İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

CANONICAL FORMS FOR FAMILIES OF ANTI-COMMUTING DIAGONALIZABLE LINEAR OPERATORS

M.Sc. Thesis by
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Programme : Mathematical Engineering

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FOREWORD

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SUMMARY

In this thesis, we examine canonical forms for families of anti-commuting diagonalizable linear operators on finite dimensional vector spaces.

We begin with basic definitions, basic concepts of linear algebra and a review of the structures of Clifford algebras. Then, we review a well-known result on the simultaneous diagonalization of a family of commuting linear operators on a finite dimensional vector space which asserts that an arbitrary family of commuting diagonalizable operators can be simultaneously diagonalized.

In Section 2, we consider an anti-commuting family $\mathcal{A}$ of diagonalizable operators on a finite dimensional vector space V. Real or complex representations of Clifford algebras are typical anti-commuting diagonalizable (over $\mathbb{C}$) families. In order to give a motivation for general case, we give a detailed construction for two and three element families of anti-commuting diagonalizable linear operators in Sections 2.1 and 2.2.

Our main result is that V has an $\mathcal{A}$-invariant direct sum decomposition into subspaces V_α such that the restriction of the family to each V_α summand either consists of a single nonzero operator or it is a representation of some Clifford algebra. This result, presented in Section 2.2, is derived directly from the fact that the squares of the operators in $\mathcal{A}$ form a commuting family of diagonalizable operators whose kernels are the same as the original family. One can then simultaneously diagonalize $\mathcal{A}$, rearrange the basis and obtain subspaces on which there are families of non-degenerate anti-commuting operators whose squares are constants.

Closing Section 2, we modify our results to a more general form of anti-commuting families and replace the diagonalizability condition by the requirement that the square of the family is diagonalizable. Then, we show that V has a direct sum decomposition such that each summand is a representation of a degenerate or non-degenerate Clifford algebra.

Since the classifications of Clifford algebras and their representations are well known, it is thus in principle possible to give a complete characterization of anti-commuting families of diagonalizable operators. In last section, we give some classifications of real and complex representations of Clifford algebras.

TERS-DEĞİŞMELİ KÖŞEGENLEŞTİRİLEBİLİR LİNEER OPERATÖR AİLELERİ İÇİN KANONİK FORMLAR

ÖZET

Bu çalışmada, sonlu boyutlu bir vektör uzayı üzerinde ters-değişmeli köşegenleştirilebilir lineer operatörler ailelerinin kanonik formlarını inceledik.

İlk bölümde Lineer cebir ve Clifford cebirlerinin temel tanımlamaları ve özellikleriyle başladık. Ardından sonlu boyutlu vektör uzayında değişmeli bir lineer operatörler ailesinin eş zamanlı köşegenleştirilmesi hakkında iyi bilinen bir sonucu verdik. Bu sonuca göre herhangi bir değişmeli köşegenleştirilebilir operatörler ailesi eş zamanlı köşegenleştirilebilir.

İkinci bölümde, sonlu boyutlu bir V vektör uzayı üzerinde ters-değişmeli köşegenleştirilebilir bir $\mathcal{A}$ operatörler ailesini ele aldık. Clifford cebirlerinin reel veya kompleks temsilleri ters-değişmeli ($\mathbb{C}$ üzerinde) köşegenleştirilebilir ailelerin tipik örneklerindendir. Genel durum için bir yön çizmesi açısından 2.1 ve 2.2. bölümlerde, iki ve üç elemanlı ters-değişmeli köşegenleştirilebilir lineer operatörler ailelerinin inşası için detaylı bir yapı verdik.

Çalışmanın ana sonucu şu şekildedir: V'nin V_α alt uzaylarına öyle bir $\mathcal{A}$-invaryant direkt toplam dekompozisyonu vardır ki $\mathcal{A}$'nın her V_α'ya kısıtlanışı ya sıfırdan farklı bir tane operatörden oluşur ya da bazı Clifford cebirlerinin bir temsilidir. İkinci bölümde sunulan bu sonuç, direkt olarak $\mathcal{A}$'daki operatörlerin karelerinin aynı çekirdeklere sahip ama değişmeli bir köşegenleştirilebilir operatörler ailesi oluşturmasından çıkarılmıştır. Bundan sonra $\mathcal{A}$'yı eş zamanlı köşegenleştirme, baz vektörlerini yeniden düzenleme işlemleri gerçekleştirilerek, kareleri sabit dejenere olmayan ters-değişmeli operatörler ailelerinin bulunduğu V'nin alt-uzayları elde edilebilir.

2. Bölüm'ü kapatırken, bulgularımızı ters-değişmeli ailelerin daha genel bir formuna modifiye ettik ve köşegenleştirilebilirlik koşulunu ailenin kendisinin değil karesinin köşegenleştirilebilir olması gerekliliği ile değiştirdik. Bu durumda V'nin öyle bir direkt toplam dekompozisyonu vardır ki toplamın her bir elemanı bir dejenere veya dejenere olmayan bir Clifford cebrinin bir temsili olur.

Clifford cebirleri ve temsillerinin sınıflandırması iyi bilindiği için, ters-değimeli köşegenleştirilebilir operatör ailelerinin tam bir karakterizasyonunu vermek prensipte mümkündür. Son bölümde, Clifford cebirlerinin reel ve kompleks temsillerinin sınıflandırması ile ilgili bilgiler verdik.

1. INTRODUCTION

1.1. Notation and Basic Definitions

In the following V is a finite dimensional real ($\mathbb{R}$) or complex ($\mathbb{C}$) vector space. Linear operators on V will be denoted by upper case Latin letters A, B etc, and the components of their matrices with respect to some basis will be denoted by A_{ij}, B_{ij} respectively. Labels of operators will be denoted by single indices from the beginning of the alphabet, for example A_a, $a = 1, \ldots, n$ denotes elements of a family of operators. A family $\mathcal{A}$ of operators is called an "anti-commuting family" if for every distinct pair of operators A and B in the family, $AB + BA = 0$. The symbol δ_{ij} denotes the Kronecker delta, that is $\delta_{ij} = 1$, if $i = j$ and zero otherwise. When we shall use partitioning of matrices, lower case letters will denote sub-matrices of appropriate size. $R(n), C(n)$ and $H(n)$ denote $n \times n$ matrices with real, complex and quaternionic entries respectively. Now, we give definitions of basic algebraic structures.

Definition 1.1.1. A group G is a set closed under a binary operation $*$, satisfying the following conditions

i. $(a * b) * c = a * (b * c),$ for all $a, b, c \in G$ (associativity).

ii. There is $e \in G$ such that $e * a = a * e = a$ for all $a \in G$ (identity).

iii. There is $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$ for all $a \in G$ (inverse).

