom. Two spin frames can be defined with these bases:

$$\zeta_a^A = \{o^A, \iota^A\} \quad A = 1,2 \quad a = 0,1 \tag{B.9}$$

$$\tilde{\zeta}_{x'}^{X'} = \{\tilde{o}^{X'}, \tilde{\iota}^{X'}\} \quad X' = 1', 2' \quad x' = 0', 1' \tag{B.10}$$

where capital Latin letters indicate spinor components and small Latin letters run over the two spinors, omikron (o) and iota (ι) respectively. The both types of indices can be raised and lowered by using Levi-Civita symbol (i.e. $o_A = o^B \varepsilon_{BA} = -\varepsilon_{AB} o^B$).

Only non-zero normalization condition is

$$o_A \iota^A = 1 \qquad \tilde{o}_{X'} \tilde{\iota}^{X'} = 1. \tag{B.11}$$

Primed and unprimed indices cannot be contracted as they define objects in different spaces.

A correspondence between complex dyad and the spinor basis can be set by using Infeld-van der Waerden connection quantities ($\sigma^\mu_{AX'}$) [73]. A representation can be chosen as,

$$\begin{aligned} l^\mu &= \sigma^\mu_{00'} = \sigma^\mu_{AX'} o^A \tilde{o}^{X'} \\ \bar{l}^\mu &= \sigma^\mu_{11'} = \sigma^\mu_{AX'} \iota^A \tilde{\iota}^{X'} \\ m^\mu &= \sigma^\mu_{01'} = \sigma^\mu_{AX'} o^A \tilde{\iota}^{X'} \\ \bar{m}^\mu &= -\sigma^\mu_{10'} = -\sigma^\mu_{AX'} \iota^A \tilde{o}^{X'} \end{aligned} \tag{B.12}$$

These definitions which relates tensor and spinor fields can also be written in matrix notation:

$$\sigma^\mu_{AX'} = \begin{pmatrix} l^\mu & m^\mu \\ -\bar{m}^\mu & \bar{l}^\mu \end{pmatrix} \qquad \sigma^{AX'}_\mu = \begin{pmatrix} \bar{l}_\mu & \bar{m}_\mu \\ -m_\mu & l_\mu \end{pmatrix} \tag{B.13}$$

Infeld-van der Waerden symbols are proportional to Pauli spin matrices [87, 88]. Therefore, the curved space gamma matrices can be written as,

$$\gamma^\mu = \sqrt{2} \begin{pmatrix} 0 & 0 & l^\mu & m^\mu \\ 0 & 0 & -\bar{m}^\mu & \bar{l}^\mu \\ \bar{l}^\mu & -m^\mu & 0 & 0 \\ \bar{m}^\mu & l^\mu & 0 & 0 \end{pmatrix} \tag{B.14}$$

B.3 Ricci Rotation Coefficients

Ricci rotation coefficients are the complex dyad components of the Levi-Civita connection [73]:

$$\gamma_{ijk} = e_{j\mu;\nu} e_i^{\mu} e_k^{\nu} = -\gamma_{jik} \tag{B.15}$$

where semicolon denotes covariant diferentiation. The Ricci rotation coefficients can be labelled as,

$$\begin{array}{ll} \kappa = \gamma_{311} = l_{\mu;\nu}\, m^{\mu} l^{\nu}, & \pi = \gamma_{231} = m_{\mu;\nu}\, \bar{l}^{\mu} l^{\nu}, \\ \tau = \gamma_{312} = l_{\mu;\nu}\, m^{\mu} \bar{l}^{\nu}, & \nu = \gamma_{232} = m_{\mu;\nu}\, \bar{l}^{\mu} \bar{l}^{\nu}, \\ \sigma = \gamma_{313} = l_{\mu;\nu}\, m^{\mu} m^{\nu}, & \mu = \gamma_{233} = m_{\mu;\nu}\, \bar{l}^{\mu} m^{\nu}, \\ \rho = -\gamma_{314} = l_{\mu;\nu}\, m^{\mu} \bar{m}^{\nu}, & \lambda = -\gamma_{234} = m_{\mu;\nu}\, \bar{l}^{\mu} \bar{m}^{\nu}, \end{array} \tag{B.16}$$

$$\begin{array}{llll} \varepsilon & = \frac{1}{2}(\gamma_{211} - \gamma_{341}) & = & \frac{1}{2}(l_{\mu;\nu}\, \bar{l}^{\mu} l^{\nu} - \bar{m}_{\mu;\nu}\, m^{\mu} l^{\nu}), \\ \gamma & = \frac{1}{2}(\gamma_{212} + \gamma_{342}) & = & \frac{1}{2}(l_{\mu;\nu}\, \bar{l}^{\mu} \bar{l}^{\nu} + \bar{m}_{\mu;\nu}\, m^{\mu} \bar{l}^{\nu}), \\ \alpha & = -\frac{1}{2}(\gamma_{214} - \gamma_{344}) & = & -\frac{1}{2}(l_{\mu;\nu}\, \bar{l}^{\mu} \bar{m}^{\nu} - \bar{m}_{\mu;\nu}\, m^{\mu} \bar{m}^{\nu}), \\ \beta & = \frac{1}{2}(\gamma_{213} + \gamma_{343}) & = & \frac{1}{2}(l_{\mu;\nu}\, \bar{l}^{\mu} m^{\nu} + \bar{m}_{\mu;\nu}\, m^{\mu} m^{\nu}), \end{array}$$

using a notation similar to the NP spin coefficients for the Lorentzian signature.

Exterior derivatives of the basis 1-forms l and m give the Ricci rotation coefficients as well,

$$\begin{array}{lll} d\,l & = & (\bar{\gamma} - \varepsilon)\, l \wedge \bar{l} + (\alpha + \bar{\beta} - \bar{\pi})\, l \wedge m + (\tau - \beta - \bar{\alpha})\, l \wedge \bar{m} \\ & & -\bar{\nu}\, \bar{l} \wedge m + \kappa\, \bar{l} \wedge \bar{m} - (\bar{\lambda} + \rho)\, m \wedge \bar{m} \\ d\,m & = & (\pi + \tau)\, \bar{l} \wedge l - (\bar{\varepsilon} + \gamma - \bar{\lambda})\, l \wedge m - \mu\, l \wedge \bar{m} \\ & & +(\varepsilon - \rho + \bar{\gamma})\, \bar{l} \wedge m + \sigma\, \bar{l} \wedge \bar{m} - (\bar{\alpha} - \beta)\, m \wedge \bar{m} \end{array} \tag{B.17}$$

B.4 Connection 1-Forms

Connection is a 1-form which satisfy Cartan's equations of structure:

$$de^{\alpha} + \omega^{\alpha}_{\ \beta} \wedge e^{\beta} = 0, \tag{B.18}$$

and it is anti-symmetric:

$$\omega_{\alpha\beta} = -\ \omega_{\beta\alpha} \tag{B.19}$$

The spinor equivalent of the connection 1-form is given as,

$$\begin{array}{lll} \omega_{\alpha\beta} & \leftrightarrow & \Gamma_{ax'\ by'} \\ & & = \Gamma_{ab}\, \varepsilon_{x'y'} + \tilde{\Gamma}_{x'y'}\, \varepsilon_{ab} \\ & \Leftrightarrow & \Gamma_{ab} \end{array} \tag{B.20}$$

These split into two, according to two spin frames (Γ_{ab} and $\tilde{\Gamma}_{x'y'}$). Using,

$$\Gamma_a^{\ b} = \Gamma_{a\ cx'}^{\ b}\, \sigma_{\mu}^{cx'}\, dx^{\mu}, \qquad \tilde{\Gamma}_{x'}^{\ y'} = \tilde{\Gamma}_{x'\ az'}^{\ y'}\, \sigma_{\mu}^{az'} dx^{\mu} \tag{B.21}$$

they can be found in terms of rotation coefficients:

$$\begin{array}{l} \Gamma_0^{\ 0} = \frac{1}{2}\left(l_{\mu;\nu}\ \bar{l}^{\mu} + m_{\mu;\nu}\ \bar{m}^{\mu}\right) dx^{\nu} = \varepsilon\, \bar{l} - \bar{\varepsilon}\, l - \alpha\, m + \bar{\alpha}\, \bar{m} \\ \Gamma_0^{\ 1} = -l_{\mu;\nu}\ m^{\mu} dx^{\nu} = -\tau\, l - \kappa\, \bar{l} + \rho\, m - \sigma\, \bar{m} \\ \Gamma_1^{\ 0} = -\bar{\Gamma}_0^{\ 1}, \qquad \Gamma_1^{\ 1} = -\Gamma_0^{\ 0} \end{array} \tag{B.22}$$

and

$$
\begin{aligned}
\tilde{\Gamma}_{0'}^{\;0'} &= \tfrac{1}{2}\left(l_{\mu;\nu}\bar{l}^{\mu} - m_{\mu;\nu}\bar{m}^{\mu}\right)dx^{\nu} = \gamma l - \bar{\gamma}\bar{l} + \beta\,\bar{m} - \bar{\beta}\,m \\
\tilde{\Gamma}_{0'}^{\;1'} &= l_{\mu;\nu}\;\bar{m}^{\mu}dx^{\nu} = -\bar{\pi}\,l - \bar{\nu}\,\bar{l} - \bar{\mu}\,m + \bar{\lambda}\,\bar{m} \\
\tilde{\Gamma}_{1'}^{\;0'} &= -\bar{\tilde{\Gamma}}_{0'}^{\;1'}, \qquad\qquad \tilde{\Gamma}_{1'}^{\;1'} = -\tilde{\Gamma}_{0'}^{\;0'}
\end{aligned} \tag{B.23}
$$

