ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

CONTROL OF AN ACTIVE MAGNETIC BEARING SYSTEM WITH SLIDING MODE CONTROLLER USING NONLINEAR DISTURBANCE OBSERVER

MSc. Thesis by
Rüstem Tolga BÜYÜKBAŞ

Department : **Control Engineering**

Programme: **Control and Automation Engineering**

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Supervisor (Chairman): Assoc. Prof. Dr. Fuat GÜRLEYEN

Members of the Examining Committee Prof. Dr. Metin GÖKAŞAN

Prof. Dr. Selçuk PAKER

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BİR AKTİF MANYETİK YATAKLAMA SİSTEMİNİN DOĞRUSAL OLMAYAN BOZUCU GÖZLEYİCİSİ KULLANARAK KAYMA YÜZEYLİ KONTROLLÖR İLE KONTROLÜ

YÜKSEK LİSANS TEZİ

Rüstem Tolga BÜYÜKBAŞ
504031133

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Tez Danışmanı : Doç. Dr. Fuat GÜRLEYEN

Diğer Jüri Üyeleri Prof. Dr. Metin GÖKAŞAN

Prof. Dr. Selçuk PAKER

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ÖNSÖZ

Tez konumu seçmemde fikirlerini esirgemeyen ve bu konuda beni yönlendirerek sonuca varmamı sağlayan, herzaman destek olan ve kolaylıklar sağlayan değerli danışmanım ve hocam Doç. Dr. Fuat GÜRLEYEN'e ve karşılıksız maddi ve manevi desteğiyle beni bugünlere getiren çok sevgili aileme, babama, anneme ve kardeşime yürekten teşekkür ederim.

Mayıs 2008 Rüstem Tolga BÜYÜKBAŞ

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ABBREVIATIONS

PID	: Proportional Integral Derivative
PI	: Proportional Integral
PD	: Proportional Derivative
NDO	: Nonlinear Disturbance Observer

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SYMBOL LIST

r	: Reference signal
$e, \dot{e}$	: Error signal and error signal derivative
u, u_c	: Control signal
y	: Output signal
$x, \dot{x}$	: State variable
x_e	: Equilibrium state
ξ, δ	: Positive real numbers
$f(x)$	: Function of nonlinear system
$g(x)$	: Function of high-order derivative terms
$V, \dot{V}$	: Lyapunov candidate function and its derivative
A	: State space matrix
M, Q	: Positive definite matrices
k_1, k_2	: Positive real numbers
x_m	: State variable of model system
ϕ	: Positive real number
F_1, F_2	: Electromagnetic force
D	: Disturbance
K_e	: Error correction matrix
ξ	: Damping ration
ω_n	: Natural frequency

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ÖZET

Bu çalismada, aktif manyetik yataklama sistemleri için düşük mertebeden doğrusal olmayan bozucu gözlemleyicisi kullanılarak kayan kipli bir kontrollör tasarlanması amaçlanmıştır. Sistemdeki bozucu etkilerin, yer çekimi ivmesi ile ortam ve mıktanıstan kaynaklanan diğer düzensizliklerin tümünü içerdiği kabul edilen nonlineer bir sistem modeli kullanılmıştır. Düşük mertebeden doğrusal olmayan bir gözleyici, ölçülebilen durum değişkenleri dışındaki tüm durum değişkenlerini ön görmektedir. Sistem düzensizlikleri de, doğrusal olmayan bu düşük mertebeli gözlemleyicinin çıkışlarından biridir. Aktif manyetik yataklama sisteminin doğrusal olmayan matematik modelini kullanarak kontrol etmek üzere önerilen kayan kipli kontrollörün kontrol kuralı elde edilmiştir. Daha sonra sistemde kestirilen bozucu etki fonksiyonu, kayan kipli kontrollörde öne sürülen kontrol işaretinin foksiyonunda kullanılır. Son olarak kayan kipli kontrollör tasarımı, elde edilen hata işaretinin istenen karakteristiği sağlaması yönünde, yeni bir kontrol işareti fonksiyonu ön görülmesiyle tamamlanır.

CONTROL OF AN ACTIVE MAGNETIC BEARING SYSTEM WITH SLIDING MODE CONTROLLER USING NONLINEAR DISTURBANCE OBSERVER

SUMMARY

In this study, it has been aimed to design a sliding mode controller in order to control an active magnetic bearing system by using a reduced-order nonlinear disturbance observer. The disturbance in the system is issued to the gravitational acceleration, friction in the environment and disturbance and uncertainties caused from electromagnet. The reduced-order nonlinear observer estimates all of the state variables rather than the measurable state variables. The estimated disturbance is one of the outputs of this reduced-order nonlinear observer. Thereafter, the control law of the sliding mode controller is extracted which is proposed to control the active magnetic bearing system using its mathematical model. Then the disturbance function estimated by the observer is applied to the function of the proposed control law. Finally, the design of the sliding mode controller is completed by defining a control signal applied to the plant in the way that the error behaviour performs a desired characteristic.

1. INTRODUCTION: THEORY AND ANALYSIS

Active magnetic bearing systems are systems where the rotor of the motor or bearing equipment are hooked without any contact and therefore cause very low energy loss and also provide very high speed [9]. Magnetic levitation and active bearing systems which can suspend objects without mechanical contact have been used in many applications such as high speed magnetic levitation vehicles, magnetic bearings for high speed machinery, flywheels, artificial hearts, magnetic vibration isolation and pointing systems and wind tunnel suspensions [12]. These active magnetic levitation and bearing systems are open-loop unstable. Feedback controllers are generally used to achieve desired stability. Nevertheless, due to the nonlinearities, the governing differential equations are linearized about various operating points and local feedback controllers are implemented to stabilize small perturbations [10].

The need for high performance accurate magnetic levitation and active magnetic bearing systems has become increasingly important due to the recent applications [8].

The most recent work in the adaptive approach concentrates on constructing estimation rules to estimate and cancel the nonlinearities of the system in issue. Regarding the robust control approach, the sliding control methodology has been investigated frequently. Usually, the sliding mode controllers based on the linear models and viewed the nonlinearities and uncertainties as disturbance to the models [11].

Due to the importance of the system differential function in controller design, the nonlinear system function of behaviour of the active magnetic system should be obtained with the most possible uncertainties covering loss and frictions in the physical mechanism. To achieve a successful controller to stabilize and control an active bearing system, an observer is designed. This observer is aimed to estimate the disturbance which is the uncertain term of the control signal suggested in sliding mode controller. As next step, the control action of the sliding mode is proposed.

1.1. Introduction to Nonlinear Systems

Systems and system representations or models may be classified onto numerous categories according to mathematical structure and physical realizability. A typical classification may be summarized in how they are commonly represented by partial differential equations or by a finite number of ordinary differential equations. They may be stochastic (random) or deterministic; linear or nonlinear; discrete or continuous; autonomous or non-autonomous.

In this classification from control point of view, although linear system theory and control design has been established well along the decades, nonlinear systems in general do not have a convenient uniform theory. Unfortunately, many classical notions developed for linear systems are not valid for nonlinear systems. Even the concept of stability for linear system theory may not be always convenient for nonlinear system design either.

In nonlinear systems, stability is strongly dependent on the magnitude of the initial conditions as well as the magnitude of any input. Moreover, nonlinear systems have generally more than one equilibrium points where some of them may be stable and some unstable. As a result, nonlinear mathematical models do not have a unique solution [14].

In some cases, nonlinear systems may be analyzed conveniently by an appropriate selection of coordinates, transformation and or state space representation. A class of nonlinear systems is one of which consists of a linear system with appropriately constrained by non-dynamic nonlinear element in the feedback regulator as shown in figure (1.1). The latter is based on the existence of a Lyapunov function.

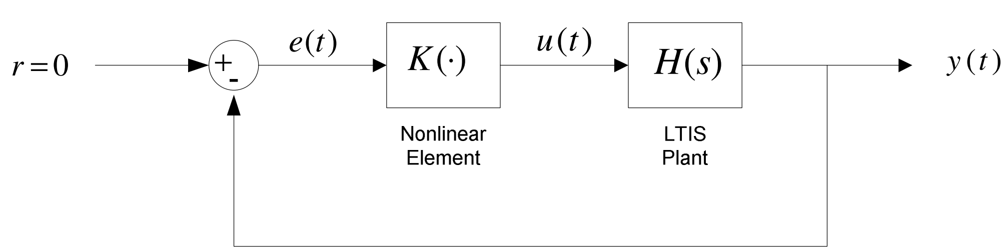


Figure 1.1: Simple nonlinear regulator [14]

More generally, a finite-state differential system is defined by the nonlinear vector differential equation

$$\frac{dx(t)}{dt} = f(x(t), u(t)) \tag{1.1}$$

where $f(.)$ is a real nonlinear mapping from $R^n \times R^m$ to R^n. The output can be given by

$$y(t) = g(x(t), u(t)) \tag{1.2}$$

with $x \in R^n$, $u \in R^m$ and $y \in R^p$. Hence, time-variant systems of this class are denoted by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \end{aligned} \tag{1.3}$$

1.2. Lyapunov's First Method

1.2.1. Equilibrium Point

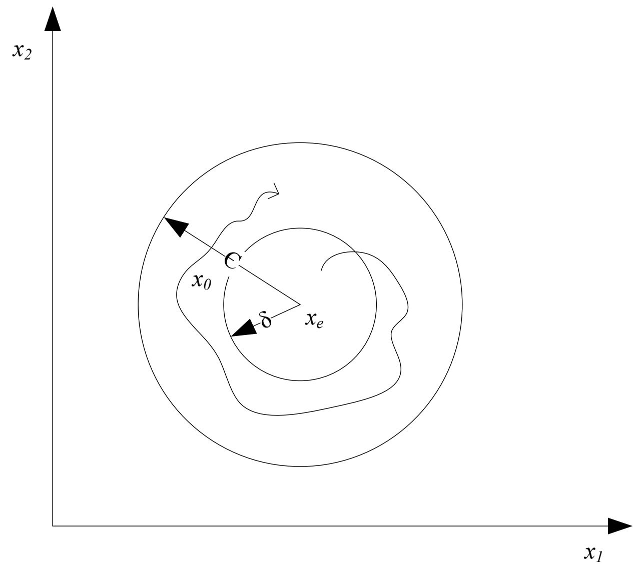
Consider a nonlinear system described by

$$\frac{dx}{dt} = f(x) \tag{1.4}$$

where the equilibrium states x_e are given by

$$f(x_e) = 0 \tag{1.5}$$

Let x_e be an isolated equilibrium point (state) in state space such that no other equilibrium point lie within its infinitesimal neighbourhood. Then the stability of x_e can be defined as in figure 1.2.



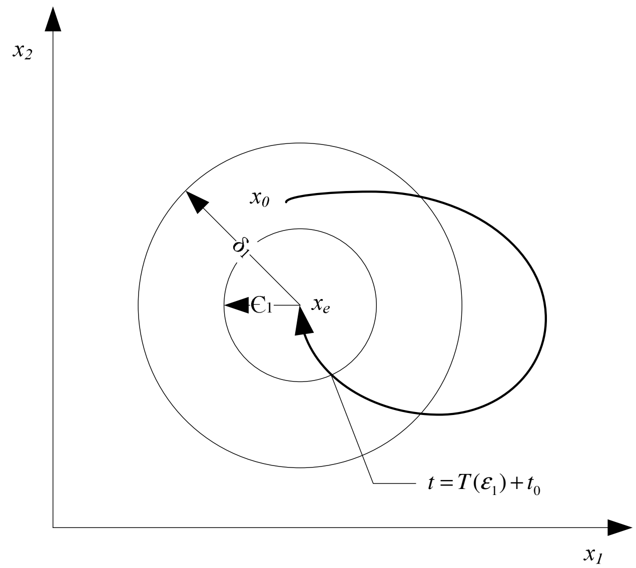
$$\varepsilon > 0, \delta > 0$$
$$\|x_0 - x_e\| < \delta \Rightarrow \|x(x_0, t) - x_e\| < \varepsilon$$

Figure 1.2: Illustration of equilibrium point stability

The system (1.4) is stable at x_e if for every initial state x_0 that is sufficiently close to x_e, the solution $x(x_0, t)$ remains near x_e.

More precisely, the equilibrium point x_e is stable if for every $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $\|x_0 - x_e\| < \delta$ implies that $\|x(x_0, t) - x_e\| < \varepsilon$ for all $t \geq t_0$. The system (1.4) is asymptotically stable at x_e if it is stable, which means also $x(t)$ approaches x_e as $t \to \infty$. The equilibrium state x_e is asymptotically stable if it is stable and convergent. Regarding figure 1.3, there exists a real number $\delta_1 > 0$, and for every $\varepsilon_1 > 0$ there exists a $T(\varepsilon_1) > 0$ such that $\|x(t_0) - x_e\| < \delta_1$ implies that $\|x(x_0, t) - x_e\| < \varepsilon_1$ for all $t \geq T + t_0$. More commonly, the definition can be also given by

$$\lim_{t \to \infty} \|x(x_0, t) - x_e\| = 0 \qquad \textbf{(1.6)}$$



$\varepsilon_1 > 0, \delta_1 > 0 T(\varepsilon_1) > 0$
such that
$\|x_0 - x_e\| < \delta_1 \Rightarrow \|x(x_0, t) - x_e\| < \varepsilon_1 \forall t \geq T(\varepsilon_1) + t_0$

Figure 1.3: Asymptotically stable equilibrium point

Asymptotic stability requires that the motion proceeds to x_e in the limit as t → ∞. Furthermore, it is the motion converges asymptotically so that the longer it gets the closer it gets to x_e [14].

1.2.2. Lyapunov's First Method and Local Stability Theorem

Consider the system (1.4) with a perturbation equation at an equilibrium state x_e, given by

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x_e)\delta x + r(x_e, \delta x) \quad \textbf{(1.7)}$$

so that

$$\lim_{\|\delta x\| \to 0} \frac{r(x_e, \delta x)}{\|\delta x\|} \quad \textbf{(1.8)}$$

Noting that the eigenvalues $\lambda_i, \quad i = 1,2,\ldots,n$ of the $n \times n$ matrix A, which is $\frac{\partial f(x_e)}{\partial x}$ in (1.4) are the solutions of the matrix determinant $\det[I\lambda_i - A] = 0$, the following results can be obtained:

- If all the eigenvalues of $\frac{\partial f(x_e)}{\partial x}$ have only negative real parts, x_e is asymptotically stable
- If one or more of the eigenvalues of $\frac{\partial f(x_e)}{\partial x}$ have positive real parts, x_e is unstable.
- If one or more eigenvalues of $\frac{\partial f(x_e)}{\partial x}$ have zero real parts and no eigenvalues with positive real parts, stability of x_e can not be ascertained by perturbation theory.

In particular, Lyapunov's first method maybe analysed for stability at isolated equilibrium points by means of its linearized autonomous equations if the highe-order terms of the Tylor series are sufficiently small [14].

Assume that the equilibrium point to be tested for stability is the origin like the nonlinear system (1.4) with (1.5). Let the elements of the Jacobian matrix

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \textbf{(1.9)}$$

exist and be continuous at the origin. As a consequence, $f(x)$ can be written

$$f(x) = Ax + g(x) \qquad \textbf{(1.10)}$$

where

$$\lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} = 0 \qquad \textbf{(1.11)}$$

As a candidate of Lyapunov function, $V(x) = x^T Mx$ is selected where M is the solution of a Lyapunov equation

$$A^T M + MA = -Q$$

with $Q = I$ and M are positive definite matrices. Then

$$\dot{V} = (x^T A^T + g^T(x))Mx + x^T(Ax + g(x))$$
$$= -x^T Qx + g^T(x)Mx + x^T Mg(x) \quad \textbf{(1.12)}$$

Using (1.11), $g(x)$ approaches zero faster than x. Thus, by keeping x sufficiently small, $\left\| g^T(x)Mx + x^T Mg(x) \right\|$ can be kept smaller than $x^T Qx = x^T x$. Hence, the local stability implies that the origin of the nonlinear system $\frac{dx}{dt} = f(x)$ is asymptotically stable if the Jacobian matrix (1.9) has all of its eigenvalues in the left half plane excluding the imaginary axis. If the linearized system has eigenvalues on the imaginary axis, the stability in the vicinity of the origin depends on the higher-order terms, *i.e.* $g(x)$ in (1.11) [2].

1.3. Lyapunov's Second Method

Lyapunov introduced an interesting direct method to investigate the stability of a solution to a nonlinear differential equation. The key idea is that the equilibrium will be stable if there can be found a real function on the state space whose level curves enclose the equilibrium such that the derivative of the state variables always points towards the interior of the level curves [1].

1.3.1. Stability and Energy

Consider the total constant energy E of a conservative system as

$$T(\dot{x}) + V(x) = E \quad \textbf{(1.13)}$$

where $T(\dot{x}) = \frac{1}{2}m(\dot{x})^2$ is kinetic energy, $V(x) = \int f(x)dx$ is potential energy stored in a forced spring with the spring force function $f(x)$, and x, $\dot{x}$ are respectively position and velocity. The Euler-Lagrange equation for such a system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \textbf{(1.14)}$$

with the Lagrangian, $L(x, \dot{x}) = T(\dot{x}) - V(x)$. Then (1.13) can be represented by normalizing by $m = 1$

$$\frac{d^2 x}{dt^2} + f(x) = 0 \quad \textbf{(1.15)}$$

Noting these, if the conservative system expressed in (1.15) has an added nonlinear damping term $h(x)\dot{x}$, the non-conservative system becomes

$$\frac{d^2x}{dt^2} + h(x)\frac{dx}{dt} + f(x) = 0 \tag{1.16}$$

where $h(x) \geq 0$. The total energy $E = H(x, \dot{x})$ decreases monotically toward an asymptotically stable equilibrium state where the potential energy becomes a minimum with zero kinetic energy. In canonical form, (1.16) is given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -f(x_1) - h(x_1)x_2 \end{aligned} \tag{1.17}$$

where $x_1 f(x_1) > 0$ and $f(0) = 0$ so that $x_{1e} = x_{2e} = 0$ is a unique equilibrium point. If the energy is normalized with respect to mass $m = 1$, and $V(x_1, x_2)$ designates the total energy as

$$V(x_1, x_2) = \frac{(x_2)^2}{2} + \int_0^{x_1} f(\sigma)d\sigma \tag{1.18}$$

with

$V(x) > 0$ for $x \neq 0$ and

$$V(0) = 0 \tag{1.19}$$

(1.19) states that the energy goes to zero at equilibrium point.

Note that the rate of change of the energy a long a trajectory which is the solution to (1.18) is given by

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt}$$

where

$$\frac{dV}{dt} = -h(x_1)(x_2)^2 \tag{1.20}$$

Therefore, the total energy $V(x)$ is dissipated along a solution path if the damping $h(x_1)$ is positive for all nonzero x_1, and x_2 is nonzero. Moreover, since $\dot{x}_2 = -f(x_1)$ is zero only when $x_e = 0$, the motion can not remain at $x_2 = 0$ unless it is an equilibrium with $x_1 = 0$ as well.

