İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

THE COUPLINGS OF ELECTROMAGNETIC AND DIRAC SPINOR FIELDS TO GRAVITY

Ph.D. Thesis by
Özcan SERT

Department : Physics Engineering

Programme : Physics Engineering

APRIL 2011

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

THE COUPLINGS OF ELECTROMAGNETIC AND DIRAC SPINOR FIELDS TO GRAVITY

Ph.D. Thesis by
Özcan SERT

Department : Physics Engineering

Programme : Physics Engineering

APRIL 2011

İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

THE COUPLINGS OF ELECTROMAGNETIC AND DIRAC SPINOR FIELDS TO GRAVITY

Ph.D. Thesis by
Özcan SERT
(509062101)

Date of submission : 4 March 2011
Date of defence examination : 11 April 2011

Supervisor (Chairman) : Prof. Dr. Neşe ÖZDEMİR (ITU)
Cosupervisor : Prof. Dr. Tekin DERELİ (Koc U.)
Members of the Examining Committee : Prof. Dr. Ayşe H. BİLGE (K. Has U.)
Prof. Dr. Mahmut HORTAÇSU (ITU)
Assoc.Prof.Dr. S.Kayhan ÜLKER (FGE)
Assist.Prof.Dr. A.Savaş ARAPOĞLU (ITU)
Assist.Prof.Dr. Aybike ÖZER (ITU)

APRIL 2011

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

ELEKTROMANYETİK VE DIRAC SPİNÖR ALANLARININ GRAVİTASYONA BAĞLANMASI

DOKTORA TEZİ
Özcan SERT
(509062101)

Tezin Enstitüye Verildiği Tarih : 4 Mart 2011
Tezin Savunulduğu Tarih : 11 Nisan 2011

Tez Danışmanı	**:**	**Prof. Dr. Neşe ÖZDEMİR (İTÜ)**
Eş Danışman	**:**	**Prof. Dr. Tekin DERELİ (Koç Ü.)**
Diğer Jüri Üyeleri	**:**	**Prof. Dr. Ayşe H. BİLGE (K. Has Ü.)**
		Prof. Dr. Mahmut HORTAÇSU (İTÜ)
		Doç. Dr. S.Kayhan ÜLKER (FGE)
		Yard.Doç.Dr. A.Savaş ARAPOĞLU (İTÜ)
		Yard.Doç.Dr. Aybike ÖZER (İTÜ)

NİSAN 2011

to my mother...

FOREWORD

Firstly, I would like to express my deep appreciation and thanks to Tekin DERELİ, for his patience, guidance, enthusiasm, encouragement, inspiration, and humor.

I am indebted to Neşe ÖZDEMİR for her willingness to share her knowledge with me numerous times during my thesis studies.

I must thank Mahmut HORTAÇSU, Ayşe H. BiLGE, Savaş ARAPOĞLU for helpful discussions and correspondence.

I thank Tolga BİRKANDAN for his friendship and engaging, educational conversations. I would also like to thank all the people of the Department of Physics at the Istanbul Technical University.

I offer my regards and thanks to all of those who supported me in any respect during the completion of the thesis.

Finally, words alone can not express the thanks I owe to my family.

This thesis is partly supported by a grant from the Turkish Technological and Scientific Research Center (TUBITAK).

APRIL 2011 Özcan SERT

TABLE OF CONTENTSs

ABBREVIATIONS

GR	**:**	General Relativity
EM	**:**	Einstein-Maxwell
ECD	**:**	Einstein-Cartan-Dirac
EH	**:**	Einstein-Hilbert
NMEM	**:**	Non-Minimal-Einstein-Maxwell

LIST OF SYMBOLS

ψ	:	Spinor field
ϕ	:	Scalar field
M	:	Manifold
g	:	Metric
∇	:	Connection
$\{x^{\mu}\}$	:	Coordinate Function
ι	:	Inner Product
$T(M)$	:	Tangent space
$T^{\star}(M)$	:	Cotangent space
η_{ab}	:	Minkowski metric
$\{e^{a}\}$	:	Orthonormal base 1-forms
$\{X_{b}\}$	:	Orthonormal Reference Frame
$\wedge$	:	Exterior Product
$\Lambda^{p}(M)$	:	p-forms space
d	:	Exterior Derivative Operator
${\Lambda^{a}}_{b}$	:	Connection 1-forms
D	:	Covariant Exterior Derivative Operator
Q_{ab}	:	Nonmetricity 1-forms
${\Omega^{a}}_{b}$	:	Metric Compatible Connection 1-forms
T^{a}	:	Torsion 2-forms
${\omega^{a}}_{b}$	:	Levi-Civita Connection 1-forms
${K^{a}}_{b}$	:	Contortion 1-forms
${q^{a}}_{b}$	:	Anti-Symmetric Connection 1-forms
${R^{a}}_{b}$	:	Curvature 2-forms
S	:	Action
L	:	Lagrange Density 4-form
$\mathscr{L}$	:	Lagrange Function
κ	:	Universal Gravitation cuopling Constant
$*$	:	Hodge Star Operator
δ	:	Infinitesimal Variation

THE COUPLINGS OF ELECTROMAGNETIC AND DIRAC SPINOR FIELDS TO GRAVITY

SUMMARY

In order to obtain new insights into gravity, we investigate the couplings of electromagnetic and spinor fields to gravity.

Firstly, after we summarized Einstein-Cartan Gravity in d-dimensions using the algebra of exterior differential forms, we investigate couplings of electromagnetism to gravity in four dimensions. We obtain the field equations of the non-minimal couplings described by a Lagrangian that involves generic RF^2-terms. We consider both theories without torsion, which is called non-minimally coupled Einstein-Maxwell theory and with torsion which is called non-minimally coupled Einstein-Cartan-Maxwell theory. In particular, we give a class of exact plane wave solutions and static, spherically symmetric magnetic monopole solutions. The solutions verify the predictions of the classical laws of electrodynamics up to high levels of accuracy. These are the laws that are usually extrapolated to describe astrophysical phenomena under extreme conditions of temperature, pressure and density. Any departures from these laws under such extreme conditions may be ascribed to new types of interactions between the electromagnetic fields and gravity.

Since major part of the known universe consists of fermions, it is important to know the effects of the fermions coupled to gravity. But it is not easy to determine the behavior of spinor fields in four dimensions. Nevertheless, in three dimensions, the system is simplified partly. Therefore, secondly, we formulate Einstein-Cartan-Dirac theory in (1+2)-dimensions using the algebra of exterior differential forms. That is, we couple a Dirac spinor to gravity and obtain the field equations by a variational principle. We determine the space-time torsion to be given algebraically in terms of the Dirac condensate field. We give circularly symmetric, stationary, exact solutions that collapse to static AdS_3 geometry in the absence of a Dirac spinor.

ELEKTROMANYETİK VE DIRAC SPİNÖR ALANLARININ GRAVİTASYONA BAĞLANMASI

ÖZET

Gravitasyon teorisiyle ilgili yeni öngörüler elde edebilmek için, elektromanyetik ve spinör alanlarının gravitasyona bağlanmalarını inceliyoruz.

İlk olarak, d-boyutta Einstein-Cartan gravitasyon teorisini özetledikten sonra diferansiyel formların dış cebirini kullanarak dört boyutta elektromanyetik alanların gravitasyona minimal olmayan bağlanmalarını düşünüyoruz. Genel RF^2 formunda terimler içeren Lagrangian tarafından tariflenen minimal olmayan bağlanmalar için alan denklemlerini elde ediyoruz. Minimal olmayacak şekilde bağlanmış burulmasız olan Einstein-Maxwell teorisi ile birlikte minimal olmayacak şekilde bağlanmış burulma da içerebilen Einstein-Cartan-Maxwell teorisini de hesaba katıyoruz. Özel olarak, bu teorilere analitik, düzlem yüzlü dalga ve küresel simetrik, durgun manyetik tek-kutup çözümleri buluyoruz. Bu çözümler klasik elektrodinamik yasalarını yüksek hassasiyetlere kadar doğrulamaktadır. Bu yasalar sıcaklık basınç ve yoğunluğun bazı uç koşullarda olduğu astrofiziksel olayları tariflemek için kullanılabilir. Bu uç koşullar altında, bu yasalardan sapmalar elektromanyetik alanlar ve gravitasyon arasında yeni etkileşim türlerine atfedilebilir.

Bilinen evrenin büyük bir kısmı fermiyonlardan oluştuğu için fermiyonların gravitasyona bağlanmalarının etkilerini bilmek önemlidir. Fakat, dört boyutta spinör alanlarının bu davranışını bilmek kolay değildir. Üç boyutta bu sistem kısmen basitleşir. Bü yüzden, ikinci olarak, (1+2)-boyutta Einstein-Cartan-Dirac teorisinin formalizmini dış diferensiyel form hesabını kullanarak veriyoruz. Yani gravitasyona Dirac spinör alanını bağlayarak varyasyon yöntemiyle alan denklemlerini elde ediyoruz. Uzay-zaman burulmasını Dirac yoğunlaşmış alanları cinsinden elde ediyoruz. Dirac spinörünün yokluğunda AdS_3 durgun metriğine dönüşen durağan çembersel simetrik tam çözümleri belirliyoruz.

1. INTRODUCTION

The predictions of the classical laws of electrodynamics have been verified to high levels of accuracy. These are the laws that are usually extrapolated to describe astrophysical phenomena under extreme conditions of temperature, pressure and density. Any departures from these laws under such extreme conditions may be ascribed to new types of interactions between the electromagnetic fields and gravity.

In this thesis, we firstly consider non-minimal couplings of gravitational and electromagnetic fields described by a Lagrangian density that involves generic RF^2-terms. Such a coupling term was first considered by Prasanna and classified by Horndeski to gain more insight into the relationship between space-time curvature and electric charge conservation. It is remarkable that a calculation in QED of the photon effective action from 1-loop vacuum polarization on a curved background contribute similar non-minimal coupling terms.

After we present required fundamental concepts for our research in the second section, in the third section we give an outline of the Einstein-Cartan theories of Gravitation in any number of dimensions considering the presence of the other fields.

In the fourth section, in order to gain more insight to the observations, we formulate non-minimally coupled Einstein-Maxwell theory which is non-minimally coupled the curvature and Maxwell tensor in form of RF^2 in four dimensions using the algebra of exterior differential forms. We derive the field equations by a first order variational principle. We will be working with the unique metric-compatible, torsion-free Levi-Civita connection at first. We impose this choice of the connection through constrained variations by the method of Lagrange multipliers. That is, we add to the Lagrangian density of the theory Lagrange multiplier 2-forms whose variation imposes the zero-torsion constraint. We also use a first order variational principle for the electromagnetic field 2-form F to impose the homogeneous Maxwell equation as a constraint. Secondly, we consider the variational field equations without the zero-torsion constraint. The resulting field equations are highly non-linear in both cases. The case with a connection with zero torsion and the case with a connection with non-zero torsion give rise to inequivalent systems of field equations. Intense gravitational fields that will be found near black holes behave as a specific kind of non-linear medium in the presence of non-minimal couplings. Conversely, one should expect new gravitational effects induced by non-minimal couplings in the vicinity of the neutron stars or magnetars where there are intense electromagnetic fields. Such new effects, if there are any, can be discussed in terms of exact solutions of the coupled field equations with appropriate isometries. Finding spherically symmetric solutions is not an easy task for such theories. Furthermore, any arbitrary non-minimal coupling may not give rise to solutions satisfying physical asymptotic conditions and observations in solar and cosmological scales. RF^2-coupled terms in the Lagrangian lead to modifications both in the Maxwell and Einstein field equations. The modifications in the Maxwell equations can be related with the polarization and magnetization in a specific medium. The non-minimal couplings also give rise important modifications to the structure of a charged black hole. These may shed light on some problems of gravity such as dark matter and dark energy without introducing a cosmological constant or any other type of scalar fields. This means that, if dark matter is not some strange matter, but, for instance the non-minimal couplings produce such effects, then the electromagnetic fields get modified at large (astrophysical) scales and thus contribute to the conventional electromagnetic energy density which may then be interpreted as the effects of dark matter. In particular, we look for static, spherical symmetric, electric and magnetic monopole solutions and plane fronted wave (pp-wave) solutions. Then, we obtain a class of asymptotically flat solutions that include new black hole candidate configurations, except for the parameter values when there is a naked essential singularity at the origin. There are two different solutions with magnetic monopole potential for non-minimally coupled Einstein-Maxwell theory; one of them has central singularity and the other has no central singularity. On the other hand, in the case of non-minimally coupled Einstein-Cartan-Maxwell theory with torsion for the same magnetic monopole potential, only one of these solutions which does not have a central singularity is allowed. This solution does not correspond to a black hole in general. We discuss the structure of these solutions. Also, we find that a class of pp-wave solutions, which is solution of both the field equations obtained from the non-minimally coupled Einstein-Maxwell theory and the non-minimally coupled Einstein-Cartan-Maxwell theory.

Even if it is considered that the matter couplings to gravity have a small effect on test particles, under some extreme conditions such as high density, small scales and near black holes, it can cause important effects. Moreover, it can be a new insight to consider the couplings in the context of astrophysical and quantum field theory. For this aim, lastly, we investigate Einstein-Cartan-Dirac theory in 1+2 dimensions differently from the theories in 1+3 dimensions.

2. THE GEOMETRY OF SPACETIME

2.1 Preliminaries

In this thesis, the space-time is denoted by $\{M, g, \nabla\}$ where M is a d-dimensional smooth and differentiable manifold, diffeomorfic to R^d, equipped with a Lorentzian metric g which is a second rank, covariant, symmetric, non-degenerate tensor and ∇ is a linear connection which defines parallel transport of vectors (or more generally tensors and spinors).

The coordinate system which is given by $\{x^\mu(p)\}$, constitutes such a coordinate reference frame $\{\frac{\partial}{\partial x^\mu}(p)\}$ or $\{\partial_\mu\}$ at any point $p \in M$. This reference frame is a set of base vectors of $T_p(M)$ tangent space. Analogously, $\{dx^\mu(p)\}$ is a coordinate reference co-frame of the cotangent space $T_p{}^*(M)$. On the manifold M, functions are (0,0) type tensors, vectors are (1,0) type contravariant tensors and co-vectors are (0,1) type covariant tensors.

2.1.1 Exterior algebra and differential forms

We will use exterior algebra throughout this thesis [1–3]. In the exterior algebra space, the basis of cotangent bundle $T_p{}^*(M)$ are called 1-forms. Any p-forms space which is denoted by $\Lambda^p(M)$ can be obtained from the antisymmetric tensor product space as $\underbrace{T_p{}^*(M) \times \ldots \times T_p{}^*(M)}_{\text{p-times}}$. Therefore, the exterior algebra space is consist of the sum of the p-forms spaces; $\bigoplus_{p=0}^{d} \Lambda^p(M)$.

Any p form $\omega \in \Lambda^p(M)$ can be written in closed 1-forms (closed means that $d(dx^\nu) = 0$) as:

$$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \qquad \textbf{(2.1)}$$

In this exterior algebra space, let's consider a real constant α, $\omega_1 \in \Lambda^p(M)$, $\omega_2 \in \Lambda^q(M)$ and $\omega_3 \in \Lambda^r(M)$. They satisfy the following properties:

1. $(\alpha\omega_1)\wedge\omega_2 = \omega_1\wedge(\alpha\omega_2) = \alpha(\omega_1\wedge\omega_2)$

2. $(\omega_1+\omega_2)\wedge\omega_3 = \omega_1\wedge\omega_3+\omega_2\wedge\omega_3$

3. $\omega_1\wedge(\omega_2\wedge\omega_3) = (\omega_1\wedge\omega_2)\wedge\omega_3$

4. $\omega_1\wedge\omega_2 = (-1)^{p.q}\omega_2\wedge\omega_1$

After the definitions, we can look at some fundamental operators in the exterior algebra.

2.1.2 Exterior derivative operator

Exterior derivative operator d is an exact derivative and maps p-forms to (p+1) forms

$$d:\Lambda^p(M)\longrightarrow\Lambda^{p+1}(M) \qquad \textbf{(2.2)}$$

The operator satisfies

1. $d(\omega_1+\omega_2) = d\omega_1+d\omega_2$

2. $d\omega = \frac{1}{p!}\frac{\partial\omega_{\mu_1\wedge\cdots\mu_p}}{\partial x^\mu}dx^\mu\wedge dx^{\mu_1}\wedge\cdots dx^{\mu_p}$

3. $d(\omega_1\wedge\omega_3) = d\omega_1\wedge\omega_3+(-1)^p\omega_1\wedge d\omega_3$

4. $d(d\omega)=0.$

2.1.3 Interior product operator

Interior product operator or contraction operator $\iota_{\widetilde{e}_a}$ is an antiderivative operator for each $\widetilde{e}_a\in T_pM$ and maps p-forms to (p-1) forms.

$$\iota_{\widetilde{e}_a} = \iota_a = \eta_{ab}\iota^b := \Lambda^p(M)\longrightarrow\Lambda^{p-1}(M) \qquad \textbf{(2.3)}$$

let's consider $\omega\ \in\Lambda^p(M)$ and scalar function f , the operator satisfies

1. $\iota_a f = 0$

2. $\iota_{fa}\omega = f\iota_a\omega$

3. $e^a\wedge\iota_a\omega = p\omega$

4. $\iota_a\iota_b\omega = -\iota_b\iota_a\omega$

5. $\iota_a(\omega_1 \wedge \omega_2) = \iota_a\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \iota_a\omega_2$.

The interior product of base vectors of tangent space wand base co-vectors of cotangent space is determined by Kronecker delta.

$$dx^\mu(\frac{\partial}{\partial x^\nu}) \equiv \iota_{\frac{\partial}{\partial x^\nu}} dx^\mu = \delta^\mu{}_\nu \tag{2.4}$$

One can choose a set of linearly independent orthonormal frames on tangent space $T_p(M)$ which is denoted by $\{\widetilde{e}_a\}$, $a = 0,1,2,3..,d-1$ and is called orthonormal reference frame. The dual basis of the orthonormal reference frame will be denoted by $\{e^a\}$. Similarly to the (**2.4**), the interior product of $\{\widetilde{e}_a\}$ and $\{e^a\}$ satisfy

$$e^a \cdot \widetilde{e}_b \equiv \iota_{\widetilde{e}_b}(e^a) = \delta^a_b \tag{2.5}$$

where $\iota_{\widetilde{e}_a} \equiv \iota_a$ is the interior product. In this research, the first half of Greek alphabet $\alpha, \beta, \ldots = \hat{0}, \hat{1}, \hat{2}, .., \hat{d} - \hat{1}$ and the second half $\mu, \nu, \ldots = \hat{1}, \hat{2}, .., \hat{d} - \hat{1}$ are coordinate (holonomic) indices. The first half of Latin alphabet $a, b, \ldots = 0,1,2,..d-1$ and the second half $i, j, \ldots = 1,2,3,..,d-1$ are frame (anholonomic) indices. The orthonormal frame $\widetilde{e}_a(p)$ is related to the coordinate frame $\partial_\alpha(p)$ via $h^\alpha{}_a(p)$ vielbein or tedrad;

$$\widetilde{e}_a(p) = h^\alpha{}_a(p)\partial_\alpha(p) \tag{2.6}$$

If $h^\alpha{}_a(p)$ is nondegenerate or $\det h^\alpha{}_a(p) \neq 0$, then $\widetilde{e}_a$ is an anholonomic base. Analogously, the co-frame 1-forms can be written in the form of exact 1-forms as

$$e^a(p) = h^a{}_\alpha(p)dx^\alpha(p) \tag{2.7}$$

Moreover, the tedrad satisfy

$$\iota_a e^b = h^\alpha{}_a(p)h^b{}_\alpha(p) = \delta^b_a. \tag{2.8}$$

In this thesis, we will use mostly the shorthand notations for exterior product of co-frames $e^a \wedge e^b \wedge \cdots = e^{ab\cdots}$ and interior product operators $\iota_a\iota_b \cdots = \iota_{ab\cdots}$ One can show that although ∂_α and ∂_β commute, $\widetilde{e}_a$ and $\widetilde{e}_b$ may not commute

$$[\widetilde{e}_a, \widetilde{e}_b] = h^\alpha{}_a\partial_\beta\widetilde{e}_b - h^\beta{}_b\partial_\alpha\widetilde{e}_a. \tag{2.9}$$

2.1.4 Hodge star operator

Hodge star operator $*$ is a linear mapping from p-forms to (d-p) forms for a d-dimensional manifold:

$$*: \ \Lambda^p(M) \longrightarrow \Lambda^{d-p}(M) \qquad \textbf{(2.10)}$$

The volume d-form is defined by

$$*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 ... e^{d-1} = \frac{1}{d!}\varepsilon_{abc...d} e^a \wedge e^b \wedge e^c ... \wedge e^d \qquad \textbf{(2.11)}$$

and the completely antisymmetric Levi-Civita tensor density is fixed by choosing $\varepsilon_{012...d-1} = +1$. The star operator has the following properties for $\alpha, \beta \ \in \Lambda^p(M)$:

1. $\alpha \wedge *\beta = \beta \wedge *\alpha \qquad$ *and* $\qquad *\beta \wedge \alpha = *\alpha \wedge \beta$
2. $*(\alpha \wedge e_a) = \iota_a * \alpha$
3. $**\alpha = \pm\alpha$

Using the above definitions and properties, we can introduce a spacetime metric. The metric which is related to the distance between two infinitesimally near points x^μ and $x^\mu + dx^\mu$ can be written via reference frames and orthonormal reference co-frames as

$$g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta = h_\alpha{}^a h_\beta{}^b \eta_{ab} dx^\alpha \otimes dx^\beta = \eta_{ab} e^a \otimes e^b \qquad \textbf{(2.12)}$$

Here $g(\widetilde{e}_a, \widetilde{e}_b) = \eta_{ab}$ is Minkowski metric which is $diag(-1,1,1,..,1)$.