B.4.1 Self-Dual Gauge

A most convenient way in dealing with 4-dimensional Riemannian manifolds that admit (anti) self-dual curvature is to find a frame where the connection also shares this property. Connection 1-form equations (B.22) and (B.23) gives the (anti) self-duality condition. (Anti) self-duality is given by the definition,

$$
{}^{\pm}\omega_{\alpha\beta} = \frac{1}{2}\left(\omega_{\alpha\beta} \pm \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\gamma\delta}\,\omega_{\gamma\delta}\right), \tag{B.24}
$$

$(+)$ signed case, being zero gives anti-self-dual gauge, while (-) signed case being zero gives self-dual gauge.

$$
{}^{-}\omega_{\alpha\beta} = 0, \qquad\qquad {}^{+}\omega_{\alpha\beta} = 0, \tag{B.25}
$$

If the spinor equivalents are considered, the gauge of the connection 1-form is given by,

$$
\Gamma_{ab} \equiv 0, \qquad\qquad \tilde{\Gamma}_{x'y'} \equiv 0 \tag{B.26}
$$

as self-dual or anti-self-dual respectively. Therefore, self-duality condition

$$
\varepsilon = \alpha = \tau = \kappa = \rho = \sigma = 0, \tag{B.27}
$$

and anti self-duality condition is given by

$$
\gamma = \beta = \pi = \mu = \nu = \lambda = 0. \tag{B.28}
$$

These are the necessary and sufficient conditions.

B.5 Basis 2-Forms

Spinor equivalents of the basis 2-forms are the wedge products of the Infeld-van der Waerden matrices of basis 1-forms.

$$
e^{a}_{\mu}\,e^{b}_{\nu}\,dx^{\mu}\wedge dx^{\nu} \;\;\leftrightarrow\;\; \sigma^{ax'}\wedge\sigma^{by'}
$$

Consequently, there are two basis 2-form sets corresponding to the two spin frames:

$$
L_{a}{}^{b} = \frac{1}{2}\sigma_{\mu\,ax'}\,\sigma_{\nu}{}^{bx'}\,dx^{\mu}\wedge dx^{\nu} = \frac{1}{2}\sigma_{ax'}\wedge\sigma^{bx'} \tag{B.29}
$$

$$
\tilde{L}_{x'}^{\;y'} = \frac{1}{2}\sigma_{ax'}\wedge\sigma^{ay'} \tag{B.30}
$$

It can be seen that,

$$
\begin{aligned}
L_{0}^{\;0} &= \tfrac{1}{2}\left(l\wedge\bar{l} + m\wedge\bar{m}\right), & \tilde{L}_{0'}^{\;0'} &= \tfrac{1}{2}\left(l\wedge\bar{l} - m\wedge\bar{m}\right), \\
L_{0}^{\;1} &= -l\wedge m, & \tilde{L}_{0'}^{\;1'} &= l\wedge\bar{m}, \\
L_{1}^{\;0} &= \bar{l}\wedge\bar{m}, & \tilde{L}_{1'}^{\;0'} &= -\bar{l}\wedge m
\end{aligned} \tag{B.31}
$$

where $L_0{}^1 = -\bar{L}_1^0$ ve $\tilde{L}_{0'}^{1'} = -\bar{\tilde{L}}_{1'}^{0'}$.

B.6 Curvature

Riemannian tensor can be decomposed as

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= C_{\mu\nu\rho\sigma} + \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma}\right) \\ &\quad -\frac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)R \end{aligned} \tag{B.32}$$

where $C_{\mu\nu\rho\sigma}$ is the conformal Weyl tensor, $R_{\mu\nu} = g^{\rho\sigma}R_{\mu\rho\nu\sigma}$ is the Ricci tensor and R is the curvature scalar. Two sets of Weyl scalars can be obtained for the complex dyad:

$$\begin{array}{ll} \Psi_0 = C_{1313} = C_{\mu\nu\sigma\tau}l^\mu m^\nu l^\sigma m^\tau & \tilde{\Psi}_0 = C_{1414} = C_{\mu\nu\sigma\tau}l^\mu \bar{m}^\nu l^\sigma \bar{m}^\tau \\ \Psi_1 = C_{1213} = C_{\mu\nu\sigma\tau}l^\mu \bar{l}^\nu l^\sigma m^\tau & \tilde{\Psi}_1 = C_{1241} = C_{\mu\nu\sigma\tau}l^\mu \bar{l}^\nu \bar{m}^\tau l^\sigma \\ \Psi_2 = C_{1324} = C_{\mu\nu\sigma\tau}l^\mu m^\nu \bar{l}^\sigma \bar{m}^\tau & \tilde{\Psi}_2 = C_{1423} = C_{\mu\nu\sigma\tau}l^\mu \bar{m}^\nu \bar{l}^\sigma m^\tau \end{array} \tag{B.33}$$

Trace-free Ricci scalars:

$$\begin{array}{llll} \Phi_{00} & = -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu = \bar{\Phi}_{22}, & \Phi_{01} & = -\frac{1}{2}R_{\mu\nu}l^\mu m^\nu = -\bar{\Phi}_{21}, \\ \Phi_{02} & = -\frac{1}{2}R_{\mu\nu}m^\mu m^\nu = \bar{\Phi}_{20}, & \Phi_{10} & = \frac{1}{2}R_{\mu\nu}l^\mu \bar{m}^\nu = -\bar{\Phi}_{12}, \\ \Phi_{11} & = -\frac{1}{2}R_{\mu\nu}l^\mu \bar{l}^\nu + 3\Lambda & & \end{array} \tag{B.34}$$

Curvature scalar:

$$\Lambda = \frac{1}{24}R \tag{B.35}$$

B.7 Curvature 2-Forms

The spinor equivalents of the curvature 2-forms can be given as

$$\begin{aligned} {\Theta_0}^0 &= -(\Psi_2 - \Lambda + \Phi_{11})\, l\wedge\bar{l} - (\Psi_2 - \Lambda - \Phi_{11})\, m\wedge\bar{m} \\ &\quad -\bar{\Psi}_1\, l\wedge m + \Psi_1\, \bar{l}\wedge\bar{m} - \Phi_{10}\, \bar{l}\wedge m - \Phi_{12}\, l\wedge\bar{m}, \\ {\Theta_0}^1 &= (\Psi_1 + \Phi_{01})\, l\wedge\bar{l} + (\Psi_1 - \Phi_{01})\, m\wedge\bar{m} + \Phi_{02}\, l\wedge\bar{m} \\ &\quad -(\Psi_2 + 2\Lambda)\, l\wedge m - \Psi_0\, \bar{l}\wedge\bar{m} + \Phi_{00}\, \bar{l}\wedge m, \\ {\Theta_1}^0 &= (\bar{\Psi}_1 - \Phi_{21})\, l\wedge\bar{l} + (\bar{\Psi}_1 + \Phi_{21})\, m\wedge\bar{m} - \Phi_{22}\, l\wedge\bar{m} \\ &\quad +\bar{\Psi}_0\, l\wedge m + (\Psi_2 + 2\Lambda)\, \bar{l}\wedge\bar{m} - \Phi_{20}\, \bar{l}\wedge m = -\bar{\Theta}_0^1 \end{aligned} \tag{B.36}$$

and

$$\begin{aligned} \tilde{\Theta}_{0'}^{0'} &= -(\tilde{\Psi}_2 - \Lambda + \Phi_{11})\, l\wedge\bar{l} + (\tilde{\Psi}_2 - \Lambda - \Phi_{11})\, m\wedge\bar{m} \\ &\quad -\tilde{\Psi}_1\, \bar{l}\wedge m + \bar{\tilde{\Psi}}_1\, l\wedge\bar{m} + \Phi_{01}\, \bar{l}\wedge\bar{m} + \Phi_{21}\, l\wedge m, \\ \tilde{\Theta}_{0'}^{1'} &= (\tilde{\Psi}_1 + \Phi_{10})\, l\wedge\bar{l} - (\tilde{\Psi}_1 - \Phi_{10})\, m\wedge\bar{m} - \Phi_{20}\, l\wedge m \\ &\quad +(\tilde{\Psi}_2 + 2\Lambda)\, l\wedge\bar{m} + \tilde{\Psi}_0\, \bar{l}\wedge m - \Phi_{00}\, \bar{l}\wedge\bar{m}, \\ \tilde{\Theta}_{1'}^{0'} &= (\bar{\tilde{\Psi}}_1 - \Phi_{12})\, l\wedge\bar{l} - (\bar{\tilde{\Psi}}_1 + \Phi_{12})\, m\wedge\bar{m} + \Phi_{22}\, l\wedge m \\ &\quad -\bar{\tilde{\Psi}}_0\, l\wedge\bar{m} - (\tilde{\Psi}_2 + 2\Lambda)\, \bar{l}\wedge m + \Phi_{02}\, \bar{l}\wedge\bar{m} = -\bar{\tilde{\Theta}}_{0'}^{1'} \end{aligned} \tag{B.37}$$

for the two spin frames.