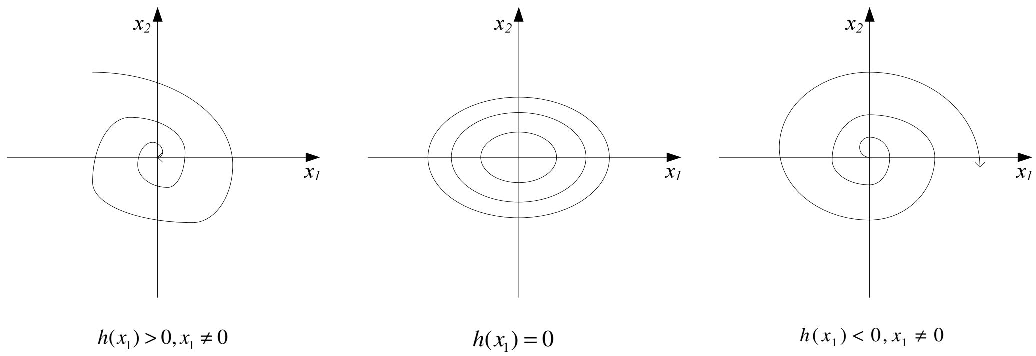


Figure 1.4: Phase-plane sketches for (1.17)

1.3.2. Lyapunov's Stability Theorem

Consider the system (1.4) with (1.5). Then the Lyapunov function $V(x)$ corresponding to this system is defined in a neighbourhood D of the origin if

- $V(x)$ is positive definite.
- $\frac{dV}{dt} = \left(\frac{\partial V}{\partial x}\right)\dot{x} = \frac{\partial V}{\partial x} f(x)$ is negative semi-definite.

Then the following theorems can be derived [14]:

- Stability Theorem: The origin is table if a Lyapunov function $V(x)$ exists throughout D, a neighbourhood of origin.
- Asymptotical Stability Theorem: The origin is asymptotically stable if a Lyapunov function $V(x)$ exists throughout D, a neighbourhood of the origin such that $\dot{V}(x)$ is negative definite.
- Instability Theorem: The origin is unstable if a $V(x)$ exists in the neighbourhood D of the origin, where $V(0) = 0$ such that $\dot{V}(x)$ is positive definite on D, and $V(x) > 0$ for $\|x\|$ arbitrary small.

- Global Asymptotic Stability Theorem: The origin is asymptotically stable in the large if it is asymptotically stable and $V(x)$ is radially unbounded such that $V(x) \to \infty$ as $\|x\| \to \infty$.

- Region of the Asymptotic Stability Theorem: As theorem (2) with $V(x) < \eta$ in D. Then the 0 is asymptotically stable and every solution with $x(t_0)$ in D approaches 0 asymptotically.

1.3.3. Lyapunov Function Generation

For linear continuous systems, a quadratic form which $V(x) = x^T Mx$ is a Lyapunov function satisfies Lyapunov equation such as $A^T M + MA = -Q$ where $M = M^T > 0$, $Q = Q^T > 0$ if the equilibrium state is asymptotically stable. Nevertheless, for nonlinear systems there is not such a methodology available. Therefore, several methods are proposed to generate Lyapunov function for nonlinear systems. One of them is Aizerman's Method which proceeds as follows to analyse stability of 0 for (1.4).

a. Linearize (1.7) at 0 to obtain

$$\dot{\delta x} = A(0)\delta x \qquad \textbf{(1.21)}$$

where

$$A(0) = \frac{\partial f}{\partial x}(0)$$

b. Select a quadratic form for (1.21) which is positive definite so that

$$V = x^T Mx \qquad \textbf{(1.22)}$$

has unspecified $m_{ij} = m_{ji}$, where M is a positive definite matrix and m_{ii} is its real positive elements where $i = 1, 2, 3\ldots n$.

c. Select a negative definite $\dot{V} = -x^T Qx$ according to the Lyapunov equation $A^T M + MA = -Q$ and (1.21), which in turn specifies q_{ii} and Q, if (1.21) is stable.

d. Solve $\dot{V}$ along $x(t)$ for the nonlinear system (1.4), and recomputed V from (1.22) and step (c).

e. Find the range for permissible parameter values for asymptotic stability.

2. SLIDING MODE CONTROL

Every control variable has a limit range. As an example, an on-off switching device can not be more than fully open or more than fully closed. In the same sense, the control voltage of a drive system can not exceed the supply voltage. Therefore, all control system design in practice must be handled with control variables that are saturated [2].

Usually, use of an actuator that is so capable to avoid saturation, it is often not economical to implement because of its characteristics such as cost, weigh or size and etc. as well. Hence, a control system using such an actuator has been over designed if the actuator is also rarely used [2].

In early 60's, researches in sliding mode control had been widely done in former USSR by Emelyanov and Barbashin and also in Yugoslavia [5]. The nature of the investigations had expanded from mostly theoretical issues in the next two decades to many industrial applications. In 1976, it was the article of Professor Utkin which provided a broad perspective of many potential applications of sliding mode control. [6].

The popularity of sliding mode control has continued increasing for the last decade due to the possibility of realization in nonlinear systems and its ability to consider robustness to modelling uncertainty and disturbance. In the nonlinear systems, the sliding mode controller tackles both the nonlinearity and the uncertainty of the system [11]. Sliding mode control has been applied in robot control, motor control, in spacecraft control and process control [9].

Many applications in frictionless bearing and high speed trains have been seen to have successfully applied sliding mode control in magnetic levitation. The performance of the sliding mode controller has achieved a superior result compared to classical controller in magnetic levitation applications [10].

In sliding mode control, modelling the nonlinear system with unknown disturbance has a big influence in the result. A proposed novel controller is designed with the help of well-defined model for the nonlinearities and finite element analysis for characterization of uncertainties [11]. According to the comparative simulations results of different controllers such as proposed novel sliding mode controller, feedback linearization, PID control and Linear-model-based sliding mode controller, the tracking errors are given in table 2.1 [11].

Table 2.1: Comparison of tracking errors with different controller types investigated in the study of Yeh, Chung and Wu [11]

	Proposed design	Feedback linearization	PID Control	Linear-model-Based sliding
RMS error	0.23 μm	0.34 μm	0.46 μm	2.4 μm
Mean error	0.004 μm	0.18 μm	0.015 μm	0.03 μm

2.1. Variable Structure Control

Variable structure control systems are a class of systems where the control law is deliberately changed during the control process according to the certain rules defined to stabilize the plant in issue. They consist of a set of continuous subsystems with a proper switching logic. The resulting control action is a discontinuous function of the system states, disturbance and reference inputs.

Consider a double integrator of a system where y is the position and $\dot{y}$ is the velocity

$$\ddot{y}(t) = u(t) \qquad \textbf{(2.1)}$$

and the effect of using feedback control law with a positive scalar k

$$u(t) = -ky(t) \qquad \textbf{(2.2)}$$

Substituting (2.1) in (2.2) and multiplying both sides with $\dot{y}$ gives

$$\dot{y}\ddot{y} = -k\dot{y}y \qquad \textbf{(2.3)}$$

Integrating (2.3) results in

$$\dot{y}^2 + ky^2 = c \qquad \textbf{(2.4)}$$

Depending on the value of k, equation (2.4) plots a circle or an ellipse. From control point of view, the control law given in (2.2) is not appropriate since the position y

and the velocity $\dot{y}$ do not converge to the origin. Although y and $\dot{y}$ remain bounded for all time the closed loop is stable, it is asymptotically not stable.

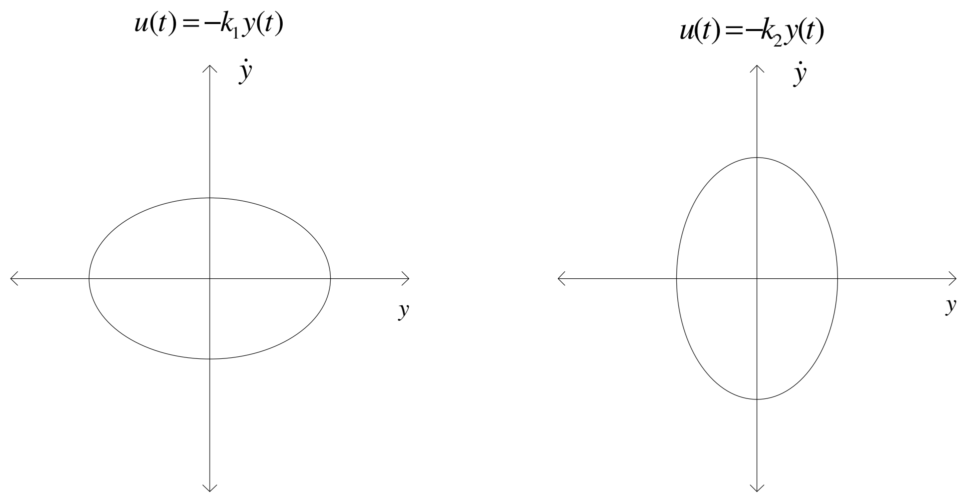


Figure 2.1: Phase portraits of simple harmonic motion.

Therefore, the control law is modified

$$u(t) = \begin{cases} -k_1 y(t) & y\dot{y} < 0 \\ -k_2 y(t) & otherwise \end{cases} \quad \textbf{(2.5)}$$

where $0 < k_1 < 1 < k_2$. This control law fits the description of variable structure control and results in the following plot in figure 2.2 by splicing together the appropriate regions of the plots in figure 2.1.

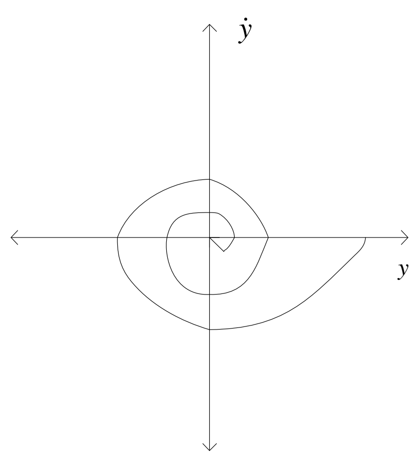


Figure 2.2: Phase portrait of the system under variable structure control

This can be verified by considering the function

$$V(y,\dot{y}) = y^2 + \dot{y}^2 \quad \textbf{(2.6)}$$

The function in (2.6) is the circle if the distance from the point $(y,\dot{y})$ to the origin and may be considered as the energy of the system. The time derivative of (2.6)

$$\dot{V}(y,\dot{y}) = 2y\dot{y} + 2\dot{y}\ddot{y} = \begin{cases} 2y\dot{y}(1-k_1) & y\dot{y} < 0 \\ 2y\dot{y}(1-k_2) & y\dot{y} > 0 \end{cases} \quad \textbf{(2.7)}$$

is always negative and the distance approaches to the origin.

A more significant expression for control law can be defined as follows

$$u(t) = \begin{cases} -1 & s(y,\dot{y}) > 0 \\ 1 & s(y,\dot{y}) < 0 \end{cases} \quad \textbf{(2.8)}$$

where the switching function is

$$s(y,\dot{y}) = my + \dot{y}, \qquad m > 0 \quad \textbf{(2.9)}$$

The switching function (2.9) crosses the origin for any value of *m* where $m\lfloor \dot{y} \rfloor < 1$ is satisfied and $s(y,\dot{y}) = 0$ as shown in figure 2.3.

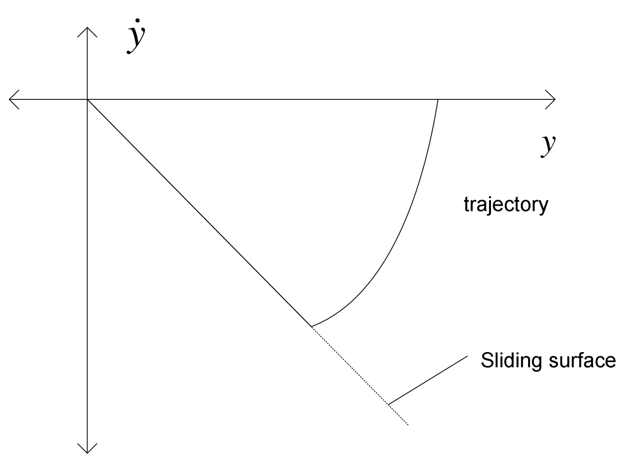


Figure 2.3: Phase portrait of sliding motion

When $m|\dot{y}| < 1$,

$$\lim_{s \to 0^+} \dot{s} < 0 \qquad \text{and} \qquad \lim_{s \to 0^-} \dot{s} > 0 \qquad \textbf{(2.10)}$$

Such dynamical behaviour is called as ideal sliding motion and the equation (2.11) is called as the sliding surface.

$$L_s = \{(y, \dot{y}) \ : \ s(y, \dot{y}) = 0\} \qquad \textbf{(2.11)}$$

2.2. Properties of Sliding Motion

The key result is that the sliding surface (2.11) is obtained and is forced to remain there. During sliding mode, the system behaves as if it is independent of the control. The control action ensures that the conditions in (2.10) are satisfied and this guarantees that $s(y, \dot{y}) = 0$. (2.10) is also expressed as

$$s\dot{s} < 0 \qquad \textbf{(2.12)}$$

which is referred to as reachability condition.

The aim is therefore to explore the relation between the control action and the switching function instead of the one between the control action and the plant output.

Consider the double integrator in (2.1) and the control law in (2.8). When $m = 1$, the control action is in figure 2.4 for closed-loop behaviour. Assume t_s when the switching surface is reached and an ideal sliding motion takes place.

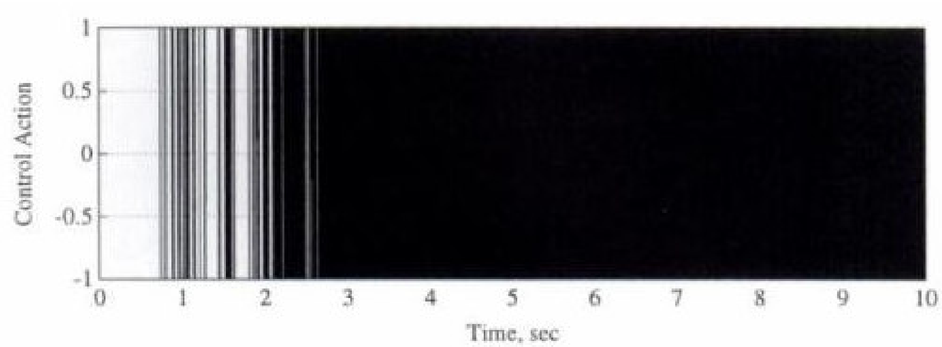


Figure 2.4: Discontinuous control action [5]

When $s(t) = 0$ for all $t > t_s$, thus $\dot{s}(t) = 0$ for all $t \geq t_s$. This implies

$$u(t) = -m\dot{y}(t) \text{ for } t \geq t_s \quad \textbf{(2.13)}$$

The control action in (2.13) is called as equivalent control action.

2.2.1. Existence of Solution and Equivalent Control

Consider the linear time invariant system with uncertainty

$$\dot{x}(t) = Ax(t) + Bu(t) + f(x,u,t) \quad \textbf{(2.14)}$$

where $A \in R^{n \times n}$ and $B \in R^{n \times m}$ with $1 \leq m < n$. $f : R \times R^n \times R^m \rightarrow R^n$ represents the bounded uncertainty. Let $s : R^n \rightarrow R^m$ be a linear function represented as

$$s(x) = Sx \quad \textbf{(2.15)}$$

where $S \in R^{m \times n}$ is full rank and is defined as hyper plane

$$S = \{x \in R^n \quad : \quad s(x) = 0\} \quad \textbf{(2.16)}$$

Let the uncertainty of system in (2.14) is identically zero and assume the systems states lay on the surface *S* define in (2.16) at the time t_s which means that $Sx(t) = 0$ and $\dot{s}(t) = S\dot{x}(t) = 0$ for all $t \geq t_s$. Thus (2.14) becomes

$$S\dot{x}(t) = SAx(t) + SBu(t) = 0 \quad \text{for all } t \geq t_s \quad \textbf{(2.17)}$$

Suppose the matrix *S* such that *SB* is a non-singular square matrix. This implies the definition that the equivalent control associated with the system (2.14) with zero uncertainty is defined to be the unique solution to the algebraic equation (2.17) and given as

$$u_{eq}(t) = -(SB)^{-1} SAx(t) \quad \textbf{(2.18)}$$

Thus a motion independent of the control action is denoted as

$$\dot{x}(t) = (I - B(SB)^{-1} S)Ax(t) \quad \text{for all } t \geq t_s \text{ and } Sx(t_s) = 0 \quad \textbf{(2.19)}$$

2.2.2. Independency of Uncertainty

Define

$$P_s \equiv (I - B(SB)^{-1} S) \quad \textbf{(2.20)}$$

as a projection operator satisfying

$$SP_s = 0 \qquad \text{and} \qquad P_s B = 0 \qquad \textbf{(2.21)}$$

Defining the uncertainty function in (2.14) as $f(x,u,t) = D\xi(x,t)$ where the matrix $D \in R^{n \times l}$ is known and $\xi : R_+ \times R^n \to R^l$ is unknown, the equivalent control (2.18) becomes

$$u_{eq}(t) = -(SB)^{-1}(SAx(t) + SD\xi(x,t)) \qquad \text{for all } t \geq t_s \qquad \textbf{(2.22)}$$

and the sliding motion satisfies

$$\dot{x}(t) = P_s Ax(t) + P_s D\xi(x,t) \quad \text{for all } t \geq t_s \text{ and } Sx(t_s) = 0 \qquad \textbf{(2.23)}$$

Consider P_s is the projection operator as in (2.20) and $R(D) \subset R(B)$. There exists a matrix of elementary column operations $R \in R^{m \times l}$ such that $D = BR$. This implies $P_s D = 0$ and results in

$$\dot{x}(t) = P_s Ax(t) \qquad \text{for all } t \geq t_s \text{ and } Sx(t_s) = 0 \qquad \textbf{(2.24)}$$

As a result, any uncertainty which can be expressed as in (2.14) where $f(x,u,t) = D\xi(x,t)$ and $R(D) \subset R(B)$ is defined as matched uncertainty and the sliding motion does not depend on the exogenous signal.

2.2.3. Reachability

Consider the system general denoted in (2.14) and its sliding motion defined in (2.9 – 2.11). Since $\frac{1}{2}\frac{d}{dt}(s^2) = s\dot{s}$,

$$V(s) = \frac{1}{2}s^2 \qquad \textbf{(2.25)}$$

follows as a Lyapunov function for the state *s*. Equations (2.10) and (2.12) do not guarantee the existence of an ideal sliding motion as they guarantee that the sliding surface is reached asymptotically.

Let the linear feedback control law be defined as

$$u(t) = -(m + \Phi)\dot{y}(t) - \Phi y(t) \qquad \textbf{(2.26)}$$

where Φ is a positive design scalar. The closed-loop motion therefore has the poles $(-m, -\Phi)$ and a direct computation reveals

$$\dot{s} = -\Phi s \qquad \text{where} \quad \dot{s}s = -\Phi s^2 \qquad \textbf{(2.27)}$$

From (2.27), it follows that

$$s(t) = s(0)e^{-\Phi t} \quad \textbf{(2.28)}$$

Therefore, if $s(0) \neq 0$ which means that the states initially do not lay on the sliding surface, then $s(t) \neq 0$ for all $t > 0$. However $s(t) \to 0$ as $t \to \infty$. A stronger condition is the η-reachability condition given by

$$s\dot{s} \leq -\eta|s| \quad \textbf{(2.29)}$$

where η is a small positive constant. Rewriting (2.29) as $\frac{1}{2}\frac{d}{dt}(s^2) = s\dot{s} \leq -\eta|s|$ and integrating it from 0 to t_s

$$|s(t_s)| - |s(0)| \leq -\eta t_s \quad \textbf{(2.30)}$$

is obtained and t_s is implied as

$$t_s \leq \frac{|s(0)|}{\eta} \quad \textbf{(2.31)}$$

2.3. Chattering Problem

There might be two possible erroneous switching curves for a second order system. If the actual switching curve is below the ideal switching curve, the switching would follow later than it would on the ideal switching curve but parallel to it. This sequence continues indefinitely, as the trajectory works its way to the origin without reaching it in a finite time.