2.1.5 Connection 1- forms

Let us take two observers each using their own reference frame to measure spacetime intervals on the manifold M. The observer O fixes $\{\widetilde{e}_a\}$ and the observer O' fixes $\{\widetilde{e}'_a\}$ reference frames at the same point $p \ \in M$. One can find ${L^{-1}}^b{}_a$ a local Lorentz matrix satisfying the transformation,

$$\widetilde{e}'_a(p) = \widetilde{e}'_b(p) {L^{-1}}^b{}_a(p). \qquad \textbf{(2.13)}$$

Similarly, the transformation of orthonormal reference co-frames is defined by

$$e'^a(p) = L^a{}_b(p) e^b(p) \qquad \textbf{(2.14)}$$

Then, the local Lorentz transformation of interior product of the frames and co-frames are

$$\tilde{e}'_a(e'^a) = \tilde{e}_c L^{-1c}{}_a (L^a{}_d e^d) = \tilde{e}_c \delta^c{}_d e^d = \tilde{e}_c e^c = \tilde{e}_a e^a. \quad \textbf{(2.15)}$$

Lorentz invariant. Let us see the local Lorentz transformation of exterior derivative of the covariant e^a basis.

$$de'^a = d(e^b L^a{}_b) = dL^a{}_b e^b + L^a{}_b de^b \quad \textbf{(2.16)}$$

Because of $dL^a{}_b e^b$ term, the transformation of de^a is not Lorentz covariant. That is; it does not transform as a tensor. We need to use connection 1-forms in order to make it Lorentz covariant and the local Lorentz transformation of the connections must be defined as

$$\Lambda^{a\prime}{}_b = L^a{}_c \Lambda^c{}_f L^{-1f}{}_b + L^a{}_c dL^{-1c}{}_b. \quad \textbf{(2.17)}$$

2.1.6 Covariant exterior derivative

For any general (p,q) type tensor $R_{a_1 \cdots a_p}{}^{b_1 \cdots b_q}$, the covariant exterior derivative operator defined as follow;

$$\begin{aligned} DR_{a_1\cdots a_p}{}^{b_1\cdots b_q} &= dR_{a_1\cdots a_p}{}^{b_1\cdots b_q} + \Lambda^{b_1}{}_c R_{a_1\cdots a_p}{}^{cb_2\cdots b_q} + \cdots + \Lambda^{b_q}{}_c R_{a_1\cdots a_p}{}^{b_1\cdots c} \\ &\quad -\Lambda^c{}_{a_1} R_{ca_2\cdots a_p}{}^{b_1\cdots b_q} - \cdots - \Lambda^c{}_{a_p} R_{a_1 a_2\cdots c}{}^{b_1\cdots b_q}. \end{aligned} \quad \textbf{(2.18)}$$

We have shown that $de^{a\prime} \neq L^a{}_b(de^b)$ does not transform as a tensor. But now we can show that the covariant exterior derivative of e^a transforms as a tensor.

$$\begin{aligned} De^{a\prime} &= de^{a\prime} + \Lambda^a{}_b{}' \wedge e^{b\prime} \\ &= d(L^a{}_b e^b) + (L^a{}_c \Lambda^c{}_f L^{-1f}{}_b + L^a{}_c dL^{-1c}{}_b) \wedge L^b{}_k e^k \\ &= dL^a{}_b \wedge e^b + L^a{}_b de^b + L^a{}_c \Lambda^c{}_f L^{-1f}{}_b L^b{}_k \wedge e^k + L^a{}_c dL^{-1c}{}_b L^b{}_k \wedge e^k \\ &= dL^a{}_b \wedge e^b + L^a{}_b de^b + L^a{}_c \Lambda^c{}_k \wedge e^k - dL^a{}_k \wedge e^k \\ &= L^a{}_b (de^b + \Lambda^b{}_k \wedge e^k) \\ De^{a\prime} &= L^a{}_b De^b \end{aligned} \quad \textbf{(2.19)}$$

and it is called local Lorentz covariant. After the fundamental definitions, we can look at the covariant exterior derivative of the metric η_{ab}, e^a orthonormal basis 1-forms and $\Lambda^a{}_b$ connection 1-forms. They are known as Cartan Structure equations. In the gauge approach to gravity η_{ab}, $e^a, \Lambda^a{}_b$ are interpreted as the generalized gauge potentials, while the nonmetricity 1-forms, torsion 2-forms and curvature 2-forms correspond to field strengths.

2.2 Nonmetricity

The covariant exterior derivative of the η_{ab} metric gives us first Cartan structure equation [4].

$$Q_{ab} = -\frac{1}{2} D\eta_{ab}. \qquad (\mathbf{2.20})$$

The symmetric 1-forms Q_{ab} are (1,2)-type tensor and called nonmetricity tensor. The Q_{ab} tensor is the symmetric part of Λ_{ab}:

$$D\eta_{ab} = d\eta_{ab} - \Lambda^{c}{}_{a}\eta_{cb} - \Lambda^{c}{}_{b}\eta_{ac} \qquad (\mathbf{2.21})$$

because of η_{ab} has real constant elements, $d\eta_{ab} = 0$ and so,

$$D\eta_{ab} = -\Lambda_{ab} - \Lambda_{ba} \qquad (\mathbf{2.22})$$

$$Q_{ab} = \frac{1}{2}(\Lambda_{ba} + \Lambda_{ab}) \qquad (\mathbf{2.23})$$

Thus, if we compare these two equations (**2.22**) and (**2.23**), we reach the equation (**2.20**). If $Q_{ab} = 0$, it is said that the connection is *metric compatible*.

Geometrically the nonmetricity tensor measures the deformation of length and angle standards under parallel transport. Technically speaking it is a measure of compatibility of the affine connection with the metric. The scalar product of vectors is, in general, not preserved under parallel transport due to the appearance of nonmetricity. Einstein's general relativity theory is formulated in spacetimes with metric compatible connection (vanishing nonmetricity tensor). But, the solutions which are symmetric teleparallelly equivalent to Einstein's general relativity (giving the same solutions with Einstein's relativity and more in the framework of symmetric teleparallel gravity) can be found considering only the nonmetricity tensor [8,9].

2.3 Torsion

The covariant exterior derivative of e^a orthonormal basis 1-forms gives T^a torsion tensor, or second Cartan structure equation [4]:

$$T^a := De^a = de^a + \Lambda^{a}{}_{b} \wedge e^b. \qquad (\mathbf{2.24})$$

T^a 2-forms are called (1,2)-type torsion tensor. Torsion can be obtained from contortion 1-forms as

$$K^a{}_b \wedge e^b = T^a. \qquad \textbf{(2.25)}$$

The full connection 1-forms can be decomposed by [5–7],

$$\Lambda_{ab} = \hat{\omega}_{ab} + K_{ab} + q_{ab} + Q_{ab} \qquad \textbf{(2.26)}$$

where $\hat{\omega}^a{}_b$ are the zero-torsion Levi-Civita connection 1-forms satisfying

$$de^a + \hat{\omega}^a{}_b \wedge e^b = 0 \qquad \textbf{(2.27)}$$

and the antisymmetric connection q_{ab} can be derived from the symmetric Q_{ab} nonmetricity tensor ;

$$q^a{}_b = -\iota^a Q_{bc} \wedge e^c + \iota_b Q^a{}_c \wedge e^c \qquad \textbf{(2.28)}$$

So, the antisymmetric part of the full connection is

$$\Lambda_{[ab]} = \hat{\omega}_{ab} + K_{ab} + q_{ab} \qquad \textbf{(2.29)}$$

and the symmetric part is

$$\Lambda_{(ab)} = Q_{ab}. \qquad \textbf{(2.30)}$$

We can see from the above equation, if $Q^a{}_b = 0$ then $q^a{}_b = 0$. In this case it is possible to decompose the connection 1-forms in a unique way:

$$\omega^a{}_b = \hat{\omega}^a{}_b + K^a{}_b \qquad \textbf{(2.31)}$$

In addition to Q_{ab}, if also K_{ab} is zero then

$$\Lambda^a{}_b \to \hat{\omega}^a_b \qquad \textbf{(2.32)}$$

Moreover, ω_{ab} satisfy the equation

$$2\omega_{ab} = -\iota_a(de_b) + \iota_b(de_a) + \iota_a\iota_b(de_c)e^c \qquad \textbf{(2.33)}$$

Analogously, $K^a{}_b$ contorsion 1-forms can be written as

$$2K_{ab} = \iota_a T_b - \iota_b T_a - (\iota_a\iota_b T_c)e^c \qquad \textbf{(2.34)}$$

The connections are dimensionless quantities. Thus, Torsion has dimension of length, $T^a = [L]$.

Physically, the torsion tensor can describe the density of intrinsic angular momentum or effects of scalar fields depending on the related theory. In Einstein's general relativity, torsion tensor is zero. But, when one considers the couplings of gravity with matter fields, the torsion tensor has to be taken into account. Additionally, theories with torsion have more degrees of freedom to comply with observations.

2.4 Curvature

The covariant exterior derivative of the connection 1-forms gives $R^a{}_b(\Lambda)$ curvature tensor or third Cartan structure equation [4].

$$R^a{}_b(\Lambda) := D\Lambda^a{}_b := d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b \tag{2.35}$$

$R^a{}_b(\Lambda)$ is a (1,3)-type Riemann curvature tensor. We can show that $R^a{}_b(\Lambda)$ transforms as a tensor;

$$\begin{aligned}
R^{a\prime}{}_b &= d\Lambda^{a\prime}{}_b + \Lambda^{a\prime}{}_c \wedge \Lambda^{c\prime}{}_b \\
&= d(L^a{}_c \Lambda^c{}_f L^{-1f}{}_b + L^a{}_c dL^{-1c}{}_b) \\
&\quad + (L^a{}_e \Lambda^e{}_f L^{-1f}{}_c + L^a{}_e dL^{-1e}{}_c) \wedge (L^c{}_g \Lambda^g{}_h L^{-1h}{}_b + L^c{}_g dL^{-1g}{}_b) \\
R^{a\prime}{}_b &= L^a{}_c R^c{}_d L^{-1d}{}_b
\end{aligned}$$

In the second line, we have used that $(dL^{-1c}{}_b)L^b{}_g = -L^{-1c}{}_b(dL^b{}_g)$.

Geometrically it is related to linear group. Now let us see the effect of curvature on vectors after parallel transport along a closed loop. If the vector does not undergo a rotation, then the space is flat. Conversely, if the vector is rotated, then the space is curved. Performing the parallel transport of a vector $A = e_a A^a$ around a closed small path one obtains the following transformation

$$\Delta A^a \simeq \frac{1}{2}\int_S R^a{}_{b\mu\nu} A^b dx^\mu \wedge dx^\nu \tag{2.36}$$

where S is the surface of the loop. If this tensor is not zero that means if any component is not zero, then space-time is curved. If this tensor is zero, then space-time is flat. Physically this is a fundamental tensor in gravitation, in particular in Einstein's general relativity.

2.5 Bianchi Identities

One can find Bianchi identities taking the covariant exterior derivative of curvature, torsion, nonmetricty tensors as

$$
\begin{aligned}
DQ_{ab} &= \frac{1}{2}(R_{ab}+R_{ba}) && \textbf{(2.37)}\\
DT^a &= R^a{}_b \wedge e^b && \textbf{(2.38)}\\
DR^a{}_b &= 0. && \textbf{(2.39)}
\end{aligned}
$$

Here we have used (**2.24**), (**2.35**), (**2.20**) and the properties of exterior algebra.

Also, one can show that the following equalities noting that lowering or raising an index in front of the covariant exterior derivative if the spacetime metric is not compatible or nonmetricity is not zero.

$$
\begin{aligned}
D*e_a &= -Q\wedge *e_a + T^b \wedge *e_{ab} && \textbf{(2.40)}\\
D*e_{ab} &= -Q\wedge *e_{ab} + T^c \wedge *e_{abc} && \textbf{(2.41)}\\
D*e_{abc} &= -Q\wedge *e_{abc} + T^d \wedge *e_{abcd} && \textbf{(2.42)}\\
D*e_{abcd} &= -Q\wedge *e_{abcd}. && \textbf{(2.43)}
\end{aligned}
$$

3. EINSTEIN-CARTAN GRAVITY

Einstein-Cartan Gravity is considered as an extension of general relativity in the presence of torsion tensor field. The source of the torsion field can be intrinsic angular momentum, scalar fields, non-minimal couplings of gravity and electromagnetic fields depending on the theory. In the following sections we will discuss also the effects of torsion for electromagnetic field coupled to gravity in four dimensions and Dirac field coupled to gravity in three dimensions.

Therefore, we will point out the difference between general relativity and Einstein-Cartan gravity in this chapter giving an outline of these two theories in arbitrary d-dimensions. We will take a metric compatible connection; that is, the nonmetricity tensor is equal to zero.

3.1 General Relativity

Einstein's theory of gravity has been formulated in (pseudo-)Riemannian space-times in four dimensions such that the structure of the space-time is characterized by the metric or co-frame uniquely and the corresponding field strength is the curvature $R^a{}_b$ written in terms of the Levi-Civita connection. That is; in Einstein gravity, in addition to the non-metricity, the torsion is zero and the zero-torsion condition can be imposed to the field equations by inserting a Lagrange multiplier term to the Lagrangian. One can generalize the theory to d-dimensions writing the following action

$$I = \int_M \left\{ \mathscr{L}_{EH} + \lambda * 1 + \widehat{\mathscr{L}_M} + T^a \wedge \lambda_a \right\} \tag{3.1}$$

where the Einstein-Hilbert Lagrangian density is defined by

$$\mathscr{L}_{EH} = -\frac{1}{2\kappa^2} \hat{R}_{ab} \wedge *(e^a \wedge e^b), \tag{3.2}$$

with κ is the gravitational coupling constant such that $\kappa^2 = \frac{8\pi G}{c^4} = 8\pi \ell_p$, $\ell_p \simeq 10^{-35} m$ is the Planck length, G is the Newton's gravitational constant. We will take $c = 1$ in this research and $\hat{}$ describe the tensors related with the Levi-Civita connection. In

(**3.1**) λ is the cosmological constant, λ_a are the Lagrange multiplier (d-2)-forms, the Lagrangian $\widehat{\mathscr{L}_M} = \mathscr{L}_M(e, \hat{\omega}, \text{matter fields})$ can be composed of metric, connection and other fields.

The field equations are obtained by making independent variations of the action with respect to the co-frame $\{e^a\}$, the connection $\{\widehat{\omega}^a{}_b\}$ and the other gauge potentials. We write the infinitesimal variations of the Lagrangian as (up to a closed form)

$$\begin{aligned} \dot{\mathscr{L}} &= \dot{e}^a \wedge \left(-\frac{1}{2\kappa^2} \hat{R}^{bc} \wedge *e_{abc} + \lambda * e_a + \hat{\tau}_a + \hat{D}\lambda_a \right) \\ &\quad + \dot{\hat{\omega}}^{ab} \wedge \left(e_{[b} \wedge \lambda_{a]} + \hat{\Sigma}_{ab} \right) + T^a \wedge \dot{\lambda}_a \end{aligned} \quad (\mathbf{3.3})$$

where the symbol $[ab]$ means that the indices a, b are antisymmetric and the variations of the matter Lagrangian yield the stress-energy (d-1)-forms

$$\hat{\tau}_a = \frac{\partial \hat{\mathscr{L}}_M}{\partial e^a} = T_{ab} * e^b \quad (\mathbf{3.4})$$

and the angular momentum (d-1)-forms

$$\hat{\Sigma}_{ab} = \frac{\partial \hat{\mathscr{L}}_M}{\partial \hat{\omega}^{ab}} = S_{ab,c} * e^c. \quad (\mathbf{3.5})$$

Therefore, in Einstein theory of gravity with matter fields, the field equations are given as

$$-\frac{1}{2\kappa^2} \hat{R}^{bc} \wedge *e_{abc} + \lambda * e_a = -\hat{\tau}_a - \hat{D}\lambda_a, \quad (\mathbf{3.6})$$

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = 2\hat{\Sigma}_{ab}. \quad (\mathbf{3.7})$$

The second equation (**3.7**), can be solved for λ_a via the interior product for any d-dimensions interestingly and the result is

$$\lambda^a = 2\iota_b \hat{\Sigma}^{ba} + \frac{1}{2} e^a \wedge \iota_{bc} \hat{\Sigma}^{cb} \quad (\mathbf{3.8})$$

If we substitute this λ_a into (**3.6**), we find the Einstein field equations:

$$-\frac{1}{2\kappa^2} \hat{R}^{bc} \wedge *e_{abc} + \lambda * e_a = -\hat{\tau}_a - 2\hat{D}\iota_a \hat{\Sigma}^{ac} - \frac{1}{2} e^c \wedge \hat{D}\iota_{ba} \hat{\Sigma}^{ab}. \quad (\mathbf{3.9})$$

3.2 Einstein-Cartan Gravitation Theory

Einstein-Cartan theory gravity is a generalization of Einstein gravitation theory. In this theory, the full connection has a torsion part and the torsion is considered independent of the co-frame. Since the torsion is introduced into the theory, the space-time is Riemann-Cartan. The field equations of Einstein-Cartan theory of gravity [10] are obtained by varying the action in d-dimensions

$$I = \int_M \{\mathscr{L}_{EC} + \lambda * 1 + \mathscr{L}_M\} \qquad \textbf{(3.10)}$$

where the Einstein-Cartan Lagrangian density

$$\mathscr{L}_{EC} = -\frac{1}{2\kappa^2} R_{ab} \wedge *(e^a \wedge e^b). \qquad \textbf{(3.11)}$$

Here the gravitational constant κ and $\mathscr{L}_M(e, \omega, \text{matter fields})$ is the Lagrangian density related with the other fields. The curvature 2-forms are decomposed as follows:

$$R^a{}_b = \hat{R}^a{}_b + \hat{D}K^a{}_b + K^a{}_c \wedge K^c{}_b \qquad \textbf{(3.12)}$$

where

$$\hat{D}K^a{}_b = dK^a{}_b + \hat{\omega}^a{}_c \wedge K^c{}_b - \hat{\omega}^c{}_b \wedge K^a{}_c.$$

Similarly to the Einstein theory of gravity, we write the infinitesimal variations as (up to a closed form)

$$\begin{aligned} \dot{\mathscr{L}} &= \dot{e}^a \wedge \left(-\frac{1}{2\kappa^2} R^{bc} \wedge *e_{abc} + \lambda * e_a + \tau_a \right) \\ &\quad + \dot{\omega}^{ab} \wedge \left(-\frac{1}{\kappa^2} * e_{abc} \wedge T^c + \Sigma_{ab} \right) \end{aligned} \qquad \textbf{(3.13)}$$

where the co-frame and connection variations of the matter Lagrangian yield the stress-energy

$$\tau_a = \frac{\partial \mathscr{L}_{\mathscr{M}}}{\partial e^a} = T_{ab} * e^b \qquad \textbf{(3.14)}$$

and the angular momentum

$$\Sigma_{ab} = \frac{\partial \mathscr{L}_{\mathscr{M}}}{\partial \omega^{ab}} = S_{ab,c} * e^c \qquad \textbf{(3.15)}$$

respectively. Therefore, the Einstein-Cartan field equations are given as

$$-\frac{1}{2\kappa^2}R^{bc}\wedge *e_{abc}+\lambda * e_a=-\tau_a, \tag{3.16}$$

$$\frac{1}{2\kappa^2}T^{c}\wedge *e_{abc}=\Sigma_{ab}. \tag{3.17}$$

We note that while the field equations of Einstein-Cartan gravity is written in terms of the full connection ω, the field equations of Einstein gravity is written in terms of $\hat{\omega}$ Levi-Civita connection.