B.8 Maxwell Field

Source-free Maxwell equations are written in two sets. φ_{AB} and $\tilde{\varphi}_{X'Y'}$ being symmetric spinors, spinor equivalence of the Maxwell field can be written in the form,

$$\begin{aligned} \varphi_1 &:= \varphi_{01} = \tfrac{1}{2} F_{\mu\nu} \left(l^\mu \bar{l}^\nu + m^\mu \bar{m}^\nu \right), & \varphi_0 &:= \varphi_{00} = F_{\mu\nu} l^\mu m^\nu \\ \tilde{\varphi}_1 &:= \tilde{\varphi}_{0'1'} = \tfrac{1}{2} F_{\mu\nu} \left(l^\mu \bar{l}^\nu - m^\mu \bar{m}^\nu \right), & \tilde{\varphi}_0 &:= \tilde{\varphi}_{0'0'} = F_{\mu\nu} \bar{m}^\mu l^\nu \\ \bar{\varphi}_0 &:= \varphi_{11}, & \bar{\tilde{\varphi}}_0 &:= \tilde{\varphi}_{1'1'} \end{aligned} \tag{B.38}$$

B.8.1 Source-Free Maxwell Equations

Maxwell equations in vacuum can be written as,

$$d\mathscr{F} = 0, \qquad d\widetilde{\mathscr{F}} = 0 \tag{B.39}$$

Here, $\mathscr{F} = \varphi_{AB} L^{AB}$, $d\widetilde{\mathscr{F}} = \tilde{\varphi}_{X'Y'} \tilde{L}^{X'Y'}$. When the definitions are used,

$$\begin{aligned} (\delta - 2\tau)\,\varphi_1 - (\bar{D} + 2\bar{\varepsilon} - \bar{\rho})\,\varphi_0 + \sigma\,\bar{\varphi}_0 &= 0 \\ (D - 2\rho)\,\varphi_1 + (\bar{\delta} + 2\alpha - \bar{\tau})\,\varphi_0 + \kappa\,\bar{\varphi}_0 &= 0 \end{aligned} \tag{B.40}$$

and

$$\begin{aligned} (\bar{\delta} + 2\bar{\pi})\,\tilde{\varphi}_1 + (\bar{D} - 2\gamma + \lambda)\,\tilde{\varphi}_0 + \bar{\mu}\,\bar{\tilde{\varphi}}_0 &= 0 \\ (D + 2\bar{\lambda})\,\tilde{\varphi}_1 - (\delta - 2\beta + \pi)\,\tilde{\varphi}_0 + \bar{\nu}\,\bar{\tilde{\varphi}}_0 &= 0 \end{aligned} \tag{B.41}$$

are obtained.

B.9 Topological Numbers

Chern formula for the Euler number is

$$\begin{aligned} \chi &= \frac{1}{32\pi^2} \int_{\mathscr{M}} \varepsilon^{abcd} \theta_{ab} \wedge \theta_{cd} \\ &\quad - \frac{1}{16\pi^2} \int_{\partial\mathscr{M}} \varepsilon^{abcd} \left(\Omega_{ab} \wedge \theta_{cd} - \frac{2}{3}\, \Omega_{ab} \wedge \Omega_c{}^e \wedge \Omega_{ed} \right) \end{aligned} \tag{B.42}$$

and the Hirzebruch signature can be given as

$$\tau = -\frac{1}{24\pi^2} \int_{\mathscr{M}} \theta_a{}^b \wedge \theta_b{}^a + \frac{1}{24\pi^2} \int_{\partial\mathscr{M}} \Omega_a{}^b \wedge \theta_b{}^a - \eta_s(\partial\mathscr{M}) \tag{B.43}$$

Here, θ_{ab} is the curvature 2-form and Ω_{ab} is the second fundamental form on the $\partial\mathscr{M}$ boundary which is defined as

$$\Omega_{ab} = \omega_{ab} - (\omega_0)_{ab} \tag{B.44}$$

where ω_{ab} is the curvature 1-form for the original metric and $(\omega_0)_{ab}$ is the curvature 1-form for the metric on the boundary. $\eta_s(\partial\mathscr{M})$ is the eta-invariant.

If the spinor equivalents of these objects are used,

$$\begin{aligned} \chi &= \frac{1}{4\pi^2} \int_{\mathscr{M}} \big[|\Psi_0|^2 + 4|\Psi_1|^2 + 3\Psi_2^2 + |\tilde{\Psi}_0|^2 \\ &\quad + 4|\tilde{\Psi}_1|^2 + 3\tilde{\Psi}_2^2 - 2(|\Phi_{00}|^2 + |\Phi_{02}|^2) \\ &\quad - 4(|\Phi_{01}|^2 + |\Phi_{11}|^2 + |\Phi_{12}|^2 - 3\Lambda^2) \big]\, l \wedge \bar{l} \wedge m \wedge \bar{m} \end{aligned} \tag{B.45}$$

$$\tau = -\frac{1}{6\pi^2}\int_{\mathcal{M}} [\,|\Psi_0|^2 + 4|\Psi_1|^2 + 3\Psi_2^2 - |\tilde{\Psi}_0|^2$$
$$-4|\tilde{\Psi}_1|^2 - 3\tilde{\Psi}_2^2\,]\, l\wedge\bar{l}\wedge m\wedge\bar{m} - \eta_s(\partial\mathcal{M}) \tag{B.46}$$

can be found.

B.10 Petrov Types

The traceless matrix constructed using the Weyl scalars

$$\Psi = \begin{pmatrix} \Psi_2 - \frac{1}{2}(\Psi_0+\overline{\Psi}_0) & -\frac{i}{2}(\Psi_0-\overline{\Psi}_0) & \Psi_1+\overline{\Psi}_1 \\ -\frac{i}{2}(\Psi_0-\overline{\Psi}_0) & \Psi_2+\frac{1}{2}(\Psi_0+\overline{\Psi}_0) & i(\Psi_1-\overline{\Psi}_1) \\ \Psi_1+\overline{\Psi}_1 & i(\Psi_1-\overline{\Psi}_1) & -2\Psi_2 \end{pmatrix} \tag{B.47}$$

is used [73]. Since this matrix is real and symmetric it can always be diagonalised in the orthonormal basis with eigenspinors. Corresponding to three distinct real eigenvalues

$$\lambda_1 = \Psi_2 - \Psi_0, \qquad \lambda_2 = \Psi_2 + \Psi_0, \qquad \lambda_3 = -2\Psi_2 \tag{B.48}$$

obeying the condition

$$\lambda_1+\lambda_2+\lambda_3 = 0$$

we have the case of algebraically general, or Petrov type I, gravitational instantons. When two roots coincide $\lambda_1 = \lambda_2$, we obtain

$$\Psi_2 = \lambda_1 = \lambda_2, \qquad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{B.49}$$

which corresponds to the algebraically special, or Petrov type D, gravitational instantons. For the anti-self-dual case a similar analysis holds for the independent primed 3×3 matrix.

C. EXAMPLES FOR THE NPInstanton PACKAGE

In this section, we will apply our program to Nutku helicoid metric and give the command definitions through this example. After that, an exercise on the four and five dimensional scalar operators, the Petrov type and the self-duality for Eguchi-Hanson metric will be done.

C.1 Example 1: Calculations for the Nutku Helicoid Metric

The Nutku helicoid metric is given as

$$\begin{aligned} ds^2 &= \frac{a^2}{2}\sinh 2x(dx^2+d\theta^2) \\ &+\frac{2}{\sinh 2x}[(\sinh^2 x+\sin^2\theta)dy^2 \\ &-\sin 2\theta dydz+(\sinh^2 x+\cos^2\theta)dz^2]. \end{aligned} \tag{C.1}$$

where $0<r<\infty$, $0\leq\theta\leq 2\pi$, y and z are along the Killing directions and are taken to be periodic coordinates on a 2-torus [12]. This metric reduces to the flat metric if we take $a=0$.

The NP legs can be chosen as,

$$l^\mu = \frac{a\sqrt{\sinh 2x}}{2}(1,i,0,0), \tag{C.2}$$

$$m^\mu = \frac{1}{\sqrt{\sinh 2x}}(0,0,\cosh(x-i\theta),i\sinh(x-i\theta)). \tag{C.3}$$

The application of the NPInstanton package on this metric will be given in the following pages along with the command definitions.

- scalaroperator() command

This command calculates the massless scalar equation, finding the scalar operator. The scalar operator is given as

$$H\varphi \equiv \frac{1}{\sqrt{|g|}}\partial_\nu \sqrt{|g|}g^{\mu\nu}\partial_\mu \varphi. \tag{C.4}$$

Here, g is the determinant of the metric. The procedure takes the scalar function (name: scalarfunction) from the input file. The output to be used thereafter:

> scalarop;

For the massive case, one can equate this object to $M^2\varphi^2$, M being the mass of the scalar particle. As an additional property, this procedure is not dependent on the metric signature and it does not use the NP objects so it can be used for any spacetime in four dimensions by extracting it from the program. The Maple output of this command is shown in Figure C.1.