Another situation is where the actual switching curve is above the ideal switching curve. In this case, the switching would occur before it reaches the ideal switching curve. The sign of the control is such that the state would move on a trajectory that returns it to the region from where it was just before. As soon as this happens, the control is switched again. Therefore, the control would switch at an infinite frequency which is called as chattering while the state slides along the switching curve [2].

2.4. Model Reference Design Approach

The model-following design has the objective to develop a control scheme which drives the plant dynamics to follow the desired dynamics of an ideal model and is developed because of the difficulties encountered in direct design of multi-variable control system using linear optimal control techniques. A linear model-following approach avoids also the difficulty of performance specification because the model specifies the design objectives where the controller is supposed to minimise the tracking error between the plant and the model. The problem of parameter variations will still remain which requires that the adaptive rules maintain the high performance. Therefore a transient response of the error dynamics can be prescribed. The approach is well suited to apply to uncertain, time-varying systems because it does not require any convergence properties [5].

Assume a linear time-invariant system defined by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.32}$$

and the corresponding ideal model by

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{2.33}$$

where $x, x_m \in R^n$ are the state vectors of the real system and ideal model, $u \in R^m$ is the control vector, $r \in R^r$ is the control input vector and A, B, A_m and B_m are the compatible dimensioned matrices. The pair (A, B) is assumed to be controllable and that the ideal mode is asymptotically stable. The tracking error is defined by

$$e(t) = x(t) - x_m(t) \tag{2.34}$$

The error is derived as

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t) \tag{2.35}$$

The dynamics of the error system can now be determined directly from equations (2.34) and (2.35)

$$\dot{e}(t) = Ax(t) - A_m x_m(t) + Bu(t) - B_m r(t) \tag{2.36}$$

Adding and subtracting the term $Ax_m(t)$ the equation (2.36) becomes

$$\dot{e}(t) = Ae(t) + (A - A_m)x_m(t) + Bu(t) - B_m r(t) \tag{2.37}$$

It is evident that for any given system and model, a perfect model-following system may be imposed to achieve. A sufficient condition is that all orders of the time derivatives of the error are zero at any time *t*. By starting with the zeroth derivative, it follows

$$x_m(t) = x(t) \tag{2.38}$$

Considering that some arbitrary term feeding forward the model states, is added to the control action gives

$$\dot{x}(t) = Ax(t) + B(u(t) + Gx_m(t)) \tag{2.39}$$

Since the first derivative of error zero,

$$Ax(t) + Bu(t) + BGx_m(t) = A_m x_m(t) + B_m r(t) \tag{2.40}$$

must hold. Thus the control expression is obtained as

$$u(t) = B^{\iota}(A_m x_m(t) + B_m r(t) - Ax(t) - BGx(t)) \tag{2.41}$$

where B^{ι} denotes the Moore – Penrose pseudo-inverse of matrix *B*. Substituting the equation (2.41) in (2.39) and rearranging yields

$$(BB^{\iota} - I)A_m x_m(t) - (BB^{\iota} - I)Ax(t) + (BB^{\iota} - I)B_m r(t) = 0 \tag{2.42}$$

Noting the equation (2.38), the equation (2.42)

$$(BB^{\iota} - I)(A - A_m) = 0 \tag{2.43}$$

$$(BB^{\iota} - I)B_m = 0 \tag{2.44}$$

The equations (2.43) and (2.44) show that all the derivatives of error will be also zero after an arbitrary time *t*. If the structure of the control signal will be defined by

$$u(t) = u_1(t) + u_2(t) \tag{2.45}$$

where

$$u_1(t) = -Ke(t) \tag{2.46}$$

$$u_2(t) = B^{\iota}(A_m - A)x(t) + B^{\iota}B_m r(t) \tag{2.47}$$

Substituting the control law (2.45) in the equation (2.37) and assuming (2.43) and (2.44) hold, then

$$\dot{e}(t) = (A_m - BK)e(t) \qquad \textbf{(2.48)}$$

If (A_m, B) is a controllable pair, the closed-loop matrix $A_m - BK$ can have an arbitrary set of eigenvalues to find suitable K. Equation (2.43) and (2.44) are the conditions for a perfect tracking and the equations (2.45) is the control law for implementing it.

If the following rank conditions hold

$$rank[B \quad A_m - A] = rank[B] \qquad \textbf{(2.49)}$$

$$rank[B \quad B_m] = rank[B] \qquad \textbf{(2.50)}$$

there exists compatibly dimensioned matrices F and G such that

$$BF = A_m - A \qquad \textbf{(2.51)}$$

$$BG = B_m \qquad \textbf{(2.52)}$$

Thus the equation (2.47) can be rewritten by

$$u_2(t) = BFx(t) + Gr(t) \qquad \textbf{(2.53)}$$

Gurleyen, Bahadir and Tekin proposed using a model reference design in their approach by linearizing the non-linear system behaviour. A differential dynamic behaviour of second order is determined to achieve as objective and is denoted by

$$\begin{bmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\varpi_n^2 & -2\xi\omega \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix} + \begin{bmatrix} 0 \\ \varpi_n^2 \end{bmatrix} x_r \qquad \textbf{(2.54)}$$

and is based on the transfer function of the second order dynamic system defined by

$$\frac{X_m}{X_r}(s) = \frac{\varpi_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad \textbf{(2.55)}$$

where x_m represents the desired state, x_r represents the reference state, ϖ_n is the undamped natural frequency being positive and ξ is the damping ratio of the second order system as positive value. Both of the parameters are pre-defined to achieve the desired performance.

In the study, the error vector $E = [e_1 \quad e_2]^T$ is defined by

$$E = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_{1m} - x_1 \\ x_{2m} - x_2 \end{bmatrix} \quad \textbf{(2.56)}$$

To extract the error dynamics, it is essential to consider the system model used in the study. The system is illustrated in section 4.1 and the dynamic behaviour of the system is given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{1}{2M}\frac{\partial L(x_1)}{\partial x_1}u(t)^2 + d(t) \end{aligned} \quad \textbf{(2.57)}$$

Therefore, the error dynamic is denoted by

$$\dot{e}(t) = Ae(t) + b\sigma(u) \quad \textbf{(2.58)}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\varpi_n^{\ 2} & -2\xi\omega_n \end{bmatrix}; \qquad b = \begin{bmatrix} 0 \\ \varpi_n^{\ 2} \end{bmatrix} \quad \textbf{(2.59)}$$

and $\sigma(u)$ is a function of control signal u_c and the control current term ($u(t) = i(t)$) of system dynamics.

$$u_c = \omega_n^{\ 2}(x_r - x_1) - 2\xi\varpi_n x_2 - d(t) \quad \textbf{(2.60)}$$

Therefore,

$$u_c = \omega_n^{\ 2}(x_r - x_1) - 2\xi\varpi_n x_2 - d(t) \quad \textbf{(2.61)}$$

$$\sigma(u) = \frac{1}{\varpi_n^{\ 2}}(u_c - \frac{1}{2M}\frac{\partial L}{\partial x_1}u^2) \quad \textbf{(2.62)}$$

By choosing the Lyapunov equation as

$$\begin{aligned} V(E) &= E^T PE \\ \dot{V}(E) &= -E^T QE + 2E^T Pb\sigma(u) \end{aligned} \quad \textbf{(2.63)}$$

the stability of the approach can be verified as long as there is a real $u(t)$ satisfying $\sigma(u) = 0$ for every $t \geq t_0$ [9].

2.5. Controllers Using Output Information

In most practical situations as mentioned before, all the state variables of the system might be neither physically possible nor economical to measure. Therefore, the approach to design the control system with uncertainties aims to use the only available output information.

Consider a system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(x,u,t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.64}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$ with $m \le p < n$. Assume that the nominal linear system (A, B, C) is known and that the input and output matrices B and C are both of full rank. The function $f : R_+ \times R^n \times R^m \to R^n$ represents the system nonlinearities and is assumed to match the condition

$$f(x,u,t) = B\xi(x,u,t) \tag{2.65}$$

where the bounded function $\xi : R_+ \times R^n \times R^m \to R^m$ satisfies

$$\|\xi(x,u,t)\| < k_1\|u\| + \alpha(y,t) \tag{2.66}$$

with some known function $f : R_+ \times R^p \to R_+$ and positive constant $k_1 < 1$.

The objective here is to develop a control law which induces and ideal sliding motion on the surface

$$S = \{x \in R^n : FCx = 0\} \tag{2.67}$$

for some selected matrix $F \in R^{m \times p}$. A control law of the form

$$u(t) = Gy(t) - v_y \tag{2.68}$$

will be searched where G is a fixed gain matrix and the discontinuous vector v_y

$$vy = \begin{cases} \rho(y,t)\dfrac{Fy(t)}{\|Fy(t)\|} & Fy \ne 0 \\ 0 & otherwise \end{cases} \tag{2.69}$$

where $\rho(y,t)$ is the positive scalar function of the outputs.

2.6. Some Other Approaches to Sliding Mode Control Design

In this section, the suggested methods in several sources are applied to active magnetic levitation systems.

Considering the system illustrated in figure 2.5; Cho, Kato and Spilman used the dynamic model defined by

$$\dot{x}(t) = B(x)u(t) - g + d(t) \quad \textbf{(2.70)}$$

where $B(x)$ is the force – distance relationship given by

$$B(x) = \frac{1}{m\left(a_1 x^2(t) + a_2 x(t) + a_3\right)} \quad \textbf{(2.71)}$$

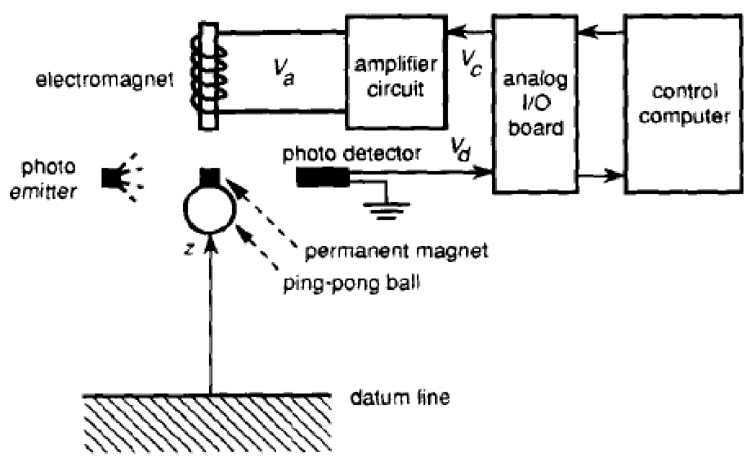


Figure 2.5: Schematic of single-axis magnetic levitation. z ($x(t)$) is the distance from object to the bottom [10].

In order to achieve a desired error dynamics, Cho, Kato and Spilman suggested the sliding surface in their study [10] as

$$S(t) = \dot{e}(t) + \lambda e(t) \quad \textbf{(2.72)}$$

Thus their objective has been to achieve $S(t) = 0$. For this, the attraction condition has been defined as $S(t)\dot{S}(t) < 0$ as in (2.12).

Remembering the reachability condition in (2.29), a control law is formulated to achieve $S(t) = 0$ on average.

$$u(t) = \left(m(a_1 x_1(t)^2 + a_2 x_1(t) + a_3)\right) \times \left(g + \ddot{x}_r(t) - \lambda(x2(t) - x_r(t)) - \eta \operatorname{sgn}(S(t))\right) \quad \textbf{(2.73)}$$

This control law results in a chattering problem due to the discontinuity of the function *sgn*(*S*(*t*)) while the control in (2.73) law nevertheless stabilizes the system. The chattering problem can be improved by using control smoothing approximation. The indefinite of *sgn*(*S*(*t*)) at *S*(*t*) = 0 can be replaced with a finite gain when the magnitude of the *S*(*t*) is smaller than some prescribed value ϕ. This can be achieved by replacing the function *sgn*(*S*(*t*)) with

$$sat\left(\frac{S(t)}{\phi}\right)=\begin{cases}\text{sgn}(S(t)) & |S(t)|\geq\phi\\ \dfrac{S(t)}{\phi} & |S(t)|<\phi\end{cases} \qquad \textbf{(2.74)}$$

The attraction guarantee of the *S*(*t*) = 0 manifold is possible only when $|S(t)|\geq\phi$. When $|S(t)|<\phi$, the attraction guarantee of the *S*(*t*) = 0 manifold may not be satisfied due to presence of the modelling errors and disturbance [10].

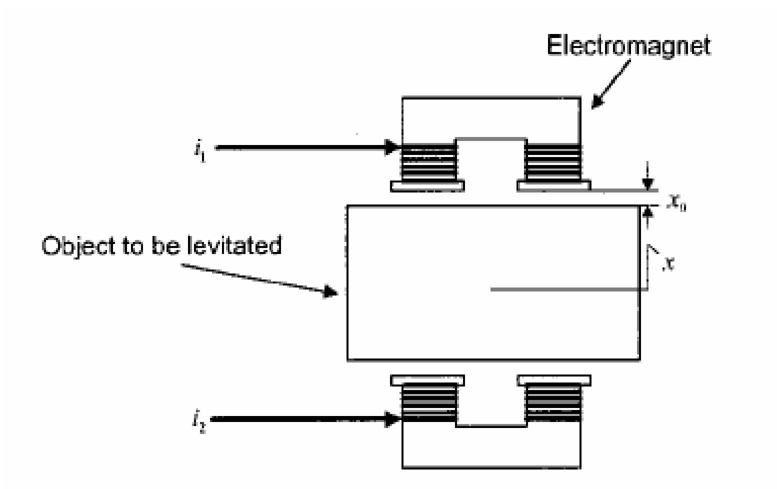


Figure 2.6: Magnetic bearing system

Yeh, Chung and Wu have worked on another model of magnetic bearing systems illustrated in figure 2.6 and proposed a controller consisting of a nominal control part that linearizes the nonlinear dynamics and the robust control part that provides robust performance against the uncertainties. There are two electromagnets and a levitated object. In this model, i_1 and i_2 are the currents input to the electromagnets, x_0 denotes the nominal air gap, x is the displacement, and μ_0 is the air permeability. The magnets are assumed to have the same pole area *A* and the same number of turns *n*.

The dynamic equation of the system is proposed by the equation in (2.75).

$$m\ddot{x} = \frac{n^2 \mu_0 A i_1{}^2}{4(x_0 - x)^2} - \frac{n^2 \mu_0 A i_2{}^2}{4(x_0 + x)^2} - mg \tag{2.75}$$

where

$$F = \frac{n^2 \mu_0 A i_1{}^2}{4(x_0 - x)^2} - \frac{n^2 \mu_0 A i_2{}^2}{4(x_0 + x)^2} \tag{2.76}$$

A bias current is applied to both coils and a control current is added or subtracted from either of the coil current. Thus, the dynamics (2.75) can be linearized and a linear controller is sufficient to maintain the stability and performance near the equilibrium point. However, the linearization is accurate only locally and the power consumption is slightly higher due to the bias voltage applied.

The control law is defined by

$$F = m\ddot{x}_r - c_0 \dot{e} - k_0 e + mg \tag{2.77}$$

where $e = x_r - x$ is the tracking error and the parameters c0 and k0 are positive constants so that the control law can lead to the following exponentially stable dynamics defined by

$$m\ddot{e} + c_0 \dot{e} + k_0 e = 0 \tag{2.78}$$

Since F is virtually control input, the control law has to be implemented by properly modulating coil currents such as

$$\begin{aligned} i_1 &= \sqrt{\frac{4F(x_0 - x)^2}{n^2 \mu_0 A}} \quad , i_2 = 0 \quad F \geq 0 \\ i_2 &= \sqrt{\frac{4F(x_0 + x)^2}{n^2 \mu_0 A}} \quad i_1 = 0 \quad F < 0 \end{aligned} \tag{2.79}$$

The feedback linearization stabilizes the system without presence of uncertainties and disturbance. Therefore the sliding mode controller is addressed to achieve the robust stability. The differential equation is defined then by

$$\ddot{x} = b(x)F - g \tag{2.80}$$

where $b(x)$ is a position dependent control gain and uncertain. Thus the control law has been chosen as

$$F = b^{-1}(x)\left[\hat{F} - k \cdot sat\left(\frac{s}{\phi}\right)\right] \quad \textbf{(2.81)}$$

In this control law

$$k \geq \beta(x)\eta + [\beta(x) - 1]|\hat{F}| \quad \textbf{(2.82)}$$

and *sat*(.) is the saturation function. *s* is the sliding surface defined by

$$s = \dot{e} + 2\lambda e + \lambda^2 \int e \quad \textbf{(2.83)}$$

with λ being strictly positive constant, ϕ the boundary layer, and η another strictly positive constant which dictates how fast the state trajectory reaches the sliding surface. $\beta(x)$ is the associated gain matrix and $\hat{F}$ is the control law when $b(x)$ is exactly known [11].

Hassan and Mohamed have used the variable structure control to stabilize a magnetic levitation system illustrated in figure 2.7 with the dynamic equation (2.84).

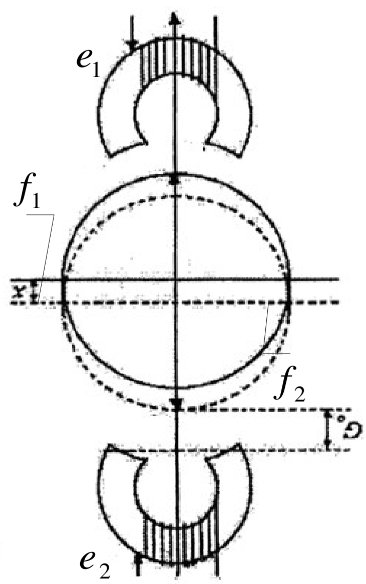


Figure 2.7: Schematic of magnetic levitation used by Hassan and Mohamed [12]

$$m\ddot{x} = f_2 - f_1 + mg - f_d \quad \textbf{(2.84)}$$

where x is the air gap deviation under the electromagnet, m is the object mass, f_1 and f_2 are the forces produced by the upper and lower electromagnets, and f_d represents disturbance or model uncertainty. The electromagnetic force for each electromagnet can be expressed in terms of flux ϕ and constant k

$$f_j = k\phi_j \tag{2.85}$$

The voltage e_j across the electromagnet has been defined by

$$e_j = N\frac{d\phi_j}{dt} + \frac{2R\phi_j g_j}{\mu_0 AN} \qquad j = 1,2 \tag{2.86}$$

where R is the coil resistance, N is the number of coil turns, A is the area of one magnet pole, g_j the air gap and G_0 the nominal air gap denoted as

$$g_1 = G_0 + x\,, \quad g_2 = G_0 - x \tag{2.87}$$

In order to control the air gap x, the dynamic equation is desired to include the voltages of electromagnets rather than electromagnetic forces and has been obtained by differentiating as

$$m\dddot{x} = \dot{f}_2 - \dot{f}_1 + \dot{f}_d \tag{2.88}$$

Differentiating (2.85) and substituting in (2.88) gives the dynamic equation including the electromagnetic forced voltages.

$$m\dddot{x} = \frac{2k\phi_2}{N}\left(e_2 - \frac{2R(G_0 - x)\phi_2}{\mu_0 AN}\right) - \frac{2k\phi_1}{N}\left(e_1 - \frac{2R(G0 + x)\phi 1}{\mu_0 AN}\right) - \dot{f}_d \tag{2.89}$$

The state variables are chosen as $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $x_4 = \phi_1$, $x_5 = \phi_2$ $u_1 = e_1$ and $u_2 = e_2$ where

$$X = A(X) + B(x)U + V(x) \tag{2.90}$$

with $X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$, $U = [u_1 \quad u_2]^T$, $D = \dot{f}_d$

The proposed design of variable structure controller by Hassan and Mohamed consists of two inputs and thus they have used two sliding surfaces S_1 and S_2. However the problem has been simplified into dealing with one sliding surface S because the objective to control the air gap of upper and lower electromagnets $\pm x$ results in the same switching surfaces S_1 and S_2 with opposite signs. The switching surface is suggested to be a linear combination of the error and its higher order derivatives such as

$$S = \ddot{e} + \lambda_1 \dot{e} + \lambda_2 e \qquad \textbf{(2.91)}$$

In the sliding surface, λ_1 and λ_2 are free design parameters such that the system is asymptotically stable. To minimize chattering, they have also denoted a reaching function of the form

$$\dot{S} = -Q\,\mathrm{sgn}(S) - KS \qquad \textbf{(2.92)}$$

where Q and K are free design parameters being positive real numbers.