4. THE COUPLINGS OF ELECTROMAGNETIC FIELDS TO GRAVITY

We can test a majority of the results of general relativity via photons coming from distant stars and galaxies. In order to verify the insight of a gravitation theory exactly, it should couple to electromagnetic fields. The Einstein-Maxwell theory is a minimally coupled theory between the electromagnetic fields and gravitation and this theory is described by the action;

$$S = \int \{-\frac{1}{2\kappa^2}\hat{R}_{ab} \wedge *(e^a \wedge e^b) + \frac{1}{2}F \wedge *F\} \tag{4.1}$$

where electromagnetic coupling constant q is absorbed into the electromagnetic field F. In this minimal theory, the spacetime geometry is modified by the electromagnetic fields. The spherically symmetric and static solution of this theory is known as Reissner-Nordström solution. Some gravitational wave solutions of the Einstein-Maxwell theory were given in [11], [12] and [14].

To extend this theory as non-minimal, the coupling terms including curvature and Maxwell tensor in the same term are inserted into the Lagrangian of Einstein-Maxwell theory. The coupling terms were first considered by Prasanna [15] . They were soon extended and classified by Horndeski [16] to gain more insight into the relationship between spacetime curvature and electric charge conservation. It is remarkable that a calculation in QED of the photon effective action from 1-loop vacuum polarization on a curved background [17] contributed similar nonminimal coupling terms. It was contemplated at about the same times that Kaluza-Klein reduction of a five-dimensional R^2-Lagrangian would induce similar non-minimal couplings in four dimensions [18]. A variation of an arbitrary Lagrangian with non-minimally coupled gravitational and electromagnetic fields in general may involve field equations of order higher than two. The nonminimal couplings in four dimensions classified by Horndeski are exactly those that involve at most second order terms. These particular combinations are obtained by reduction of the Euler-Poincaré Lagrangian in five dimensions to four dimensions [19], [20].

Recently, in more detail, such a 3-parameter nonminimally coupled Einstein-Maxwell theory was applied to the spherical symmetric models in [21], [22] and the cosmological models in [23]. Later, Balakin *et al.* have also extended the nonminimal theory to presence of axion fields, which is the non-minimal 10-parameter Einstein-Maxwell-axion model [24]. They considered the model with pp wave metric in the Bondi *et al.* form. They have shown that the non-minimal coupling of the photon and axions to gravitational field generally may lead to the birefringence effect and optical activity.

In this chapter, we formulate a general 6-parameter nonminimal extended Einstein-Maxwell theory and Einstein-Cartan-Maxwell theory that are linear in the curvature and quadratic in the electromagnetic field; using the algebra of exterior differential forms without torsion and with torsion. We derive the field equations of the model according to the first order variation method and we look for plane-fronted wave solutions in Ehlers-Kundt form and static, spherically symmetric solutions. Consequently, although the structure of Maxwell field equations is modified by the coupling terms, the modifying part vanishes and the Maxwell equations are left the same as vacuum for the pp-wave metric solutions. But, Einstein and Einstein-Cartan field equations allow a class of nontrivial solutions. Additionally, the energy-momentum transported by the pp waves is modified by the nonminimal coupling terms. We have shown the difference between Einstein-Maxwell and Einstein-Cartan-Maxwell theory for pp-wave and static spherically symmetric metric.

4.1 Non-minimally Coupled Einstein-Maxwell Theory

Non-minimally coupled Lagrangian density $\hat{\mathscr{L}}_{NM} = \mathscr{L}_{NM}(A, e, \hat{\omega})$ can include couplings of curvature and Maxwell tensor such as $R^n F^m$ in any invariant order (n,m=1,2,.. are not indices, they describe the order of a tensor). In this study, we will use a first order formalism. We will use the electromagnetic field 2-forms F for which the homogeneous field equation $dF = 0$ is imposed by the variation of the Lagrange multiplier 2-form μ. We will start with the following action with constraint;

$$I = \int_M \left\{ \frac{1}{2\kappa^2} \hat{R}^{ab} \wedge {}^*e_{ab} + \lambda * 1 - \frac{1}{2} F \wedge *F + \hat{\mathscr{L}}_{NM} + T^a \wedge \lambda_a + \mu \wedge dF \right\} \qquad \mathbf{(4.2)}$$

which has Einstein-Hilbert Lagrangian density, cosmological constant λ, Maxwell Lagrangian density, non-minimally coupled Lagrangian density and constraint terms respectively. Here κ is the gravitational constant, λ_a and μ are Lagrange multiplier 2-forms. That is, we will take $\{e^a\}$ and $\{\hat{\omega}^a{}_b\}$ to be the fundamental field variables, F is the electromagnetic field 2-form. We write the infinitesimal variations of the Lagrangian as (up to a closed form)

$$\begin{aligned}\dot{\mathscr{L}} &= \dot{e}^a \wedge \left(\frac{1}{2\kappa^2}\hat{R}^{bc} \wedge *e_{abc} + \lambda * e_a + \hat{\tau}_a + \hat{D}\lambda_a \right) + \dot{\hat{\omega}}^{ab} \wedge \left(e_{[a} \wedge \lambda_{b]} + \hat{\Sigma}_{ab} \right) \\ &\quad + \dot{A} \wedge (-d * F + \frac{\partial \hat{\mathscr{L}}_{NM}}{\partial A}) + \dot{\lambda}_a \wedge T^a + \dot{\mu} \wedge dF \end{aligned} \quad (\mathbf{4.3})$$

where the symbol $[ab]$ means that the indices a, b are antisymmetric. We can write the stress-energy 3-forms $\hat{\tau}_a$ related with the Levi-Civita connection from the above variation as

$$\hat{\tau}_a = {}^{Max}\tau_a + {}^{NM}\hat{\tau}_a, \quad (\mathbf{4.4})$$

where the Maxwell stress-energy tensor and the non-minimally coupled stress-energy tensor are

$${}^{Max}\tau_a = \frac{1}{2}(\iota_a F \wedge *F - F \wedge \iota_a * F) \quad (\mathbf{4.5})$$

$${}^{NM}\hat{\tau}_a = \frac{\partial \hat{\mathscr{L}}_{NM}}{\partial e^a}. \quad (\mathbf{4.6})$$

The angular momentum 3-forms are found from the above variation (**4.3**) as

$$\hat{\Sigma}_{ab} = \frac{\partial \hat{\mathscr{L}}_{NM}}{\partial \hat{\omega}^{ab}} = \hat{S}_{ab,c} * e^c. \quad (\mathbf{4.7})$$

After solving the λ_a's as (**3**.**8**), the Einstein field equations and the Maxwell equations turn out to be

$$-\frac{1}{2\kappa^2}\hat{R}^{bc} \wedge *e_{abc} - \lambda * e_a = -\hat{\tau}_a - 2\hat{D}\iota^b\hat{\Sigma}_{ba} - \frac{1}{2}e_a \wedge \hat{D}\iota_{bc}\hat{\Sigma}^{cb}. \quad (\mathbf{4.8})$$

$$dF = 0, \qquad -d * F + \frac{\partial \hat{\mathscr{L}}_{NM}}{\partial A} = 0 \quad (\mathbf{4.9})$$

with $T^a = 0$.

In this section, we will consider only the following Lagrangian density as non-minimally coupled electromagnetic fields to gravity:

$$\begin{aligned}\hat{\mathscr{L}}_{NM} = &= \frac{c_1}{2}\hat{R}_{ab}F^{ab} \wedge *F + \frac{c_2}{2}\iota^a F \wedge \hat{R}_a \wedge *F + \frac{c_3}{2}\hat{R}F \wedge *F \\ &\quad + \frac{c_4}{2}\hat{R}_{ab}F^{ab} \wedge F + \frac{c_5}{2}\iota^a F \wedge \hat{R}_a \wedge F + \frac{c_6}{2}\hat{R}F \wedge F \end{aligned} \quad (\mathbf{4.10})$$

where $c_i's$ are phenomenological coupling constants and we assume that the cosmological constant is zero. Through the research we will show the interior products of the electromagnetic tensor 2-form $F = \frac{1}{2}F_{ab}e^{ab}$ and the curvature 2-forms $R_{ab} = \frac{1}{2}R_{ab,cd}e^{cd}$ with the co-frame e^a as

$$\iota_a F = F_{ab}e^b = F_a \qquad 1-form, \tag{4.11}$$

$$\iota_{ba}F = F_{ab} \qquad 0-form, \tag{4.12}$$

$$\iota_a R^{ab} = R^{ab}{}_{,ad}e^d = R^b \qquad Ricci\ 1-form, \tag{4.13}$$

$$\iota_{ba}R^{ab} = R^{ab}{}_{,ab} = R \qquad curvature\ scalar,\ 0-form. \tag{4.14}$$

The first term in the (**4.10**) has been considered firstly by Prasanna [15]. For the six non-minimally coupled terms the stress-energy tensors ${}^i\tau^c$ can be found as:

$$\begin{aligned} {}^1\tau^c &= -\frac{1}{4}(4F^{ac}\iota^b F \wedge *\hat{R}_{ab} + \iota^c F \wedge *\hat{R}_{ab}F^{ab} - \hat{R}_{ab}F^{ab} \wedge \iota^c * F \\ &\quad + \iota^c \hat{R}_{ab}F^{ab} \wedge *F - F \wedge \iota^c * \hat{R}_{ab}F^{ab}) \end{aligned} \tag{4.15}$$

$$\begin{aligned} {}^2\tau^c &= \frac{1}{4}[2\hat{R}F^c \wedge *F - 2F^c \wedge \hat{R}_a \wedge \iota^a * F + 2F_{ab}\iota^c \hat{R}^{ab} \wedge *F \\ &\quad + 2\iota^c \hat{R}^{ba} \wedge F_a \wedge \iota_b * F + F^{ac}\hat{R}_a \wedge *F - \iota^c \hat{R}_a \iota^a F \wedge *F \\ &\quad - F^c \wedge *(F^a \wedge \hat{R}_a) + F^a \wedge \hat{R}_a \wedge \iota^c * F + F \wedge \iota^c * (F^a \wedge \hat{R}_a)] \end{aligned} \tag{4.16}$$

$$\begin{aligned} {}^3\tau^c &= -\frac{1}{2}[2\iota^c \hat{R}^b \iota_b F \wedge *F + 2\iota^c \hat{R}^b F \wedge \iota_b * F + \iota^c F \wedge *\hat{R}F \\ &\quad - \hat{R}F \wedge \iota^c * F] \end{aligned} \tag{4.17}$$

$${}^4\tau^c = c_4[F^{ac}\hat{R}_a \wedge F - F^{ac}\hat{R}_{ab} \wedge F^b] \tag{4.18}$$

$$\begin{aligned} {}^5\tau^c &= \frac{c_5}{2}[-F^{ca}\hat{R}^a \wedge F + F^c \hat{R} \wedge F - F^c \wedge \hat{R}^a \wedge F_a - F_{ab}\iota^c \hat{R}^{ba} \wedge F \\ &\quad - F_a \iota^c \hat{R}^a \wedge F - F_a \wedge \iota^c \hat{R}^{ba} \wedge F_b] \end{aligned} \tag{4.19}$$

$${}^6\tau^c = -2c_6(\iota^c \hat{R}_b)F^b \wedge F \tag{4.20}$$

and the angular momentum tensor $\hat{\Sigma}^{ab}$:

$$\begin{aligned} \hat{\Sigma}^{ab} &= \frac{c_1 - c_2 + c_3}{2}\hat{D}(F^{ab} * F) + \frac{2c_3 - c_2}{4}\hat{D}(F^b \wedge \iota^a * F - F^a \wedge \iota^b * F) \\ &\quad - \frac{c_3}{2}\hat{D}(F \wedge \iota^{ab} * F) + \frac{c_4 - c_5 + 2c_6}{2}\hat{D}F^{ab}F + \frac{c_5 - 2c_6}{2}\hat{D}F^a \wedge F^b \end{aligned} \tag{4.21}$$

Additionally, the Maxwell field equations read

$$dF = 0 \tag{4.22}$$

$$d\{-*F + c_1 F^{ab} * \hat{R}_{ab} + \frac{c_2}{2}[\hat{R}_a \wedge \iota^a * F - \hat{R} * F + *(F^a \wedge \hat{R}_a)]$$
$$+c_3 \hat{R} * F + \frac{c_4}{2}[2F_a \wedge \hat{R}^a + 2F_{ab}\hat{R}^{ab}] + \frac{c_4 - c_5 + 2c_6}{2} F\hat{R}\} = 0 \quad (\mathbf{4.23})$$

Trace of the co-frame equation (**4.8**) is

$$\frac{1}{\kappa^2}\hat{R} * 1 - \lambda * 1 - c_1 F \wedge F^{ab} * \hat{R}_{ab} - c_2 F^a \wedge \hat{R}_a \wedge *F - c_3 \hat{R} F \wedge *F$$
$$-c_4 F_{ab}\hat{R}^{ab} \wedge F + c_5 \hat{R} F \wedge F - 2c_6 \hat{R}_a \wedge F^a \wedge F + e_a \wedge \hat{D}\lambda^a = 0. \quad (\mathbf{4.24})$$

4.2 Electromagnetic Constitutive Equations

In general, one encodes the effects of non-minimal couplings of electromagnetic fields to gravity into the definition of a constitutive tensor. Maxwell's equations for an electromagnetic field F in an arbitrary medium can be written as,

$$dF = 0 \quad , \quad *d*G = J \quad (\mathbf{4.25})$$

where G is called the excitation 2-form and J is the source electric current density 1-form. The effects of gravitation and electromagnetism on matter are described by G and J. To close this system we need electromagnetic constitutive relations relating G and J to F. Here we consider only the source-free interactions, so that $J = 0$. Then we take a simple linear constitutive relation

$$G = \mathscr{Z}(F) \quad (\mathbf{4.26})$$

where $\mathscr{Z}$ is a type-(2,2) constitutive tensor. For the above theory we have

$$G = F - c_1 R_{ab} F^{ab} - c_2 \iota^a F \wedge R_a - c_3 RF - c_4 R_{ab} F^{ab}$$
$$-c_5 \iota^a F \wedge R_a - c_6 RF. \quad (\mathbf{4.27})$$

With these definitions, the non-minimal Einstein-Maxwell Lagrangian simply becomes

$$L = \frac{1}{2\kappa^2} R * 1 + \lambda * 1 - \frac{1}{2} F \wedge *G + \lambda_a \wedge T^a. \quad (\mathbf{4.28})$$

The electric field e and magnetic induction field b associated with F are defined with respect to an arbitrary unit, future-pointing time-like 4-velocity vector field U ("inertial observer") by

$$\mathbf{e} = \iota_U F \quad , \quad \mathbf{b} = \iota_U * F. \quad (\mathbf{4.29})$$

Since g(U,U)=-1 we have

$$F = \mathbf{e} \wedge \widetilde{U} - *(\mathbf{b} \wedge \widetilde{U}) \quad \textbf{(4.30)}$$

where $\widetilde{U} \in T^*M$. Likewise, the electric displacement field **d** and the magnetic field **h** associated with G are defined with respect to U as

$$\mathbf{d} = \iota_U G \qquad , \qquad \mathbf{h} = \iota_U * G. \quad \textbf{(4.31)}$$

Thus

$$G = \mathbf{d} \wedge \widetilde{U} - *(\mathbf{h} \wedge \widetilde{U}). \quad \textbf{(4.32)}$$

It is sometimes convenient to work in terms of polarization 1-form $\mathbf{p} = \mathbf{d} - \mathbf{e}$ and magnetization $\mathbf{m} = \mathbf{b} - \mathbf{h}$. More details about this concepts can be found in [25, 26].

4.3 Non-minimally Coupled Einstein-Cartan-Maxwell Theory

The field equations of Einstein-Cartan theory considering non-minimally coupled electromagnetic fields with gravity are obtained by varying the action without any constraints on torsion;

$$I = \int_M \left\{ \frac{1}{2\kappa^2} R^{ab} \wedge {}^*e_{ab} + \lambda * 1 - \frac{1}{2} F \wedge *F + \mathscr{L}_{NM} + dF \wedge \mu \right\} \quad \textbf{(4.33)}$$

where the first term is the Einstein-Cartan Lagrangian density and the non-minimally coupled Lagrangian density $\mathscr{L}_{NM}(A, e, \omega)$ now can include torsion more generally from the previous theory. In general, $\mathscr{L}_{NM}(A, e, K)$ can include couplings of curvature, electromagnetic and torsion tensors such as $R^m F^n T^l$ in any non-minimal invariant order $(n, m, l = 0, 1, 2, 3..)$. At the lowest order one can consider the direct coupling $R^a \wedge *F_a$, which is zero in the absence of torsion because of Bianchi identity, which may give interesting insights in the presence of torsion. Recently, the effects of some non-minimally couplings such as $TF\partial F$ on the Maxwell equations have been investigated in [27]. We consider the couplings in the form RF^2 again.

Similar to the Einstein gravitation theory, here we write the infinitesimal variations as (up to a closed form)

$$\begin{aligned} \dot{\mathscr{L}} &= \dot{e}^a \wedge \left(\frac{1}{2\kappa^2} R^{bc} \wedge *e_{abc} + \lambda * e_a + \tau_a \right) + \dot{\omega}^{ab} \wedge \left(\frac{1}{2\kappa^2} * e_{abc} \wedge T^c + \Sigma_{ab} \right) \\ &\quad + \dot{A} \wedge \left(-d * F + \frac{\partial \mathscr{L}_{NM}}{\partial A}\right) \end{aligned} \quad \textbf{(4.34)}$$

Now we can write τ_a the stress-energy 3-forms as

$$\tau_a = {}^{Max}\tau_a + {}^{NM}\tau_a \tag{4.35}$$

where the non-minimally coupled stress-energy tensors and the angular momentum 3-forms are

$$^{NM}\tau_a = \frac{\partial \mathscr{L}_{NM}}{\partial e^a} \tag{4.36}$$

$$\Sigma_{ab} = \frac{\partial \mathscr{L}_{NM}}{\partial \omega^{ab}} = S_{ab,c} * e^c. \tag{4.37}$$

The Einstein-Cartan field equations and the Maxwell equations of the model above are given by

$$-\frac{1}{2\kappa^2} R^{bc} \wedge *e_{abc} - \lambda * e_a = \tau_a, \tag{4.38}$$

$$\frac{1}{2\kappa^2} * e_{abc} \wedge T^c = -\Sigma_{ab}. \tag{4.39}$$

$$dF = 0, \qquad -d * F + \frac{\partial \mathscr{L}_{NM}}{\partial A} = 0. \tag{4.40}$$

For the non-minimally coupled terms (**4.10**), the stress energy tensors ${}^i\tau^c$ can be found from (**4.36**)

$$\begin{aligned} {}^1\tau^c &= -\frac{1}{4}(4F^{ac}\iota^b F \wedge *R_{ab} + \iota^c F \wedge *R_{ab}F^{ab} - R_{ab}F^{ab} \wedge \iota^c * F \\ &\quad + \iota^c R_{ab}F^{ab} \wedge *F - F \wedge \iota^c * R_{ab}F^{ab}) \end{aligned} \tag{4.41}$$

$$\begin{aligned} {}^2\tau^c &= \frac{1}{4}[2RF^c \wedge *F - 2F^c \wedge R_a \wedge \iota^a * F + 2F_{ab}\iota^c R^{ab} \wedge *F \\ &\quad + 2\iota^c R^{ba} \wedge F_a \wedge \iota_b * F + F^{ac}R_a \wedge *F - \iota^c R_a \iota^a F \wedge *F \\ &\quad - F^c \wedge *(F^a \wedge R_a) + F^a \wedge R_a \wedge \iota^c * F + F \wedge \iota^c * (F^a \wedge R_a)] \end{aligned} \tag{4.42}$$

$$\begin{aligned} {}^3\tau^c &= -\frac{1}{2}[2\iota^c R^b \iota_b F \wedge *F + 2\iota^c R^b F \wedge \iota_b * F + \iota^c F \wedge *RF \\ &\quad - RF \wedge \iota^c * F] \end{aligned} \tag{4.43}$$

$${}^4\tau^c = c_4[F^{ac}R_a \wedge F - F^{ac}R_{ab} \wedge F^b] \tag{4.44}$$

$$\begin{aligned} {}^5\tau^c &= \frac{c_5}{2}[-F^{ca}R^a \wedge F + F^c R \wedge F - F^c \wedge R^a \wedge F_a - F_{ab}\iota^c R^{ba} \wedge F \\ &\quad - F_a \iota^c R^a \wedge F - F_a \wedge \iota^c R^{ba} \wedge F_b] \end{aligned} \tag{4.45}$$

$${}^6\tau^c = -2c_6(\iota^c R_b)F^b \wedge F \tag{4.46}$$

and the angular momentum tensor Σ^{ab} (**4.37**) becomes

$$\begin{aligned}\Sigma^{ab} &= \frac{c_1-c_2+c_3}{2}D(F^{ab}*F)+\frac{2c_3-c_2}{4}D(F^b\wedge\iota^a*F-F^a\wedge\iota^b*F)\\ &-\frac{c_3}{2}D(F\wedge\iota^{ab}*F)+\frac{c_4-c_5+2c_6}{2}D(F^{ab}F)+\frac{c_5-2c_6}{2}D(F^a\wedge F^b)\end{aligned}$$