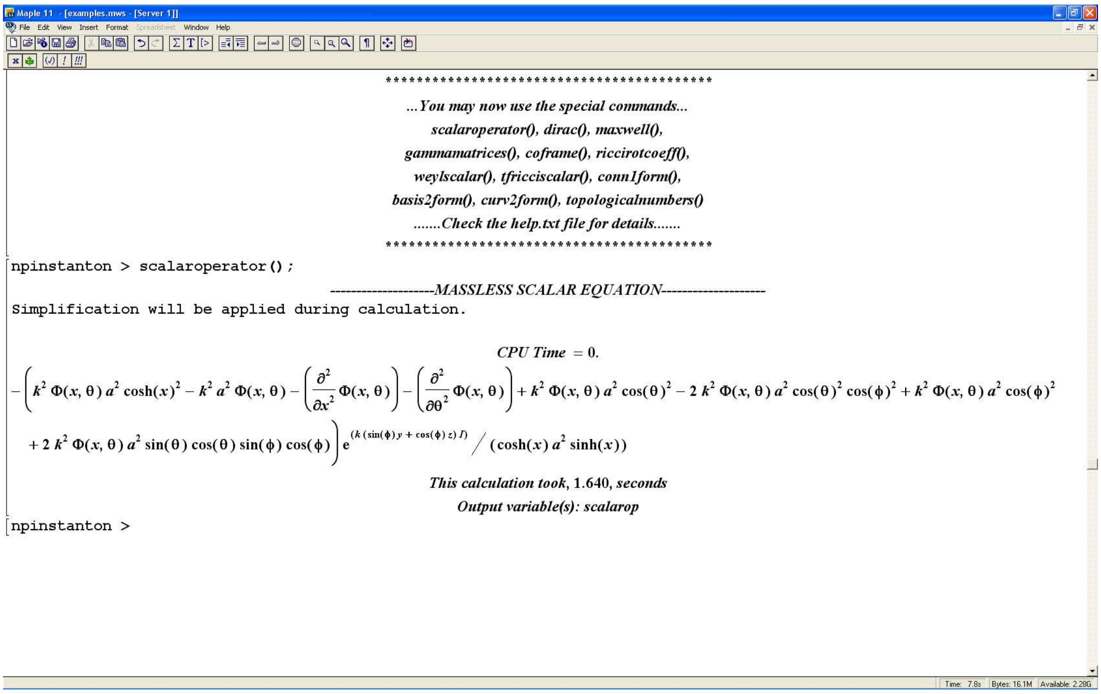

Figure C.1: Maple Session for the scalaroperator() Command

- dirac() command:

This command calculates the massless Dirac equations,

$$\gamma^{\mu}\nabla_{\mu}\psi = 0 \tag{C.5}$$

where

$$\nabla_{\mu} = \partial_{\mu} - \Gamma_{\mu} \tag{C.6}$$

and Γ_{μ} are the spin connections. The procedure takes the spinor vector components from the input file under these names: spinorcomponent1 (ψ_1), spinorcomponent2 (ψ_2), spinorcomponent3 (ψ_3) and spinorcomponent4 (ψ_4). The output to be used thereafter is the components of the "dirac" vector as (i=1, 2, 3, 4):

> dirac[i];

For the massive case, one can equate these objects to $\frac{M}{i}\psi$ vector, M being the spinor mass and $i \equiv \sqrt{-1}$. The output of this command is given partially in Figure C.2.

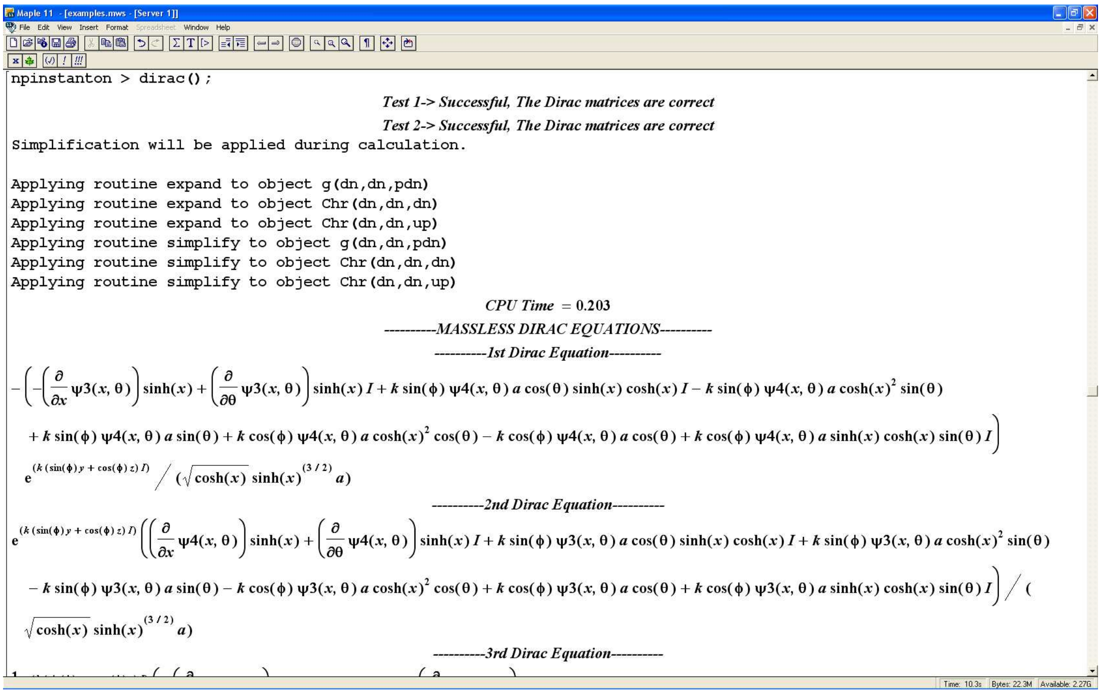

Figure C.2: Maple Session for the dirac() Command

- maxwell() command:

This command calculates the source-free Maxwell equations using the equations (B.38-B.41). F_{ij}'s in the output are the usual Maxwell field matrix components. The output is set to be "maxwell" vector whose components are the source-free Maxwell equations as (i=1, 2, 3, 4):

> maxwell[i];

The output of this command is given partially in Figure C.3.

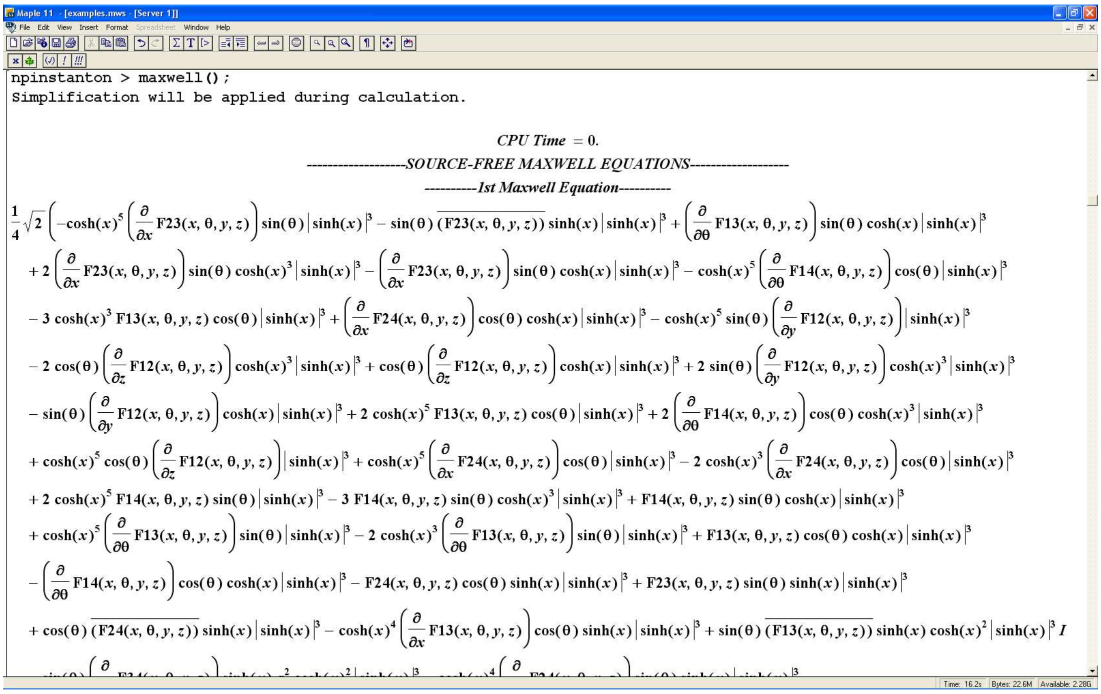

Figure C.3: Maple Session for the maxwell() Command

- gammamatrices() command:

This command finds the covariant and contravariant Dirac γ matrices and tests them with

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \tag{C.7}$$

anticommutation relation. The output is "gamma_contr" (contravariant γ matrices: γ^{μ}) and "gamma_covar" (covariant γ matrices: γ_{μ}) vectors as (i=1, 2, 3, 4):

> gamma_contr[i];
> gamma_covar[i];

The calculation of the Dirac γ matrices could be put into the dirac() procedure but further calculations involving higher dimensions may need these matrices. To find the γ matrix for the extra dimension, one can use these results and look for the γ matrix of the extra dimension using the anticommutation relations as in reference [15]. Therefore the procedure is a separate one. Figure C.4 shows the output for the Nutku helicoid case.