G is called abelian, if $*$ is a commutative operation, i.e $a * b = b * a$ for all $a, b \in G$ [4].

Definition 1.1.2. A ring R is a set with two binary operations, addition $+$ and multiplication $\cdot$ such that

i. R is an abelian group with addition.

ii. Multiplication is associative.

iii. $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(a+b) \cdot c = (a \cdot c) + (b \cdot c)$ holds for all $a, b, c \in R$.

A commutative division ring is called a field [4].

In this thesis, we will work exclusively with either the field of real or complex numbers, $\mathbb{R}$ and $\mathbb{C}$. In addition, we will also use the divison ring of quaternions which is sometimes called a skew-field.

Definition 1.1.3. A vector space V over a field F is an abelian group under addition with scalar multiplication of each element of V, i.e. vectors, by each element of F, i.e. scalars, on the left satisfying the following conditions:

i. $av \in V$.

ii. $a(bv) = (ab)v$.

iii. $(a+b)v = av + bv$.

iv. $a(v+w) = av + aw$.

v. $1v = v$.

for all $a, b \in F$ and for all $v, w \in V$ where scalar multiplication is a function: $F \times V \to V$ [4].

Definition 1.1.4. An algebra is a vector space V over a field F, with a binary operation of multiplication of vectors in V satisfying the following three conditions:

i. $(au)v = a(uv) = u(av)$.

ii. $(u+v)w = uw + vw$.

iii. $u(v+w) = uv + uw$.

for all $a \in F$ and for all $u, v, w \in V$ [4]. V is a division algebra over F if V has a multiplicative identity and it contains a multiplicative inverse for every nonzero element in V.

Definition 1.1.5. Let R be a ring. An R-module is an abelian group M with multiplication of each element of M by each element of R on the left satisfying

i. $(rv) \in M$.

ii. $r(u+v) = ru + rv$.

iii. $(r+s)u = ru + su$.

iv. $(rs)u = r(su)$,

for all $r, s \in R$ and for all $u, v \in M$ [4].

Definition 1.1.6. Let V be a vector space over a field F and let v_i, $i = 1, \ldots, n$ be vectors in V. Then $X = c_1 v_1 + \cdots + c_n v_n$ where c_i's are in F, is called a linear combination of the vectors v_i. If U is a subset of V and if every vector of V is a linear combination of the vectors in U, then we say that the vectors of U span V.

Definition 1.1.7. Let V be a vector space over a field F and U be a subset of V. If $c_1 v_1 + \cdots + c_n v_n = 0$ where c_i's are in F and v_i's are in U, implies that $c_1 = \cdots = c_n = 0$, then v_i's are called linearly independent.

Definition 1.1.8. Let V be a vector space over a field F and U be a subset of V. If U is linearly independent and if it spans V then it is called a basis of V. If V is finite dimensional, then the number of vectors in any basis is the same and this common number is called the dimension of V [4]. Note that, the definition of linear combination involves a finite number of vectors. Thus, if U is a basis for V, then every vector in V should be written as a finite linear combination of the vectors in U.

Next, we define linear operators.

Definition 1.1.9. Let V and W be vector spaces over the field F. A function $L : V \to W$ satisfying

$$L(au + v) = a(L(u)) + L(v), \quad \forall a \in F, \quad \forall u, v \in V \tag{1.1}$$

is called a linear transformation of V into W. If especially $W = V$, then L is called a linear operator on V [1].

Definition 1.1.10. Let L be a linear operator on a vector space V over a field F. If for $a \in F$, there is a non-zero vector $v \in V$ satisfying the equation

$$(L - aI)v = 0 \tag{1.2}$$

then a is called a characteristic value of L and v is called a characteristic vector corresponding to a [1].

Lastly in this section, we define "quadratic form"s which will be used in defining Clifford algebras.

Definition 1.1.11. A homogeneous polynomial of degree two in a number of variables with coefficients from a field k is called a quadratic form over k and the associated bilinear form of a quadratic form q is defined by

$$2q(v,w) = q(v+w) - q(v) - q(w). \qquad \textbf{(1.3)}$$

1.2. Basic Properties of Linear Operators

In this section, we give properties on diagonalizability which will be useful in construction of theorems in Section 2.

Definition 1.2.1. Let D be a linear operator on a finite dimensional vector space V. If there is a basis of V such that each basis vector is a characteristic vector of D, then D is diagonalizable [1].

Definition 1.2.2. Let L be a linear operator on a vector space V over a field F. The subset defined by

$$Ker(L) = \{v \in V : L(v) = 0\} \qquad \textbf{(1.4)}$$

where 0 is the zero vector, is called the kernel of L.

Proposition 1.2.3. *Let A be a linear operator on a vector space V. If A is diagonalizable, $Ker(A^2) = Ker(A)$.*

Proof. We choose a basis with respect to which A is diagonal. Then eigenvalues of A^2 are squares of eigenvalues of A. Hence, obviously $Ker(A^2) = Ker(A)$.

Definition 1.2.4. The minimal polynomial p for a linear operator L over a field F is uniquely determined by the following three conditions

i. p is monic over F which means that it has 1 as the highest coefficient.

ii. $p(L) = 0$.

iii. p has the smallest degree among polynomials satisfying (ii).

[1]. In this thesis we are concerned with minimal polynomials over $\mathbb{C}$.

Definition 1.2.5. Let $M_n(F)$ be a family of $n \times n$ matrices over a field F and let $A, B \in M_n(F)$. A is said to be similar to B if there exists a nonsingular matrix $C \in M_n(F)$ such that $B = C^{-1}AC$. A Jordan block J (over $\mathbb{C}$) is a lower (upper) triangular matrix which has the form

$$J = \begin{bmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda & 0 \\ 0 & 0 & \cdots & 1 & \lambda \end{bmatrix}. \tag{1.5}$$

A Jordan matrix is a direct sum of Jordan blocks and a Jordan matrix which is similar to a matrix A is called the Jordan canonical form of A [5].

The following proposition will be used in order to derive Corollary 2.3.1

Proposition 1.2.6. *Let A be a non-diagonalizable linear operator. Then A^2 is diagonalizable (over $\mathbb{C}$) if and only if each Jordan block J_i of A is either diagonal or it satisfies $J_i^2 = 0$.*

Proof. Assume that A has a non-diagonal Jordan block J with eigenvalue λ of size n as in (**1.5**). Then J^2 has the following form

$$J^2 = \begin{bmatrix} \lambda^2 & 0 & \cdots & 0 & 0 \\ 2\varepsilon\lambda & \lambda^2 & 0 & \cdots & 0 \\ 1 & 2\lambda & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \lambda^2 & 0 \\ 0 & 0 & 1 & 2\lambda & \lambda^2 \end{bmatrix}. \tag{1.6}$$

Recall that if J^2 is diagonalizable, then its minimal polynomial has to be a product of factors that are linear over the complex numbers. From (**1.6**) above it is clear that if J^2 is diagonalizable then $J^2 - \lambda^2$ should be zero. The first subdiagonal consists of the terms λ, so we should take λ equal to zero. Then the second subdiagonal consists of 1 and $J^2 = 0$, hence the size of the Jordan blocks should be at most 2. The proof of the converse is obvious.