Substituting the necessary equations into (2.92), the expression of the control signals can be obtained such as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{2kx_4}{mN} \\ \frac{2kx_5}{mN} \end{pmatrix} \left[\begin{pmatrix} -\frac{2kx_4}{mN} & \frac{2kx_5}{mN} \end{pmatrix} \begin{pmatrix} -\frac{2kx_4}{mN} \\ \frac{2kx_5}{mN} \end{pmatrix} \right]^{-1} \cdot$$

$$\left\{ \frac{4kR\left(x_5^{\,2}(G_0 - x_1) - x_4^{\,2}(G_0 + x_1)\right)}{m\mu_0 A N_2} + Q\,\mathrm{sgn}(S) + KS + \dddot{x}_d + \lambda_1(\ddot{x}_d - x_3) + \lambda_2(\dot{x}_d - x_2) - \frac{1}{m}D \right\}$$

(2.93)

Except the disturbance D, all the quantities are known in equation (2.93) according to the model of Hassan and Mohamed. To cope the problem due to the disturbance, they suggest replacing D with a conservative known quantity D_c, guaranteeing the reaching condition. Being D_L and D_U the lower and upper bounds respectively,

$$D_L \le D \le D_U$$

the conservative disturbance D_c is chosen according to the statements

- when $S < 0$, $D > D_c$ is desired, so let $D_c = D_L$
- when $S > 0$, $D < D_c$ is desired, so let $D_c = D_U$

which is denoted as

$$D_c = \frac{D_U + D_L}{2} + \frac{D_U - D_L}{2}\,\mathrm{sgn}(S) \qquad \textbf{(2.94)}$$

When the equation (2.94) is substituted with D in (2.93), the ultimate control signal can be obtained. As a result, robust stability against parameter perturbation is achieved [12].

3. NONLINEAR OBSERVER

In designing of control systems, all state variables assumed to be available and measurable for feedback by measuring with several kinds of sensor devices. However, in practise, current sensor technology and associated technical and cost limitations do not always allow the observation of full vector of systems state variable. On the other hand, there might not be necessary to measure other measurable state variables, either. These state variables can be linearly related to the other ones which are available to measure by conventional sensor devices.

Estimation of state variables is commonly called as observation. A device or a tool that estimates or observes the state variables is called as state observer. If the observer observes all the state variables of the system, it is called a full-order state observer. There might be some cases where the output variables are observable and linearly related to state variables. Thus, it is only necessary to estimate n – m state variables, where n is the dimension of the state vector and m is the dimension of the output vector. If an observer estimates n – m state variables of such a system, it is called reduced-order state observer.

3.1. Observability

Consider the system described by the following equations:

$$\dot{x} = Ax + Bu \quad \textbf{(3.1)}$$

$$y = Cx + Du \quad \textbf{(3.2)}$$

The system is said to be observable if every state $x(t_0)$ can be determined from the observation of the output y(t) over a finite time interval, $t_0 \leq t \leq t_1$. Therefore, the system is completely observable if every transition of the state eventually affects every element of the output vector.

According to the equations (3.1) and (3.2), the state x(t) and y(t) can be found as in the followings:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \quad \textbf{(3.3)}$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du \quad \textbf{(3.4)}$$

The matrices A, B, C and D are known and u(t) is also known, the last two terms on the right-hand side if the equation (3.4) are known quantities. Therefore, they may be subtracted from the observed value of y(t). Hence, to investigate the observability, it suffices to consider matrices A and C. Let x be an n – dimensional vector and y an m – dimensional output vector. Assume that A is an n × n and C is an m × n matrix. Considering the system with the following equations;

$$\dot{x} = Ax \quad \textbf{(3.5)}$$

$$y = Cx \quad \textbf{(3.6)}$$

y(t) becomes

$$y(t) = Ce^{At}x(0) \quad \textbf{(3.7)}$$

From the mathematical computation by using Cayley-Hamilton Theorem, e^{At} can be obtained as in the following.

$$e^{At} = \sum_{k=0}^{n-1}\alpha_k(t)A^k \quad \textbf{(3.8)}$$

By using (3.7) and (3.8), y(t) becomes

$$y(t) = \sum_{k=0}^{n-1}\alpha_k(t)CA^k x(0) = \alpha_0(t)Cx(0) + \alpha_1(t)CAx(0) + \ldots + \alpha_{n-1}(t)CA^{n-1}x(0) \quad \textbf{(3.9)}$$

According to the equation (3.9), given the output y(t) over a time interval, x(0) can be uniquely determined. Therefore, if the rank of the mn × n matrices

$$\begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix} \text{ or } \begin{bmatrix} C^T & A^TC^T & . & . & (A^T)^{n-1}C^T \end{bmatrix}$$

is n, the system described in (3.1) and (3.2) is completely observable.

3.2. Full-Order State Observer

Consider a plant with dynamic equation (3.1) and (3.2) where $D = 0$.

The observer is a subsystem to reconstruct the state vector of the plant to be controlled. The mathematical model of the observer is basically the same as that of the plant, except that an additional term is included where there is an estimation error to compensate for inaccuracies in matrices A and B and the lack of the initial error. The estimation error, which is also observation error, is the difference between the measured output and the estimated output. Initial error is the difference between errors of initial state and initial estimated state. Considering that $\tilde{x}$ is the observed (estimated) state vector, the mathematical model of the observer is defined in (3.10).

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - C\tilde{x}) = (A - K_e C)\hat{x} + Bu + K_e y \qquad \textbf{(3.10)}$$

The estimated output will be $C\tilde{x}$. K_e is the observer gain matrix, which is a weighing matrix to the correction term involving the difference between the measured output y and the estimated output $C\tilde{x}$.

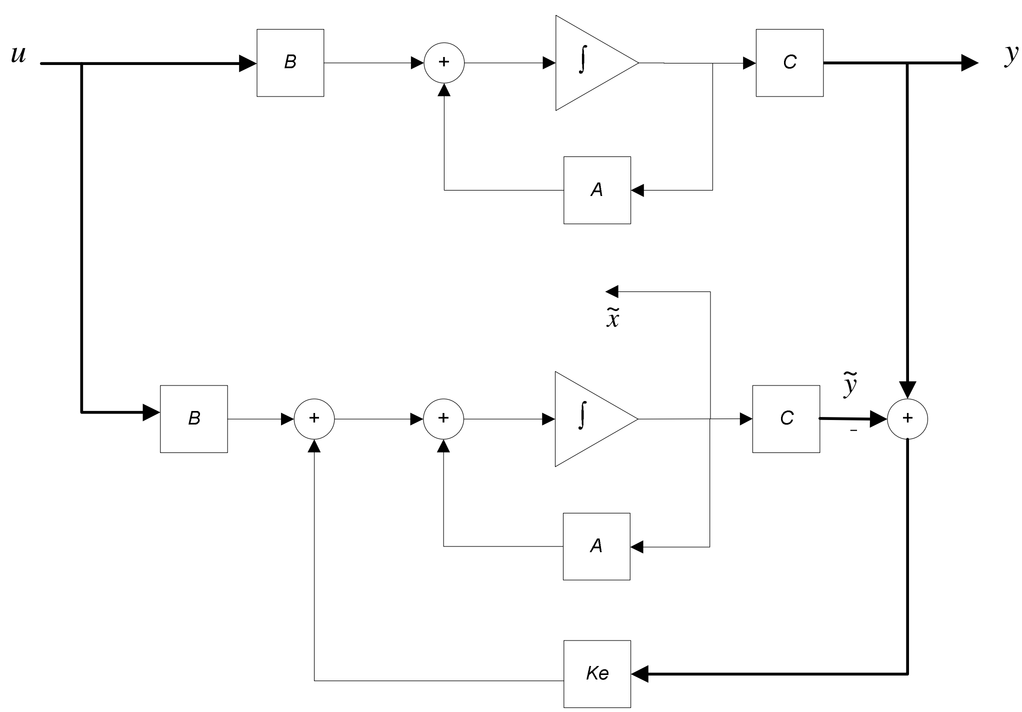


Figure 3.1: Full-state order observer figure

$x-\tilde{x}$ is defined as error vector e

$$e = x - \tilde{x} \tag{3.11}$$

and in order to obtain the observer error equation, one should subtract the equation (3.10) from the equation (3.1).

$$\dot{x} - \dot{\tilde{x}} = Ax + Bu - (A - K_e C)\hat{x} - Bu - K_e(Cx - C\tilde{x}) = (A - K_e C)(x - \tilde{x})$$

$$\dot{e} = (A - K_e C)e \tag{3.12}$$

From (3.12), the dynamic behaviour of the error vector is determined by the eigenvalues of matrix $A - K_eC$. If matrix $A - K_eC$ is a stable matrix, the error vector will converge to zero for any initial error vector e(0). This implies that $\tilde{x}(t)$ will converge to x(t) regardless of the values of x(0) and $\tilde{x}(0)$.

3.3. State Observer Gain Matrix

Designing a full-state observer becomes that of determining the observer gain K_e such that the error dynamics defined by equation (3.12) are asymptotically stable with sufficient speed response. Therefore, the design of full-order observer becomes that of determining an appropriate K_e such that $A - K_eC$ has desired eigenvalues [4].

On the other hand, let a_n be the coefficients of characteristic polynomial $|sI - A|$ such that

$$|sI - A| = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0 \tag{3.13}$$

Define α_n as the coefficients and μ_n as the desired eigenvalues of the characteristic polynomial of the equation (3.12) such as

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1}s + \alpha_n = 0 \tag{3.14}$$

Then let a new state vector $\hat{x}$ be defined such as

$$x = Q\hat{x} \tag{3.15}$$

where Q is the transition matrix and is defined as

$$Q = NW \tag{3.16}$$

$$N = \begin{bmatrix} C^T & A^T C^T & . & . & (A^T)^{n-1} C^T \end{bmatrix} \tag{3.17}$$

and

$$W = W^T = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{3.18}$$

The transition matrix approach is used in pole placement in designing a control system. Here the same approach is applied to obtain the gain matrix K_e. According to the transition matrix approach, the characteristic equation of the system (3.12) can be written as in equation (3.19).

$$|sI - A + K_e C| = |Q^{-1}(|sI - A + K_e C|)Q| = |sI - Q^{-1}AQ + Q^{-1}K_e CQ| = 0 \tag{3.19}$$

Noting that the eigenvalues of $A - KC$ and $A^T - C^T K^T$ are the same and that $Q^{-1}AQ$ and $Q^{-1}A^T Q$ result in the same matrix, the equation can be rewritten and solved as in the following.

$$\begin{aligned} &|sI - Q^{-1}A^T Q + Q^{-1}C^T K_e^{\ T} Q| \\ &= \left| sI - \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \delta_n & \delta_{n-1} & \cdot & \cdot & \delta_1 \end{bmatrix} \right| \\ &= \begin{vmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & \cdots & s + a_1 + \delta_1 \end{vmatrix} \\ &= s^n + (a_1 + \delta_1)s^{n-1} + \cdots + (a_{n-1} + \delta_{n-1})s + (a_n + \delta_n) = 0 \end{aligned} \tag{3.20}$$

The equations (3.14) and (3.20) give the same characteristic polynomial where

$$a_n + \delta_n = \alpha_n$$
.
.
$$a_2 + \delta_2 = \alpha_2$$
$$a_1 + \delta_1 = \alpha_1$$

From (3.20), it can be seen that

$$K_e{}^T Q = [\delta_n \quad \delta_{n-1} \quad \cdots \quad \delta_1]Q = [\alpha_n - a_n \quad \alpha_{n-1} - a_{n-1} \quad \cdots \quad \alpha_1 - a_1]Q \qquad \textbf{(3.21)}$$

Thus considering the property $\left(K_e{}^T Q\right)^T = Q^T K_e$ and $Q = NW$, K$_e$ can be obtained as in the following.

$$K_e = \left(Q^T\right)^{-1}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} = \left(WN^T\right)^{-1}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_1 - a_1 \end{bmatrix} \qquad \textbf{(3.22)}$$

Besides using transition matrix, another way of determining observer gain matrix is Ackermann's formula which is basically used in designing control system with state feedback. There is a matrix $\phi(A)$ defined as

$$\phi(A) = \alpha_n I + \alpha_{n-1} A + \cdots + \alpha_1 A^{n-1} + A^n \neq 0 \qquad \textbf{(3.23)}$$

$$K_e = \phi(A^T)^T (N^T)^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \textbf{(3.24)}$$

Referring to figure 3.1, one should notice that the feedback signal through the observer gain matrix K$_e$ serves as a correction signal to the plant model to account for the unknowns in the plant. If significant unknowns are involved, the feedback signal through the matrix K$_e$ should be relatively large. Nevertheless, if the output signal is contaminated significantly by disturbance and noises, the output might not be reliable and K$_e$ should be relatively small.

3.4. Nonlinear Observers

A major application of nonlinear observers is to provide an estimate of the process state for use in implementation of a nonlinear feedback control law. In such an application, estimation of the state of the process under control is only the means to another objective – stable closed-loop control. If the ultimate objective is achieved, inaccuracy in the estimate of the state of the process is hardly objectionable.

In the contrary, an estimate might also be needed for other purpose than closed-loop control. If only a rough estimate is acceptable, there is a great deal of latitude in design.

3.4.1. Nonlinear Full-Order Observer

A plant which is consisting a dynamic system like in equation (3.1), can be expressed

$$\dot{x} = Ax + Bu = f(x,u) \quad \textbf{(3.25)}$$

The observer of the system in (3.25) given by

$$y = g(x,u) \quad \textbf{(3.26)}$$

is another dynamic system, the state of which is donated by $\tilde{x}$, excited by the output y of the plant, having the property where the error is expressed as in (3.11) and converges to zero [2].

$$e = x - \tilde{x}$$

One way of obtaining an observer is to imitate the procedure applied in linear systems, namely to construct a model of the original system in (3.25) and force it with the "residual":

$$r = y - \tilde{y} = y - g(\tilde{x},u) \quad \textbf{(3.26)}$$

Thus the observer becomes

$$\dot{\tilde{x}} = f(\tilde{x},u) + \kappa(y - g(\tilde{x},u)) \quad \textbf{(3.27)}$$

where $\kappa()$ is a suitably chosen nonlinear function. A block-diagram representation of a general nonlinear observer is shown in figure 3.2.

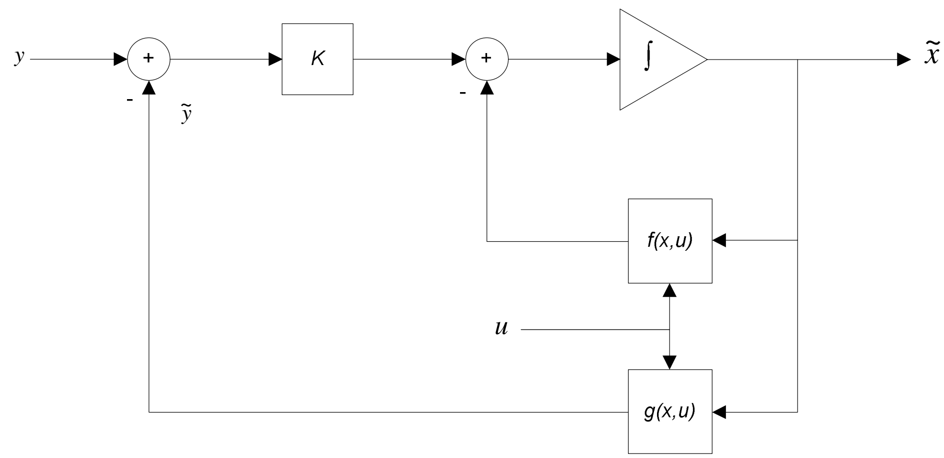


Figure 3.2: General structure of nonlinear observer

The expression of dynamic behaviour of the error e is given by

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\tilde{x}} \\ &= f(x,u) - f(\tilde{x},u) - \kappa(g(x,u) - g(\tilde{x},u)) \\ &= f(x,u) - f(x-e,u) + \kappa(g(x-e,u) - g(x,u)) \end{aligned} \qquad \textbf{(3.28)}$$

By the proper choice of $\kappa()$ the error equation in (3.28) can be made asymptotically stable, so that the equilibrium state is reached where $\dot{e}$ goes to zero when the nonlinear functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ used in the observer are exactly the same as in equations (3.25) and (3.26). Any discrepancy between the corresponding functions will generally prevent $\dot{e}$ from vanishing and therefore will lead a steady state error. Since the mathematical model of physical system is always an approximation, the steady state error will not go to zero in practise. For the same reason, the control law that goes to the plant should be applied in the same manner in the observation [2].

The function $\kappa()$ must be selected to ensure the asymptotically stability of the origin (e = 0). According to the theorem of Lyapunov's first method, the origin is asymptotically stable if the Jacobian matrix of the dynamics corresponds to an asymptotically stable linear system. For the dynamics of the error, the Jacobian matrix is obtained by deriving with respect to e at e = 0 and is given by

$$A_c(x) = \frac{\partial f}{\partial x} - K\left(\frac{\partial g}{\partial x}\right) \qquad \textbf{(3.29)}$$

3.4.2. Design Approaches for Nonlinear Reduced Order Observer Design

The design approach mentioned in the previous section can be stated in more general observer error linearization [8]. Consider a system

$$\begin{aligned} \dot{x} &= f(x,w) \\ y &= h(x,w) \end{aligned} \qquad \textbf{(3.30)}$$

that represents the dynamics of a process, where x is the state vector, y is the vector of measurements and w is the vector of unmeasurable process or sensor disturbance. The dynamics of the disturbance is governed by the system

$$\dot{w} = s(w) \qquad \textbf{(3.31)}$$

The problem of state and disturbance estimation can become a state estimation problem when the system is considered like:

$$\begin{aligned} \dot{x} &= f(x,w) \\ y &= h(x,w) \\ \dot{w} &= s(w) \end{aligned} \qquad \textbf{(3.32)}$$

where $\begin{bmatrix} x \\ w \end{bmatrix}$ is the extended state vector of the system which must be estimated with an appropriately designed observer [8]. In the consideration of the system (3.32), $f : R^n \times R^l \to R^n$, $s : R^l \to R^l$, $h : R^n \times R^l \to R^\rho$ are real analytic functions with $f(0,0) = 0$, $s(0) = 0$ and $h(0,0) = 0$. The number of states of the system (3.32) $(n+l)$ and the number of measurements is ρ.