We can write (**4.39**) in another form in terms of contortion as;

$$\frac{1}{\kappa^2}K^c{}_m\wedge e^m\wedge *e_{abc}+\hat{D}\hat{\Gamma}_{ab}-K^c{}_m\wedge\hat{\Gamma}_{cb}-K^c{}_n\wedge\hat{\Gamma}_{an}=0 \qquad (\mathbf{4.47})$$

where we have used that

$$\Sigma^{ab} = 2D\Gamma^{ab}=2D\hat{\Gamma}_{ab} \qquad (\mathbf{4.48})$$

and

$$\begin{aligned}\Gamma^{ab} &= (c_1-c_2+c_3)(F^{ab}*F)+\frac{2c_3-c_2}{2}(F^b\wedge\iota^a*F-F^a\wedge\iota^b*F)\\ &-c_3(F\wedge\iota^{ab}*F)+(c_4-c_5+2c_6)(F^{ab}F)+(c_5-2c_6)(F^a\wedge F^b).\end{aligned}$$

It is very complicated to solve the above expression algebraically in terms of $\hat{\Gamma}_{ab}$. But, for a given $\hat{\Gamma}_{ab}$, we have twenty four unknowns which are the components of K_{ab} 1-forms and twenty four differential equations. Firstly, we have to find the connections K_{ab} satisfying the equation. After, we have to replace the previous Levi-Civita connection $\hat{\omega}$ with $\omega_{ab}=\hat{\omega}_{ab}+K_{ab}$ because of the non-zero contortion K_{ab} . The Einstein-Cartan-Maxwell field equations can be obtained from the co-frame variation of (**4.33**) for the non-minimally coupled electromagnetic fields to gravity as

$$\frac{G^a}{\kappa^2}-\lambda*e^a=\tau^a+\sum_i c_i\,{}^i\tau^a \qquad (\mathbf{4.49})$$

$$dF=0 \qquad (\mathbf{4.50})$$

$$\begin{aligned}&d\{-*F+c_1F^{ab}*R_{ab}+\frac{c_2}{2}[R_a\wedge\iota^a*F-R*F+*(F^a\wedge R_a)]\\ &+c_3R*F+\frac{c_4}{2}2F_a\wedge R^a+2F_{ab}R^{ab}+\frac{c_4-c_5+2c_6}{2}FR\}=0 \qquad (\mathbf{4.51})\end{aligned}$$

4.4 Conformally Extended, Nonminimally Coupled Einstein-Maxwell Theory

Let's consider two manifolds M and M' with co-frames e^a and e'^a. If we can find the following transformations;

$$e^a \to e^{a'} = e^{\sigma} e^a, \quad , \quad \phi \to \phi' = e^{-\sigma}\phi, \tag{4.52}$$

we can say these two manifolds are conformal to each other and these are conformal transformations. Here $\sigma(x)$ is a conformal factor. Specially, $\sigma = constant$ corresponds to a scale transformation which is a global transformation. In order to extend the nonminimally coupled Einstein-Maxwell theory conformally, we write the following Lagrangian,

$$\begin{aligned} L = \quad & \frac{\phi}{2}\hat{R} * 1 - \frac{\omega}{2\phi} d\phi \wedge * d\phi - \frac{1}{2} F \wedge * F + \frac{1}{\phi}\mathscr{L}_{\mathscr{N}\mathscr{M}} \\ & + T^a \wedge \lambda_a + \mu \wedge dF \end{aligned} \tag{4.53}$$

where we consider the six nonminimally coupled terms in (**4.10**) as $\mathscr{L}_{\mathscr{N}\mathscr{M}}$. A conformally invariant non-minimally coupled Einstein-Maxwell theory is achieved (for the case $\omega = -\frac{3}{2}$) by considering

$$\begin{aligned} L = \quad & \frac{\phi}{2} R_{ab} \wedge * e^{ab} - \frac{\omega}{2\phi} d\phi \wedge * d\phi - \frac{1}{2} F \wedge * F + \frac{\gamma}{2\phi} C_{ab} \wedge F^{ab} * F \\ & + T^a \wedge \lambda_a + \mu \wedge dF \end{aligned} \tag{4.54}$$

where ϕ is the dilaton field and

$$C_{ab} = R_{ab} - \frac{1}{2}(e_a \wedge \mathscr{R}_b - e_b \wedge \mathscr{R}_a) + \frac{1}{6}\mathscr{R} e_{ab} \tag{4.55}$$

are the Weyl conformal curvature 2-forms $(c_1 = c_2 = \gamma,\ c_3 = \frac{\gamma}{3},\ c_4 = c_5 = c_6 = 0)$.

Thus, the field equations for (**4.53**) are written as[1]

$$\phi G^a = \tau^a[d\phi] + \tau^a[F] + \sum_i c_i\, {}^i\tau^a[F, \hat{R}] + \hat{D}\lambda^a \tag{4.56}$$

$$dF = 0 \tag{4.57}$$

$$\begin{aligned} d[- * F + \frac{c_1}{\phi} F^{ab} * \hat{R}_{ab} + \frac{c_2}{2\phi}[\hat{R}_a \wedge \iota^a * F - \hat{R} * F + *(\iota^a F \wedge \hat{R}_a)] \\ + \frac{c_3}{\phi}\hat{R} * F + \frac{c_4}{2\phi}[2F_a \wedge \hat{R}^a + 2F_{ab}\hat{R}^{ab}] + \frac{c_4 - c_5 + 2c_6}{2\phi} F\hat{R}] = 0 \end{aligned} \tag{4.58}$$

[1] here $i = 1,2,3,4,5,6$

where

$$\hat{G}^c = -\frac{1}{2}\hat{R}_{ab}\wedge *e^{abc} \quad \textbf{(4.59)}$$

$$\tau^c[F] = \frac{1}{2}(\iota^c F\wedge *F - F\wedge \iota^c * F) \quad \textbf{(4.60)}$$

$$\tau^c[\phi] = \frac{\omega}{2\phi}(\iota^c d\phi\wedge *d\phi + d\phi\wedge \iota^c * d\phi) \quad \textbf{(4.61)}$$

$$\begin{aligned} {}^1\tau^c[F,\hat{R}] &= -\frac{1}{4\phi}(4F^{ac}\iota^b F\wedge *\hat{R}_{ab} + \iota^c F\wedge *\hat{R}_{ab}F^{ab} - \hat{R}_{ab}F^{ab}\wedge \iota^c * F \\ &\quad + \iota^c \hat{R}_{ab}F^{ab}\wedge *F - F\wedge \iota^c * \hat{R}_{ab}F^{ab}) \quad \textbf{(4.62)} \\ {}^2\tau_c[F,\hat{R}] &= \frac{1}{4\phi}[2\hat{R}F_c\wedge *F - 2F_c\wedge \hat{R}_a\wedge \iota^a * F + 2F_{ab}\iota_c\hat{R}^{ab}\wedge *F \\ &\quad + 2\iota_c\hat{R}^{ba}\wedge F_a\wedge \iota_b * F + F_{ac}\hat{R}^a\wedge *F - \iota_c\hat{R}_a\iota^a F\wedge *F \\ &\quad - F_c\wedge *(F^a\wedge \hat{R}_a) + F^a\wedge \hat{R}_a\wedge \iota_c * F + F\wedge \iota_c * (F^a\wedge \hat{R}_a)] \quad \textbf{(4.63)} \\ {}^3\tau_c[F,\hat{R}] &= -\frac{1}{2\phi}[2\iota_c\hat{R}^b\iota_b F\wedge *F + 2\iota_c\hat{R}^b F\wedge \iota_b * F + \iota_c F\wedge *\hat{R}F \\ &\quad - \hat{R}F\wedge \iota_c * F] \quad \textbf{(4.64)} \end{aligned}$$

$$\begin{aligned} {}^4\tau_c[F,\hat{R}] &= \frac{c_4}{\phi}[F_{ac}\hat{R}^a\wedge F - F_{ac}\hat{R}^{ab}\wedge F_b] \\ {}^5\tau_c[F,\hat{R}] &= \frac{c_5}{2\phi}[-F_{ca}\hat{R}^a\wedge F + F_c\hat{R}\wedge F - F_c\wedge \hat{R}^a\wedge F_a - F_{ab}\iota_c\hat{R}^{ba}\wedge F \\ &\quad - F_a\iota_c\hat{R}^a\wedge F - F_a\wedge \iota_c\hat{R}^{ba}\wedge F_b] \quad \textbf{(4.65)} \\ {}^6\tau_c[F,\hat{R}] &= -\frac{2c_6}{\phi}(\iota_c\hat{R}_b)F^b\wedge F \quad \textbf{(4.66)} \\ \hat{\Sigma}^{ac} &= \frac{c_1 - c_2 + c_3}{\phi}F^{ac} * F + \frac{2c_3 - c_2}{2\phi}(F^c\wedge \iota^a * F - F^a\wedge \iota^c * F) - \frac{c_3}{\phi}F\wedge \iota^{ac} * F \\ &\quad + \frac{(c_4 - c_5 + 2c_6)}{\phi}F^{ac}F + \frac{(c_5 - 2c_6)}{\phi}F^a\wedge F^c] + \phi * e^a\wedge e^c \quad \textbf{(4.67)} \end{aligned}$$

we have also the scalar field ϕ equation;

$$\begin{aligned} &\frac{1}{2}\hat{R} * 1 + \frac{\omega}{2\phi^2}d\phi\wedge *d\phi + \omega d * \frac{d\phi}{\phi} - \frac{1}{2\phi^2}[c_1 F\wedge F^{ab} * \hat{R}_{ab} + c_2 F^a\wedge \hat{R}_a\wedge *F \\ &+ c_3\hat{R}F\wedge *F + c_4 F_{ab}\hat{R}^{ab}\wedge F + c_5\iota^a F\wedge \iota^b\hat{R}_{ba}F + c_6\hat{R}F\wedge F] = 0 \quad \textbf{(4.68)} \end{aligned}$$

producting with 2ϕ

$$\begin{aligned} &\phi\hat{R} * 1 + \frac{\omega}{\phi}d\phi\wedge *d\phi + 2\omega\phi d * \frac{d\phi}{\phi} - \frac{1}{\phi}[c_1 F\wedge F^{ab} * \hat{R}_{ab} + c_2 F^a\wedge \hat{R}_a\wedge *F \\ &+ c_3\hat{R}F\wedge *F + c_4 F_{ab}\hat{R}^{ab}\wedge F + c_5\iota^a F\wedge \iota^b\hat{R}_{ba}F + c_6\hat{R}F\wedge F] = 0 \quad \textbf{(4.69)} \end{aligned}$$

Trace of the co-frame equation is,

$$\begin{aligned} &\phi\hat{R} * 1 - \frac{\omega}{\phi}d\phi\wedge *d\phi - \frac{1}{\phi}[c_1 F\wedge F^{ab} * \hat{R}_{ab} + c_2 F^a\wedge \hat{R}_a\wedge *F + c_3\hat{R}F\wedge *F \\ &+ c_4 F_{ab}\hat{R}^{ab}\wedge F - c_5\hat{R}F\wedge F + 2c_6\hat{R}_a\wedge F^a\wedge F] + e^c\wedge \hat{D}\lambda_c = 0 \quad \textbf{(4.70)} \end{aligned}$$

subtract the trace eq. from ϕ eq.

$$2\omega d*d\phi - \frac{1}{\phi}[(c_5+c_6)\hat{R}F\wedge F + (c_5+2c_6)F^a\wedge\hat{R}_a\wedge F] - e^c\wedge\hat{D}\lambda_c = 0 \quad \textbf{(4.71)}$$

Lagrange multiplier λ_a can be solved again;

$$\lambda^c = \imath_a\hat{D}\hat{\Sigma}^{ac} + \frac{1}{2}\imath_{ba}\hat{D}\hat{\Sigma}^{ab}\wedge e^c. \quad \textbf{(4.72)}$$

Lastly, the field equations of the conformally invariant non-minimally coupled Einstein-Maxwell Lagrangian (**4.54**) is found to be

$$\phi G^a = \tau^a[d\phi] + \tau^a[F] + \gamma({}^1\tau^a + {}^2\tau^a + \frac{1}{3}\,{}^3\tau^a) + \hat{D}[\imath_b\hat{D}\hat{\Sigma}^{ba} + \frac{1}{2}\imath_{bc}\hat{D}\hat{\Sigma}^{cb}\wedge e^a], \textbf{(4.73)}$$

$$\begin{aligned} & d[-*F + \frac{\gamma}{\phi}F^{ab}*\hat{R}_{ab} + \frac{\gamma}{2\phi}(\hat{R}_a\wedge\imath^a*F - \hat{R}*F + *(F^a\wedge\hat{R}_a)) \\ & + \frac{\gamma}{3\phi}\hat{R}*F] = 0, \end{aligned} \quad \textbf{(4.74)}$$

$$dF = 0, \quad \textbf{(4.75)}$$

$$2\omega d*d\phi - e_a\wedge\hat{D}[\imath_b\hat{D}\hat{\Sigma}^{ba} + \frac{1}{2}\imath_{bc}\hat{D}\hat{\Sigma}^{cb}\wedge e^a] = 0 \quad \textbf{(4.76)}$$

where $\hat{\Sigma}^{ab}$

$$\hat{\Sigma}^{ac} = \frac{\gamma}{3\phi}[F^{ac}*F + \frac{1}{2}(F^c\wedge\imath^a*F - F^a\wedge\imath^c*F) - F\wedge\imath^{ac}*F] + \phi*e^{ac}. \quad \textbf{(4.77)}$$

5. EXACT SOLUTIONS

5.1 Plane Fronted Wave Solutions

Gravitational waves describing the propagation of gravitational radiation predicted by Albert Einstein based on Einstein's general relativity. They are known as fluctuations of curvature of spacetime and they can be produced by binary star systems or black holes. The linearized gravitational wave solutions of general relativity is well known. However, the linearized solutions may cause inadequate information. So, we look for the exact solutions describing plane fronted waves with parallel rays (pp-waves). The following calculations of this section can be found partly in [28]. A generic pp-wave metric (in Ehlers-Kundt form) [11, 12] is given by,

$$g = 2dudv + dx^2 + dy^2 + 2H(u,x,y)du^2. \qquad \textbf{(5.1)}$$

H is the metric disturbance which is a smooth function to be determined[1]. According to the pp-wave metric the two surfaces *u and v* are constant or plane wave surfaces and the metric of the surfaces is $(dx^2 + dy^2)$. For the metric (**5.1**), a convenient choice of orthonormal co-frames is going to be used:

$$e^0 = \frac{H-1}{\sqrt{2}}du + dv, \quad e^1 = dx, \quad e^2 = dy, \quad e^3 = \frac{H+1}{\sqrt{2}}du + dv. \qquad \textbf{(5.2)}$$

We may also exploit the advantages of complex coordinates in transverse plane by letting

$$g = 2dudv + 2dzd\bar{z} + 2H(u,z,\bar{z})du^2 \qquad \textbf{(5.3)}$$

where

$$z = \frac{x+iy}{\sqrt{2}} \quad , \quad \bar{z} = \frac{x-iy}{\sqrt{2}}. \qquad \textbf{(5.4)}$$

We firstly determine the unique Levi-Civita connection using that torsion is zero

$$\hat{\omega}^{01} = -\hat{\omega}^{13} = \frac{H_x}{2}(e^3 - e^0), \quad \hat{\omega}^{02} = -\hat{\omega}^{23} = \frac{H_y}{2}(e^3 - e^0). \qquad \textbf{(5.5)}$$

[1] If the metric disturbance H is quadratic in (x,y), then one can find a transformation [29] from the metric (**6.36**) to the metric in Bondi *et al.* form.

We calculate the Einstein tensor 3-forms which are defined as $\hat{G}_a = -\frac{1}{2\kappa^2}\hat{R}^{bc}\wedge *e_{abc}$ for the connection

$$\hat{G}_0 = -\hat{G}_3 = \frac{H_{xx}+H_{yy}}{2\kappa^2} * (e^3 - e^0), \quad \hat{G}_1 = 0 = \hat{G}_2. \tag{5.6}$$

We consider an electromagnetic potential 1-form in direction du given as $A = a(u,x,y)du$ or $A = a(u,z,\bar{z})du$ for pp-waves. Then

$$\begin{aligned} F &= dA \\ &= a_x dx \wedge du + a_y dy \wedge du \\ &= a_z dz \wedge du + a_{\bar{z}} d\bar{z} \wedge du \end{aligned} \tag{5.7}$$

and the Maxwell stress-energy 3-forms turn out to be

$$^{Max}\tau_0 = -^{Max}\tau_3 = -\frac{{a_x}^2 + {a_y}^2}{2} * (e^3 - e^0) = -a_z a_{\bar{z}} * (e^3 - e^0) \tag{5.8}$$

$$^{Max}\tau_1 = {}^{Max}\tau_2 = 0. \tag{5.9}$$

After a lengthy calculation it is found that the non-minimal invariants give a nontrivial contribution to the non-minimally coupled Einstein-Maxwell theory only via $\hat{D}\lambda_a$

$$\hat{D}\lambda_0 = -\hat{D}\lambda_3 = \frac{c_2 - c_1}{2}\left(({a_x}^2)_{xx} + 2(a_x a_y)_{xy} + ({a_y}^2)_{yy}\right) * (e^3 - e^0), \tag{5.10}$$

$$\hat{D}\lambda_1 = 0 = \hat{D}\lambda_2, \tag{5.11}$$

The all other expressions are zero;

$$^{NM}\hat{\tau}_a = 0. \tag{5.12}$$

Now we put all these terms together and write the non-minimally coupled Einstein-Maxwell equations as

$$H_{xx} + H_{yy} = -\kappa^2({a_x}^2 + {a_y}^2) + \kappa^2(c_2 - c_1)\left(({a_x}^2)_{xx} + 2(a_x a_y)_{xy} + ({a_y}^2)_{yy}\right), \tag{5.13}$$

$$a_{xx} + a_{yy} = 0. \tag{5.14}$$

These equations can be written in an invariant form on the transverse xy-plane [12], [13]:

$$\begin{aligned} \Delta H &= \kappa^2|\nabla a|^2 - \kappa^2(c_2 - c_1)Hess(a) \\ &\quad -2\kappa^2(c_2 - c_1)\left(\Delta(a\Delta a) - a\Delta(\Delta a) + (\Delta a)^2\right) \end{aligned} \tag{5.15}$$

$$\Delta a = 0 \tag{5.16}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \textbf{(5.17)}$$

is the 2-dimensional Laplacian,

$$|\nabla a|^2 = \left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2 \quad \textbf{(5.18)}$$

is the norm-squared of the 2-dimensional gradient and

$$Hess(a) = \begin{vmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{vmatrix} = a_{xx}a_{yy} - (a_{xy})^2 \quad \textbf{(5.19)}$$

is the 2-dimensional Hessian operator. In terms of complex coordinates, (**5.16**) simply become

$$\begin{aligned} H_{z\bar{z}} &= -\kappa^2 a_z a_{\bar{z}} + \kappa^2 (c_2 - c_1) a_{zz} a_{\bar{z}\bar{z}}, \\ a_{z\bar{z}} &= 0. \end{aligned} \quad \textbf{(5.20)}$$

A non-trivial solution that depends on the coupling constant $(c_2 - c_1)$ is obtained by letting

$$a(u,z,\bar{z}) = f_1(u)z + \bar{f}_1(u)\bar{z} + f_2(u)z^2 + \bar{f}_2(u)\bar{z}^2. \quad \textbf{(5.21)}$$

while $f_1(u), \bar{f}_1(u)$ arbitrary functions demonstrate the polarization states of photon in vacuum, $f_2(u), \bar{f}_2(u)$ demonstrate the polarization by the presence of nonminimal coupling terms. Then

$$\begin{aligned} \frac{1}{\kappa^2} H(u,z,\bar{z}) &= f_3(u)z^2 + \bar{f}_3(u)\bar{z}^2 - |f_1(u)|^2|z|^2 - |f_2(u)|^2|z|^4 - f_1(u)\bar{f}_2(u)\bar{z}|z|^2 \\ &\quad - f_2(u)\bar{f}_1(u)z|z|^2 + 4(c_2 - c_1)|f_2(u)|^2|z|^2. \end{aligned} \quad \textbf{(5.22)}$$

We note that the non-minimal coupling $c_2 - c_1$ between the gravitational and electromagnetic waves is carried in the last term on the right hand side of the expression above and affects only the space-time metric. Both the polarization $p = 0$ and the magnetization $m = 0$ identically in the pp-wave geometry. We write

$$A = \mathscr{A}_{1+} + \mathscr{A}_{1-} + \mathscr{A}_{2+} + \mathscr{A}_{2-} \quad \textbf{(5.23)}$$

where

$$\mathscr{A}_{1+} = f_1(u)z du = \bar{\mathscr{A}}_{1-} \quad , \quad \mathscr{A}_{2+} = f_2(u)z^2 du = \bar{\mathscr{A}}_{2-} \quad \textbf{(5.24)}$$

and introduce $z = re^{i\theta}$ to show that

$$\mathscr{L}_{\frac{1}{i}\frac{\partial}{\partial\theta}} \mathscr{A}_{1\pm} = \pm \mathscr{A}_{1\pm} \qquad \mathscr{L}_{\frac{1}{i}\frac{\partial}{\partial\theta}} \mathscr{A}_{2\pm} = \pm 2 \mathscr{A}_{2\pm}. \tag{5.25}$$

$\mathscr{L}_X$ denotes the Lie derivative along the vector field X. Hence $\mathscr{A}_{1\pm}, \mathscr{A}_{2\pm}$ are null photon helicity eigen-tensors. Similarly, the metric tensor decomposes as

$$g = \eta + \mathscr{G}_0 + \mathscr{G}_{1+} + \mathscr{G}_{1-} + \mathscr{G}_{2+} + \mathscr{G}_{2-} \tag{5.26}$$

where η is the metric of Minkowski spacetime and

$$\mathscr{G}_{1+} = -\bar{f}_1(u) f_2(u) z|z|^2 du \otimes du = \bar{\mathscr{G}}_{1-} \tag{5.27}$$

$$\mathscr{G}_{2+} = \bar{f}_3(u) z^2 du \otimes du = \bar{\mathscr{G}}_{2-} \tag{5.28}$$

$$\mathscr{G}_0 = \left(-|f_1(u)|^2 - |f_2(u)|^2|z|^4 + 4(c_2 - c_1)|f_2(u)|^2|z|^2\right) du \otimes du. \tag{5.29}$$

The $\mathscr{G}_{1\pm}, \mathscr{G}_{2\pm}$ are null g-wave helicity eigen-tensors for linearized gravitation about $\eta + \mathscr{G}_0$:

$$\mathscr{L}_{\frac{1}{i}\frac{\partial}{\partial\theta}} \bar{\mathscr{G}}_{1\pm} = \pm \bar{\mathscr{G}}_{1\pm} \qquad \mathscr{L}_{\frac{1}{i}\frac{\partial}{\partial\theta}} \bar{\mathscr{G}}_{2\pm} = \pm 2 \bar{\mathscr{G}}_{2\pm}. \tag{5.30}$$

The helicity of the electromagnetic fields must have ± 1. This means that $f_2 = \bar{f}_2 = 0$. These two helicity components correspond to the classical concepts of right-handed and left-handed circularly polarized light.