1.3. Properties of Commuting and Anti-Commuting Families of Diagonalizable Linear Operators

In this section, we give certain properties of commuting and anti-commuting families of diagonalizable linear operators. We begin with the following remark

Remark 1.3.1. If A is a diagonalizable operator on a vector space V and V has an A-invariant direct sum decomposition, then from Lemma 1.3.10 in [5] the restriction of A to each invariant subspace is also diagonalizable. Furthermore if we have a family of commuting (anti-commuting) operators on V and V has an $\mathcal{A}$ invariant direct sum decomposition, then the restriction of the family to each summand is again a commuting (anti-commuting) family of diagonalizable operators.

Now, we give Theorem 1.3.2 whose proof is adopted from [5].

Theorem 1.3.2. *Let $\mathcal{D}$ be a family of diagonalizable operators on a finite dimensional vector space V and $A,B \in \mathcal{D}$. Then A and B commute if and only if they are simultaneously diagonalizable.*

Proof. Assume that

$$AB = BA \tag{1.7}$$

holds. By a choice of basis we may assume that A is diagonal, that is $A_{ij} = \lambda_i \delta_{ij}$, $i = 1,...,k$. From (**1.7**) we have

$$(\lambda_i - \lambda_j)B_{ij} = 0, \tag{1.8}$$

that is B_{ij} is nonzero unless $\lambda_i = \lambda_j$. Rearranging the basis, we have a decomposition of V into eigenspaces of A. This decomposition is B invariant, on each subspace A is constant, B is diagonalizable, hence they are simultaneously diagonalizable.
Conversely, assume that A and B are simultaneously diagonalizable. Then, there is a basis with respect to which their matrices are diagonal. Since diagonal matrices commute, it follows that the operators A and B commute.

Remark 1.3.3. If A is diagonalizable, then A^2 is also diagonalizable.Also if the pair (A,B) anti-commutes then the pairs (A,B^2) and (A^2,B^2) commute, since

$$AB^2 = -B(AB) = B^2A, \quad A^2B^2 = A(AB^2) = A(B^2A) = (B^2A)A = B^2A^2.$$

Thus given a family $\{A_1,\ldots,A_N\}$ of anti-commuting diagonalizable operators the families $\{A_1,A_2^2\ldots,A_N^2\}$ and $\{A_1^2,\ldots,A_N^2\}$ are commuting diagonalizable families, hence they are both simultaneously diagonalizable.

Remark 1.3.4. The family $\mathcal{A}$ of anti-commuting diagonalizable linear operators on an n-dimensional vector space V is necessarily linearly independent and it contains $N < n^2$ elements. Because if B is a linear combination of the A_i $i = 1,\ldots,k$, i.e, $B = \sum_i^k c_i A_i$ and B anti-commutes with each of the A_j's in this summation it is necessarily zero. Furthermore, since the anti-commuting family cannot include the identity matrix, it follows that $N < n^2$ [6].

1.4. Clifford Algebras

One of the most interesting examples of anti-commuting families of linear operators is the Clifford algebras that are defined below.

Definition 1.4.1. Let V be a vector space over a field k and q be a quadratic form on V. Then the associative algebra with unit, generated by the vector space and the identity 1 subject to the relations

$$v \cdot v = -q(v)1, \tag{1.9}$$

for any $v \in V$ is called a Clifford algebra and denoted by $Cl(V,q)$.

If the characteristic of k is not 2, then (**1.9**) can be replaced by

$$v \cdot w + w \cdot v = -2q(v,w), \tag{1.10}$$

for all $v,w \in V$ [2].

Every non-degenerate quadratic form on $V = \mathbb{R}^n$ is equivalent to $q(v) = v_1^2 + \cdots + v_r^2 + \cdots - v_{r+1}^2 - \cdots - v_{r+s}^2$, $n = r+s$ and every non-degenerate quadratic form on $V = \mathbb{C}^n$ is equivalent to $q(v) = v_1^2 + \cdots + v_n^2$. Hence, $Cl(\mathbb{R}^n,q)$ are called the real Clifford algebras and denoted by $Cl(r,s)$ while $Cl(\mathbb{C}^n,q)$ are called the complex Clifford algebras and denoted by $Cl_c(n)$.

If $\{e_1,\ldots,e_n\}$ is an orthonormal basis for V, then by (**1.10**) the generators of $Cl(r,s)$

satisfy

$$e_i^2 = -1, \quad i = 1, \dots r; \quad e_{r+i}^2 = 1, \quad i = 1, \dots s; \quad e_i e_j + e_j e_i = 0, \quad i \neq j,$$

and the generators of $Cl_c(n)$ satisfy

$$e_i^2 = -1, \quad i = 1, \dots n; \quad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

Then the set

$$\{e_{i_1} e_{i_2} \dots e i_k | \quad 1 \leq i_1 < i_2 < \dots < i_k \leq n\} \cup \{1\} \tag{\textbf{1.11}}$$

spans both $Cl(r,s)$ and $Cl_c(n)$. Hence they are both 2^n-dimensional vector spaces.

Definition 1.4.2. Let V be an $(n+p)$-dimensional vector space over the field k and $\langle , \rangle$ be a degenerate symmetric bilinear form (in characteristic not 2) on V. Then the associative algebra with unit, generated by the vector space V and the identity 1 subject to the relations in (**1.10**) for any $v \in V$ is called a degenerate Clifford algebra.

If $\{e_1, \dots, e_n, \dots, f_1, \dots, f_p\}$ is an orthonormal basis for V, where $\{f_1, \dots, f_p\}$ are basis vectors for the restriction of $\langle , \rangle$ to the degenerate subspace of V while the rest refers to the non-degenerate subspace of V, then the relations below

$$\langle e_i, e_j \rangle = \pm \delta_{ij}, \quad 1 \leq i, j \leq n,$$
$$\langle f_i, f_j \rangle = 0, \quad 1 \leq i, j \leq p. \tag{\textbf{1.12}}$$

stand for the relations in (**1.10**) [7].