As a special case of the problem, let the disturbances affect only in an additive way expressed like in equation [8] (3.33).

$$\begin{aligned} x &= f(x,w) \\ \dot{w} &= s(w) \\ y &= h(x) + q(w) \end{aligned} \qquad \textbf{(3.33)}$$

Considering the system (3.32), a locally analytic mapping $z = \theta(x,w)$ from $R^n \times R^l$ to $R^{n+l-\rho}$ is selected that maps the system (3.32) into

$$\dot{z} = Az + \beta(y) \qquad \textbf{(3.34)}$$

In (3.34), A is a $(n+l-\rho)\times(n+l-\rho)$ matrix and $\beta : R^\rho \to R^{n+l-\rho}$ is a real analytic function with $\beta(0) = 0$ [8]. With the aid of such a mapping, the equation (3.35) can be used as observer dynamics and the system's state $\begin{bmatrix} x \\ w \end{bmatrix}$ can be reconstructed from the solution of the set of algebraic equations

$$\begin{aligned} \theta(x,w) &= z \\ h(x,w) &= y \end{aligned} \tag{3.35}$$

The equations in (3.35) are solvable in x and w. Thus the reduced-order nonlinear observer takes the form

$$\dot{\tilde{z}} = A\tilde{z} + \beta(y) \tag{3.36}$$

$$\begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} \text{ is the solution of } \begin{cases} \theta(\tilde{x},\tilde{w}) = \tilde{z} \\ h(\tilde{x},\tilde{w}) = y \end{cases} \tag{3.37}$$

It becomes that the unknown immersion map θ must satisfy the following system of singular partial differential equation:

$$\frac{\partial \theta}{\partial x}(x,w) \cdot f(x,w) + \frac{\partial \theta}{\partial w} \cdot s(w) = A \cdot \theta(x,w) + \beta(h(x,w)) \tag{3.38}$$

Therefore, the problem is reduced to the study of partial differential equations (3.38) and the properties of the solutions.

<u>Proposition:</u>

Consider the system (3.32) with the new expression (3.34) that is mapped by the θ in (3.35).

$x = f(x,w)$; $f : R^n \times R^l \to R^n$

$y = h(x,w)$; $h : R^n \times R^l \to R^\rho$

$\dot{w} = s(w)$; $s : R^l \to R^l$

$\dot{z} = Az + \beta(y)$; $\beta : R^\rho \to R^{n+l-\rho}$

with $f(0,0) = 0$, $s(0) = 0$, $h(0,0) = 0$, $\beta(0) = 0$ and

$$F = \frac{\partial f}{\partial x}(0,0),\ P = \frac{\partial f}{\partial w}(0,0),\ S = \frac{\partial s}{\partial w}(0,0),\ H = \begin{bmatrix} \frac{\partial h}{\partial x}(0,0) & \frac{\partial h}{\partial w}(0,0) \end{bmatrix},\ B = \frac{\partial \beta}{\partial y}(0).$$

Denote by $\sigma(F)$ and $\sigma(S)$ the spectra of S and F respectively.

Assume:

1) There exists a $(n+l-\rho)\times(n+l)$ matrix T such that $T\begin{bmatrix} F & P \\ 0 & S \end{bmatrix} = AT + BH$ and $\begin{bmatrix} T \\ H \end{bmatrix}$ is invertible.

2) All the eigenvalues of A are non-resonant with $\sigma(F)\cup\sigma(S)$, which means i.e. no eigenvalue λ_i of A is of the form $\lambda_j = \sum_{i=1}^{n+l} m_i k_i$ where $k_i \in \sigma(F)\cup\sigma(S)$ and m_i nonnegative integers not all zero.

3) 0 does not lie in the convex hull of $\sigma(F)\cup\sigma(S)$.

Then there exists a unique analytic solution $z=\theta(x,w)$ to the partial differential equation (3.37) locally around $(x,w)=(0,0)$. The solution has the property that $\begin{bmatrix} \frac{\partial\theta}{\partial x}(0,0) & \frac{\partial\theta}{\partial w}(0,0) \end{bmatrix} = T$ and so, $\begin{bmatrix} \theta \\ h \end{bmatrix}$ is a local diffeomorphism.

Assumptions 1 and 2 of Proposition imply that $\left(H \quad \begin{bmatrix} F & P \\ 0 & S \end{bmatrix} \right)$ is an observable pair. On the other hand, if $\left(H, \quad \begin{bmatrix} F & P \\ 0 & S \end{bmatrix} \right)$ is an observable pair, it is always possible to find matrices A, B, T which satisfy the matrix equation of Assumption 1, with $\begin{bmatrix} T \\ H \end{bmatrix}$ invertible and A having prescribed eigenvalues. It is important to note that the transformed estimation error mapped by θ follows linear dynamics governed by the arbitrarily selected matrix A:

$$\frac{d}{dt}\left[\theta(x,w)-\theta(\tilde{x},\tilde{w})\right] = A\left[\theta(x,w)-\theta(\tilde{x},\tilde{w})\right] \quad \textbf{(3.39)}$$

3.4.3. Simple Design for Nonlinear Reduced Order Observer

As discussed above, an observer can have the same dynamic order as the plant they observe, irrespective of the number of observations. In the absence of the noise, the observations can be used to determine some of the state variables, thereby reducing the number of state variables that must be included in the observer.

Some of the state variables can be measured directly and should be implemented to the control law, so they do not need to be estimate which might increase the cost in computation. In practise, a control based on reduced-order observer can be more robust than one using full-order observer. Hence to develop the equations for reduced-order observer, it is convenient to assume the state variables in two groups one of which are observed directly and the other one not at all. All further simplification is that the observed quantities are themselves the state variables in the first group. In equations, this all means that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.40}$$

where the observation is

$$y = x_1 \tag{3.41}$$

This is scarcely less general than the case

$$y = g(x_1, u)$$

provided that this expression can be solved for x_1 as a function of y and u:

$$x_1 = \psi(y, u) = \tilde{y}$$

$\tilde{y}$ is used as the observation.

Corresponding to the partitioning of the state vector as in (3.40), the dynamic equations can be written as:

$$\dot{x}_1 = f_1(x_1, x_2, u) \tag{3.42}$$

$$\dot{x}_2 = f_2(x_1, x_2, u) \tag{3.43}$$

For the estimate of the substate x_1 the observation itself can be used.

$$\tilde{x}_1 = y \tag{3.44}$$

while the other substate x_2 is estimated by using an observer in the form

$$\tilde{x}_2 = Ky + z \tag{3.45}$$

where z is the state of a dynamic system of the same order as the dimension of the subvector x_2 and is given by

$$\dot{z} = \phi(y, \tilde{x}_2, u) \tag{3.46}$$

A block diagram representation of the observer having the structure of (3.44 – 3.46) is given in figure 3.3.

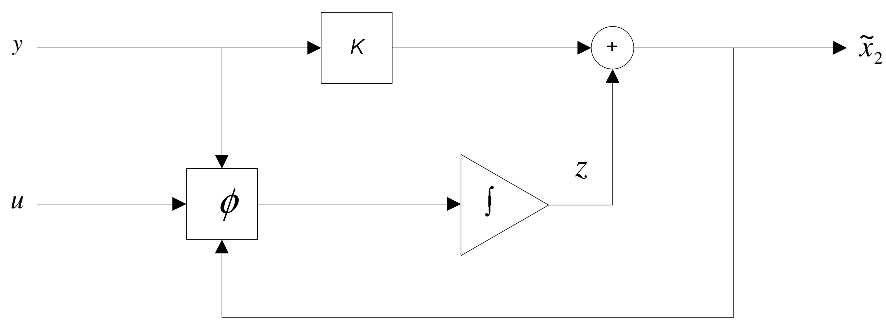


Figure 3.3: Reduced-order nonlinear observer

The object of the observer design is the determination of the observer gain matrix K and the nonlinear function ϕ. As the full-order state observer, these are to be selected such that:

- The steady state error in estimating x_2 converges to zero, independent of x and u. (The error in estimating x_1 is already zero when $\tilde{x}_1 = y$.)

- The observer is asymptotically stable.

As in the case of the full-order state observer, it will proceed with the equation (3.11). Using (3.42), (3.43) and (3.45), the following equation (3.47) is obtained.

$$\dot{e} = \dot{x}_2 - \dot{\tilde{x}}_2 = f_2(y, x_2, u) - K \cdot f_1(y, x_2, u) - \phi(y, x_2 - e, u) \tag{3.47}$$

In order for the right-hand side of (3.47) to vanish when $e = 0$, it is necessary that the function ϕ satisfies

$$\phi(y, x_2, u) = f_2(y, x_2, u) - K \cdot f_1(y, x_2, u) \tag{3.48}$$

for all values of y, x_2 and u. To achieve asymptotically stability, the linearized system

$$\dot{e} = A(x_2) \cdot e \tag{3.48}$$

with

$$A(x_2) = \frac{\partial \phi}{\partial x_2} = \frac{\partial f_2}{\partial x_2} - K\,\frac{\partial f_1}{\partial x_2} \tag{3.49}$$

4. APPLICATION

4.1. Dynamical Model of Active Magnetic Bearing System

The schematic diagram of the single-axis magnetic bearing system used in the case application is illustrated in figure 4.1. The system consists of a magnetic levitation object and an electromagnet. The magnetically levitated object can be realised in physical application by a ping pong ball with a magnet attached on and hence can attractive force. The attractive force can be controlled by the electromagnet mounted directly above the levitated object where the electromagnet can be controlled any industrial controller or computer system.

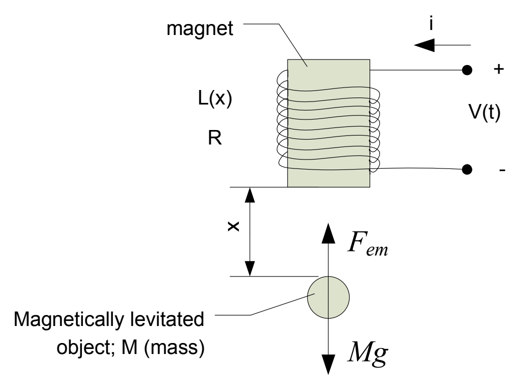


Figure 4.1: Schematic diagram of single-axis magnetic levitation system

A force balance analysis in the vertical plane implies the equation of motion in equation (4.1). In the equation, M is the mass of the levitation object in grams, x is the distance between the top of the levitation ball from a bottom point of electro magnet in millimetres, g is the gravity, and F_{em} is the magnetic levitation force in millinewtons.

$$M\ddot{x} = -F_{em} + Mg \quad \textbf{(4.1)}$$

The magnetic force between the solenoid and the permanent magnet can be determined by considering the magnet field between them as a function of the separation distance. The resistance of electromagnet is represented as *R*, and the self-inductance in *L(x)*. According to the electromagnetic field theory, the approximated self-inductance of the electromagnet's coil is determined by the function (4.2) depending on the distance *x*.

$$L(x) = L_1 + \frac{L_0}{(1+\frac{x}{a})} \tag{4.2}$$

The relation with the function (4.2) is derived from the curve which is obtained by minimum square method of the measured values *L(x)* for different *x*. Since the air gap is large enough, the magnetic circuit is not saturated and the system is electrically linear. In electrically linear magnetic systems, the magnetic energy stored is shown in the function (4.3).

$$W_m(i,x) = \frac{1}{2}L(x)i^2 \tag{4.3}$$

The force in the positive magnetic direction of *x* is expressed in (4.4).

$$F_{em}(i,x) = -\frac{\partial W_m(i,x)}{\partial x} = -\frac{1}{2}\frac{\partial L(x)}{\partial x}i^2 = \frac{1}{2a}\frac{L_0}{\left(1+\frac{x}{a}\right)^2}i^2 \tag{4.4}$$

Using (4.4) in (4.1), the dynamic of active magnetic bearing system is obtained as in (4.5).

$$M\ddot{x} = -F_{em} + Mg = -\frac{1}{2a}\frac{L_0}{\left(1+\frac{x}{a}\right)^2}i^2 + Mg \tag{4.5}$$

The nonlinearity of the system is seen in terms of the controlled current i^2 as multiplied factor and $\left(1+\frac{x}{a}\right)^2$ in the denominator in equation (4.5).

The electromagnetic voltage of the model shown in figure (4.1) is denoted as

$$V(t) = Ri + L(x)\frac{\partial i}{\partial t} + \frac{\partial L(x)}{\partial x}\frac{dx}{dt}i \tag{4.6}$$

In the control system, the controlled current i is selected as control signal of the system (plant). In the physical system, the voltage $V(t)$ is the unit used to compare the input reference signal and the output signal. The output distance x is measured by a photoelectric sensor and the information is sent in electrical voltage.

In fluid dynamics, there is another effective force called as resistance (drag) that resists the movement of a solid object through the fluid (liquid or gas). Resistance is made up of friction forces, which act in the opposite direction parallel to the object's surface and also pressure forces that act in the perpendicular direction to the object's surface. Therefore, the levitation ball in the system shown in figure (5.1) is under the effect of resistance (air friction in the air, liquid friction in the liquid in the opposite direction of movement) denoted as

$$f_d\left(\frac{dx}{dt}\right) = \frac{1}{2}\rho\left(\frac{dx}{dt}\right)^2 AC_d \tag{4.7}$$

where f_d is the friction force, ρ is the density of the fluid, $\frac{dx}{dt}$ is the speed of the object relative to the fluid, A is the reference area and C_d is the resistance coefficient [13].

Moreover, the other disturbance based from the nature of electromagnetic power and unknown disturbance are assumed as additive terms to acceleration of gravity in disturbance function $d(t)$. Adding (4.7) with other disturbance terms, (4.5) becomes

$$M\ddot{x} = -F_{em} + Mg = -\frac{1}{2a}\frac{L_0}{\left(1+\frac{x}{a}\right)^2}i^2 + Mg - f_d(\frac{dx}{dt}) \tag{4.8}$$

Denoting d(t) as

$$\begin{aligned} d(t) &= g + \frac{1}{M}f_d(\frac{dx}{dt}) \\ u(t) &= i(t) \\ x_1(t) &= x(t) \\ x_2(t) &= \frac{dx_1(t)}{dt} \end{aligned} \tag{4.9}$$

the system can be represented in state space equation as

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{L_0}{2Ma}\frac{u(t)^2}{\left(1+\frac{x_1(t)}{a}\right)^2} + d(t) \qquad \textbf{(4.10)}$$

$$y(t) = x_1(t)$$

This approach represents an active magnetic bearing system with one electromagnet and its dynamics is defined as in (4.8)

$$M\ddot{x} = -F_{em} + Mg = -\frac{1}{2a}\frac{L_0}{\left(1+\frac{x}{a}\right)^2}i^2 + Mg - f_d\left(\frac{dx}{dt}\right)$$

The system can be constructed by two electromagnet one of which is placed above and the other one below the levitated object shown in figure 4.2.

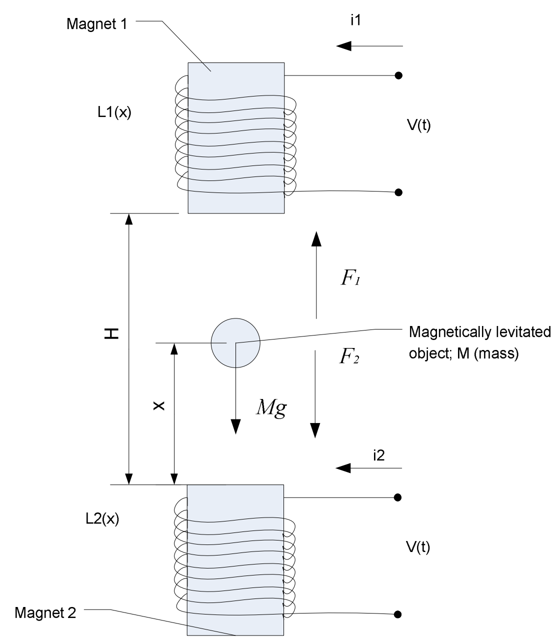


Figure 4.2: Active magnetic bearing system with 2 magnets

According to the movement direction, the dynamics is implied by

$$M\ddot{x} = F_1 - F_2 - Mg \qquad \textbf{(4.11)}$$

where F_1 and F_2 are electromagnetic forces, H is the distance between two opposite magnets and x is the position of the levitated object away from the magnet (magnet 2 in figure 4.2) below. In this model, the movement is considered to be in the opposite direction of the gravity.

$$\ddot{x} = \frac{1}{M}(F_1 - F_2) - g - \frac{1}{M}\sum f_d(\frac{dx}{dt})$$
$$\ddot{x} = \frac{1}{M}(F_1 - F_2) - d(t) \quad where \quad d(t) = g + \frac{1}{M}\sum f_d(\frac{dx}{dt}) \tag{4.12}$$

4.2. Sliding Mode Controller Design

4.2.1. Control of Asymptotically Stable Error Dynamics

Define the error vector

$$E = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} x_r - x_1 \\ \dot{x}_r - x_2 \end{bmatrix} \tag{4.13}$$

In the same manner, $\ddot{e} = \ddot{x}_r - \dot{x}_2$ implies

$$\ddot{e} = \ddot{x}_r - \dot{x}_2 = \ddot{x}_r - \ddot{x}_1 = \ddot{x}_r - \frac{1}{2M}\frac{\partial L(x_1)}{\partial x}u^2 - d \tag{4.14}$$

In order to make a form such as $\dot{X} = AX + BU$, the equation (4.14) can be rewritten

$$\dot{E} = AE + BU(x,u)$$
$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_{e1} & -k_{e2} \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}U(x,u) \tag{4.15}$$

The sliding surface can be determined from the equation (4.15) and becomes as

$$\ddot{e} + k_{e1}\dot{e} + k_{e2}e = U(x,u) \tag{4.16}$$

Let us decide the matrix $K_e = [k_{e1} \quad k_{e2}]$ which makes the sliding surface asymptotically stable by assigning $U(x, u) = 0$.

Substituting (4.14) in (4.15) gives

$$U(x,u) = K_e E + \ddot{x}_r - \frac{1}{2M}\frac{\partial L(x)}{\partial x}u^2 - d \tag{4.17}$$

In (4.17), except the term including control signal (current of magnet), a control action u_c can be suggested such as

$$u_c = K_e E + \ddot{x}_r - d \quad \textbf{(4.18)}$$

where $U(x,u) = u_c - \frac{1}{2M}\frac{\partial L(x)}{\partial x}u^2$ **(4.19)**

The sliding mode control schema to make the error dynamics asymptotically stable is illustrated in figure 4.3.