While $f_1(u), \bar{f}_1(u)$ arbitrary functions demonstrate the polarization states of photon in vacuum, $f_2(u), \bar{f}_2(u)$ demonstrate the polarization states to see the effects of the nonminimal coupling terms. Then

$$\begin{aligned} \frac{1}{\kappa^2} H(u, z, \bar{z}) &= f_3(u) z^2 + \bar{f}_3(u) \bar{z}^2 + |f_1(u)|^2 |z|^2 + |f_2(u)|^2 |z|^4 + f_1(u) \overline{f}_2(u) \bar{z} |z|^2 \\ &\quad + f_2(u) \overline{f}_1(u) z |z|^2 - 4(c_2 - c_1)|f_2|^2 |z|^2. \end{aligned} \tag{5.31}$$

These solutions describe parallelly propagating plane fronted gravitational and electromagnetic waves that do not interact with each other in the Einstein-Maxwell theory. Here if only the standard degrees of polarization (± 1 for the photon and ± 2 for the graviton) are kept, no contribution arises from the non-minimal coupling constants c_1, c_2. It is interesting to note that if c_1, c_2 are kept they bring in ± 2 polarization degrees of freedom for the photon together with ± 1 polarization degrees of freedom for the graviton. The notion of a partially massless (spin-2) photon had been introduced before by Deser and Waldron [30], [31]. On the other hand, the partially massive (spin-2) graviton here is new and it may find some observational evidence in future.

Additionally, Energy-momentum transported by the exact plane wave is given by

$$\Sigma^0 = \Sigma^3 = \frac{1}{\sqrt{2}}[|f_1(u)|^2 z^2 + 4(c_2 - c_1)|f_1(u)|^2 + 2f_1(u)\overline{f}_2(u)z + 2f_2(u)\overline{f}_1(u)\overline{z} + f_2(u)^2] * du \quad \textbf{(5.32)}$$

5.1.1 The non-zero torsion case:

Now we give up the zero-torsion constraint in the action of nonminimal Einstein-Maxwell theory and calculate the contortion 1-forms from (**4.47**) as

$$K^{31} = K^{01} = -\frac{\kappa^2(c_2-c_1)}{2}[a_y a_{yx} + 2a_x a_{xx} + a_x a_{yy}](e^3 - e^0) \quad \textbf{(5.33)}$$

$$K^{32} = K^{02} = -\frac{\kappa^2(c_2-c_1)}{2}[a_x a_{xy} + 2a_y a_{yy} + a_y a_{xx}](e^3 - e^0) \quad \textbf{(5.34)}$$

and the other components are zero. We find the non-zero torsion components from this contortion using (**2.25**)

$$T^0 = T^3 = \frac{\kappa^2(c_2-c_1)}{2}[a_y a_{yx} + 2a_x a_{xx} + a_x a_{yy}]e^1 \wedge (e^3 - e^0) + \frac{\kappa^2(c_2-c_1)}{2}[a_x a_{xy} + 2a_y a_{yy} + a_y a_{xx}]e^2 \wedge (e^3 - e^0). \quad \textbf{(5.35)}$$

Then the full connection 1- forms are to be

$$\omega^{01} = \omega^{31} = \frac{1}{2}[H_x - \kappa^2(c_2-c_1)(a_y a_{yx} + 2a_x a_{xx} + a_x a_{yy})](e^3 - e^0), \quad \textbf{(5.36)}$$

$$\omega^{02} = \omega^{32} = \frac{1}{2}[H_y - \kappa^2(c_2-c_1)(a_x a_{xy} + 2a_y a_{yy} + a_y a_{xx})](e^3 - e^0). \quad \textbf{(5.37)}$$

For the full connection the Einstein-Cartan tensor 3-forms which are defined by $G_a = -\frac{1}{2\kappa^2}R^{bc} \wedge *e_{abc}$ become

$$G_0 = -G_3 = \{\frac{H_{xx}+H_{yy}}{2\kappa^2} - \frac{c_2-c_1}{2}\left((a_x{}^2)_{xx} + 2(a_x a_y)_{xy} + (a_y{}^2)_{yy}\right)\} * (e^3 - e^0),$$
$$G_1 = 0 = G_2. \quad \textbf{(5.38)}$$

When we put all these terms together to write the non-minimally coupled Einstein-Cartan-Maxwell equations, we find remarkably that they are the same equations with the non-minimally coupled Einstein-Maxwell equations

$$H_{xx} + H_{yy} - \kappa^2(c_2-c_1)\left((a_x{}^2)_{xx} + 2(a_x a_y)_{xy} + (a_y{}^2)_{yy}\right) = -\kappa^2(a_x{}^2 + a_y{}^2),$$
$$a_{xx} + a_{yy} = 0. \quad \textbf{(5.39)}$$

5.2 Static Spherically Symmetric Solutions

We look for static spherically symmetric solutions of the field equations of non-minimally coupled Einstein-Cartan-Maxwell theory. We start with the following metric

$$g \;=\; -f(r)^2dt^2 + f(r)^{-2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.40}$$

the convenient choice of orthonormal co-frames:

$$e^0 = f(r)dt \;\;,\;\; e^1 = f(r)^{-1}dr \;\;,\;\; e^2 = rd\theta \;\;,\;\; e^3 = r\sin\theta d\phi \tag{5.41}$$

under this choice the metric (**5.40**) becomes

$$g \;=\; -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 \tag{5.42}$$

and the exterior derivative of the co-frames (**5.41**) can be calculated as

$$de^0 = f'e^{10} \;\;,\;\; de^1 = 0 \;\;,\;\; de^2 = \frac{f}{r}e^{12} \;\;,\;\; de^3 = \frac{f}{r}e^{13} + \frac{\cot\theta}{r}e^{23} \tag{5.43}$$

the Levi-Civita connection 1-forms are to be

$$\hat{\omega}^0{}_1 = f'e^0 \;\;,\;\; \hat{\omega}^2{}_1 = \frac{f}{r}e^2 \;\;,\;\; \hat{\omega}^3{}_1 = \frac{f}{r}e^3 \;\;,\;\; \hat{\omega}^3{}_2 = \frac{\cot\theta}{r}e^3 \tag{5.44}$$

from the definition of curvature (**2.35**), each component of curvature 2-form:

$$\begin{aligned} \hat{R}^{01} &= \frac{(f^2)''}{2}e^{10} \;\;,\;\; \hat{R}^{02} = \frac{(f^2)'}{2r}e^{20} \;\;,\;\; \hat{R}^{03} = \frac{(f^2)'}{2r}e^{30}, \\ \hat{R}^{21} &= \frac{(f^2)'}{2r}e^{12} \;\;,\;\; \hat{R}^{31} = \frac{(f^2)'}{2r}e^{13} \;\;,\;\; \hat{R}^{32} = \frac{1}{r^2}(1-f^2)e^{32}. \end{aligned} \tag{5.45}$$

and Ricci 1-forms from $\iota_a\hat{R}^{ab} = \hat{R}^b$:

$$\begin{aligned} \hat{R}^0 &= -[\frac{(f^2)''}{2} + \frac{(f^2)'}{r}]e^0, \quad \hat{R}^1 = -[\frac{(f^2)''}{2} + \frac{(f^2)'}{r}]e^1 \\ \hat{R}^2 &= -[\frac{f^2-1}{r^2} + \frac{(f^2)'}{r}]e^2, \quad \hat{R}^3 = -[\frac{f^2-1}{r^2} + \frac{(f^2)'}{r}]e^3 \end{aligned} \tag{5.46}$$

the scalar of curvature:

$$\begin{aligned} \hat{R} &= \iota_a\hat{R}^a = \iota_0\hat{R}^0 + \iota_1\hat{R}^1 + \iota_2\hat{R}^2 + \iota_3\hat{R}^3 \\ \hat{R} &= -(f^2)'' - 4\frac{(f^2)'}{r} - 2\frac{f^2-1}{r^2} \end{aligned} \tag{5.47}$$

Thus the Einstein tensor can be calculated as

$$\hat{G}^0 = -[\frac{(f^2)'}{r} - \frac{1-f^2}{r^2}]e^{123}, \quad \hat{G}^1 = -[\frac{(f^2)'}{r} - \frac{1-f^2}{r^2}]e^{023}$$
$$\hat{G}^2 = [\frac{(f^2)''}{2} + \frac{(f^2)'}{r}]e^{013}, \quad \hat{G}^3 = -[\frac{(f^2)''}{2} + \frac{(f^2)'}{r}]e^{012}. \quad \textbf{(5.48)}$$

In general, to solve the field equations of non-minimally coupled Einstein-Maxwell theory is not easy. So, we look at the special solutions such as the Coulomb potential and magnetic monopole potential.

5.2.1 Coulomb potential

We will consider Coulomb potential as an electromagnetic potential 1-form satisfying the Maxwell equation $dF = 0$;

$$A = h(r)dt. \quad \textbf{(5.49)}$$

We can calculate the following components of electromagnetic field

$$F = dA = \frac{1}{2}F_{ab}e^{ab} = h'e^{10}, \qquad {}^*F = h'{}^*e^{10} = h'e^{23},$$
$$F^0 = \iota^0 F = h'e^1, \qquad F^1 = \iota^1 F = h'e^0, \qquad F^{01} = h'$$
$$F^{ab}R_{ab} = -h'(f^2)''e^{10}. \quad \textbf{(5.50)}$$

Thus, the Maxwell energy momentum tensor,

$$\tau^0 = \frac{1}{2}h'^2e^{123}, \quad \tau^1 = \frac{1}{2}h'^2e^{023}, \quad \tau^2 = \frac{1}{2}h'^2e^{013}, \quad \tau^3 = -\frac{1}{2}h'^2e^{012} \quad \textbf{(5.51)}$$

and energy momentum-tensor of the nonminimally coupled terms respectively from **(4.15)**-**(4.20)**.

$${}^1\hat{\tau}^0 = h'^2(f^2)''e^{123}, \quad {}^1\hat{\tau}^1 = h'^2(f^2)''e^{023},$$
$${}^1\hat{\tau}^2 = \frac{1}{2}h'^2(f^2)''e^{013} \quad {}^1\hat{\tau}^3 = -\frac{1}{2}h'^2(f^2)''e^{012} \quad \textbf{(5.52)}$$

$${}^2\hat{\tau}^0 = -h'^2[(f^2)'' + \frac{3}{2}\frac{(f^2)'}{r}]e^{123}, \quad {}^2\hat{\tau}^1 = -h'^2[(f^2)'' + \frac{3}{2}\frac{(f^2)'}{r}]e^{023}$$

$${}^2\hat{\tau}^2 = -\frac{1}{2}h'^2[(f^2)'' + \frac{(f^2)'}{r}]e^{013}, \quad {}^2\hat{\tau}^3 = \frac{1}{2}h'^2[(f^2)'' + \frac{(f^2)'}{r}]e^{012} \quad \textbf{(5.53)}$$

$${}^3\hat{\tau}^0 = h'^2[(f^2)'' + 3\frac{(f^2)'}{r} + \frac{f^2-1}{r^2}]e^{123}, \quad {}^3\hat{\tau}^1 = h'^2[(f^2)'' + 3\frac{(f^2)'}{r} + \frac{f^2-1}{r^2}]e^{023}$$

$${}^3\hat{\tau}^2 = \frac{h'^2}{2}[(f^2)'' + 2\frac{(f^2)'}{r}]e^{013}, \quad {}^3\hat{\tau}^3 = -\frac{h'^2}{2}[(f^2)'' + 2\frac{(f^2)'}{r}]e^{012} \quad \textbf{(5.54)}$$

$$^4\hat{\tau}^a = {}^5\hat{\tau}^a = {}^6\hat{\tau}^a = 0 \quad (\mathbf{5.55})$$

the Lagrange multiplier λ_a can be found from (**3.7**) as

$$\begin{aligned}
\lambda^0 &= -2fh'[(c_1-c_2+c_3)h'' + (c_1-\frac{c_2}{2})\frac{h'}{r}]e^{23} \\
\lambda^1 &= 0 \\
\lambda^2 &= -2fh'[(2c_3-c_2)h'' - \frac{c_2}{2}\frac{h'}{r}]e^{03} \\
\lambda^3 &= 2fh'[(2c_3-c_2)h'' - \frac{c_2}{2}\frac{h'}{r}]e^{02} \quad (\mathbf{5.56})
\end{aligned}$$

and the covariant exterior derivative of them

$$\begin{aligned}
\hat{D}\lambda^0 &= -[(c_1-c_2+c_3)(f^{2'}h'h'' + 2f^2h''^2 + 2f^2h'h''') + (2c_1-c_2)(\frac{f^2h'^2}{r^2} \\
&\quad + \frac{f^{2'}h'^2}{2r}) + (8c_1-6c_2+4c_3)\frac{f^2h'h''}{r}]e^{123} \quad (\mathbf{5.57})
\end{aligned}$$

$$\begin{aligned}
\hat{D}\lambda^1 &= -[(c_1-c_2+c_3)f^{2'}h'h'' + (c_1-\frac{c_2}{2})\frac{f^{2'}h'^2}{r} + (4c_3-2c_2)\frac{f^2h'h''}{r} \\
&\quad -c_2\frac{f^2h'^2}{r^2}]e^{023} \quad (\mathbf{5.58})
\end{aligned}$$

$$\begin{aligned}
\hat{D}\lambda^2 &= [(2c_3-c_2)(f^{2'}h'h'' + f^2h''^2 + f^2h'h''') + (2c_3-2c_2)\frac{f^2h'h''}{r} \\
&\quad -\frac{c_2}{2}\frac{f^{2'}h'^2}{r}]e^{013} \quad (\mathbf{5.59})
\end{aligned}$$

$$\begin{aligned}
\hat{D}\lambda^3 &= -[(2c_3-c_2)(f^{2'}h'h'' + f^2h''^2 + f^2h'h''') + (2c_3-2c_2)\frac{f^2h'h''}{r} \\
&\quad -\frac{c_2}{2}\frac{f^{2'}h'^2}{r}]e^{012} \quad (\mathbf{5.60})
\end{aligned}$$

The total field equations are

$$\frac{1}{\kappa^2}\hat{G}^0 - \lambda e^{123} = h'^2[\frac{1}{2} + (c_1-c_2+c_3)f^{2''} + (3c_3-\frac{3c_2}{2})\frac{f^{2'}}{r} + c_3\frac{f^2-1}{r^2}]e^{123} + D\lambda^0$$

$$\frac{1}{\kappa^2}\hat{G}^1 - \lambda e^{023} = h'^2[\frac{1}{2} + (c_1-c_2+c_3)f^{2''} + (3c_3-\frac{3c_2}{2})\frac{f^{2'}}{r} + c_3\frac{f^2-1}{r^2}]e^{023} + D\lambda^1$$

$$\frac{1}{\kappa^2}\hat{G}^2 + \lambda e^{013} = [\frac{h'^2}{2} + \frac{(c_1-c_2+c_3)}{2}h'^2f^{2''} + (c_3-\frac{c_2}{2})\frac{h'^2f^{2'}}{r}]e^{023} + D\lambda^2$$

$$\frac{1}{\kappa^2}\hat{G}^3 - \lambda e^{012} = [\frac{h'^2}{2} + \frac{(c_1-c_2+c_3)}{2}h'^2 f^{2''} + (c_3 - \frac{c_2}{2})\frac{h'^2 f^{2'}}{r}]e^{023} + D\lambda^3$$

These differential equation system can be reduced to the following form; the difference between the zeroth and first equations:

$$c_1 H + (2c_1 - c_2)H'r + \frac{c_1-c_2+c_3}{2}H''r^2 = 0 \qquad \textbf{(5.61)}$$

and we can solve it for H(r) as

$$H(r) = C_1 + C_2 r^{\frac{-c_1-c_2+3c_3+b_1}{2(c_1-c_2+c_3)}} + C_3 r^{\frac{-c_1-c_2+3c_3-b_1}{2(c_1-c_2+c_3)}} \qquad \textbf{(5.62)}$$

where $b_1 = c_1^2 + 2c_1c_2 - 14c_1c_3 + c_2^2 + 2c_2c_3 + c_3^2$ the first equation:

$$\begin{aligned} -\frac{1}{\kappa^2}[\frac{(f^2)'}{r} - \frac{1-f^2}{r^2}] &= \lambda + H[\frac{1}{2} + (c_1-c_2+c_3)f^{2''} + (3c_3-c_1-c_2)\frac{f^{2'}}{r} \\ &+ \frac{(c_2+c_3)f^2 - c_3}{r^2}] - \frac{c_1-c_2+c_3}{2}f^{2'}H' - (2c_3-c_2)\frac{f^2H'}{r} \end{aligned}$$

the second equation:

$$\begin{aligned} \frac{1}{\kappa^2}[\frac{(f^2)''}{2} + \frac{(f^2)'}{r}] &= -\lambda + [\frac{H}{2} + \frac{(c_1-c_2+c_3)}{2}Hf^{2''} + (c_3-c_2)\frac{(Hf^2)'}{r}] \\ &+ \frac{2c_3-c_2}{2}(H'f^2)' \end{aligned}$$

where $H = h'^2$ and there is also the Maxwell equation from (**4.23**)

$$(1 + (c_1-c_2+c_3)f^{2''} + (4c_3-2c_2)\frac{f^{2'}}{r} + 2c_3\frac{f^2-1}{r^2})h' = q/r^2 \qquad \textbf{(5.63)}$$

We have not found any analytic solutions to these three differential equations which has one unknown function.

5.2.2 Magnetic monopole potential

Now we consider the solutions with magnetic monopole potential to this theory as

$$A = k_0(1 - cos(\theta))d\phi \qquad \textbf{(5.64)}$$

where

$$k_0 = \frac{1}{4\pi}\int_S F \qquad \textbf{(5.65)}$$

We calculate the all required expressions to solve the field equations of the non-minimal Einstein-Maxwell theory for the above potential. We point out that the

0^{th} and 1^{st} components of all the Einstein equations have the same results, except of $D\lambda^a$.