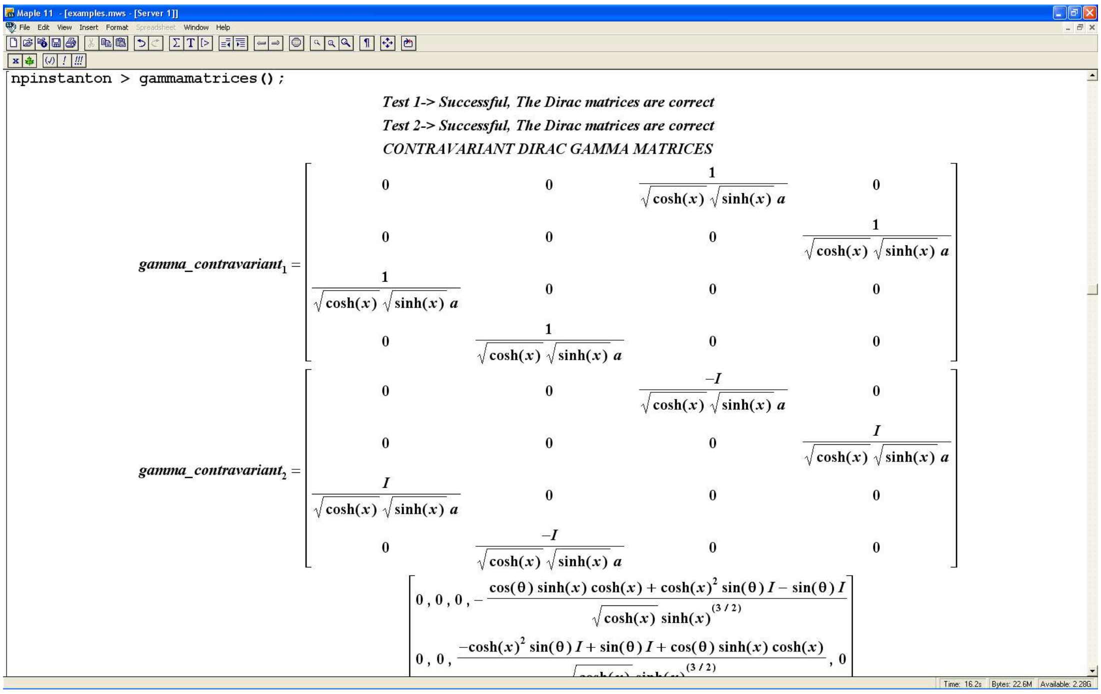

Figure C.4: Maple Session for the gammamatrices() Command

- coframe() command:

This command calculates the coframe $l \equiv$L $(= l_\mu dx^\mu)$ and $m \equiv$M $(= m_\mu dx^\mu)$. The output variables are the following, "_bar" denoting the complex conjugate:

> L;
> M;
> L_bar;
> M_bar;

The co-frame for the metric is calculated by coframe() command as shown in Figure C.5.

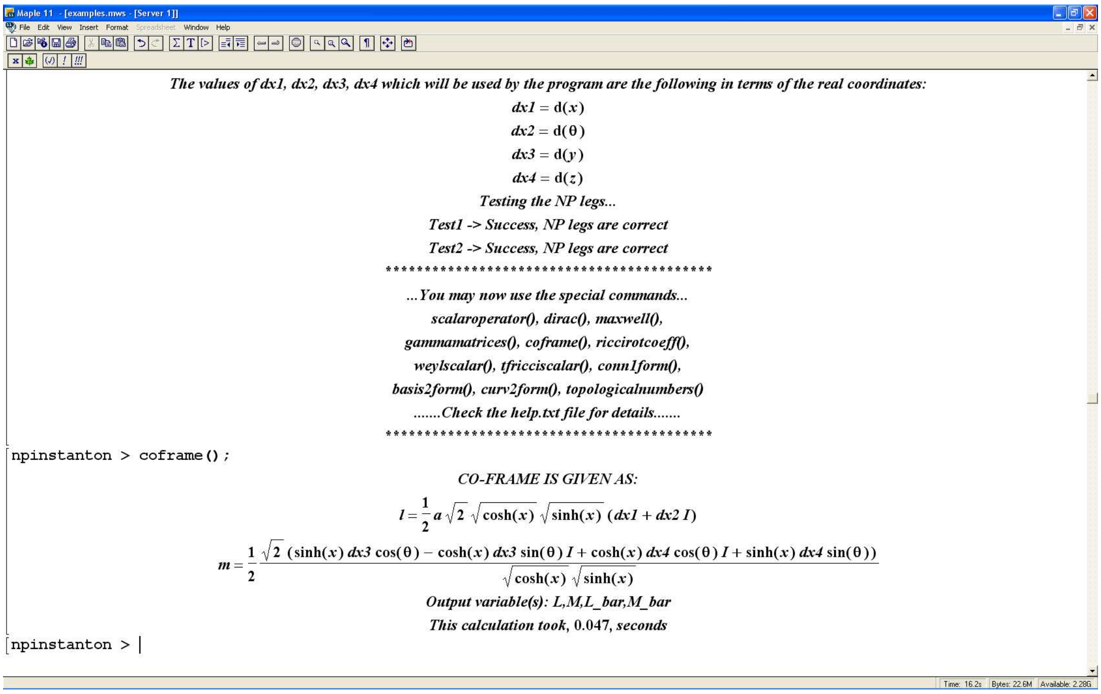

Figure C.5: Maple Session for the coframe() Command

- riccirotcoeff() command:

This procedure calculates the Ricci rotation coefficients using equations in (B.3). The output can be used later by calling:

> npkappa; > nptau; > npsigma;
> nprho; > nppi; > npnu;
> npmu; > nplambda; > npgamma;
> npepsilon; > npalpha; > npbeta;

in the meaning of the Ricci rotation coefficients κ, τ, σ, ρ, π, ν, μ, λ, γ, ε, α, β respectively. Figure C.6 shows that we have two non-zero Ricci rotation coefficient for this metric.

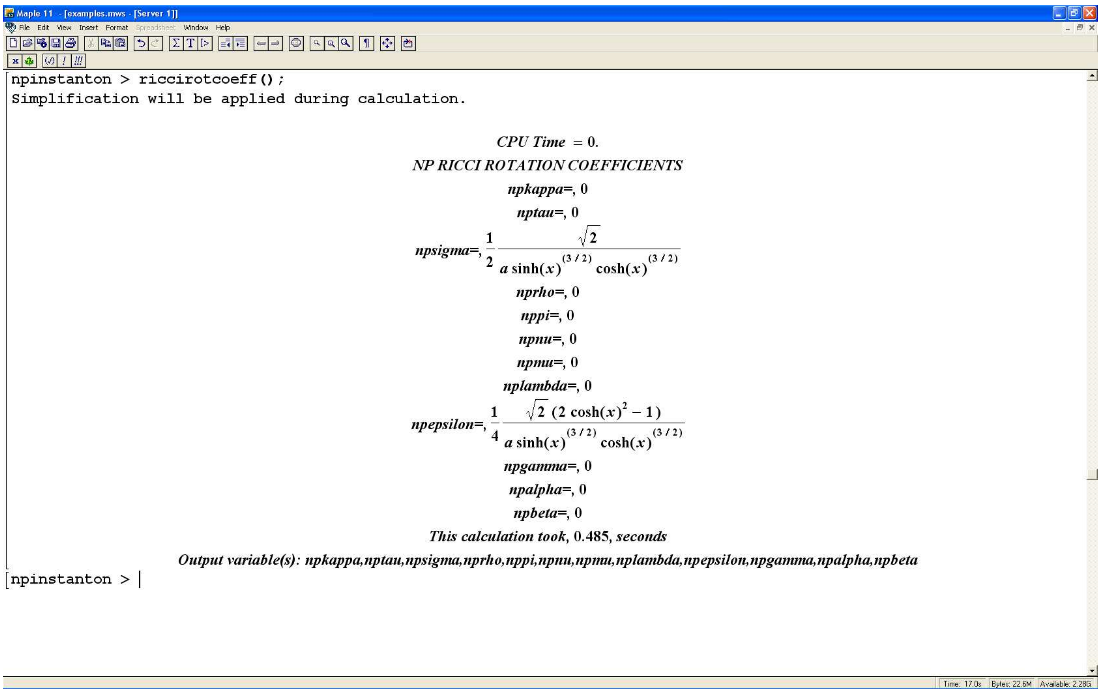

Figure C.6: Maple Session for the riccirotcoeff() Command

- weylscalar() command:

This command calculates the Weyl scalars using equation B.33. The output variables are:

> weylscalar0; > weylscalar1; > weylscalar2, > weylscalar3;
> weylscalar4; > weylscalartilde0; >weylscalartilde1;
> weylscalartilde2; > weylscalartilde3; > weylscalartilde4;

for Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4, $\tilde{\Psi}_0$, $\tilde{\Psi}_1$, $\tilde{\Psi}_2$, $\tilde{\Psi}_3$, $\tilde{\Psi}_4$ respectively. "tilde" means the variable has a tilde in the meaning of a different spin frame.

Another property of this procedure is that it finds out the Petrov class of the spacetime according to equations (B.48-B.49). Petrov classification of Euclidean spacetimes were first studied by Hacyan [89] and then by Karlhede [90].

Weyl scalars can be calculated by using weylscalar() command as shown in Figure C.7.