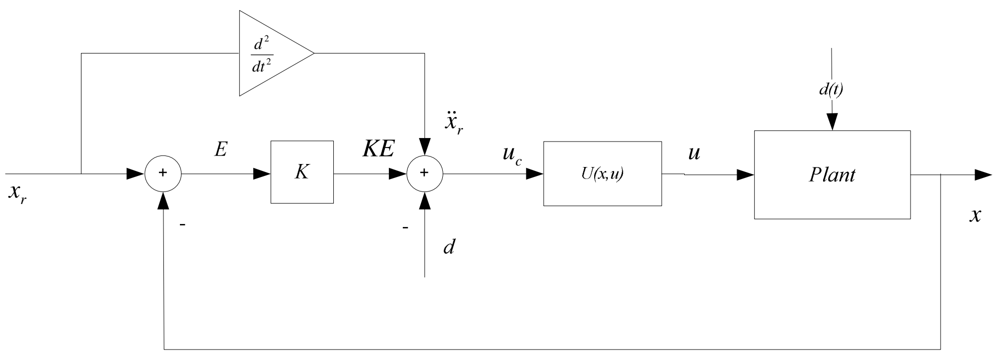


Figure 4.3: Sliding mode control schema where error dynamics is supposed to be asymptotically stable

The equation (4.19) satisfies $U(x,u) = 0$ only when

$$\text{sgn}(\frac{\partial L(x)}{\partial x}) \cdot \text{sgn}(u_c) = 1 \quad \textbf{(4.20)}$$

It is obvious from the model with the self-inductance function defined in (4.2), that it is decreasing as time passes.

$$\text{sgn}\left(\frac{\partial L(x)}{\partial x}\right) = -1 \quad \textbf{(4.21)}$$

which implies also

$$\text{sgn}(u_c) = -1 \quad \textbf{(4.22)}$$

(4.19) and (4.22) imply the control signal $u(x_1(t), x_2(t))$

$$i = u(x_1, x_2) = \pm\sqrt{M\frac{(a+x)^2}{aL_o}(\text{sgn}(u_c)-1)u_c} \quad \textit{for} \quad u_c < 0$$
$$i = \pm\sqrt{-2M\frac{(a+x)^2}{aL_o}u_c} \quad \textit{where} \quad u_c < 0 \qquad \textbf{(4.23)}$$

Equation (4.23) gives the expression of control by one electromagnet in the system. Considering that the system has two electromagnets shown in figure 4.2; the control law relation will be denoted by the equation (4.24).

$$U(x,i_1,i_2) = u_c - \frac{1}{M}F_1 + \frac{1}{M}F_2 \qquad \textbf{(4.24)}$$

Since the model of the system in figure (4.2) has the movement direction in the opposite of the gravity, the control signal is defined as

$$u_c = K_e E + \ddot{x}_r + d \qquad \textbf{(4.25)}$$

The equation becomes as in (4.26) and (4.27) in terms of x_1 and x_2.

$$U(x,i_1,i_2) = u_c + \frac{1}{2M}\frac{\partial L_1(x)}{\partial x_1}i_1^{\ 2} - \frac{1}{2M}\frac{\partial L_2(x)}{\partial x_2}i_2^{\ 2} \qquad \textbf{(4.26)}$$

where $x_1 = H - x$ and $x_2 = x$

$$U(x,i_1,i_2) = u_c - \frac{1}{2M}\frac{aL_1}{(x_1+a)^2}i_1^{\ 2} + \frac{1}{2M}\frac{aL_2(x)}{(x_2+a)^2}i_2^{\ 2}$$
$$U(x,i_1,i_2) = u_c - \frac{1}{2M}\frac{aL_1}{(H-x+a)^2}i_1^{\ 2} + \frac{1}{2M}\frac{aL_2(x)}{(x+a)^2}i_2^{\ 2} \qquad \textbf{(4.27)}$$

As mentioned before that the relation in equation (4.19) satisfies $U(x,u) = 0$ only when

$$\text{sgn}(\frac{\partial L_1(x)}{\partial x}) \cdot \text{sgn}(u_c) = -1 \quad \textit{when} \quad i_2 = 0$$
$$\text{sgn}(\frac{\partial L_2(x)}{\partial x}) \cdot \text{sgn}(u_c) = 1 \quad \textit{when} \quad i_1 = 0 \qquad \textbf{(4.28)}$$

Then the following control laws are obtained;

$$i_1 = \pm\sqrt{-2M\,\frac{(a+H-x)^2}{aL_1}u_c} \quad where \quad u_c \geq 0 \quad and \quad i_2 = 0$$

$$and \qquad \textbf{(4.29)}$$

$$i_2 = \pm\sqrt{2M\,\frac{(a+x)^2}{aL_2}u_c} \quad where \quad u_c < 0 \quad and \quad i_1 = 0$$

For simulation, the following values are chosen for the system:

L = 1 H; a = 0.008

M_n = 0.01 kg (nominal mass)

D = [-10; +10] m/s^2 (randomized)

The coefficients of PD controller are;

K_p = 6e+6; Kd = 3 e+4

Simulations are performed with initial conditions starting from the point 0 m where the top of the magnet below the object and point 0.1 m where the upper magnet is located. The reference position is determined as 0.04 m.

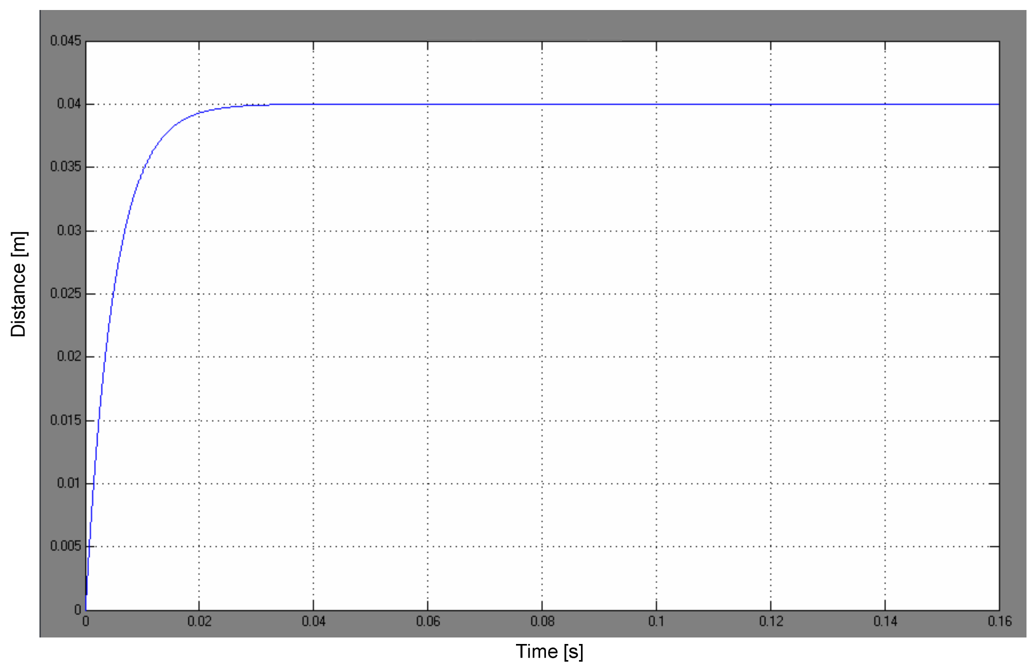


Figure 4.4: Position response of an object with nominal mass from 0 m

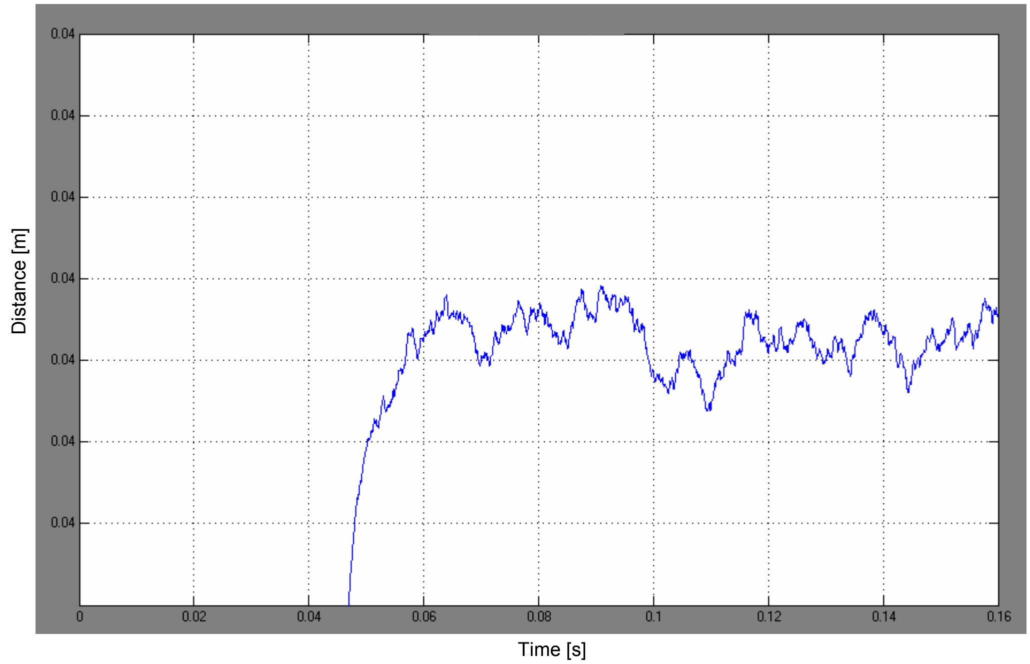


Figure 4.5: Position response in figure 4.4 under disturbance without observer

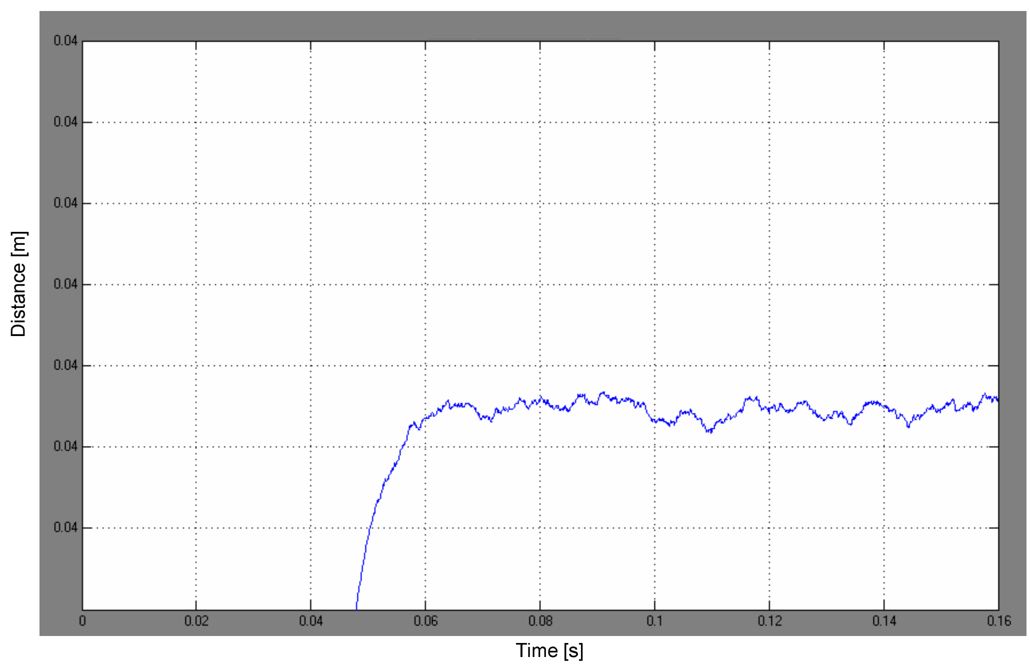


Figure 4.6: Position response in figure 4.4 under disturbance when it is estimated

When the object is placed at point 0 m, the upper electromagnet requires a current value of 59 A. In simulation mode, neither saturation module nor circuit breaker is used. Using a limiter to push a maximum certain amount of current in realization is recommended. In figure 4.7, and 4.8, the values of current flowing through the upper and lower magnets are given respectively.

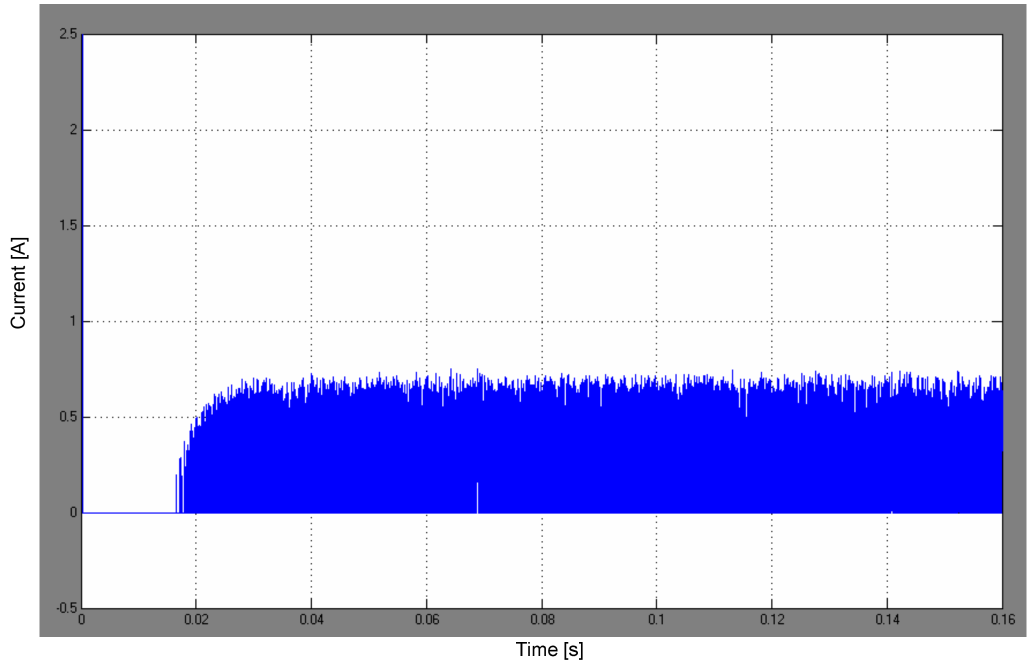


Figure 4.7: Current of upper magnet for the load of nominal mass

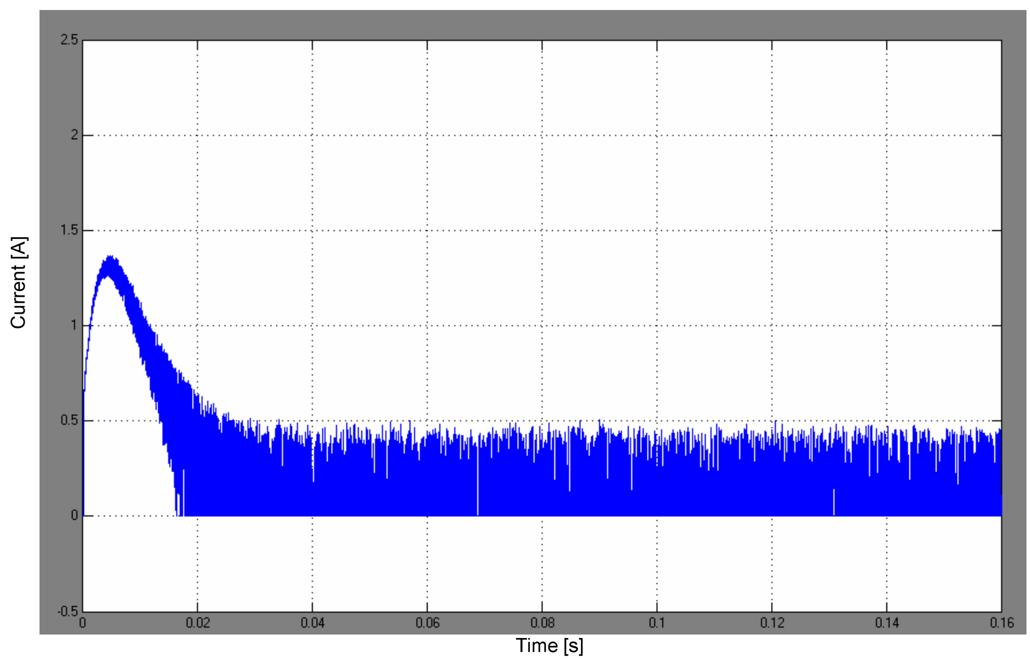


Figure 4.8: Current of lower magnet for the load of nominal mass

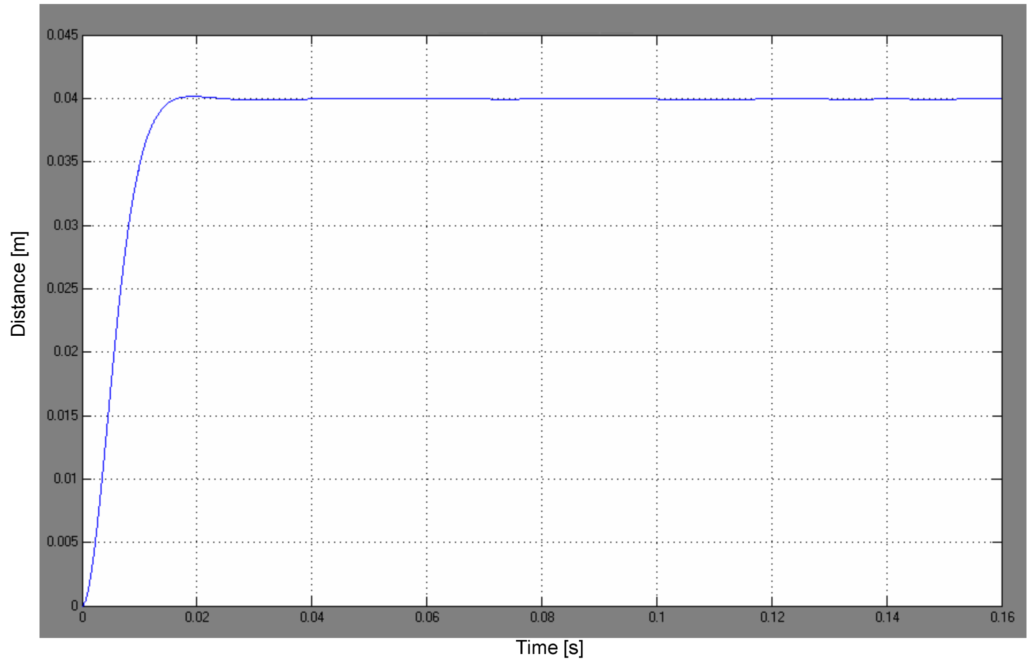


Figure 4.9: Position response of an object with limited mass of 50 times heavier. There is no limiter for current.

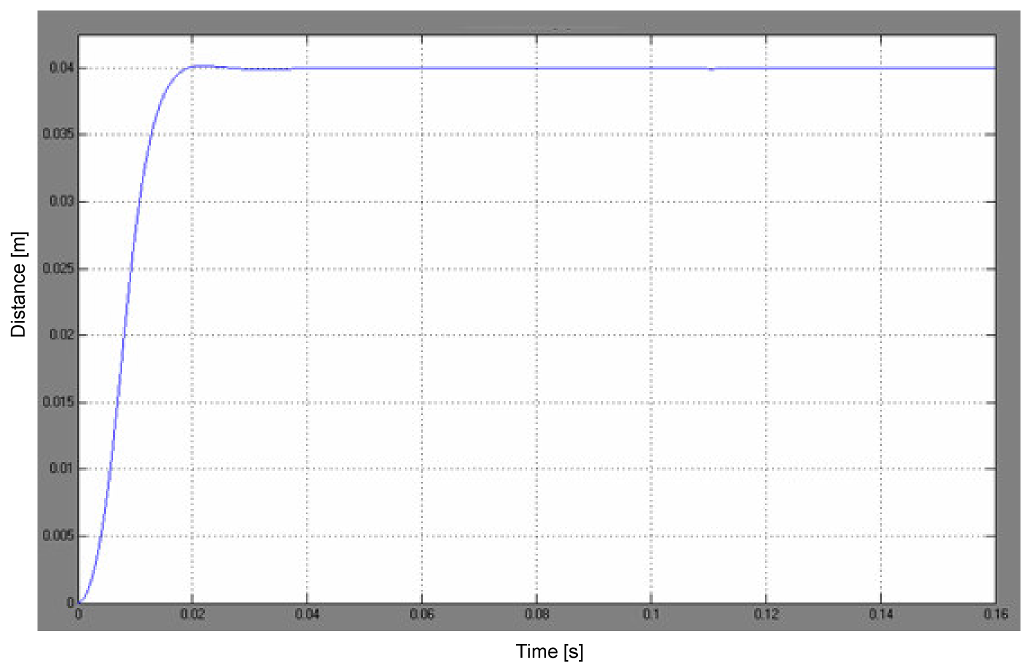


Figure 4.10: Position response of an object with limited mass of 50 times heavier. There is limiter for current.