Definition 1.4.3. Let K be a division algebra containing the field F. A K-representation of the Clifford algebra $Cl(V,q)$ is an F-algebra homomorphism p from $Cl(V,q)$ into the algebra of linear transformations on a finite dimensional vector space W over K. Here, W is a $Cl(V,q)$-module over K and p is an F-linear map, i.e. $p(a_1 v_1 + \dots + a_n v_n) = a_1 p(v_1) + \dots + a_n v_n$, $a_i \in F$, $v_i \in W$ for $i = 1, \dots, n$, satisfying

$$p(\psi\phi) = p(\psi) \circ p(\phi) \tag{\textbf{1.13}}$$

[2].

2. FAMILIES OF ANTI-COMMUTING DIAGONALIZABLE LINEAR OPERATORS

2.1. A Pair of Anti-Commuting Diagonalizable Linear Operators

In order to comprehend the structure of anti-commuting families of linear operators, we begin with a detailed examination of a family containing two anti-commuting operators [6].

Let $\mathcal{A}$ be a family of two anti-commuting diagonalizable linear operators on an n-dimensional vector space V and $A, B \in \mathcal{A}$. One can always choose a basis with respect to which either A or B is diagonal. Hence, without loss of generality we may assume that A is diagonal, i.e. $A_{ij} = \lambda_i \delta_{ij}$ and substituting this into the equation

$$AB + BA = 0, \tag{2.1}$$

we obtain

$$(\lambda_i + \lambda_j) B_{ij} = 0. \tag{2.2}$$

Thus if $\lambda_i = \lambda_j = 0$, B_{ij} is free. Otherwise, B_{ij} is nonzero only if $(\lambda_i + \lambda_j) = 0$, that is for a pair of eigenvalues with the same absolute value but opposite sign. This suggests that we should group the eigenvalues of A in three sets

$$\{0\}, \quad \{\mu_1, \dots, \mu_l\}, \quad \{\lambda_1, -\lambda_1, \dots, \pm\lambda_k, -\lambda_k\},$$

where $\mu_i + \mu_j \neq 0$ for $i, j = 1, \dots, l$. The corresponding eigenspaces will be denoted respectively by

$$Ker(A), \quad U_{A,1}, \dots, U_{A,l}, \quad W_{A,1}^{+}, W_{A,1}^{-}, \dots, W_{A,k}^{+}, W_{A,k}^{-}. \tag{2.3}$$

It is easy to see that $Ker(A)$ is B invariant, that is

$$B(Ker(A)) \subset Ker(A). \tag{2.4}$$

For if $AX = 0$, then

$$A(BX) = -BAX = -B(AX) = 0. \tag{2.5}$$

From (**2.2**) it readily follows that B restricted to the $U_{A,i}$ is identically zero for $i = 1,\dots,l$. Furthermore

$$B(W_{A,i}^{+}) \subset W_{A,i}^{-}, \quad B(W_{A,i}^{-}) \subset W_{A,i}^{+},$$

since if $AX = \lambda X$, then $A(BX) = -BAX = -\lambda(BX)$.

If the dimensions of the subspaces $W_{A,i}^{\pm}$ are not equal, that is $dim(W_{A,i}^{+}) \neq dim(W_{A,i}^{-})$, then the restriction of B to $W_{A,i} = W_{A,i}^{+} \oplus W_{A,i}^{-}$ is necessarily singular. Because if B were nonsingular on either $W_{A,i}^{\pm}$, it would map a linearly independent set to a linearly independent set. But this is impossible if the dimensions are different. However, the restriction of B can be singular even if the dimensions are equal. On the other hand if B is nonsingular, then necessarily

$$dim(W_{A,i}^{+}) = dim(W_{A,i}^{-}),$$

since bases of $W_{A,i}^{+}$ are mapped to bases of $W_{A,i}^{-}$ and vice versa. Thus for each $W_{A,i}$ we have a direct sum decomposition

$$W_{A,i} = (Ker(B) \cap W_{A,i}^{+}) \oplus (Ker(B) \cap W_{A,i}^{-}) \oplus \tilde{W}_{A,i}^{+} \oplus \tilde{W}_{A,i}^{-}, \tag{\textbf{2.6}}$$

where the $\tilde{W}_{A,i}^{\pm}$ are subspaces of equal dimension on which B is nonsingular. It follows that the restrictions of A and B to $W_{A,i}$ have the following block forms

$$A|_{W_{A,i}} = \begin{bmatrix} \lambda_i & 0 & 0 & 0 \\ 0 & -\lambda_i & 0 & 0 \\ 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & -\lambda_i \end{bmatrix}, \quad B|_{W_{A,i}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & B_2 & 0 \end{bmatrix},$$

where the first two diagonal blocks may have different dimensions but the last two diagonal blocks in the restriction of A and the submatrices B_1 and B_2 in the restriction of B are square matrices of the same dimension.

Incorporating this decomposition into (**2.6**), we can drop the tilde and we have a direct sum decomposition of V adopted to the pair of anti-commuting diagonalizable operators A and B as follows

$$V = (Ker(A) \cap Ker(B)) \oplus U_A \oplus U_B \oplus W_{A,1}^{+} \oplus W_{A,1}^{-} \oplus \cdots \oplus W_{A,k}^{+} \oplus W_{A,k}^{-}, \tag{\textbf{2.7}}$$

where $B|_{U_A} = 0$, $A|_{U_B} = 0$, and both A and B are nonsingular on the $W_{A,i}^{\pm}$, and $dim(W_{A,i}^{+}) = dim(W_{A,i}^{-})$ for $i = 1,\dots,k$.

Next we will determine the forms of B_1 and B_2. To simplify the notation now let us fix i and let $W = W_{A,i} = W^+ \oplus W^-$. Recall that A and B are both nonsingular on the subspace W and diagonalize A and B^2 simultaneously as in Remark 1.3.3. We choose a basis $\{X_1, \ldots, X_m\}$ for W^+, the $+\lambda$ eigenspace of A. Thus

$$AX_i = \lambda X_i, \quad B^2 X_i = \eta_i X_i, \quad i = 1 \ldots, m \tag{2.8}$$

and we define

$$Y_i = BX_i. \tag{2.9}$$

Then

$$AY_i = A(BX_i) = -B(AX_i) = -\lambda(BX_i) = -\lambda Y_i. \tag{2.10}$$

Thus Y_i belongs to the $-\lambda$ eigenspace of A. Furthermore

$$BY_i = B^2 X_i = \eta_i X_i. \tag{2.11}$$

It follows that with respect to the basis $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m\}$, the matrices of A, B and B^2 are as below.