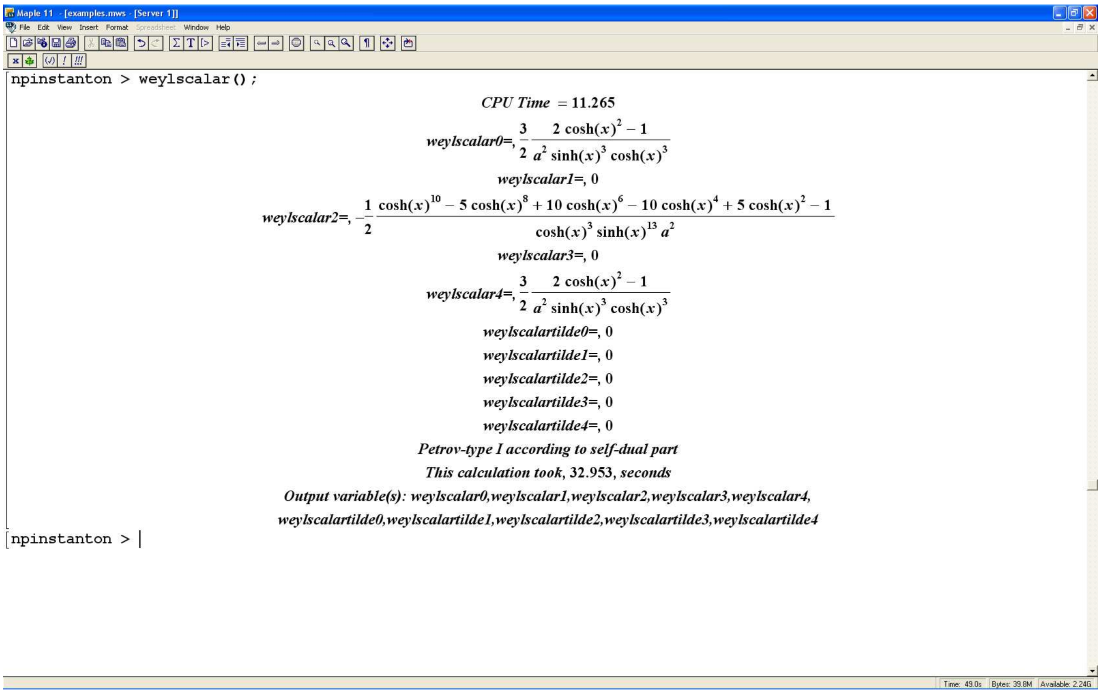

Figure C.7: Maple Session for the weylscalar() Command

- tfricciscalar() command:

This command calculates the trace-free Ricci scalars using equation (B.34) and sets them to variables for a later use:

> tfricciscalar00; > tfricciscalar01; > tfricciscalar02;
> tfricciscalar10; > tfricciscalar11; > tfricciscalar12;
> tfricciscalar20; > tfricciscalar21; > tfricciscalar22;
> scalofcurv;

for $\mathbf{\Phi}_{00}$, $\mathbf{\Phi}_{01}$, $\mathbf{\Phi}_{02}$, $\mathbf{\Phi}_{10}$, $\mathbf{\Phi}_{11}$, $\mathbf{\Phi}_{12}$, $\mathbf{\Phi}_{20}$, $\mathbf{\Phi}_{21}$, $\mathbf{\Phi}_{22}$, scalar of curvature $\mathbf{\Lambda}$ respectively. Trace-free Ricci scalars and the scalar of curvature are found by using tfricciscalar() command as shown in Figure C.8.

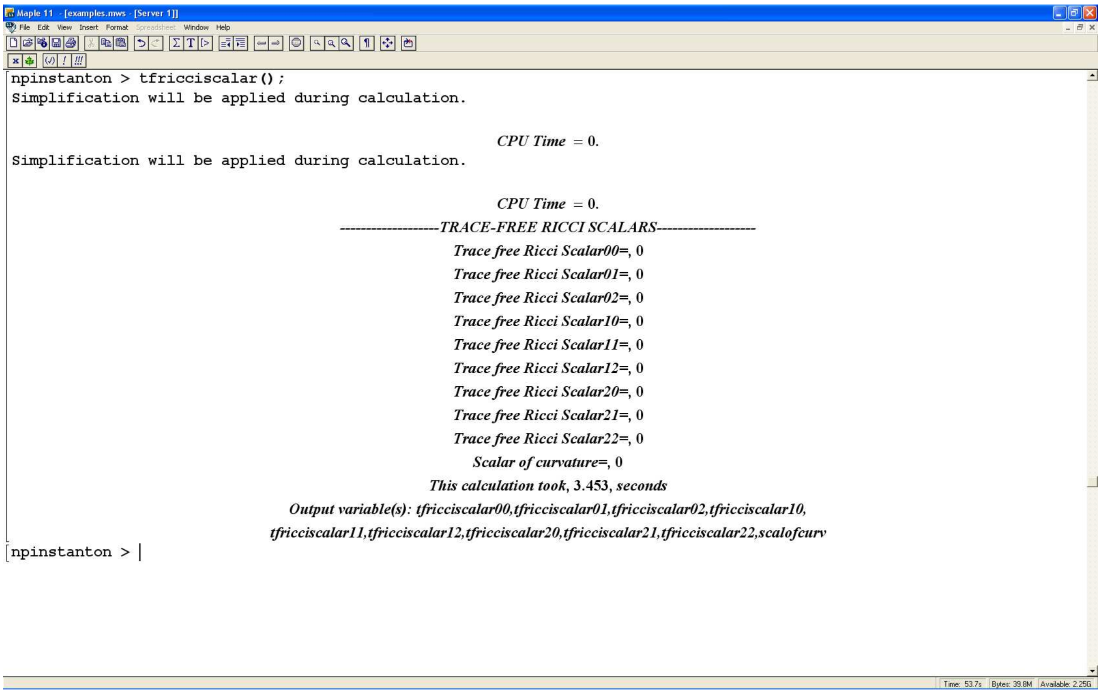

Figure C.8: Maple Session for the tfricciscalar() Command

- conn1form() command:

This procedure calculates the spinor equivalent of the connection 1-forms given by equations (B.22) and (B.23). The output can be reached by calling

> GAMMA00; > GAMMA01; > GAMMA10; > GAMMA11;
> GAMMAtilde0pr0pr; > GAMMAtilde0pr1pr;
> GAMMAtilde1pr0pr; > GAMMAtilde1pr1pr;

for $\Gamma^0_{\ 0}$, $\Gamma^1_{\ 0}$, $\Gamma^0_{\ 1}$, $\Gamma^1_{\ 1}$, $\tilde{\Gamma}^{0'}_{\ 0'}$, $\tilde{\Gamma}^{1'}_{\ 0'}$, $\tilde{\Gamma}^{0'}_{\ 1'}$, $\tilde{\Gamma}^{1'}_{\ 1'}$ respectively where "tilde" means the variable has a tilde and "pr" means "prime" in the meaning of a different spin frame.

Another property of the procedure is that, it finds out the gauge by checking the necessary and sufficient conditions for (anti-)self-duality namely, $\Gamma_{ab} \equiv 0$ implies self duality and $\tilde{\Gamma}_{x'y'} \equiv 0$ implies anti-self-duality. The Maple output is given in Figure C.9.

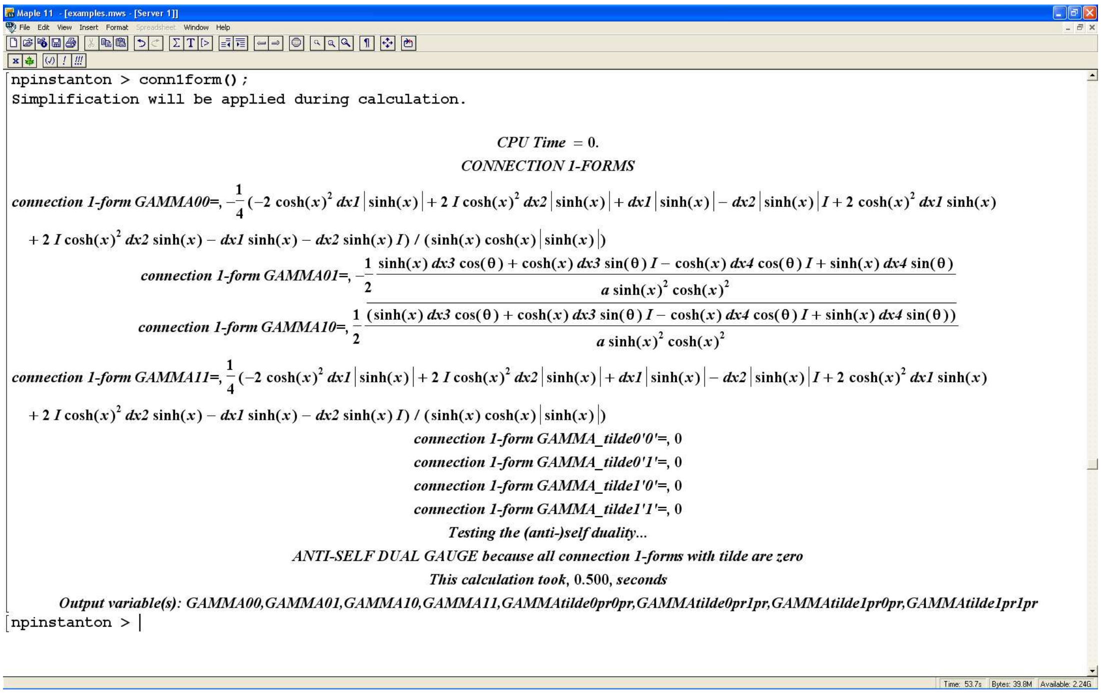

Figure C.9: Maple Session for the conn1form() Command

- basis2form() command:

This command finds the basis 2-forms using the definitions in equation (B.31). The output can be reached by

> L00; > L01; > L10;
> Ltilde0pr0pr; > Ltilde0pr1pr; > Ltilde1pr0pr;

for L^0_0, L^1_0, L^0_1, $\tilde{L}^{0'}_{0'}$, $\tilde{L}^{1'}_{0'}$, $\tilde{L}^{0'}_{1'}$ respectively. Here, "tilde" means the variable has a tilde and "pr" means "prime" in the meaning of a different spin frame. Figure C.10 shows the Maple session for this command.