In order to realize in real world there is a limiter in the system so that the magnet do not pull current of large values. By limiting the current value, the oscillation around the reference does not exist any more.

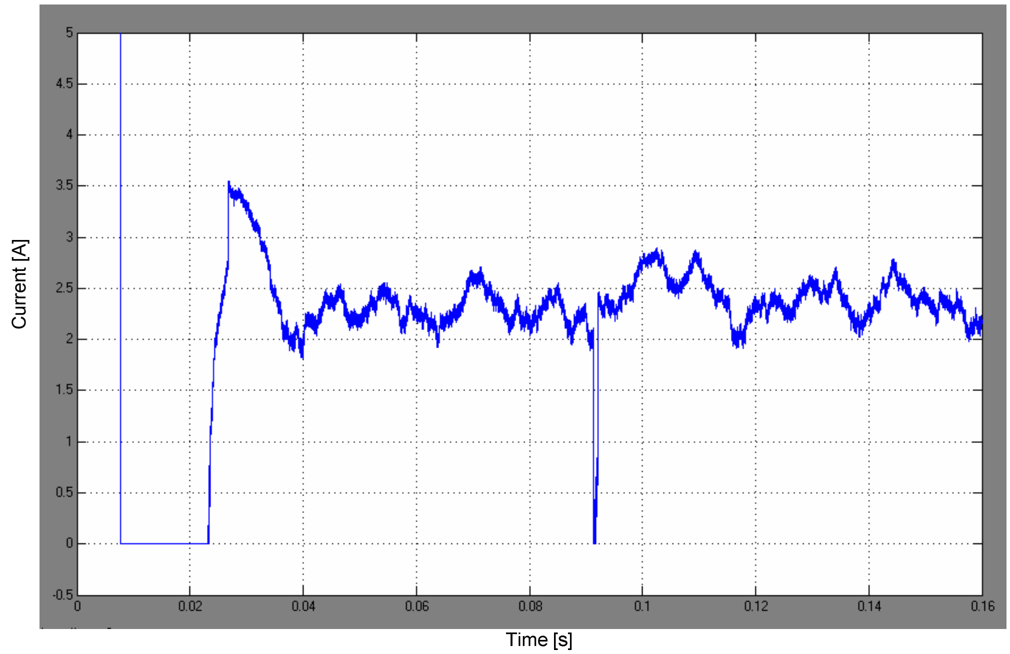


Figure 4.11: Current of upper magnet for load of 50 times of nominal mass

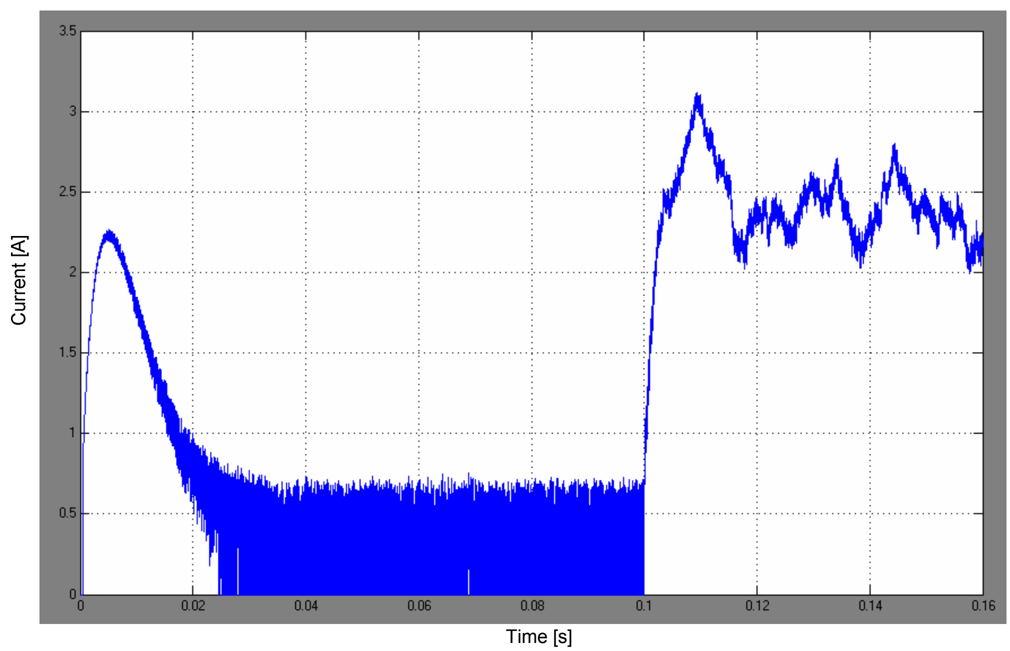


Figure 4.12: Current of upper magnet for load where mass is increased 50 times at 0.1 s

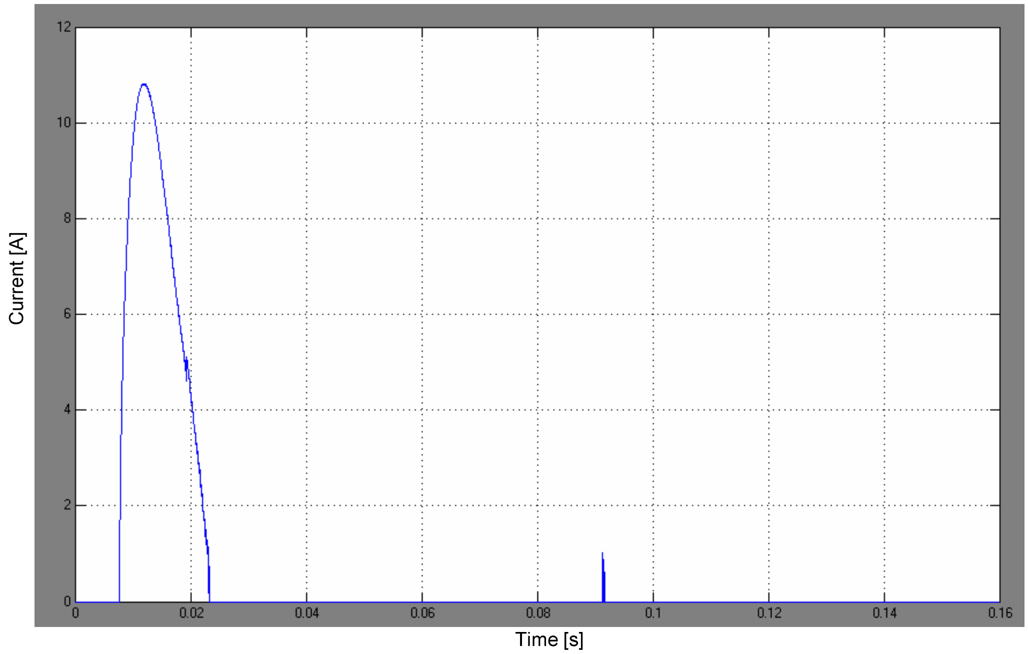


Figure 4.13: Current of lower magnet for load of 50 times heavier than nominal mass

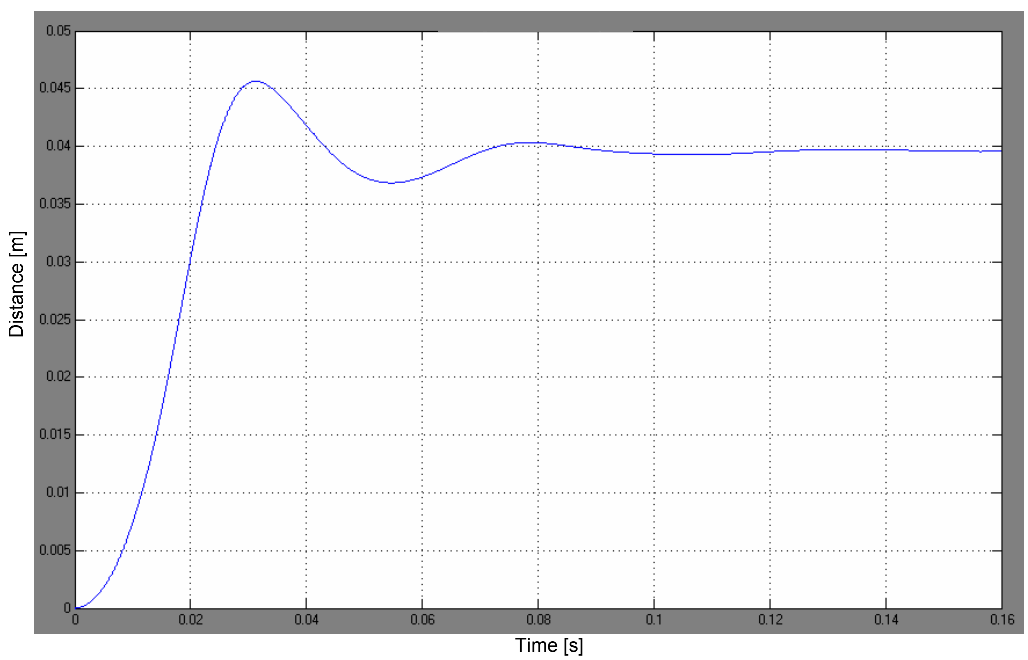


Figure 4.14: Transient response of overloaded system with 200 times heavier load

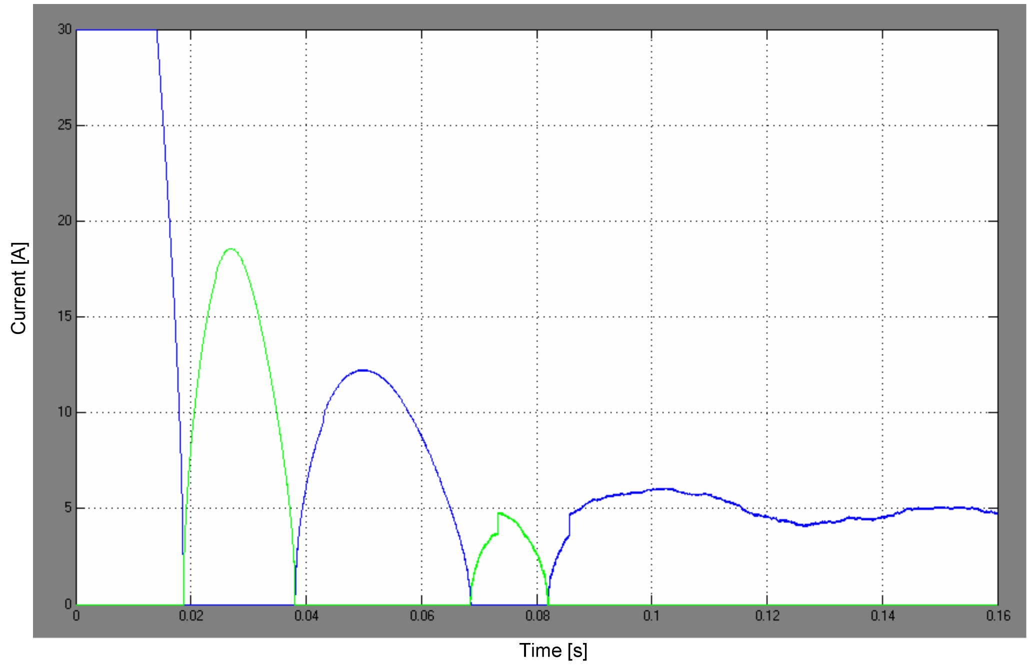


Figure 4.15: Currents for overloaded system whose response is shown in figure 4.14. Blue: upper magnet; green: lower magnet

Consider that the object with nominal mass is let free fall down from the point 0.1 m.

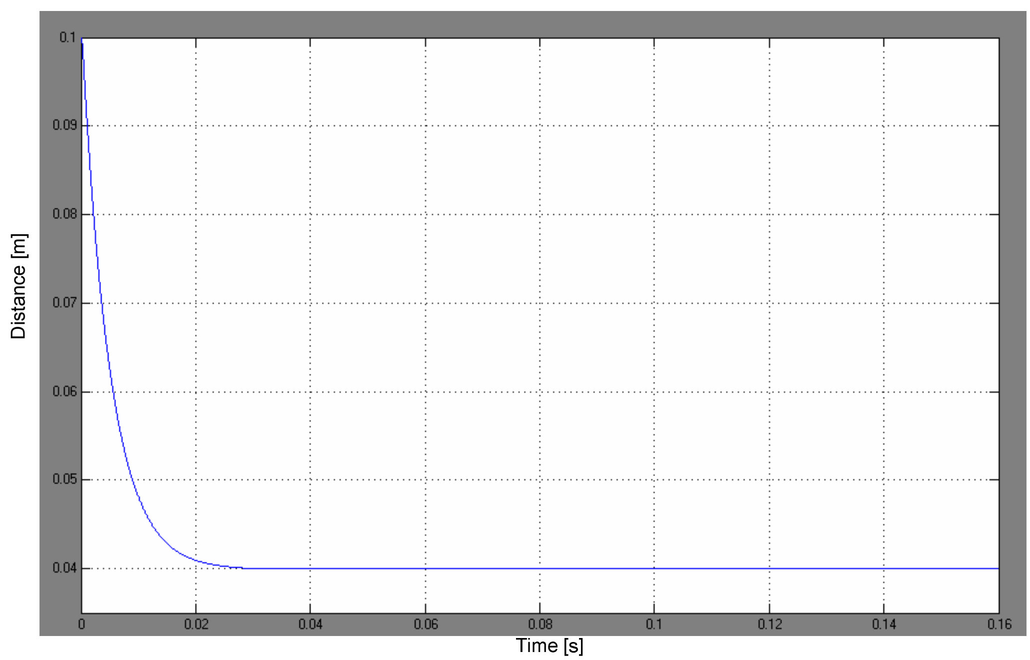


Figure 4.16: Position response of an object with nominal mass falling down from the upper magnet (x(0) = 0.1 m)

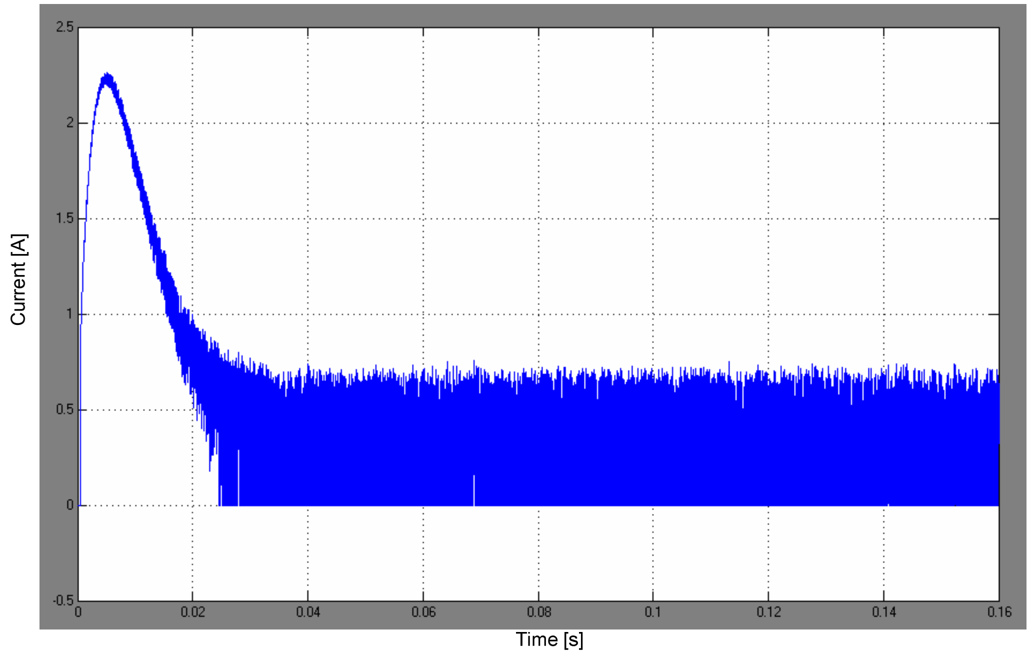


Figure 4.17: Current of upper magnet for the load of nominal mass falling down from the upper magnet

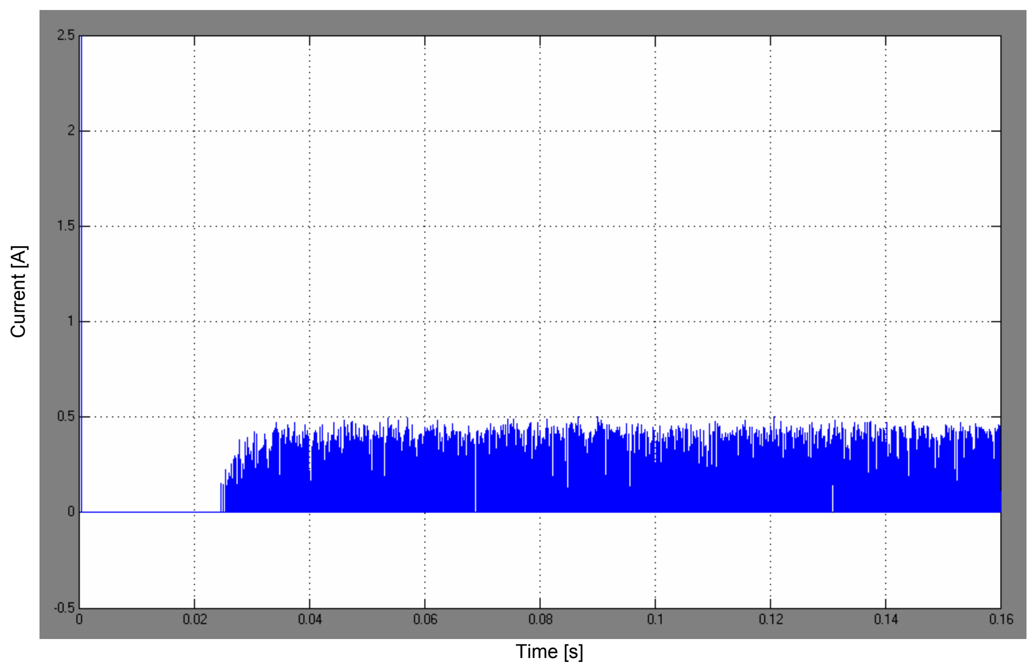


Figure 4.18: Current of lower magnet for the load of nominal mass falling down from the upper magnet

4.2.2. Control with Linearization

As mentioned in chapter 3.4, the active magnetic bearing system is desired to behave like a model of known dynamic system of second degree.