$$A|_{W_{A,i}} = \begin{bmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{bmatrix}, \quad B|_{W_{A,i}} = \begin{bmatrix} 0 & D \\ I & 0 \end{bmatrix}, \quad B^2|_{W_{A,i}} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix},$$

where all submatrices are square, I is the identity matrix and D is a diagonal matrix. If B^2 has q distinct eigenvalues $d_1, \ldots, d_q$ with eigenspaces of dimensions m_i, we can rearrange the basis so that

$$A|_{W_{A,i}} = \begin{bmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix},$$

$$B|_{W_{A,i}} = \begin{bmatrix} 0 & d_1 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d_q \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad B^2|_{W_{A,i}} = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_q & 0 \\ 0 & 0 & \cdots & 0 & d_q \end{bmatrix},$$

where λ and d_i's denote constant matrices of approporiate size. Hence W has a direct sum decomposition

$$W = W_1 \oplus \cdots \oplus W_q \tag{2.12}$$

such that A^2 and B^2 restricted to each summand are constant matrices.
Recall that in the discussion above we have used W to denote an eigenspace of A on which B was nonsingular.Repeating this, for each eigenspace of A, we have a direct sum decomposition of V such that, except for their common kernel, on a summand either A of B is nonzero or they are both nonsingular and have constant squares. Relabeling these subspaces, we have now

$$V = (Ker(A) \cap Ker(B)) \oplus U_A \oplus U_B \oplus W_1 \oplus \cdots \oplus W_s \tag{2.13}$$

where on each W_i A and B are anti-commuting diagonalizable matrices whose squares are constants. But this is exactly a representation of a 2-dimensional Clifford algebra. We summarize these results in the following theorem.

Theorem 2.1.1. *Let A and B be anti-commuting non-singular diagonalizable operators on a finite dimensional vector space V. Then there is a direct sum decomposition of V on which both A^2 and B^2 are nonzero constant matrices and on each summand they form a representation of a 2-dimensional Clifford algebra.*

In a family of N anti-commuting diagonalizable operators, once we fix an element A and define the subspaces as in (**2.5**), on $Ker(A)$ we have at family of at most $(N-1)$ anti-commuting diagonalizable operators and the rest of the family is zero on the the direct sum of the $U_{A,i}$'s. It is thus only on the direct sum of the $W^{\pm}_{A,i}$'s that we may have nontrivially a family of N anti-commuting operators.
We tried to proceed by refining this splitting by adding additional members of the family, hoping to prove that given an N-element anti-commuting family, V has a direct sum decomposition such that on each summand either we have an $(N-k)$-family of anti-commuting matrices, or a representation of an N-dimensional Clifford algebra, but this approach became cumbersome and we preferred the proof given in the next sections [6].

2.2. Families of Anti-Commuting Diagonalizable Linear Operators

In the previous section we examined the structure of a two-element family of anti-commuting diagonalizable linear operators and before dealing with the general case, we will go on with the three-element family in this section [6].

Let $\mathcal{A}$ be a family of three anti-commuting diagonalizable linear operators on an n-dimensional vector space V and $A,B,C \in \mathcal{A}$. At that point, instead of adding an operator to the previous case of two-element families, we choose to continue with a decomposition by firstly separating the common kernels and then we aim to observe a similar structure to the previous case.

We claim the following direct sum decomposition of V exists for the case of a three-element family:

$$V = (Ker(A) \cap Ker(B) \cap Ker(C)) \oplus U_A \oplus U_B \oplus U_C \oplus W_1 \oplus \cdots \oplus W_t \quad \textbf{(2.14)}$$

where $B|_{U_A} = 0$, $C|_{U_A} = 0$, $A|_{U_B} = 0$, $C|_{U_B} = 0$, $A|_{U_C} = 0$, $B|_{U_C} = 0$ and on each W_i A, B and C are anti-commuting nonsingular diagonalizable operators whose squares are constant. Here, we have a family of at most two diagonalizable operators on the direct sums of U_A, U_B and U_C and we name the direct sum where we have a family of three diagonalizable operators as W.

Since the triples (A,B^2,C^2) and (A^2,B^2,C^2) commute, they are simultaneously diagonalizable. In section 3.1 we proved that we can choose a basis with respect to which A and B are anti-commuting diagonalizable matrices whose squares are constants on each $\tilde{W}_j$ for $W = \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_s$. Also, this direct sum is invariant under the whole family. It is easy to see that $\tilde{W}_j$ is C invariant, that is

$$C\tilde{W}_j \subset \tilde{W}_j. \quad \textbf{(2.15)}$$

For if $AX^{\pm} = \pm\lambda X^{\pm}$, then

$$A(CX^{\pm}) = -CAX^{\pm} = -C(AX^{\pm}) = C(\mp\lambda X^{\pm}), \quad \textbf{(2.16)}$$

where $X^{\pm}$ are eigenvectors corresponding to eigenvalues $\pm\lambda$. Hence $C\tilde{W}_j \subset \tilde{W}_j$. Next, we will show that there exists t for which

$$W = W_1 \oplus \cdots \oplus W_t, \quad t \geq s \quad \textbf{(2.17)}$$

On each $\tilde{W}_j$ any C anti-commuting with A and B has the form

$$C|_{\tilde{W}_j} = \begin{bmatrix} 0 & -d_j c \\ c & 0 \end{bmatrix}, \quad C^2|_{\tilde{W}_j} = \begin{bmatrix} -d_j c^2 & 0 \\ 0 & -d_j c^2 \end{bmatrix},$$

where d_j is an eigenvalue of B and c^2 is diagonal. If C^2 has r distinct eigenvalues $e_1, \ldots, e_r$ with eigenspaces of dimensions n_j, we can rearrange the basis so that

$$A|_{\tilde{W}_j} = \begin{bmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix}, \quad B|_{\tilde{W}_j} = \begin{bmatrix} 0 & d_j & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d_j \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

$$B^2|_{\tilde{W}_j} = \begin{bmatrix} d_j & 0 & \cdots & 0 & 0 \\ 0 & d_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_j & 0 \\ 0 & 0 & \cdots & 0 & d_j \end{bmatrix}, \quad C^2|_{\tilde{W}_j} = \begin{bmatrix} e_1 & 0 & \cdots & 0 & 0 \\ 0 & e_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & e_r \end{bmatrix},$$

where $\tilde{W}_j = W_1 \oplus \cdots \oplus W_r$. This means that W has a direct sum decomposition

$$W = W_1 \oplus \cdots \oplus W_t, t \geq s$$

such that on each W_i A^2, B^2 and C^2 are constant and the direct sum (**2.17**) is verified.

Let $\mathcal{A} = \{A_i\}_{i=1\ldots N}$ be a family of N anti-commuting diagonalizable linear operators on an n-dimensional vector space V. Using remark 1.3.3, we diagonalize the squared family simultaneously, and rearranging the basis vectors we construct subspaces that are common kernels of N, $N-1$, $N-2$ etc. of the A_i^2's. Since A_i's and A_i^2's have the same kernel it follows that V has a decomposition into subspaces on which we have nonsingular families of anti-commuting operators.