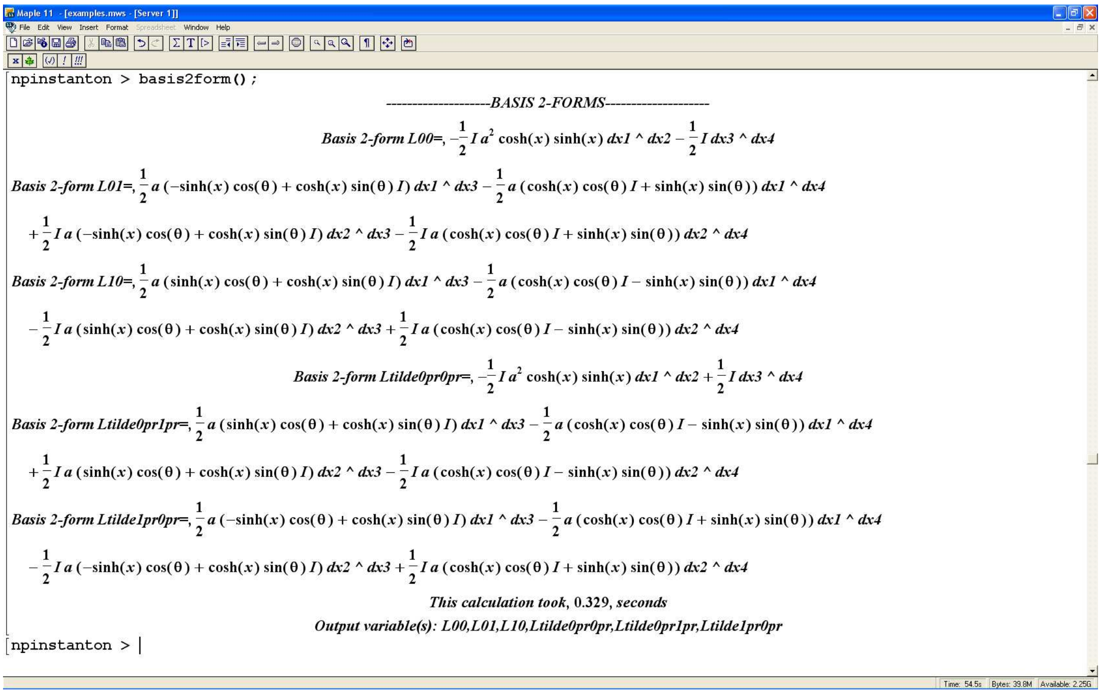

Figure C.10: Maple Session for the basis2form() Command

- curv2form() command:

This command calculates the curvature 2-forms using equations (B.36) and (B.37) and sets them to these variables:

> Theta00; > Theta01; > Theta10;
> Thetatilde0pr0pr; > Thetatilde0pr1pr; > Thetatilde1pr0pr;

for Θ_0^0, Θ_0^1, Θ_1^0, $\tilde{\Theta}_{0'}^{0'}$, $\tilde{\Theta}_{0'}^{1'}$, $\tilde{\Theta}_{1'}^{0'}$ respectively where "tilde" means the variable has a tilde and "pr" means "prime" in the meaning of a different spin frame.

Curvature 2-forms are calculated by curv2form() command as shown in Figure C.11.

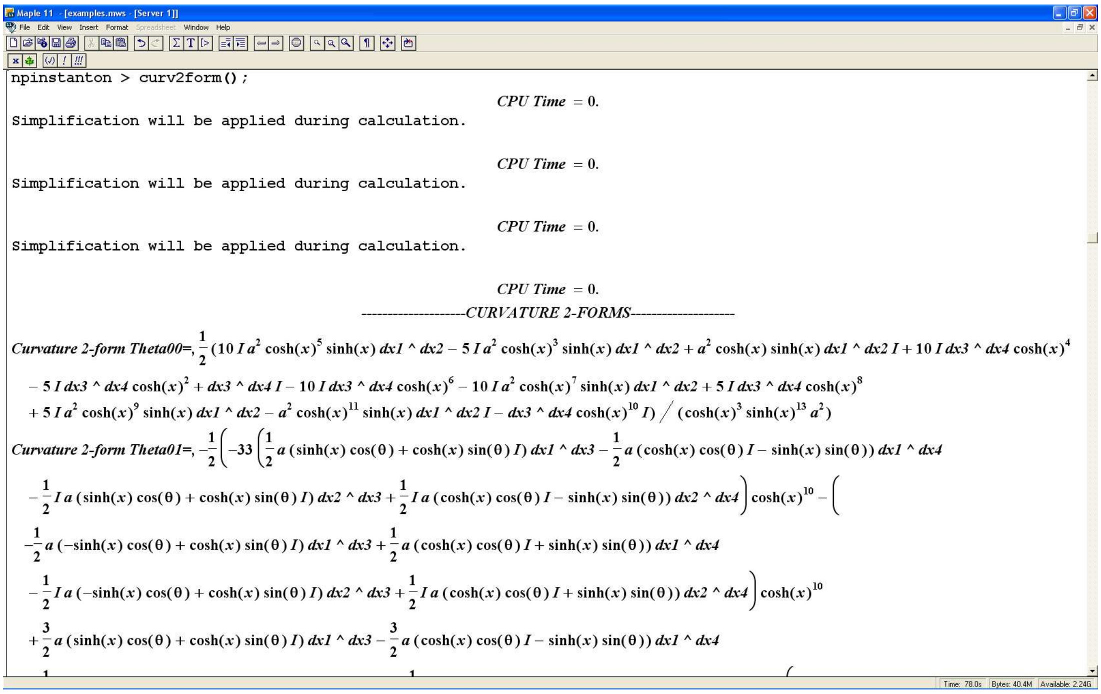

Figure C.11: Maple Session for the curv2form() Command

- topologicalnumbers() command:

This command calculates the integrands of the Euler number and the Hirzebruch signature curvature part integrals using the relations (B.45) and (B.46) namely,

$$\begin{aligned}
\chi &= \frac{1}{4\pi^2}\int_{\mathscr{M}} [\,|\Psi_0|^2 + 4|\Psi_1|^2 + 3\Psi_2^2 + |\tilde{\Psi}_0|^2 \\
&\quad + 4|\tilde{\Psi}_1|^2 + 3\tilde{\Psi}_2^2 - 2(|\Phi_{00}|^2 + |\Phi_{02}|^2) \\
&\quad - 4(|\Phi_{01}|^2 + |\Phi_{11}|^2 + |\Phi_{12}|^2 - 3\Lambda^2)\,]\, l\wedge\bar{l}\wedge m\wedge\bar{m}
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
\tau &= -\frac{1}{6\pi^2}\int_{\mathscr{M}} [\,|\Psi_0|^2 + 4|\Psi_1|^2 + 3\Psi_2^2 - |\tilde{\Psi}_0|^2 \\
&\quad - 4|\tilde{\Psi}_1|^2 - 3\tilde{\Psi}_2^2\,]\, l\wedge\bar{l}\wedge m\wedge\bar{m} - \eta_s(\partial\mathscr{M})
\end{aligned} \tag{C.9}$$

$\eta_s(\partial\mathscr{M})$ being the eta-invariant and this value will not be taken into consideration in the program. The output can be reached by calling

> eulernumber_integrand;
> hirzebruch_signature_integrand;

These numbers have a special importance for the Atiyah-Patodi-Singer index theorem of operators on manifolds with boundary [36–40]. The Maple session for this command is shown in Figure C.12.

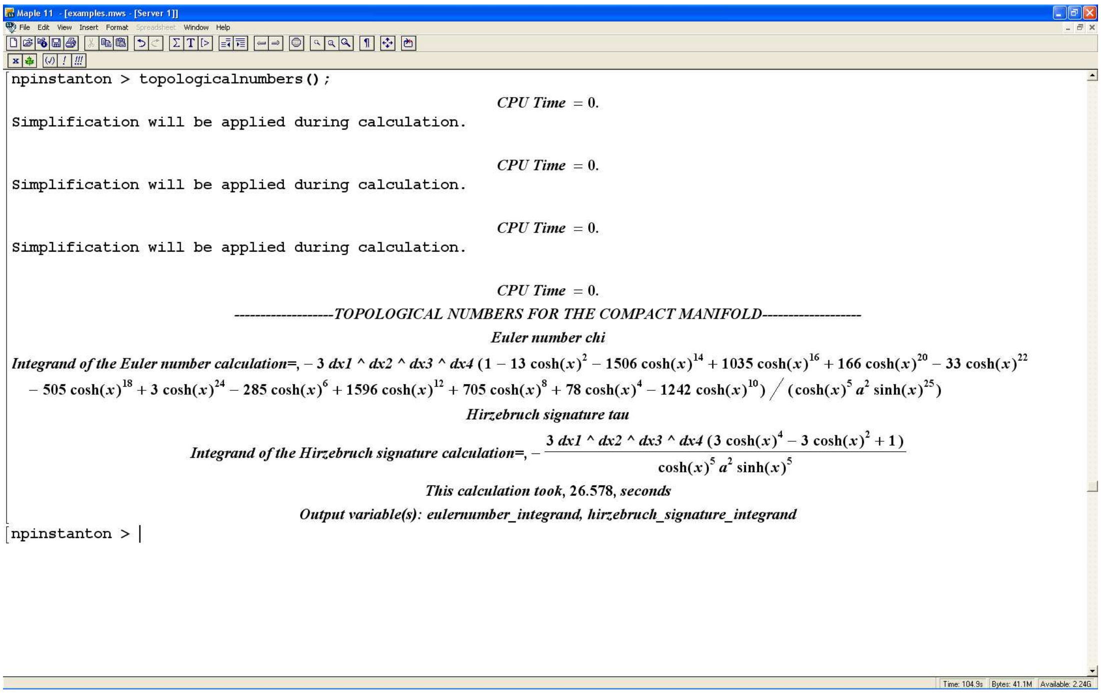

Figure C.12: Maple Session for the topologicalnumbers() Command

C.2 Example 2: Calculations for the Eguchi-Hanson Metric

Eguchi-Hanson instanton [11] is the most similar to the Yang-Mills instanton of Belavin et al. [91] and the dyad was given in equations (4.1) and (4.2). We run our program in Maple and type the name of the input file: eguchihanson