λ^a and $D\lambda^a$ components can be found as:

$$\begin{aligned}
\lambda^0 &= -\frac{2f}{r^5}(2c_3 - \frac{c_2}{2})k_0^2 e^{23} \\
\lambda^1 &= 0 \\
\lambda^2 &= -\frac{f}{r^5}(c_1 - \frac{5c_2}{2} + 4c_3)k_0^2 e^{03} \\
\lambda^3 &= \frac{f}{r^5}(c_1 - \frac{5c_2}{2} + 4c_3)k_0^2 e^{02}
\end{aligned} \quad (\mathbf{5.66})$$

$$\begin{aligned}
\hat{D}\lambda^0 &= [-\frac{6f^2}{r^6}(2c_3 - \frac{c_2}{2}) - \frac{2ff'}{r^5}(c_3 - \frac{c_2}{4})]k_0^2 e^{123} \\
\hat{D}\lambda^1 &= [-\frac{2f^2}{r^6}[(c_1 - \frac{5c_2}{2} + 4c_3) - \frac{2ff'}{r^5}(c_3 - \frac{c_2}{4})]k_0^2 e^{023} \\
\hat{D}\lambda^2 &= [-\frac{2f^2}{r^6}2(c_1 - \frac{5c_2}{2} + 4c_3) + \frac{2ff'}{r^5}(c_1 - \frac{5c_2}{2} + 4c_3)]k_0^2 e^{013} \\
\hat{D}\lambda^3 &= [\frac{2f^2}{r^6}[2(c_1 - \frac{5c_2}{2} + 4c_3) - \frac{2ff'}{r^5}(c_1 - \frac{5c_2}{2} + 4c_3)]k_0^2 e^{012}
\end{aligned} \quad (\mathbf{5.67})$$

Thus we can write the total field equations respectively;

The 0^{th} component:

$$\begin{aligned}
& \frac{f^{2'}}{r} - \frac{1-f^2}{r^2} + \frac{k^2}{2r^4} - \frac{k^2 c_3 f^{2'}}{r^5} \\
& -\frac{(4c_2 - 13c_3 - c_1)k^2 f^2 + (c_1 - c_2 + c_3)k^2}{r^6} = 0
\end{aligned} \quad (\mathbf{5.68})$$

where $k = \kappa^2 k_0$.

The 1^{st} component:

$$\begin{aligned}
& \frac{f^{2'}}{r} - \frac{1-f^2}{r^2} + \frac{k^2}{2r^4} - \frac{k^2 c_3 f^{2'}}{r^5} \\
& -\frac{(-4c_2 + 7c_3 + c_1)k^2 f^2 + (c_1 - c_2 + c_3)k^2}{r^6} = 0
\end{aligned} \quad (\mathbf{5.69})$$

The 2^{nd} component:

$$\begin{aligned}
& \frac{f^{2'}}{r} + \frac{f^2}{2} - \frac{(c_3 f^{2''} + 1)k^2}{2r^4} - \frac{(7c_3 - 4c_2 + c_1)k^2 f^{2'}}{r^5} \\
& -\frac{(8c_2 - 14c_3 - 2c_1)k^2 f^2 - 2(c_1 - c_2 + c_3)k^2}{r^6} = 0
\end{aligned} \quad (\mathbf{5.70})$$

and the Maxwell equation is automatically satisfied for this potential.

$$d * F = 0 \quad (\mathbf{5.71})$$

We can see from 0^{th} and 1^{st} components[2].

$$c_1 - 4c_2 + 10c_3 = 0 \tag{5.72}$$

- If one choose $c_3 = 0,\ c_1 = 4c_2$, the above system is reduced to the following two differential equations;

$$\begin{aligned} \frac{f^{2\prime\prime}}{2} + \frac{f^{2\prime}}{r} - \frac{k^2}{2r^4} + 6\frac{k^2 c_2}{r^6} &= 0 \\ \frac{f^{2\prime}}{r} + \frac{f^2 - 1}{r^2} + \frac{k^2}{2r^4} - 3\frac{k^2 c_2}{r^6} &= 0 \end{aligned} \tag{5.73}$$

As it is given by Balakin in [32], the solution of these differential equations is

$$f(r)^2 = 1 - \frac{2m}{r} + \frac{k^2}{2r^2} - \frac{c_2 k^2}{r^4}. \tag{5.74}$$

The solution is asymptotically flat and a generalization of Reissner-Nordström solution to the non-minimal case. The metric function f^2 has a central essential singularity for arbitrary mass and charge. That is, f^2 has at least one positive real solution for r, thus spacetime has one or more horizons.

- If $c_1 = 6c_3,\ c_2 = 4c_3$, the above system turns out to the following two differential equations;

$$\begin{aligned} \frac{f^{2\prime\prime}}{2} + \frac{f^{2\prime}}{r} - \frac{(1 + c_3 f^{2\prime\prime})k^2}{2r^4} + \frac{3k^2 c_3 f^{2\prime}}{r^5} - \frac{6c_3(f^2 - 1)k^2}{r^6} &= 0 \\ \frac{f^{2\prime}}{r} - \frac{1 - f^2}{r^2} + \frac{k^2}{2r^4} - \frac{k^2 c_3 f^{2\prime}}{r^5} + \frac{3c_3 k^2 (f^2 - 1)}{r^6} &= 0 \end{aligned} \tag{5.75}$$

the solution of this differential system [33] is

$$f^2(r) = 1 + \frac{k^2 r^2 - 4mr^3}{2(r^4 - c_3 k^2)} \tag{5.76}$$

This solution is also asymptotically flat and it is regular at $r = 0$, $f^2(0) = 1$, provided that $c_3 \neq 0$. There is a singularity at $r^4 = k^2 c_3$ and there is at least one horizon.

[2]When we take $c_1 = q_3\ c_2 = -q_2\ c_3 = q_1$, we obtain the same results with [32]

5.2.3 The non-zero torsion case:

In the beginning of this chapter, we have shown that we can consider also theory with torsion. For a given electromagnetic potential we can determine the contortion uniquely from (**4.47**) as

$$K^0{}_1 = \frac{fk^2\left(-3\,{c_6}^2k^2 - 2\,r^4c_6 - r^4c_5 + c_5\,k^2c_3 + r^4c_3\right)e^0}{r\,(r^8 + 2\,r^4c_6\,k^2 + {c_6}^2k^4)} \qquad (\mathbf{5.77})$$

$$K^1{}_2 = K^1{}_3 = \frac{fk^2\,(c_3 + c_6)\,e^3}{r\,(r^4 + c_6\,k^2)} \qquad (\mathbf{5.78})$$

and using (**2.25**) we can determine the components of the torsion

$$T^0 = \frac{fk^2\left(-3\,{c_6}^2k^2 - 2\,r^4c_6 - r^4c_5 + c_5\,k^2c_3 + r^4c_3\right)e^{01}}{r\,(r^8 + 2\,r^4c_6\,k^2 + {c_6}^2k^4)} \qquad (\mathbf{5.79})$$

$$T^2 = \frac{fk^2\,(c_3 + c_6)\,e^{12}}{r\,(r^4 + c_6\,k^2)} \qquad (\mathbf{5.80})$$

$$T^3 = \frac{fk^2\,(c_3 + c_6)\,e^{13}}{r\,(r^4 + c_6\,k^2)} \qquad (\mathbf{5.81})$$

where $c_5 = c_1 - c_2 + c_3$ and $c_6 = c_3 - \frac{c_2}{2}$.

When we calculate the Einstein-Cartan-Maxwell field equations (**4.49**)-(**4.51**), we find very long and complicate differential equations for the above magnetic monopole potential and static spherically symmetric metric. Therefore, we think it is not useful to give the expressions explicitly here. So, we look for solutions for some special values of c_i.

- If $c_1 = 6c_3,\ c_2 = 4c_3$, the above system is reduced to (**5.75**) with $T^a = 0$ interestingly and the solution is (**5.76**) again.

- But, if $c_3 = 0,\ c_1 = 4c_2$ the above system with torsion is not consistent and there is no solution. The absence of the solution gives us the difference between Einstein-Maxwell theory and Einstein-Cartan-Maxwell theory for this choice. We can see while the solution (**5.76**) is regular at $r = 0$ for $c_3 \neq 0$, the solution (**5.74**) has a singularity at this point. Thus, when we allow the torsion, there is no solution which has central singularity, while there is a solution which has central singularity (**5.76**) to the Einstein-Maxwell field equations.

6. EINSTEIN-CARTAN-DIRAC THEORY

Although general relativity describes the motion of bodies at the scale of solar system very successfully, it is inadequate in Planck densities and cosmological scales. Therefore, for a natural extension of general relativity to those scales, the effects of coupling fermions to gravity should be considered.

On the other hand, Einstein-Cartan theory [10] is a generalization of Einstein's theory of gravity allowing space-time (in arbitrary number of dimensions) to have torsion in addition to curvature and relating torsion to the density of intrinsic angular momentum of matter. In this theory, torsion is considered as a nonpropagating field. The Einstein and Einstein-Cartan theories give exactly the same results in empty space. Since all tests of general relativity are based on the idea of Einstein's field equations in vacuum, the Einstein-Cartan gravity is consistent with the idea in that case. Initial expectation of Trautman was that the intrinsic angular momentum may influence the occurrence of singularities in gravitational collapse or cosmology but that didn't turn out to be the case. The theory of gravity which has torsion and spin was given independently by Sciama and Kibble. Actually, simple theory of supergravity is equivalent to Einstein-Cartan-Dirac theory with massless, anticommuting Rarita-Schwinger field [38].

One can couple gravity with the Dirac fields in the formulation of Einstein-Cartan-Dirac theory. In Einstein-Cartan-Dirac theory, torsion depends on the spin and the energy-momentum tensor which is non-symmetric (because of torsion). The effects of torsion can be significant only at high densities of matter. Nevertheless, they may contribute at much smaller densities than the Planck density at which quantum gravitational effects are believed to dominate. Moreover, it can be a new insight to consider the couplings in the context of astrophysical and quantum field theory. Field theories often provide easy ways to check the ideas that are difficult to prove in actual (1+3)-dimensions. However, it is not surprising to uncover other new ideas as well that are specific to (1+2)-dimensions. Topologically massive gravity [34] or BTZ black holes [35] are some of the best known examples to the latter case. Other aspects and the literature may be found in Ref. [36]. An extension of the BTZ solution with torsion is discussed by Garcia et al [37] where the field equations are derived from an action that includes topological Chern-Simons terms. Despite the existence of many solutions of general relativity in the Einstein or Einstein-Maxwell theory, there are a few solutions for Einstein-Cartan-Dirac theory.

We investigate Einstein-Cartan-Dirac theory in order to obtain the behevior of the space-time metric in the precense of a Dirac spinor field. Firstly, we will obtain the field equations of Einstein-Cartan theory using the variational principle. Therefore, we will point out the difference between General relativity and Einstein-Cartan gravity giving an outline of these two theories in arbitrary d-dimensions. In order to reach this theory, the Einstein-Hilbert action is extended to Einstein-Cartan action which includes torsion tensor. We will couple a Dirac spinor to Einstein-Cartan gravity and obtain the field equations by a variational principle in (1+2)-dimensions using the algebra of exterior differential forms. We will determine the torsion tensor in terms of Dirac field couplings in three dimensions. It is interesting to note that the same Einstein-Cartan-Dirac field equations can also be obtained from an action by zero-torsion constrained variations using the method of Lagrange multipliers. Thus, we will show that the equivalence between the Einstein-Dirac theory and Einstein-Cartan-Dirac theory in three dimensions. We will consider through this paper the metric compatible connection. That is; the nonmetricity tensor is equal to zero.

The space-time torsion is determined algebraically in terms of the quadratic spinor invariant associated with a Dirac condensate field. We then looked for rotating, circularly symmetric solutions. We found a particular class of solutions that possess an essential curvature singularity at the origin r=0. The mass and the intrinsic angular momentum of this configuration can be identified. It is remarkable that in the absence of the Dirac condensate field the metric collapses to the regular AdS3 metric. We note that for our solution the Dirac condensate field determines completely the rotation of the metric of the space-time.

6.1 Variational Field Equations

We consider a Dirac spinor field ψ which has two components in 1+2 dimensional space-time in spinor representation of the Lorentz group as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.1}$$

and the conjugate spinor field

$$\bar{\psi} = \psi^\dagger \gamma_0 = (-\psi_2^* \;\; \psi_1^*) \tag{6.2}$$

where ψ_1 and ψ_2 are complex, odd Grassmann valued functions. We use a real (i.e. Majorana) realization of the gamma matrices $\{\gamma_a\}$ given explicitly as

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3}$$

To write the hermitian Dirac Lagrangian let us start the following Lagrangian for the Dirac fields ψ, $\bar{\psi}$,

$$L_D[\overline{\psi}, e, \omega] = i\overline{\psi} \wedge *\gamma \wedge \nabla\psi + im\overline{\psi}\psi * 1 \tag{6.4}$$

where m represents the mass of the fermionic field. The exterior covariant derivatives of the spinor fields are defined to be

$$\nabla\psi \equiv d\psi + \frac{1}{2}\omega^{ab}\sigma_{ab}\psi \quad , \quad \overline{\nabla\psi} \equiv d\bar{\psi} - \frac{1}{2}\omega^{ab}\bar{\psi}\sigma_{ab} \tag{6.5}$$

with

$$\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{2} * e_{abc}\gamma^c. \tag{6.6}$$

in terms of spinor connection ω_{ab}. We set $*\gamma = \gamma_a * e^a$. We see from (**6.4**) that the first term is not hermitian. But, the second term is hermitian since $(\gamma_0)^\dagger = -\gamma_0$.

$$(im\overline{\psi}\psi)^\dagger = im\overline{\psi}\psi \tag{6.7}$$

In order to make the first term in (**6.4**) hermitian, hermitian conjugate of this is inserted to the Lagrangian

$$\mathscr{O} = \frac{1}{2}(\mathscr{O} + \mathscr{O}^\dagger). \tag{6.8}$$

thus,

$$
\begin{aligned}
L_D &= \frac{1}{2}[i\overline{\psi}*\gamma\wedge\nabla\psi+(i\overline{\psi}\wedge*\gamma\wedge\nabla\psi)^{\dagger}]+im\overline{\psi}\psi*1\\
&= \frac{i}{2}[\overline{\psi}*\gamma\wedge\nabla\psi-(\overline{\psi}\wedge*\gamma\wedge\nabla\psi)^{\dagger}]+im\overline{\psi}\psi*1\\
&= \frac{i}{2}[\overline{\psi}*\gamma\wedge\nabla\psi-(\nabla\psi)^{\dagger}\wedge(*\gamma)^{\dagger}(\psi^{\dagger}\gamma_0)^{\dagger}]+im\overline{\psi}\psi*1\\
&= \frac{i}{2}[\overline{\psi}*\gamma\wedge\nabla\psi-(\nabla\psi)^{\dagger}\wedge(\gamma_a)^{\dagger}(\gamma_0)^{\dagger}\psi*e^a]+im\overline{\psi}\psi*1
\end{aligned}
$$

where we have used the properties of Dirac matrices such as ${\gamma_0\gamma_a}^{\dagger}\gamma_0=\gamma_a$, $\gamma_0\gamma_0=-1$ and ${\gamma_0}^{\dagger}=-\gamma_0$.

We will continue with taking independent variations according to the fields of the hermitian Dirac Lagrangian 3-form in three dimensions.

$$
L_D=\frac{i}{2}[\overline{\psi}*\gamma\wedge\nabla\psi-\overline{\nabla\psi}\wedge*\gamma\psi]+im\overline{\psi}\psi*1 \qquad \textbf{(6.9)}
$$

- The co-frame variation of the Dirac Lagrangian:

$$
\delta_e L_D=\frac{i}{2}(\overline{\psi}\delta*\gamma\wedge\nabla\psi-\overline{\nabla\psi}\wedge\delta*\gamma\psi)+im\overline{\psi}\psi\delta*1 \qquad \textbf{(6.10)}
$$

substituting $*\gamma=\gamma_a*e^a$ and $\delta*1=\delta e^a\wedge*e_a$ in the above expression the variation can be recast into the form

$$
\begin{aligned}
\delta_e L_D &= \delta*e^b\wedge\frac{i}{2}(\overline{\psi}\gamma_b\nabla\psi-\overline{\nabla\psi}\gamma_b\psi)+\delta e^a\wedge im\overline{\psi}\psi*e_a\\
&= \delta e^a\wedge*(e^b\wedge e_a)\wedge\frac{i}{2}(\overline{\psi}\gamma_b\nabla\psi-\overline{\nabla\psi}\gamma_b\psi)+\delta e^a\wedge im\overline{\psi}\psi*e_a\\
&= \delta e^a\wedge\frac{i}{2}[\overline{\psi}*(\gamma\wedge e_a)\nabla\psi+\overline{\nabla\psi}*(\gamma\wedge e_a)\psi]+\delta e^a\wedge im\overline{\psi}\psi*e_a
\end{aligned}
$$

Thus, the energy-momentum 2-form of a spinor field have been derived such as

$$
\hat{\tau}_a[\psi]=\frac{i}{2}[\overline{\psi}*(\gamma\wedge e_a)\nabla\psi+\overline{\nabla\psi}*(\gamma\wedge e_a)\psi]+im\overline{\psi}\psi*e_a \qquad \textbf{(6.11)}
$$

- The connection variation of the Dirac Lagrangian:

$$
\begin{aligned}
\delta_\omega L &= \frac{i}{2}[\overline{\psi}*\gamma\wedge\delta\nabla\psi+\delta\overline{\nabla\psi}\wedge*\gamma\psi]\\
&= \overline{\psi}*\gamma\wedge\frac{i}{4}\delta\omega^{ab}\sigma_{ab}\psi-\frac{i}{4}(-\overline{\psi}\delta\omega^{ab}\sigma_{ab}*\gamma\psi)\\
&= \delta\omega^{ab}\wedge\frac{i}{4}\overline{\psi}(*\gamma\sigma_{ab}+\sigma_{ab}*\gamma)\psi\\
&= \delta\omega^{ab}\wedge\frac{i}{4}\overline{\psi}(\gamma_c\sigma_{ab}+\sigma_{ab}\gamma_c)\psi*e^c
\end{aligned}
$$

by using the properties of the Dirac matrices the variation

$$\begin{aligned}\delta_{\omega}L &= \delta\omega^{ab}\wedge\frac{i}{4}\overline{\psi}\psi\varepsilon_{abc}*e^{c} \\ &= -\delta\omega^{ab}\wedge\frac{i}{4}\overline{\psi}\psi e_{ab}\end{aligned} \tag{6.12}$$

from here the spinor angular momentum 2-form can be identified as

$$\Sigma_{ab} = -\frac{i}{4}\overline{\psi}\psi e_{ab} \tag{6.13}$$

- The $\overline{\psi}$ field variation of the Dirac Lagrangian:

$$\delta_{\overline{\psi}}L = \frac{i}{2}[\delta\overline{\psi}*\gamma\wedge\nabla\psi - \overline{\nabla(\delta\psi)}\wedge*\gamma\psi] + \delta\overline{\psi}im\psi*1 \tag{6.14}$$

The second term in the parenthesis

$$\begin{aligned}\overline{\nabla(\delta\psi)}\wedge*\gamma\psi &= d(\delta\overline{\psi})\wedge*\gamma\psi - \frac{1}{2}\delta\overline{\psi}\omega^{ab}\sigma_{ab}\wedge*\gamma\psi \\ &= -\delta\overline{\psi}[*\gamma\wedge d\psi + d*\gamma\psi + \frac{1}{2}\omega^{ab}\sigma_{ab}\wedge*\gamma\psi]\end{aligned}$$

using this property $\sigma_{ab}\gamma_c = \gamma_c\sigma_{ab} + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b$

$$\begin{aligned}\overline{\nabla(\delta\psi)}\wedge*\gamma\psi &= -\delta\overline{\psi}[d(*\gamma\psi) + \frac{1}{2}\omega^{ab}\wedge(*\gamma\sigma_{ab} + \gamma_a*e_b - \gamma_b*e_a)\psi] \\ &= -\delta\overline{\psi}*\gamma\wedge\nabla\psi - \delta\overline{\psi}(d*e^a + \omega^{ab}*e_b)\gamma_a\psi \\ &= -\delta\overline{\psi}*\gamma\wedge\nabla\psi + \delta\overline{\psi}T\wedge*\gamma\psi\end{aligned} \tag{6.15}$$

where in the last step we have used

$$D*e^a = d*e^a + \omega^{ab}\wedge*e_b = T^b*e^a{}_b \tag{6.16}$$

$$= -T\wedge*e^a \tag{6.17}$$

Thus from (**6.14**) non-linear Dirac equation is obtained in the Riemann-Cartan space-times

$$\delta_{\overline{\psi}}L = \delta\overline{\psi}\frac{i}{2}[2*\gamma\wedge\nabla\psi - *\gamma\wedge T\psi + 2m\psi*1] \tag{6.18}$$

and for the ψ field after same process we find that

$$\delta_{\psi}L = \frac{i}{2}[-2\overline{\nabla\psi}\wedge*\gamma + *\gamma\wedge T\psi + 2m\psi*1]\delta\psi \tag{6.19}$$