Then, diagonalizing the squared family simultaneously and rearranging the eigenvectors, we have subspaces on which all squared members are constant and nonzero. But an anti-commuting family whose squares are constant is just a representation of some Clifford algebra. Hence we have the following theorem which is the generalized form of Theorem 2.1.1.

Theorem 2.2.1. *Let $\{A_i\}_{i=1\ldots N}$ be a family of finitely many anti-commuting non-singular diagonalizable operators on a finite dimensional vector space V. Then* *there is a direct sum decomposition of V such that on each summand all matrices of the operators $A_1^2, \ldots, A_N^2$ are nonzero constants and the family is a representation of an N-dimensional Clifford algebra.*

In fact we can prove a stronger result, which is related to the representations of degenerate Clifford algebras. We will give this result in the following section.

2.3. Families of Anti-Commuting Square Diagonalizable Linear Operators

In previous sections, we examined the structure of a family of anti-commuting diagonalizable linear operators, where the direct sum decomposition of the underlying vector space was based on the simultaneous diagonalizability of the families of the squared operators. Thus, in this section we consider an anti-commuting family $\mathcal{A}$ of operators whose squares are diagonalizable.

This generalization arises in the following two contexts:

Firstly, the discussion on diagonalizability usually considered as diagonalizability over complex numbers. If we are interested in the diagonalizability over the real numbers, any linear operator whose square is proportional to negative of identity operator is not diagonalizable over $\mathbb{R}$. Thus if we are interested in diagonalizability over the real numbers, we should allow families that involve elements which are not diagonalizable but whose squares are diagonalizable.

Secondly, non-zero operators whose squares are zero are used in a number of places. For example, "Grassmann numbers" used in many physical theories form an algebra where the squares of the elements are zero. Also, degenerate Clifford algebras contain elements whose squares are zero [7].

Most of our proof on simultaneous block diagonalization is based on the diagonalizability of the squared family. By the remarks above, it looks like that a square diagonalizable family of anti-commuting linear operators is more basic than a diagonalizable family [6].

When we use Proposition 1.2.6 and review Theorem 2.2.1, we can omit the condition for the family $\{A_i\}_{i=1\ldots N}$ to be necessarily diagonalizable. Instead we assume that the family $\{A_i^2\}_{i=1\ldots N}$ is diagonalizable. Then, since operators A_i^2 form a commuting family, they are simultaneously diagonalizable and we can split V into a direct sum of subspaces on which each A_i^2 is equal to a constant, which may be zero. By definition of degenerate Clifford algebras, Corollary 2.3.1 is obvious.

Corollary 2.3.1. *Let $\{A_i\}_{i=1\ldots N}$ be an anti-commuting family of finitely many linear operators satisfying $\{A_i^2\}_{i=1\ldots N}$ is diagonalizable. Then there is a direct sum decomposition of V such that on each summand all matrices of the operators $A_1^2,\ldots,A_N^2$ are constant, including zero. This is just the representation of a degenerate Clifford algebra.*

3. CANONICAL FORMS OF REPRESENTATIONS OF CLIFFORD ALGEBRAS

We have seen that an anti-commuting family of diagonalizable linear operators can be simultaneously block diagonalized and on each invariant subspace the family reduces to a representation of some Clifford algebra, up to multiplicative constants. If the family is diagonalizable over the complex numbers and we search for possibly complex canonical forms, we can use more or less straightforward construction of complex representations of Clifford algebras, discussed in Section 3.1. However, if the square of some of the elements are negative, then over the real numbers the family is not diagonalizable, but it is square diagonalizable. But the discussion of the previous section shows that the reduction to the block diagonal form still works and on each invariant subspace we have a representation of some non-degenerate Clifford algebra. These constructions discussed in Section 3.2, are nontrivial for Clifford algebras which contain no generator with a positive square.

3.1. Canonical Forms over Complex Numbers

When we allow canonical forms over the complex numbers, we can always assume that there are two anti-commuting operators A and B such that they can be simultaneously put to a canonical form. Then the rest of the family which commutes with both of these two has to be of a specific form expressed in terms of matrices in the half dimension. In this construction, since we can use complex numbers, whether the squares of the elements are positive or not is irrelevant [6].

Lemma 3.1.1. *Let A and B be trace zero anti-commuting linear operators on a $2n$-dimensional vector space V, with minimal polynomials $A^2 + \lambda^2 I = 0$ and $B^2 + \mu^2 I = 0$ where λ and μ are real or pure imaginary constants and I is the identity. Then there* *is a basis for V with respect to which*

$$A = \lambda \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad B = \mu \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \tag{3.1}$$

where I denotes $n \times n$ identity matrices.

Proof. Since A and B are trace free, the $\pm\lambda$ eigenspaces of A are n-dimensional. Let $\{X_1, \ldots, X_n\}$ be a basis for $+\lambda$ eigenspace and let $Y_\alpha = -(1/\mu^{-1})BX_\alpha$. Then it can be seen that

$$AY_\alpha = -\lambda Y_\alpha \tag{3.2}$$

and

$$BX_\alpha = -\mu Y_\alpha, \quad BY_\alpha = \mu X_\alpha. \tag{3.3}$$

Then passing to the basis $Z_\alpha = X_\alpha + iY_\alpha$ and $T_\alpha = X_\alpha - iY_\alpha$ we can complete the proof.

Then, if we have a family of N operators, once we put any two of them into the canonicals forms above, the requirement that the remaining anti-commute with these two determines the rest as below.

Lemma 3.1.2. *Let $\{A_\alpha\}_{\alpha=1,\ldots,N}$ be a set of trace zero anti-commuting linear operators on a $2n$-dimensional vector space V with minimal polynomials $A_\alpha^2 + \lambda_\alpha^2 I = 0$, where the λ_α's are real or pure imaginary constants and I is the identity. Then there is an orthonormal basis of V with respect to which*

$$A_1 = \lambda_1 \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad A_2 = \lambda_2 \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} a_\alpha & 0 \\ 0 & -a_\alpha \end{bmatrix} \tag{3.4}$$

where the a_α are $n \times n$ matrices with minimal polynomials

$$a_\alpha^2 + \lambda_\alpha^2 I = 0. \tag{3.5}$$

Proof. Since A_α are trace zero the λ_α eigenspaces are $n-$dimensional. Then one can take $A = i\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \otimes I$. Let $X_1, \ldots, X_n$ be an orthonormal basis for the $+i$ eigenspace of A, i.e. $AX_i = iX_i$ and define $Y_i = BX_i$. Going on computations $AY_i = A(BX_i) = -B(AX_i) = -iBX_i = -iY_i$ and $BY_i = B^2X_i = -X_i$ which gives us the desired canonical forms of B and C.