We can find the scalar operator by using scalaroperator() command. We can easily find the solution for the scalar equation with one extra dimension added if this dimension is a Killing direction. In the example, we just added $-k_t^2\Phi(r,\theta)$ to the equation after dividing by the exponential part which comes from the solution of the Killing directions. Here, k_t is the eigenvalue corresponding to the extra dimension. The scalar equation can be solved in terms of hypergeometric equations in four dimensional case. We see that the addition of the extra dimension results in a more singular equation which can be solved in terms of confluent Heun equations as the solution of the radial part. The Maple session of this procedure is shown in Figure C.13.

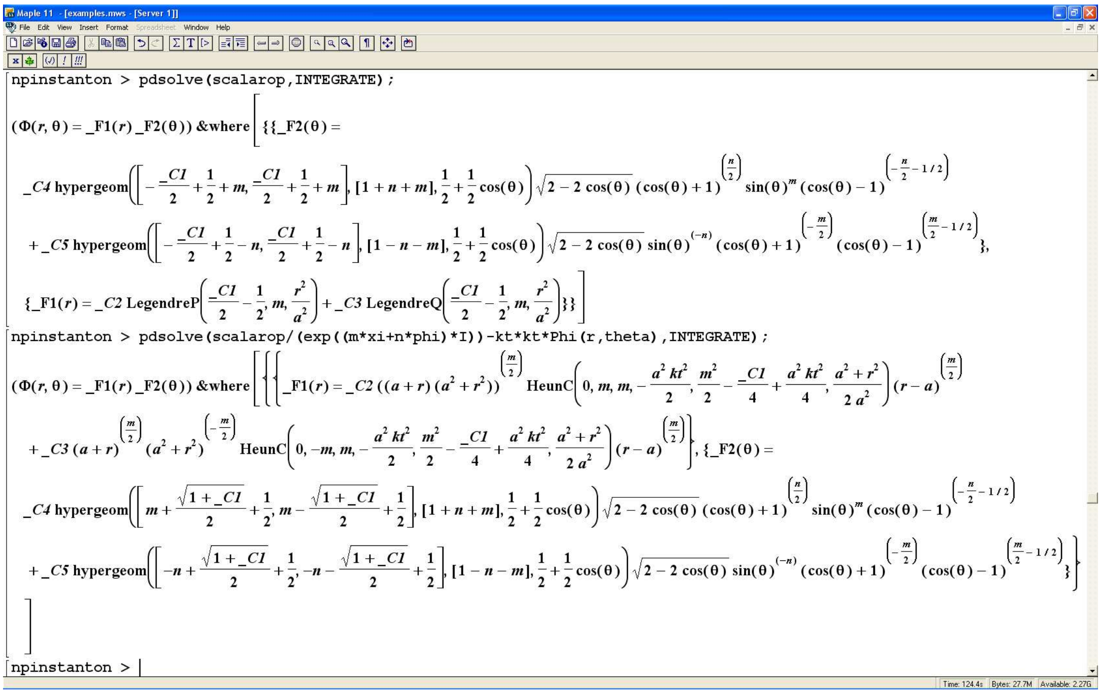

Figure C.13: Scalar Operator for Eguchi-Hanson Metric

Now, let us calculate the connection 1-forms by conn1form() command. The Maple output of this command is given in Figure C.14. Then the gauge is found by the program as self-dual.

To find the Weyl scalars and the Petrov class, one can run weylscalar() command. The Maple output of this command is given in Figure C.15. After the Weyl scalars are shown, the Petrov type is found to be Petrov-type D.

The whole results agree with the literature [90].

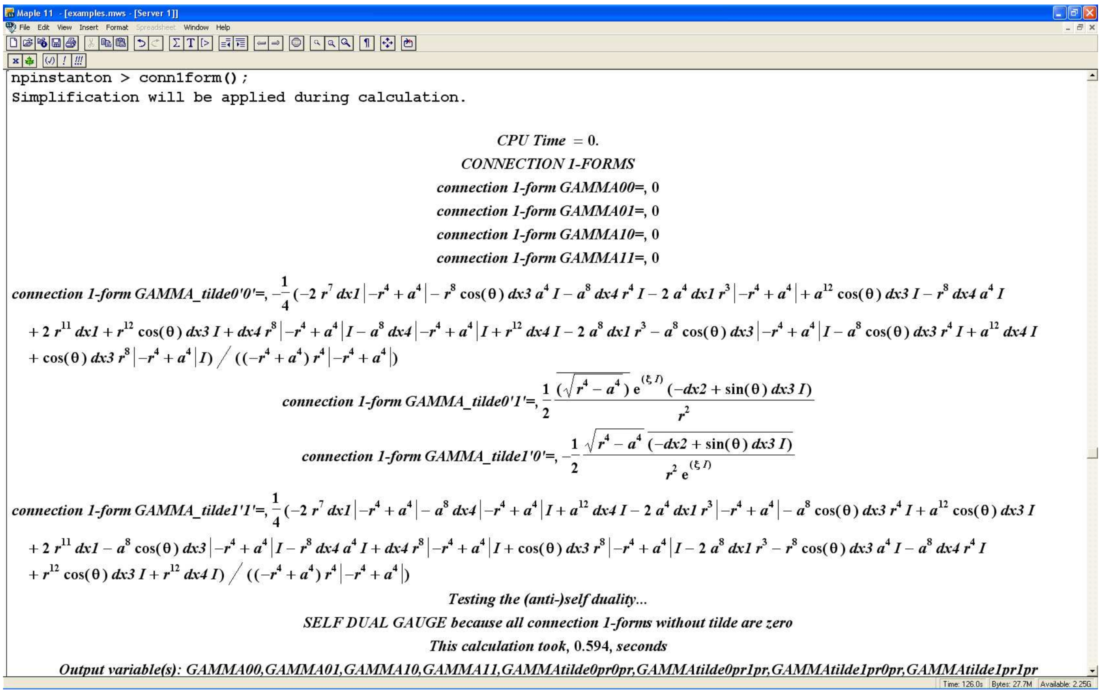

Figure C.14: conn1form() Command for the Eguchi-Hanson Case

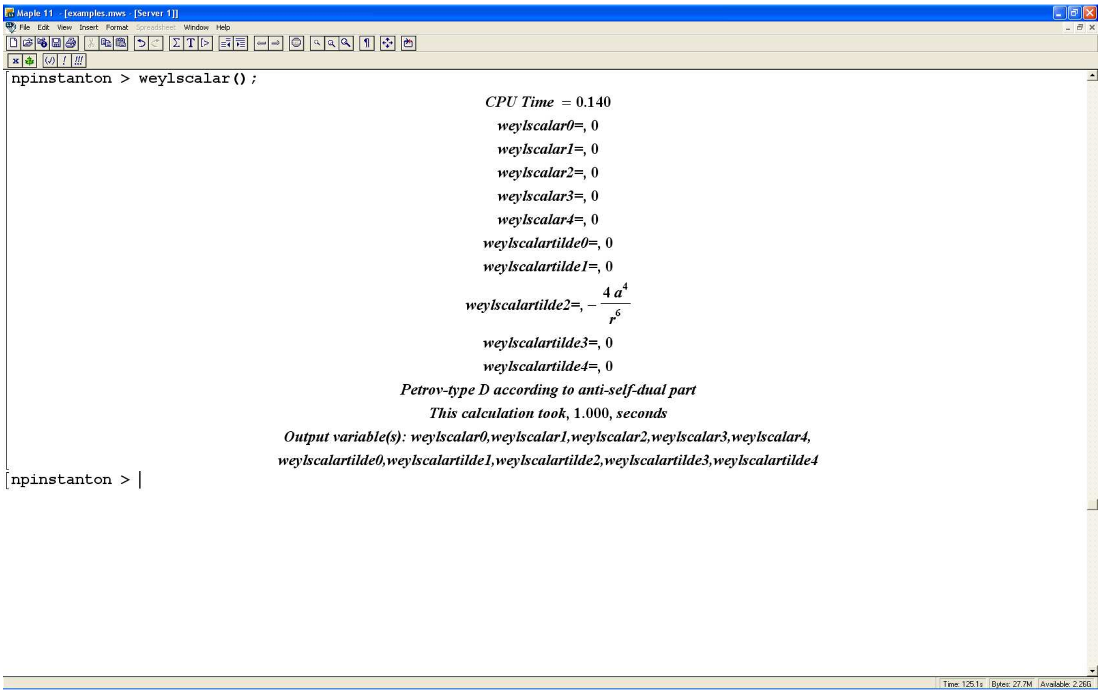

Figure C.15: weylscalar() Command for the Eguchi-Hanson Case

CURRICULUM VITAE

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