Thus, the infinitesimal variations of the Dirac Lagrangian (**6.9**) are found to be (up to a closed form)

$$
\begin{aligned}
\dot{\mathscr{L}}_D &= \dot{e}^a \wedge \left\{ \frac{i}{2} {*e^b}_a \wedge (\bar{\psi}\gamma_b \nabla\psi - \overline{\nabla\psi}\gamma_b\psi) + im\bar{\psi}\psi * e_a \right\} \\
&+ \frac{1}{2}\dot{\omega}^{ab} \wedge \left\{ \frac{i}{2}\bar{\psi}(*\gamma\sigma_{ab} + \sigma_{ab} * \gamma)\psi \right\} \\
&+ i\dot{\bar{\psi}} \left\{ *\gamma \wedge \nabla\psi + \frac{1}{2} * e^a_b \wedge T^b \gamma_a \psi + m * \psi \right\} \\
&- i \left\{ \overline{\nabla\psi} \wedge *\gamma - \frac{1}{2} * e^a_b \wedge T^b \bar{\psi}\gamma_a - m * \bar{\psi} \right\} \dot{\psi}. \qquad (\mathbf{6.20})
\end{aligned}
$$

Here we use the notation $\delta L = \dot{L}$. In order to obtain the field equations of the Einstein-Cartan-Dirac theory, we substitute these variations into (**3.16**) and **3.17**). Thus, we obtain

$$
\begin{aligned}
R_{ab} &= \kappa\lambda e_a \wedge e_b + im\kappa\bar{\psi}\psi e_a \wedge e_b \\
&+ i\frac{\kappa}{2} e_a \wedge (\bar{\psi}\gamma_b \nabla\psi) - i\frac{\kappa}{2} e_b \wedge (\bar{\psi}\gamma_a \nabla\psi) \\
&+ i\frac{\kappa}{2} e_b \wedge (\overline{\nabla\psi}\gamma_a\psi) - i\frac{\kappa}{2} e_a \wedge (\bar{\nabla}\psi\gamma_b\psi), \qquad (\mathbf{6.21})
\end{aligned}
$$

$$T_a = i\frac{\kappa}{2}\bar{\psi}\psi * e_a \qquad (\mathbf{6.22})$$

We note that the torsion 2-forms (**6.22**) satisfy $*e^a_b \wedge T^b = 0$ and the Dirac equation (**6.18**) simplifies to

$$*\gamma \wedge \nabla\psi + m\psi * 1 = 0. \qquad (\mathbf{6.23})$$

6.2 Equivalence of Einstein-Cartan-Dirac and Einstein-Dirac Theories

When we solve contortion from (**6.22**) using (**2.25**) and substitute it into the field equations (**6.21**) and (**6.23**) we can rewrite the field equations explicitly. In order to show this the con-torsion 1-forms can be calculated as

$$K_{ab} = -\frac{i}{4}\overline{\psi}\psi * e_{ab} = -\frac{\tau}{2} * e_{ab} \qquad (\mathbf{6.24})$$

where $\tau(r) = \frac{i\kappa}{2}\overline{\psi}\psi$ is a new radial function.

Noting that the full connection $\omega_{ab} = \hat{\omega}_{ab} + K_{ab}$, Einstein tensor and energy-momentum tensor can be written as

$$\begin{aligned}\hat{G}_c &= -\frac{1}{2\kappa}[\hat{R}^{ab}\wedge *e_{abc} + \hat{\nabla}K^{ab}\wedge *e_{abc} + K^a{}_f \wedge K^{fb}\wedge *e_{abc}] \\ &= \hat{G}_c - \frac{1}{2\kappa}(d\tau\wedge e_c + \frac{\tau^2}{2} * e_c) \qquad \textbf{(6.25)}\\ \tau_c[\psi,\omega] &= \frac{i}{2}[\overline{\psi} * (\gamma\wedge e_c)\wedge\nabla\psi + \overline{\nabla\psi}\wedge *(\gamma\wedge e_c)\psi] + im\overline{\psi}\psi * e_c \\ &= \frac{i}{2}[\overline{\psi} * (\gamma\wedge e_c)\wedge\hat{\nabla}\psi + \overline{\hat{\nabla}\psi}\wedge *(\gamma\wedge e_c)\psi] + im\overline{\psi}\psi * e_c \\ &\quad + \frac{i}{4}\overline{\psi}\psi * (e_f\wedge e_c)\wedge K^{ab}(\gamma^f\sigma_{ab} + \sigma_{ab}\gamma_f) \\ &= \tau_c[\psi,\hat{\omega}] + \frac{\tau^2}{\kappa} * e_c \qquad \textbf{(6.26)}\end{aligned}$$

These field equations (**6.21**) and (**6.23**) can be rewritten in terms of the Levi-Civita connection only [39].

$$\begin{aligned}\hat{R}_{ab} = \kappa\lambda e_a\wedge e_b - \kappa * e_{abc}\hat{\tau}^c + i\frac{\kappa}{4}d(\bar{\psi}\psi)\wedge *(e_a\wedge e_b) - \frac{3\kappa}{16}(\bar{\psi}\psi)^2 e_a\wedge e_b, \\ *\gamma\wedge\bar{\nabla}\psi + m\psi * 1 + i\frac{3\kappa}{8}(\bar{\psi}\psi)\psi * 1 = 0 \qquad \textbf{(6.27)}\end{aligned}$$

It is interesting to note that the Einstein-Cartan-Dirac equations (**6.27**) can be obtained from an action by zero-torsion constrained variations using the method of Lagrange multipliers [40]. To this end, we consider a modified Dirac Lagrangian density 3-form

$$\mathscr{L}'_D = \frac{i}{2}\left(\bar{\psi} * \gamma\wedge\hat{\nabla}\psi - \overline{\hat{\nabla}\psi}\wedge *\gamma\psi\right) + im\bar{\psi}\psi * 1 - \frac{3\kappa}{16}(\bar{\psi}\psi)^2 * 1 \qquad \textbf{(6.28)}$$

together with the constraint term

$$\mathscr{L}_{constraint} = (de^a + \omega^a{}_b\wedge e^b)\wedge\lambda_a \qquad \textbf{(6.29)}$$

where $\{\lambda_a\}$ are the Lagrange multiplier 1-forms. The variation of the total action

$$I = \int_M \left(\mathscr{L}_{EC} + \mathscr{L}'_D + \mathscr{L}_{constraint}\right) \qquad \textbf{(6.30)}$$

with respect to the Lagrange multipliers imposes the constraint that the connection 1-forms are Levi-Civita. Then we firstly find the connection variation equations under this constraint for the Lagrange multiplier 1-forms from the connection $\hat{\omega}$ variation of (**6.30**)

$$\frac{i}{4}\overline{\psi}\psi e_{ab} + \lambda_a\wedge e_b = 0 \qquad \textbf{(6.31)}$$

We can solve the Lagrange multiplier λ_c from the Levi-Civita connection equation and find,

$$\lambda_a = -\frac{\tau}{2\kappa} e_a \tag{6.32}$$

and the covariant derivative of it

$$\hat{D}\lambda_a = -\frac{d\tau}{2\kappa} \wedge e_a = -\frac{1}{2\kappa} \hat{\nabla} K^{ab} \wedge *e_{abc} \tag{6.33}$$

The Einstein and Dirac field equations of the constraint Lagrangian are found as;

$$\begin{aligned} \hat{G}_c + \lambda * e_c = -\tau_c[\psi, \hat{\omega}] - \hat{D}\lambda_c + \frac{3\kappa}{16}(\overline{\psi}\psi)^2 * e_c, \\ *\gamma \wedge \bar{\nabla}\psi + m\psi * 1 + i\frac{3\kappa}{8}(\bar{\psi}\psi)\psi * 1 = 0 \quad . \end{aligned} \tag{6.34}$$

where

$$\tau_c[\psi, \hat{\omega}] = \frac{i}{2}[\overline{\psi} * (\gamma \wedge e_c) \wedge \nabla(\hat{\omega})\psi + \overline{\nabla(\hat{\omega})\psi} \wedge *(\gamma \wedge e_c)\psi] + im\overline{\psi}\psi * e_c \tag{6.35}$$

If we rewrite the Einstein-Dirac field equations (**6.34**) explicitly and compare with the equations of Einstein-Cartan-Dirac theory (**6.27**), we find field equations of these two theories are equivalent.

6.3 Stationary, Circularly Symmetric Solutions

We will seek the solutions of the field equations in terms of the local coordinates (t, r, ϕ) given by the metric tensor,

$$g = -f(r)^2 dt^2 + h(r)^2 dr^2 + r^2(d\phi + a(r)dt)^2 \tag{6.36}$$

In the orthonormal co-frames

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \tag{6.37}$$

where we choose

$$e^0 = f(r)dt, \qquad e^1 = h(r)dr, \qquad e^2 = r(d\phi + a(r)dt). \tag{6.38}$$

leads to the Levi-Civita connection 1-forms

$$\hat{\omega}^0{}_1 = \alpha e^0 - \frac{\beta}{2} e^2 \ , \ \hat{\omega}^0{}_2 = -\frac{\beta}{2} e^1 \ , \ \hat{\omega}^1{}_2 = -\gamma e^2 - \frac{\beta}{2} e^0 \tag{6.39}$$

where the new unknown functions

$$\alpha = \frac{f'}{fh}, \quad \gamma = \frac{1}{rh}, \quad \beta = \frac{a'r}{fh} \tag{6.40}$$

are introduced to simplify the calculations with denoting the derivative $\frac{d}{dr}$.

The exterior derivatives of the co-frames become

$$de^0 = \frac{f'}{fh}e^{10}, \qquad de^1 = 0, \qquad de^2 = -\frac{a'r}{fh}e^{01} + \frac{1}{rh}e^{12} \tag{6.41}$$

On the other hand, assuming $i\frac{\kappa}{2}\bar{\psi}\psi = \tau(r)$, we calculate the contortion 1-forms

$$K^0{}_1 = \frac{\tau}{2}e^2 \ , \ K^0{}_2 = -\frac{\tau}{2}e^1 \ , \ K^1{}_2 = -\frac{\tau}{2}e^0. \tag{6.42}$$

We can write the full connection 1-forms as

$$\omega^0{}_1 = \alpha e^0 - \frac{\beta-\tau}{2}e^2 \ , \ \omega^0{}_2 = -\frac{\beta+\tau}{2}e^1 \ , \ \omega^1{}_2 = -\gamma e^2 - \frac{\beta+\tau}{2}e^0 \tag{6.43}$$

As we know GR is based on a space which has real metric functions. But here we find the metric functions can take even Grassman numbers. If we want to generalize GR to the superspace considering the Dirac fields, the metric necessarily involves the even Grassman valued functions. ECD theory is defined here similarly. We can extend the real metric as $\bar{g} = g$ and the (**6.42**) satisfy the requirement. It is need for consistency of the theory with SUGRA and QFT. However, the all following results are also correct for ordinary complex Dirac spinors without any other assumptions.

Using this expressions as a first step towards a solution, we take a Dirac spinor that depends only on r and work out (**6.23**) in components as follows:

$$\psi_1{}' + \frac{h}{2}(\alpha+\gamma)\psi_1 + \frac{h}{4}(\beta+3\tau+4m)\psi_2 = 0 \tag{6.44}$$

$$\psi_2{}' + \frac{h}{2}(\alpha+\gamma)\psi_2 + \frac{h}{4}(\beta+3\tau+4m)\psi_1 = 0. \tag{6.45}$$

It is important to remember that while α, β, γ are the metric functions, τ is the function of torsion tensor. Since differential equations depend on both ψ_1 and ψ_2 simultaneously, we take the combinations $\psi_+ = \psi_1 + \psi_2$ and $\psi_- = \psi_1 - \psi_2$ and write a decoupled system of equations

$$\begin{aligned} \psi_+' + (k_1+k_2)\psi_+ &= 0 \\ \psi_-' + (k_1-k_2)\psi_- &= 0 \end{aligned} \tag{6.46}$$

where we set

$$k_1 = \frac{h}{2}(\alpha+\gamma) \quad , \quad k_2 = \frac{h}{4}(\beta+3\tau+4m).$$

The formal solution to these equations are given by

$$\begin{aligned}
\psi_1 &= e^{-\int^r k_1 dr}\left(\xi_+ e^{-\int^r k_2 dr} + \xi_- e^{\int^r k_2 dr}\right) \\
\psi_2 &= e^{-\int^r k_1 dr}\left(\xi_+ e^{-\int^r k_2 dr} - \xi_- e^{\int^r k_2 dr}\right) \qquad (\mathbf{6.47})
\end{aligned}$$

where ξ_+ and ξ_- are complex, odd Grassmann valued constants. It can easily be verified

$$\tau(r) = i\kappa(\xi_-^*\xi_+ - \xi_+^*\xi_-)e^{-2\int^r k_1 dr}. \qquad (\mathbf{6.48})$$

The function of torsion is a real function with even Grassmann numbers. This is the general solution of the radial Dirac equation with torsion for stationary circularly symmetric metric. Now let us solve the co-frame equations for (**6.36**). The Einstein tensor related with Levi-Civita connection,

$$\hat{G}^0 = \frac{1}{\kappa}[(\frac{\beta'}{2g}+\gamma\beta)e^{01} - (\frac{\gamma'}{g}+\gamma^2+\frac{\beta^2}{4})e^{12}] \qquad (\mathbf{6.49})$$

$$\hat{G}^1 = -\frac{1}{\kappa}(\gamma\alpha+\frac{\beta^2}{4})e^{02} \qquad (\mathbf{6.50})$$

$$\hat{G}^2 = \frac{1}{\kappa}[(\frac{\alpha'}{g}+\alpha^2-\frac{3\beta^2}{4})e^{01} + (\frac{\beta'}{2g}+\beta\gamma)e^{12}] \qquad (\mathbf{6.51})$$

Non-Riemannian part of the Einstein tensor related with torsion or spinor fields

$$-\frac{1}{2\kappa}[\hat{D}K_{af}\wedge *e^{af0} + K_a{}^f\wedge K_{fb}\wedge *e^{ab0}] = \frac{1}{\kappa}[\frac{\tau'}{2g}e^{01}+\frac{\tau^2}{4}e^{12}] \qquad (\mathbf{6.52})$$

$$-\frac{1}{2\kappa}[\hat{D}K_{af}\wedge *e^{af1} + K_a{}^f\wedge K_{fb}\wedge *e^{ab1}] = \frac{1}{\kappa}\frac{\tau^2}{4}e^{02} \qquad (\mathbf{6.53})$$

$$-\frac{1}{2\kappa}[\hat{D}K_{af}\wedge *e^{af2} + K_a{}^f\wedge K_{fb}\wedge *e^{ab2}] = \frac{1}{\kappa}[-\frac{\tau^2}{4}e^{01}-\frac{\tau'}{2g}e^{12}] \qquad (\mathbf{6.54})$$

and the energy-momentum tensor for the Dirac fields

$$\tau^0 = \frac{\tau\alpha}{\kappa}e^{01} + (\frac{i}{2g}\Xi - \frac{\tau^2}{\kappa} - \frac{2m\tau}{\kappa})e^{12} \qquad (\mathbf{6.55})$$

$$\tau^1 = -(\frac{\tau^2}{\kappa}+\frac{2m\tau}{\kappa})e^{02} - \frac{i}{2g}(\psi_1{}^*\psi_1' - \psi_1{}^{*\prime}\psi_1 + \psi_2{}^*\psi_2' - \psi_2{}^{*\prime}\psi_2)e^{12} \qquad (\mathbf{6.56})$$

$$+\frac{i}{2g}(\psi_2{}^*\psi_1' + \psi_1{}^*\psi_2' - \psi_2'^*\psi_1 - \psi_1{}^{*\prime}\psi_2)e^{01} \qquad (\mathbf{6.57})$$

$$\tau^2 = -\frac{\tau\gamma}{\kappa}e^{12} + (-\frac{i}{2g}\Xi + \frac{\tau^2}{\kappa} + \frac{2m\tau}{\kappa} + \frac{\beta\tau}{\kappa})e^{01} \qquad (\mathbf{6.58})$$

where $\Xi = -\psi_1{}^*\psi_1' + \psi_1{}^{*\prime}\psi_1 + \psi_2{}^*\psi_2' - \psi_2{}^{*\prime}\psi_2$. We arrange the equations such that the left hand side of the following equation system represents torsion-less contributions and cosmological constant term, and the right hand side describes with torsion part of Einstein tensor in addition to energy momentum tensor of Dirac spinors. We next work out the Einstein field equations (**6.21**) that after simplifications reduce to the following system of coupled first order differential equations:

$$\frac{\beta'}{2g} + \beta\gamma = -\frac{\tau'}{2g} - \tau\alpha \tag{6.59}$$

$$\frac{\gamma'}{g} + \frac{\beta^2}{4} + \gamma^2 + \lambda\kappa = \frac{3\tau^2}{4} + \frac{\beta\tau}{2} \tag{6.60}$$

$$\frac{\alpha'}{g} - \frac{3\beta^2}{4} + \alpha^2 + \lambda\kappa = \frac{3\tau^2}{4} - \frac{\beta\tau}{2} \tag{6.61}$$

$$-\frac{\beta^2}{4} - \alpha\gamma - \lambda\kappa = \frac{3\tau^2}{4} + 2m\tau \tag{6.62}$$

$$\frac{\beta'}{2g} + \beta\gamma = \frac{\tau'}{2g} + \tau\gamma \tag{6.63}$$

At this point, to be able to find an explicit solution we fix a negative cosmological constant

$$\kappa\lambda = -\frac{1}{l^2} < 0 \tag{6.64}$$

and restrict our attention to those cases for which

$$\gamma = \alpha = \frac{1}{rh} \quad , \quad \tau = \beta = \frac{\beta_0}{r^2}. \tag{6.65}$$

We then integrate for the metric functions

$$f(r) = \frac{r}{l} \quad , \quad h(r) = \frac{l}{r\sqrt{1 - \frac{2m\beta_0 l^2}{r^2} - \frac{\beta_0^2 l^2}{r^4}}}, \tag{6.66}$$

and

$$a(r) = \frac{1}{2l}\arcsin\left(\frac{m}{\sqrt{m^2 + \frac{1}{l^2}}}\right) - \frac{1}{2l}\arcsin\left(\frac{m + \frac{\beta_0}{r^2}}{\sqrt{m^2 + \frac{1}{l^2}}}\right). \tag{6.67}$$

It now remains to integrate for the Dirac spinor and we find

$$\psi_\pm = \frac{\xi_\pm \ell}{r}\left|\frac{r^2}{\ell^2} - m\beta_0 + \sqrt{\frac{r^4}{\ell^4} - \frac{2m\beta_0 r^2}{\ell^2} - \frac{\beta_0^2}{\ell^2}}\right|^{\mp\frac{ml}{2}} e^{\pm\frac{1}{2}\arcsin\left(\frac{m+\frac{\beta_0}{r^2}}{\sqrt{m^2+\frac{1}{l^2}}}\right)} \tag{6.68}$$

where $-\lambda\kappa = \ell^{-2}$, $Q := k\sqrt{\ell^{-2}r^4 - 2\beta_0\ell^2 mr^2 - \beta_0^2\ell^4}$, $k = \pm 1$ and $\beta_0 = i\kappa(\xi_2{}^*\xi_1 - \xi_1{}^*\xi_2)$ is an even Grassmann number. Here dimensions are $[\ell] = [\kappa] = L$ and $[\beta_0] = [m] = [\xi_i] = L^{-1}$.