3.2. Canonical Forms over Real numbers

If the family consists of matrices whose squares are negative, then the constructions above fail. We shall describe a procedure for the construction of canonical forms. The list of Clifford algebras that can be represented at a given dimension are presented in Table 3.1 adopted from [3].

Table 3.1: Representations of Clifford algebras on different dimensions

R	**R(2)**	**R(4)**		**R(8)**				**R(16)**
$Cl(0,0)$ $R(1)$	$Cl(1,0)$ $C(1)$	$Cl(2,0)$ $H(1)$	$Cl(3,0)$ $H(1)\oplus H(1)$	$Cl(4,0)$ $H(2)$	$Cl(5,0)$ $C(4)$	$Cl(6,0)$ $R(8)$	$Cl(7,0)$ $R(8)\oplus R(8)$	$Cl(8,0)$ $R(16)$
$Cl(0,1)$ $R(1)\oplus R(1)$	$Cl(1,1)$ $R(2)$	$Cl(2,1)$ $C(2)$	$Cl(3,1)$ $H(2)$	$Cl(4,1)$ $H(2)\oplus H(2)$	$Cl(5,1)$ $H(4)$	$Cl(6,1)$ $C(8)$	$Cl(7,1)$ $R(16)$	$Cl(8,1)$ $R(16)\oplus R(16)$
$Cl(0,2)$ $R(2)$	$Cl(1,2)$ $R(2)\oplus R(2)$	$Cl(2,2)$ $R(4)$	$Cl(3,2)$ $C(4)$	$Cl(4,2)$ $H(4)$	$Cl(5,2)$ $H(4)\oplus H(4)$	$Cl(6,2)$ $H(8)$	$Cl(7,2)$ $C(16)$	$Cl(8,2)$ $R(32)$
$Cl(0,3)$ $C(2)$	$Cl(1,3)$ $R(4)$	$Cl(2,3)$ $R(4)\oplus R(4)$	$Cl(3,3)$ $R(8)$	$Cl(4,3)$ $C(8)$	$Cl(5,3)$ $H(8)$	$Cl(6,3)$ $H(8)\oplus H(8)$	$Cl(7,3)$ $H(16)$	$Cl(8,3)$ $C(32)$
$Cl(0,4)$ $H(2)$	$Cl(1,4)$ $C(4)$	$Cl(2,4)$ $R(8)$	$Cl(3,4)$ $R(8)\oplus R(8)$	$Cl(4,4)$ $R(16)$	$Cl(5,4)$ $C(16)$	$Cl(6,4)$ $H(16)$	$Cl(7,4)$ $H(16)\oplus H(16)$	$Cl(8,4)$ $H(32)$
$Cl(0,5)$ $H(2)\oplus H(2)$	$Cl(1,5)$ $H(4)$	$Cl(2,5)$ $C(8)$	$Cl(3,5)$ $R(16)$	$Cl(4,5)$ $R(16)\oplus R(16)$	$Cl(5,5)$ $R(32)$	$Cl(6,5)$ $C(32)$	$Cl(7,5)$ $H(32)$	$Cl(8,5)$ $H(32)$
$Cl(0,6)$ $H(4)$	$Cl(1,6)$ $H(4)\oplus H(4)$	$Cl(2,6)$ $H(8)$	$Cl(3,6)$ $C(16)$	$Cl(4,6)$ $R(32)$	$Cl(5,6)$ $R(32)\oplus R(32)$	$Cl(6,6)$ $R(64)$	$Cl(7,6)$ $C(64)$	$Cl(8,6)$ $H(64)$
$Cl(0,7)$ $C(8)$	$Cl(1,7)$ $H(8)$	$Cl(2,7)$ $H(8)\oplus H(8)$	$Cl(3,7)$ $H(16)$	$Cl(4,7)$ $C(32)$	$Cl(5,7)$ $R(64)$	$Cl(6,7)$ $R(64)\oplus R(64)$	$Cl(7,7)$ $R(128)$	$Cl(8,7)$ $C(128)$
$Cl(0,8)$ $R(16)$	$Cl(1,8)$ $C(16)$	$Cl(2,8)$ $H(16)$	$Cl(3,8)$ $H(16)\oplus H(16)$	$Cl(4,8)$ $H(32)$	$Cl(5,8)$ $C(64)$	$Cl(6,8)$ $R(128)$	$Cl(7,8)$ $R(128)\oplus R(128)$	$Cl(8,8)$ $R(256)$

In all constructions we shall need tensor products with the following matrices

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.6}$$

This construction is not unique, but our algorithm gives a precise way of construction for each case. First we note that if the family contains at least one element whose square is negative and another element whose square is positive, then Lemma 3.1.1 is valid for real numbers λ and μ. This is described as Case 1.

Case 1. Representations of $Cl(r,s)$ with $r \geq 1$, $s \geq 1$: Given A and B with $A^2 + \lambda^2 = 0$ and $B^2 - \mu^2 = 0$, By Lemma 3.1.1 ,we can choose a basis such that

$$A = \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.7}$$

where 1 denotes the identity matrix in the half dimension. Then any other member of the family is of the form

$$C_a = \mu \begin{bmatrix} c_a & 0 \\ 0 & -c_a \end{bmatrix}, \tag{3.8}$$

where $c_a^2 = \pm\gamma_a^2$, according as the square of C_a is positive or negative. Thus the representations of $Cl(r,s)$ on R^{2N} are obtained from the representations of $Cl(r-1,s-1)$ on R^N by tensoring with the matrix $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. That is we move diagonally backward on Table 1.

The construction above ends up either with $Cl(r,0)$ or $Cl(0,s)$. Note that $Cl(0,1)$ is just R, thus we may start with $s \geq 2$.

Case 2. Representations of $Cl(0,s)$ for $s \geq 2$: In this case, we can use the intermediate steps in Lemma 5.1 and choose a basis such that

$$B_1 = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.9}$$

Then any other member of the family B_a with $B_a^2 = \lambda_a^2$, is of the form

$$B_a = \lambda_a \begin{bmatrix} 0 & b_a \\ -b_a & 0 \end{bmatrix}, \tag{3.10}$$

but now, $b_a^2 = -\lambda_a^2$, that is they form a representation of the Clifford algebra $Cl(s-2,0)$ on the half dimension.

The problem is thus reduced to the construction of simultaneous canonical forms for the real representations of $Cl(r,0)$. These constructions differ for $r = 8d$, $r = 8d+1$, $r = 8d+3$ and $r = 8d+7$. We start with $r = 1$, $r = 3$ and $r = 7$ and then give the algorithm for the construction in general.