In order to understand the physical meaning of this solution we write down the metric

$$g = -\frac{r^2}{l^2}dt^2 + \frac{l^2 dr^2}{r^2\left(1 - \frac{2m\beta_0 l^2}{r^2} - \frac{\beta_0^2 l^2}{r^4}\right)} + r^2(d\phi + a(r)dt)^2. \tag{6.69}$$

Firstly we observe that in the absence of a Dirac condensate ($\beta_0 = 0$) the above metric collapses to the AdS_3 metric

$$g_0 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\phi^2. \tag{6.70}$$

Even when $\beta_0 \neq 0$, the metric $g \to g_0$ asymptotically as $r \to \infty$. Secondly we note a metric singularity at

$$r_c = \left\{ \begin{array}{lcl} l\sqrt{m\beta_0 + \beta_0\sqrt{m^2 + \frac{1}{l^2}}} & , & \beta_0 > 0 \\ l\sqrt{|\beta_0|\sqrt{m^2 + \frac{1}{l^2}} - m|\beta_0|} & , & \beta_0 < 0 \end{array} \right\}. \tag{6.71}$$

This is a coordinate singularity as evidenced by a further calculation of the curvature scalar

$$\mathscr{R} = -\frac{6}{l^2} + \frac{4m\beta_0}{r^2} \tag{6.72}$$

and the quadratic curvature invariant

$$*(R_{ab} \wedge *R^{ab}) = \frac{6}{l^4} - \frac{8m\beta_0}{l^2 r^2} + \frac{\beta_0^2(8m^2 - \frac{4}{l^2})}{r^4} + \frac{16m\beta_0^2}{r^6} + \frac{8\beta_0^4}{r^8} \tag{6.73}$$

that are regular at $r = r_c$. However, these curvature invariants exhibit an essential singularity at $r = 0$. Such a configuration resembles to a black hole for which the essential curvature singularity at the coordinate origin is hidden behind an event horizon. A global extension of the above solution is tedious and will not be attempted here. Instead we will check the quasi-local conserved quantities associated with our solution at a distance $r > r_c$. A comprehensive discussion of the conserved quasi-local quantities for gravitating systems within the framework of general relativity may be found in [41, 42]. For calculational details in $(1+2)$ dimensions we again refer to Ref. [43].

The quasi-local angular momentum is a constant

$$J(r) = \frac{r^3}{f(r)h(r)}\frac{da}{dr} = \beta_0, \tag{6.74}$$

the quasi-local energy is

$$E(r) = \frac{1}{h_0(r)} - \frac{1}{h(r)} \tag{6.75}$$

where the first term describes the contribution of the background "empty" spacetime. Using the metric functions in (**6.69**),

$$E(r) = \frac{r}{l} - \sqrt{\frac{r^2}{l^2} - 2m\beta_0 - \frac{\beta_0^2}{r^2}} \simeq \frac{\beta_0 ml}{r}, \tag{6.76}$$

and the quasi-local mass is determined by the expression

$$M(r) \;\; = \;\; 2f(r)E(r) - J(r)a(r). \tag{6.77}$$

We can calculate the quasi-local mass for the system as;

$$\begin{aligned} M(r) \;\; &= \;\; 2\frac{l^2}{r^2} - 2\frac{l^2}{r^2}\sqrt{1 - \frac{2m\beta_0 l^2}{r^2} - \frac{\beta_0^2 l^2}{r^4}} \\ &\quad + \frac{1}{2l}\arcsin\left(\frac{m}{\sqrt{m^2+\frac{1}{l^2}}}\right) - \frac{1}{2l}\arcsin\left(\frac{m+\frac{\beta_0}{r^2}}{\sqrt{m^2+\frac{1}{l^2}}}\right) \\ &\simeq \;\; 2m\beta_0. \end{aligned} \tag{6.78}$$

in the limit as $r \to \infty$.

The results of this chapter is submitted for publication [44]. Also, a similar model has been considered [45] where the space-time torsion was introduced independent of a Dirac spinor. The static solutions were discussed there rather than the stationary solutions as given here.

7. CONCLUSION

Firstly, we have investigated gravitation theories considering non-minimally coupled electromagnetic fields to gravity. We have derived the field equations of the theories with torsion and without torsion using exterior algebra of differential forms by the first order variation procedure. We investigated static spherically symmetric and pp-wave solutions. We found an exact magnetic monopole solution for the field equations of Einstein-Cartan-Maxwell theory. The torsion affects the singularity of the solution and do not allow a singularity at the central point. We give also a class of exact plane fronted wave solutions in Brinkmann form to the field equations for a generic 6-parameter non-minimally coupled terms. Maxwell field equations are not changed by the nonminimal coupled terms for the pp-wave metric. But, Einstein field equations allow a class of nontrivial solutions by the presence of the first two non-minimal couplings of electromagnetic fields to gravity. Additionally, The energy-momentum transported by the plane-fronted waves is modified by the nonminimal coupled terms.

Secondly, we have formulated the Einstein-Cartan-Dirac theory in (1+2)-dimensions using the algebra of exterior differential forms. We coupled a Dirac spinor to Einstein-Cartan gravity and obtained the field equations by a variational principle. The space-time torsion is given algebraically in terms of the quadratic spinor invariant associated with a Dirac condensate field. We then looked for rotating, circularly symmetric solutions. We found a particular class of solutions that possess an essential curvature singularity at the origin $r = 0$. The mass and the intrinsic angular momentum of this configuration can be identified. It is remarkable that in the absence of the Dirac condensate field the metric collapses to the regular AdS_3 metric.

REFERENCES

[1] **Dereli, T.,** 1984. Differential Forms and Maxwell Equations, Lectures Notes, TÜBİTAK Graduate Summer School, 18-28 Sep.

[2] **Flanders, H.,** 1963. *Differential Forms with Applications to the Physical Sciences,* ISBN 0122596501, Academic Press, New York, USA.

[3] **Thring, W.,** 1997. *Classical Mathematical Physics: Dynamical Systems and Field Theories,* (3^{rd} Edition), ISBN 0-397-94843-0, 539p., Springer-Verlag.

[4] **Cartan, E.**, 1923. *On manifolds with an affine connection and the theory of general relativity,* edited 1986, Bibliopolis, Italy.

[5] **Dereli, T., Önder, M., Tucker, R.W.,** 1995: Solutions for neutral axi-dilaton gravity in four dimensions, *Class. Quant. Grav.*, **12**, L25.

[6] **Dereli, T., Önder, M., Schray, J., Tucker, R. W., Wang, C.**, 1996. Non-Riemannian Gravity and the Einstein-Proca system, *Class. Quant. Grav.*, **13**, L103.

[7] **Hehl, F. W., McCrea, J. D., Mielke, E. W., Ne'eman, Y.,** 1995: Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identies, World Spinors and Breaking of Dilaton Invariance, *Phys. Rep.,* **12**, L31.

[8] **Adak M., Sert Ö.,** 2005. A solution to symmetric teleparallel gravity, *Turk. J. Phys.*, **29**, 1-7.

[9] **Adak M., Kalay M., Sert Ö.,** 2006. Lagrange formulation of the symmetric teleparallel gravity, *Int. J. Mod. Phys. D*, **15**, 619.

[10] **Trautman, A.,** 1972a, b, c, 1973a. On the Einstein-Cartan equations. I-IV, *Bull. Acad. Polon. Scie.*, **20**, 185, 503, 895, ibid.**21** 345.

[11] **Ehlers, J., Kundt W.** 1962. *Gravitation: An Introduction to Current Research*, edited by L. Witten, Wiley, New York.

[12] **Aichelburg, P. C.**, 1971. Remark On The Superposition Principle For Gravitational Waves, *Acta Phys. Austr.*, **34**, 279.

[13] **Dereli, T. and Tucker, R. W.,** 2004. On the energy-momentum density of gravitational plane waves, *Class. Q. Grav.*, **21** , 1459-1464.

[14] **Stephani, H., Kramer, D., Maccallum, M., Hoenselaers, C., and Herlt, E.,** 2005. *Exact Solutions to Einstein's Field Equations*, Second ed., Chambridge University Press, UK.

[15] **Prasanna, A. R.,** 1971. A new invariant for electromagnetic fields in curved space-time, *Phys. Lett.*, **A37**, 331.

[16] **Horndeski, G. W.,**Conservation of charge and Einstein-Maxwell field equations, 1976. *J. Math. Phys.* **17** 1980.

[17] **I. T. Drummond and S. J. Hathrell** , 1980. QED vacuum polarization in a background gravitational field and its effect on the velocity of photons, *Phys. Rev. D* **22**, 343.

[18] **Buchdahl, H. A.,** 1979. On a Lagrangian for non-minimally coupled gravitational and electromagnetic fields, *J. Phys. A*, **12**, 1037.

[19] **Müller-Hoissen, F.,** 1988. Non-minimal coupling from dimensional reduction of the Gauss-Bonet action, *Class. Q. Grav.*, **5**, L35.

[20] **Dereli, T., Üçoluk, G.,** 1990. Kaluza-Klein reduction of generalized theories of gravity and nonminimal gauge couplings, *Class. Q. Grav.*, **7**, 1109.

[21] **Balakin, A. B.,** 2005. Non-minimal coupling for the gravitational and electromagnetic fields: a general system of equations, *Class. Q. Grav.*, **22**, 1867.

[22] **Balakin, A.B., Bochkarev, V.V., Lemos, J.P.S.** 2008. Nonminimal coupling for the gravitational and electromagnetic fields: Black hole solutions and solitons, *Phys. Rev. D*, **77**, 084013.

[23] **Balakin, A. B., Zimdahl, W.** 2005. Anisotropic cosmological models with nonminimally coupled magnetic field, *Phys. Rev. D* **71**, 124014.

[24] **Balakin, A.B., Ni, W.T.,** 2010. Non-minimal coupling of photons and axion, *Grav. Cosm.* **27**, 055003, p.23.

[25] **Dereli, T., Gratus, J. and Tucker, R.W.,**, 2007. New perspectives on the relevance of gravitation for the covariant description of electromagnetically polarizable media, *J. Phys. A: Math. Theor.* **40**, 5695.

[26] **Dereli, T., Gratus, J. and Tucker R.W.**, 2007. The covariant description of electromagnetically polarizable media, *Physics Letters A* **361**, 3, 190-193.

[27] **Das, M., Mohanty, S., Prasanna, A.R.,** 2009. Constraints on background torsion from birefringence of CMB polarization, arXiv:0908.0629v1.

[28] **T. Dereli, Ö. Sert**, 2011. Non-minimally Coupled Gravitational and Electromagnetic Fields: pp-Wave Solutions, *Phys. Rev. D* 83, 065005, arXiv:1101.1177.

[29] **Misner, C. W., Thorne, K. S. and Wheeler, J. A.**, 1973. Gravitation, SanFrancisco: Freeman.

[30] **S. Deser, A. Waldron,** 2006. Partially massless spin-2 electrodynamics, *Phys. Rev. D* **74**, 084036.

[31] **Yu. M. Zinoviev,** 2009. On massive spin 2 electromagnetic interactions, *Nucl. Phys. B* **821**, 431.

[32] **Balakin, A. B., Dehnen, H., Zayats, A.E.,** 2009. Non-minimal monopoles of the Dirac type as realization of the censorship conjecture, *Phys. Rev. D*, **79**, 024007.

[33] **Balakin, A. B., Zayats, Alexei, E.,** 2008. Ray optics in the field of a nonminimal Dirac monopole, *Grav. Cosm.* **14** 86-94, arXiv:0710.0606.

[34] **Deser, S., Jackiw, R., Templeton, S.,** 1982. Three-Dimensional Massive Gauge Theories, *Ann. Phys.* **140**, 372.

[35] **Banados, M., Teitelboim, C., Zanelli, J.,** 1992. Black hole in three-dimensional spacetime, *Phys. Rev. Lett.*, **69**, 1849.

[36] **Carlip, S.,** 1998. *Quantum Gravity in (2+1)-Dimensions*, Cambridge University Press, UK.

[37] **Garcia, A. A., Hehl, F. W., Heinicke, C., Macias, A.**, 2003. Exact vacuum solution of a (1+2)-dimensional Poincaré gauge theory: BTZ solution with torsion, *Phys. Rev. D*, **67**, 124016.

[38] **Trautman, A.,** 2006. Einstein-Cartan theory, in Encyclopedia of Mathematical Physics Volume:2, edited by J.-P. Francoise, G. L. Naber, Tsou S.T., Elsevier, p. 189, arXiv:gr-qc/0606062 and references therein.

[39] **Hehl, F. W., Datta, B. K.**, 1971. Nonlinear spinor equation asymmetric connection in general relativity, *J. Math. Phys.* **12**, 1334.

[40] **Dereli, T., Tucker, R. W.**, 1988. Gravitational Interactions in 2+1 Dimensions, *Class. Quant. Grav.* **5**, 951.

[41] **Brown, J. D. York Jr., J. W.**, 1993. Quasilocal energy and conserved charges derived from the gravitational action, *Phys. Rev. D*, **47**, 1407.

[42] **Brown, J. D., Creighton, J., Mann, R. B.**, 1994. Temperature, energy, and heat capacity of asymptotically anti-de Sitter black holes, *Phys. Rev. D* **50**, 6394.

[43] **Dereli, T., Obukhov, Y.** , 2000. General analysis of self-dual solutions for the Einstein-Maxwell-Chern-Simons theory in (1+2) dimensions, *Phys. Rev. D* , **62**, 0240013.

[44] **Dereli, T., Özdemir, N., Sert, Ö.**, 2010. Einstein-Cartan-Dirac theory in (1+2)-dimensions, arXiv:1002.0958v1 [gr-qc].

[45] **Hortaçsu, M., Özçelik, H.T., Özdemir, N.,** 2008. 2+1 dimensional solution of Einstein Cartan equations, arXiv:0807.4413[gr-qc].

APPENDIX

APPENDIX A: Derivation of Field Equations from a General Action.

A. DERIVATION OF FIELD EQUATIONS FROM A GENERAL ACTION

Let M be an n-dimensional manifold and $\alpha, \beta \in \Lambda^p(M)$ where $\Lambda^p(M)$ denotes any p-form on M. Then we would like to find the extremum of the following action integral

$$I[\alpha,\beta,e^a] = \int_M \alpha \wedge *\beta \qquad \textbf{(A.1)}$$

by varying it with respect to the dependent variables [1]; α, β and e^a [2].

$$\delta I = \int_M \delta\alpha \wedge *\beta + \alpha \wedge \delta * \beta \qquad \textbf{(A.2)}$$

Here, variation of the second term on the right hand side needs some calculations to be given in detail, because it contains Hodge star.

$$\begin{aligned} \alpha \wedge \delta * \beta &= \alpha \wedge \delta(\frac{1}{p!}\beta_{i_1\cdots i_p} * e^{i_1\cdots i_p}) \\ &= \alpha \wedge \frac{1}{p!}(\delta\beta_{i_1\cdots i_p}) * e^{i_1\cdots i_p} + \alpha \wedge \frac{1}{p!}\beta_{i_1\cdots i_p}\delta * e^{i_1\cdots i_p} \end{aligned} \qquad \textbf{(A.3)}$$

First, use the identity in the first term

$$\theta \wedge *\vartheta = \vartheta \wedge *\theta \qquad \textbf{(A.4)}$$

where θ , $\vartheta \in \Lambda^p(M)$

$$\alpha \wedge \delta * \beta = \frac{1}{p!}(\delta\beta_{i_1\cdots i_p})e^{i_1\cdots i_p} \wedge *\alpha + \alpha \wedge \frac{1}{p!}\beta_{i_1\cdots i_p}\delta * e^{i_1\cdots i_p} \; . \qquad \textbf{(A.5)}$$

By making the use of the equality

$$\begin{aligned} \delta\beta &= \delta(\frac{1}{p!}\beta_{i_1\cdots i_p}e^{i_1\cdots i_p}) \\ &= \frac{1}{p!}(\delta\beta_{i_1\cdots i_p})e^{i_1\cdots i_p} + (\delta e^{i_1}) \wedge \frac{1}{(p-1)!}\beta_{i_1\cdots i_p}e^{i_2\cdots i_p} \\ &= \frac{1}{p!}(\delta\beta_{i_1\cdots i_p})e^{i_1\cdots i_p} + (\delta e^{i_1}) \wedge (\iota_{i_1}\beta) \qquad \textbf{(A.6)} \\ \frac{1}{p!}(\delta\beta_{i_1\cdots i_p})e^{i_1\cdots i_p} &= \delta\beta - (\delta e^a) \wedge (\iota_a\beta) \qquad \textbf{(A.7)} \end{aligned}$$

and the equality

$$\begin{aligned} \frac{1}{p!}\beta_{i_1\cdots i_p}\delta * e^{i_1\cdots i_p} &= \frac{1}{p!}\beta_{i_1\cdots i_p}\delta\left[\frac{1}{(n-p)!}\varepsilon^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n}e^{i_{p+1}\cdots i_n}\right] \\ &= (\delta e^{i_{p+1}}) \wedge \frac{1}{p!(n-p-1)!}\varepsilon^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n}\beta_{i_1\cdots i_p}e^{i_{p+2}\cdots i_n} \\ &= (\delta e^a) \wedge (\iota_a * \beta) \end{aligned} \qquad \textbf{(A.8)}$$

[1] Since these variables are dependent of chart coordinates x^μ, we call them as "dependent variables".
[2] These kinds of Lagrangian contains co-frame via the Hodge star.

in (**A.5**) and then substituting the results into (**A.2**) we obtain

$$\begin{aligned}\delta I &= \delta\int_M \alpha\wedge *\beta \\ &= \int_M \delta\alpha\wedge *\beta+\delta\beta\wedge *\alpha-\delta e^a\wedge[(\iota_a\beta)\wedge *\alpha-(-1)^p\alpha\wedge(\iota_a *\beta)]\end{aligned} \quad \textbf{(A.9)}$$

where α , $\beta\in\Lambda^p(M)$.

- Special Case: $\alpha=\beta=F=dA$

We encounter these kinds of Lagrangians, especially in electromagnetic theory and symmetric teleparallel gravity models. In this case (**A.9**) becomes

$$\delta I=\int_M(\delta dA)\wedge(2*dA)-\delta e^a\wedge[(\iota_aF)\wedge *F-(-1)^pF\wedge(\iota_a *F)] \quad \textbf{(A.10)}$$

where $A\in\Lambda^{p-1}(M)$. Since variation and exterior derivative commute with each other

$$\delta d=d\delta \quad \textbf{(A.11)}$$

the equation may be rewritten

$$\begin{aligned}\delta I &= \int_M(d\delta A)\wedge(2*dA)-\delta e^a\wedge[(\iota_aF)\wedge *F-(-1)^pF\wedge(\iota_a *F)] \\ &= \int_M(\delta A)\wedge(-1)^p(2d*F)+d(\delta A\wedge 2*F) \\ &\qquad -\delta e^a\wedge[(\iota_aF)\wedge *F-(-1)^pF\wedge(\iota_a *F)]\end{aligned} \quad \textbf{(A.12)}$$

By applying the Stoke's theorem, the second term on the right hand side can be written

$$\int_M d(\delta A\wedge 2*F)=\int_{\partial M}\delta A\wedge 2*F=0 \quad \textbf{(A.13)}$$

because the boundary condition is $\delta A|_{\partial M}=0$ where ∂M is the boundary of M. Thus

$$\begin{aligned}\delta I &= \delta\int_M dA\wedge *dA \\ &= \int_M(\delta A)\wedge(-1)^p(2d*F)-\delta e^a\wedge[(\iota_aF)\wedge *F \\ &\qquad -(-1)^pF\wedge(\iota_a *F)]\end{aligned} \quad \textbf{(A.14)}$$

where $F=dA\in\Lambda^p(M)$.

CURRICULUM VITA

Candidate's Full Name	**:**	Özcan SERT
Place and date of birth	**:**	Denizli, 13/08/1982.
Universities and colleges attended:	**:**	Pamukkale University, M.Sc., in Physics , 2005. Pamukkale University, B.S., in Physics, 2002.

Publications :

1. Adak M., **Sert Ö.** , 2005. "A solution to symmetric teleparallel gravity" Turk. J. Phys., 29, 1-7.

2. Adak M., Kalay M., **Sert Ö.**, 2006. "Lagrange formulation of the symmetric teleparallel gravity", *Int. J. Mod. Phys. D*, **15**, 619-634.

3. T. Dereli, N. Ozdemir, **O. Sert**, 2010. Einstein-Cartan-Dirac Theory in (1+2)-Dimensions, arXiv:1002.0958.

4. T. Dereli, **Ö. Sert**, 2011. Non-minimally Coupled Gravitational and Electromagnetic Fields: pp-Wave Solutions, *Phys. Rev. D* 83, 065005, arXiv:1101.1177.

5. T. Dereli, **Ö. Sert**, 2011. Non-minimal $lnRF^2$ Couplings of Electromagnetic Fields to Gravity: static, spherically symmetric solutions, *EPJ C* **71**, 3, 1589, DOI: 10.1140/epjc/s10052-011-1589-2 , arXiv:1102.3863.

Conferences:

1. Dereli, T., **Sert, Ö**. "Non-minimally Couplings of Electromagnetic Fields to Gravity" , 8. Wokshop on Quantization, Dualities and Integrable Systems, 23-25 Nisan 2009, Ankara Üniversitesi, Tandoğan Kampüsü, Ankara.

2. Dereli T., Özdemir N., **Sert Ö.**, "1+2 Boyutta Einstein-Cartan-Dirac Teorisi", Mahmut Hortaçsu'ya Saygı Toplantısı, 22 Ocak 2010 İ.T.Ü. Maslak Kampüsü, İstanbul.