Case 3a. Representation of $Cl(1,0)$ on R^2: This is a complex representation on R^2, that is if

$$A = \mu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.11}$$

then the complex structure J is equal to A since it commutes with itself. We note that there are exactly two matrices with positive squares that commute with A. These are τ and σ.

$$\tau = \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma = \mu \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.12}$$

Case 3b. Representations of $Cl(2,0)$ and $Cl(3,0)$ on R^4: Recall that $Cl(2,0)$ and $Cl(3,0)$ are isomorphic respectively to H and $H \oplus H$ hence their representations are quaternionic. Let A_1, A_2 and A_3 be as below.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

We will show that the representations of the generators of $Cl(2,0)$ and $Cl(3,0)$ will be put respectively to the forms $\{A_1,A_2\}$ and $\{A_1,A_2,A_3\}$ simultaneously. For this, let X be any nonzero vector in R^4, and choose a basis such that

$$X_1 = X, \quad X_2 = -A_1X, \quad X_3 = -A_2X, \quad X_4 = -A_1A_2X. \tag{3.13}$$

It can be checked that with respect to this basis, A_1 and A_2 have the forms above, hence they can be put to the canonical forms above simultaneously. Then it can be checked that the product A_1A_2 anti-commutes with both $A-1$ and A_2, hence we can choose the representation of $Cl(3,0)$ as $\{A_1,A_2,A_3\}$.

Note that

$$A_1 = \sigma \otimes \varepsilon, \quad A_2 = \varepsilon \otimes 1, \quad A_3 = \tau \otimes \varepsilon. \tag{3.14}$$

The quaternionic structure is given by the following three matrices which commute

Table 3.2: Action of generators of $Cl(7,0)$ on basis vectors

	$\mathbf{X}$	$\mathbf{A_1X}$	$\mathbf{A_2X}$	$\mathbf{A_3X}$	$\mathbf{A_4X}$	$\mathbf{A_5X}$	$\mathbf{A_6X}$	$\mathbf{A_7X}$
$\mathbf{A_1}$	A_1X	$-X$	$-A_3X$	A_2X	$-A_5X$	A_4X	A_7X	$-A_6X$
$\mathbf{A_2}$	A_2X	A_3X	$-X$	$-A_1X$	$-A_6X$	$-A_7X$	A_4X	A_5X
$\mathbf{A_3}$	A_3X	$-A_2X$	A_1X	$-X$	$-A_7X$	A_6X	$-A_5X$	A_4X
$\mathbf{A_4}$	A_4X	A_5X	A_6X	A_7X	$-X$	$-A_1X$	$-A_2X$	$-A_3X$
$\mathbf{A_5}$	A_5X	$-A_4X$	A_7X	$-A_6X$	A_1X	$-X$	A_3X	$-A_2X$
$\mathbf{A_6}$	A_6X	$-A_7X$	$-A_4X$	A_5X	A_2X	$-A_3X$	$-X$	A_1X
$\mathbf{A_7}$	A_7X	A_6X	$-A_5X$	$-A_4X$	A_3X	A_2X	$-A_1X$	$-X$

with the A_i's.

$$J_1 = 1 \otimes \varepsilon, \quad J_2 = \varepsilon \otimes \tau, \quad J_3 = \varepsilon \otimes \sigma \tag{3.15}$$

Case 3c. Representations of $Cl(4,0)$, $Cl(5,0)$, $Cl(6,0)$ and $Cl(7,0)$ on R^8: We shall describe how to construct canonical forms for representations of $Cl(7,0)$. This construction is based on Proposition 4.7, in [3], quoted below.

Proposition 1. *Let A_i, $i = 1, \dots 7$ be an anti-commuting set of matrices with squares -1 satisfying $A_1 \dots A_7 = 1$. Then, the subgroup generated by the matrices*

$$M_1 = A_1A_2A_3, \quad M_2 = A_1A_4A_5, \quad M_3 = A_2A_4A_6 \tag{3.16}$$

is an abelian subgroup and has exactly one eigenvector X with an eigenvalue 1.

Then, starting with this specific X, we choose the basis

$$\{X, A_1X, A_2X, A_3X, A_4X, A_5X, A_6X, A_7X\}. \tag{3.17}$$

Note that for the representation of $Cl(3,0)$, the product of 3 matrices was a scalar. In this case, the product of triples of A_i's are not scalars, but their action on this specific X is a scalar. It follows that the action of the A_i's on these basis elements can be given as in Table 3.2.

One can check that the corresponding matrices are represented as tensor products as follows.

$A_1 = -\sigma \otimes (\sigma\varepsilon) = -\sigma \otimes A_1,$

$A_2 = -\sigma \otimes (\varepsilon \otimes 1) = -\sigma \otimes A_2,$

$A_3 = -\sigma \otimes (\tau \otimes \varepsilon) = -\sigma \otimes A_3,$

$A_4 = -\varepsilon \otimes (1 \otimes 1) = -\varepsilon \otimes 1,$

$A_5 = -\tau \otimes (1 \otimes \varepsilon) = -\tau \otimes J_1,$

$A_6 = -\tau \otimes (\varepsilon \otimes \sigma) = -\tau \otimes J_3,$

$A_7 = -\tau \otimes (\varepsilon \otimes \tau) = -\tau \otimes J_2.$

Case 3d. Representation of $CL(8,0)$ on R^{16}: The construction of these representations is straightforward. Represent on of the elements by $A_1 = \varepsilon \otimes 1$ and the remaining 7 by $\sigma \otimes A_i$ where A_i's are as above.

For the general case, one can use Lemma 4.11 and the Table 2 in [3]

4. CONCLUSIONS

In this thesis, families of anti-commuting diagonalizable linear operators on a finite dimensional vector space are examined. A well-known property of commuting diagonalizable linear operators on finite dimensional vector spaces, which is simultaneous diagonalization, inspired us for analyzing the structure of anti-commuting diagonalizable linear operators. It is also well known that real or complex representations of Clifford algebras are typical anti-commuting diagonalizable (over $\mathbb{C}$) families.

Later on, we proved that if $\mathcal{A}$ is a family of of anti-commuting diagonalizable linear operators on a finite dimensional vector V, then V has an $\mathcal{A}$-invariant direct sum decomposition into subspaces such that the restriction of the family to each summand either consists of a single nonzero operator or it is a representation of some Clifford algebra. Indeed we have shown that the diagonalizability condition can be replaced by the requirement that the square of the family is diagonalizable.

Lastly, we discussed classifications of real and complex representations of Clifford algebras which can be used to derive a complete characterization of families of anti-commuting diagonalizable operators.

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