Let x_r be the reference state (position), x_{1m} be the state (position) of the known dynamic reference system and x_1 be the actual state (position) measured. x_{1m} can be also considered as output of the model. Thus the transfer function of the system should be obtained as

$$\frac{X_m}{X_r}(s) = \frac{\varpi_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

and the state space as

$$\begin{bmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\varpi_n^2 & -2\xi\omega \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix} + \begin{bmatrix} 0 \\ \varpi_n^2 \end{bmatrix} x_r$$

The error dynamics is proposed the same as in section (4.2.1) being

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\varpi_n^2 & -2\xi\omega \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u) \qquad \textbf{(4.30)}$$

Let one electromagnet be in the system shown in figure 4.1. Then $\sigma(u)$ turns out to be

$$\sigma(u) = \frac{1}{\varpi_n^2}\left(u_c - \frac{1}{M}\frac{\partial L(x)}{\partial x} u^2 \right) \qquad \textbf{(4.31)}$$

and the control signal u_c can be denoted as

$$u_c = \varpi_n^2 (x_r - x) - 2\xi\varpi_n x - d \qquad \textbf{(4.32)}$$

The control scheme can be seen in figure 4.19. Only with a difference in expression of control signal, the control law is similar and the equation (4.23) implies as well.

$$i_2 = \pm\sqrt{2M\frac{(a+x)^2}{aL_2}u_c} \quad where \quad u_c < 0 \quad and \quad i_1 = 0$$

When there are two magnets in the system shown in figure 4.2, control signal expression $\sigma(u)$ becomes

$$\sigma(u) = \frac{1}{\varpi_n^{\ 2}}\left(u_c + \frac{1}{M}\frac{\partial L_1(x)}{\partial x_1}i_1^{\ 2} - \frac{1}{M}\frac{\partial L_2(x)}{\partial x_2}i_2^{\ 2}\right) \qquad \textbf{(4.33)}$$

and the control laws are the same as in (4.29)

$$i_1 = \pm\sqrt{-2M\frac{(a+H-x)^2}{aL_1}u_c} \quad where \quad u_c \geq 0 \quad and \quad i_2 = 0$$

and

$$i_2 = \pm\sqrt{2M\frac{(a+x)^2}{aL_2}u_c} \quad where \quad u_c < 0 \quad and \quad i_1 = 0$$

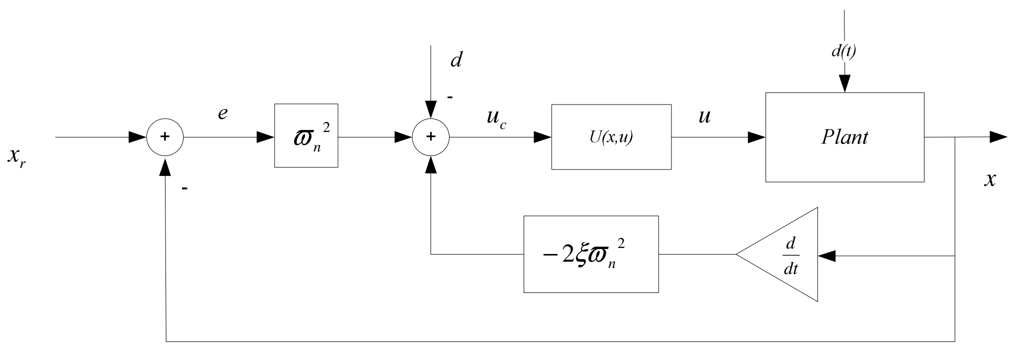


Figure 4.19: Sliding mode control schema for linearized system

In this approach, the position response (figure 4.20) of the load with nominal mass and the current values (figure 4.21 and 4.22) of the control system are plotted for ω_n = 300 and ξ = 0.95. The reference position is set to 0.05 m.

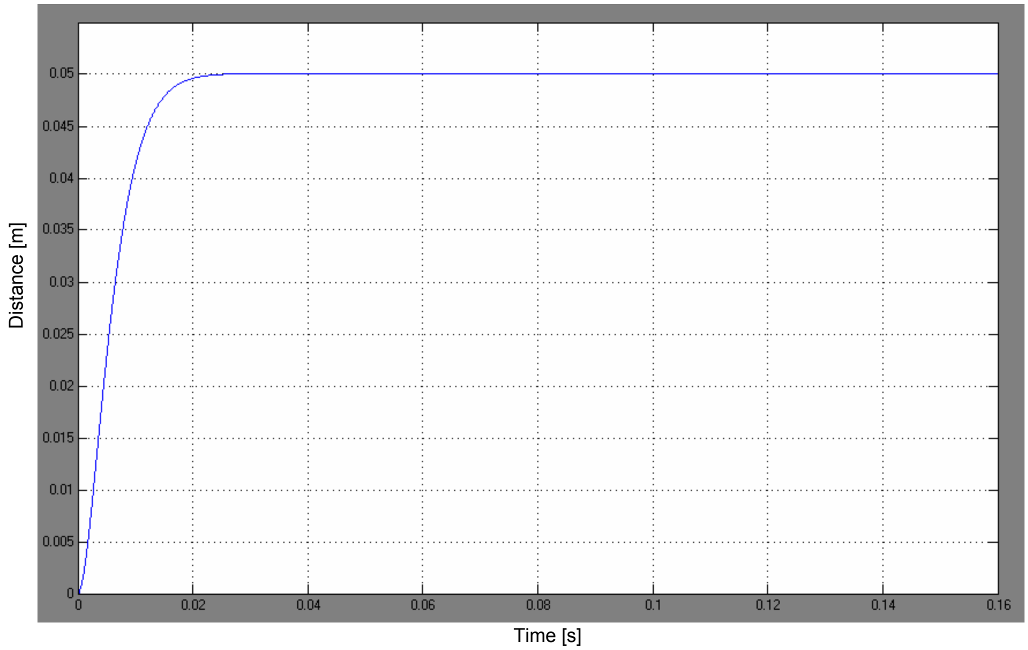


Figure 4.20: Position response of an object with nominal mass

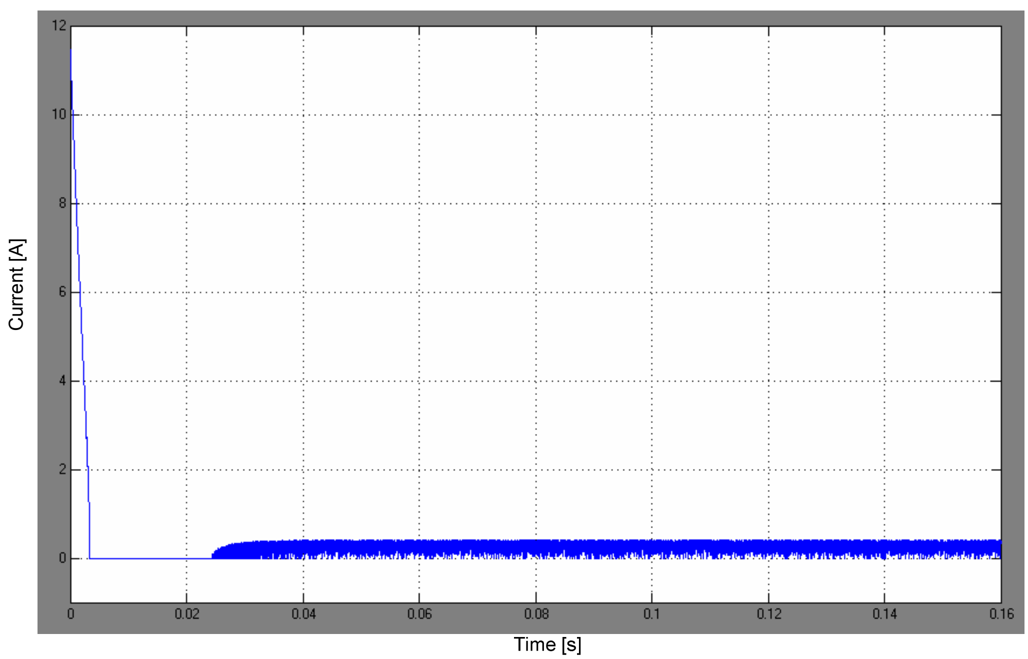


Figure 4.21: Current of upper magnet for the load of nominal mass

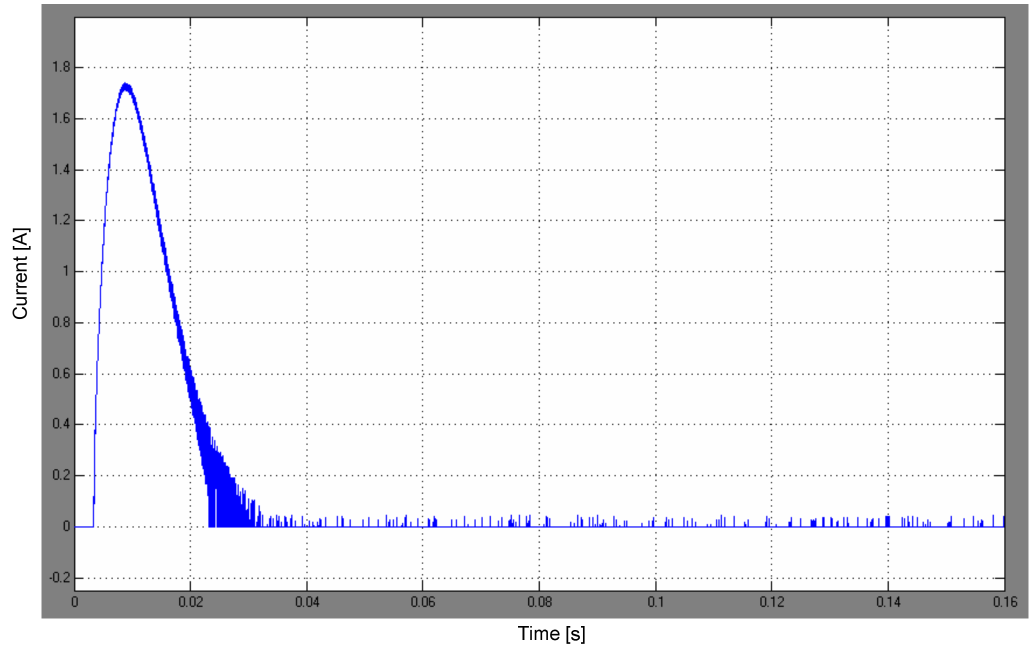


Figure 22: Current of lower magnet for the load of nominal mass

When the load is increased 50 times in mass, the response of the system is obtained with a significant overshoot and steady state error (see in figure 4.22) which is subject to be suppressed. This can be achieved by using a regular PID controller. P-control is already managed by multiplying the error signal with $\omega_n = 3000$.

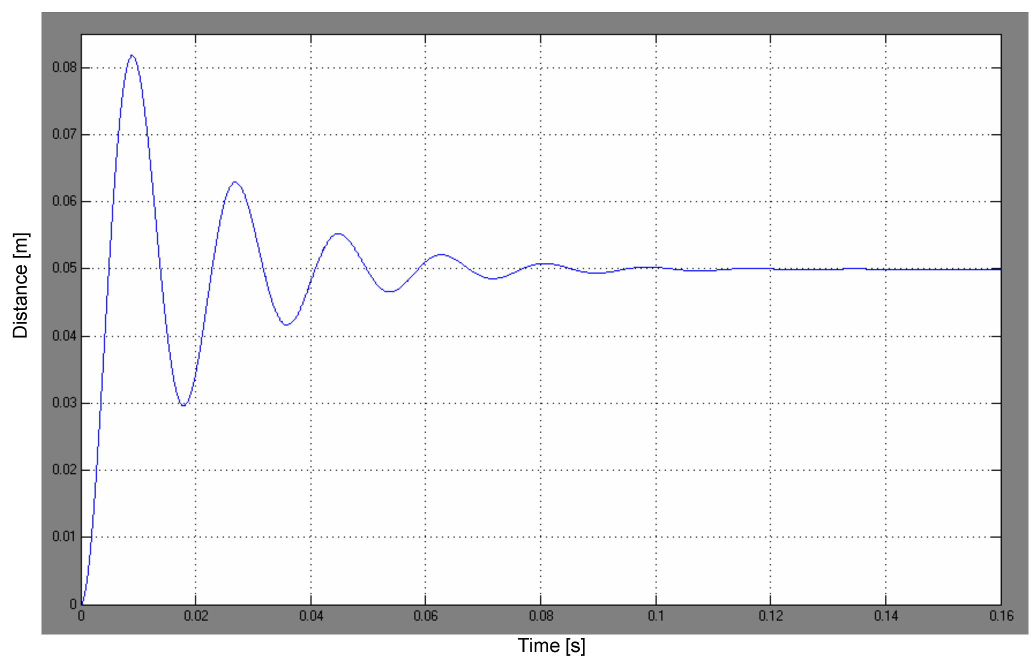


Figure 4.23: Position response for the load of 0.5 kg (50 times heavier) without PID

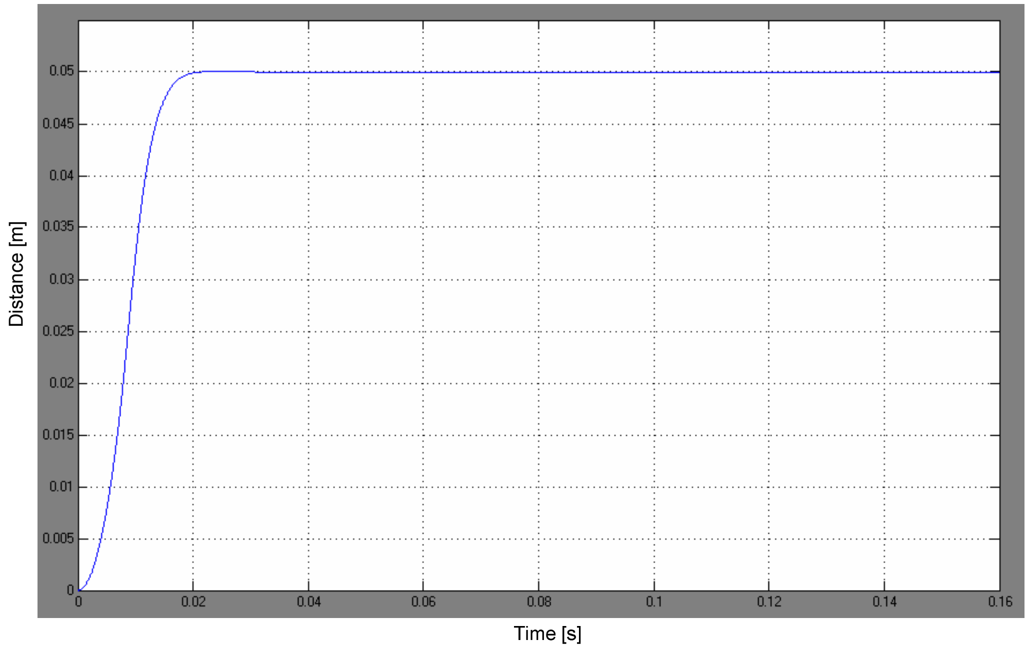


Figure 4.24: Position response for the load of 0.5 kg (50 times heavier) with PID

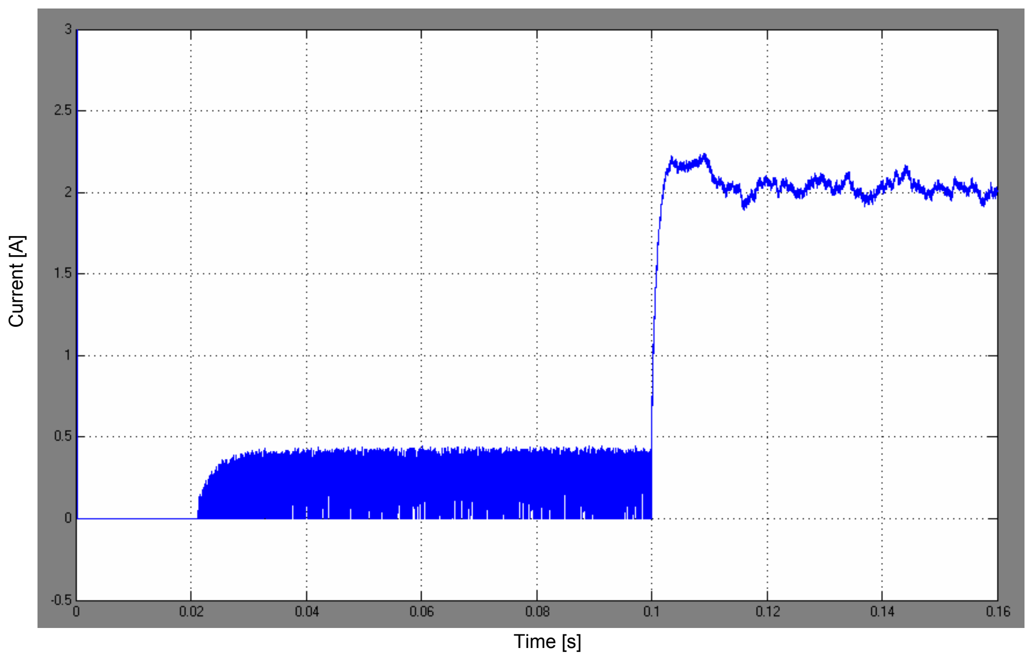


Figure 4.25: Current of upper magnet where the mass of the load is increased 50 times more at 0.1 s.

Similarly when the mass of the load is increased by 200 times, the steady state error becomes significant nevertheless the system stays still stable (figure 4.26).

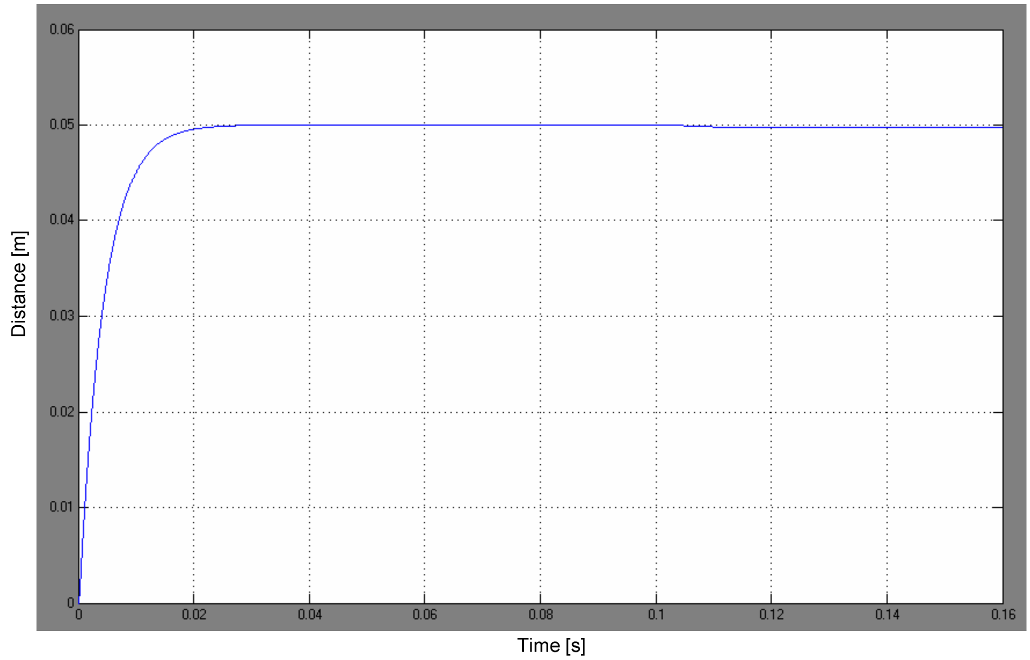


Figure 4.26: Overloaded system with 200 times heavier load

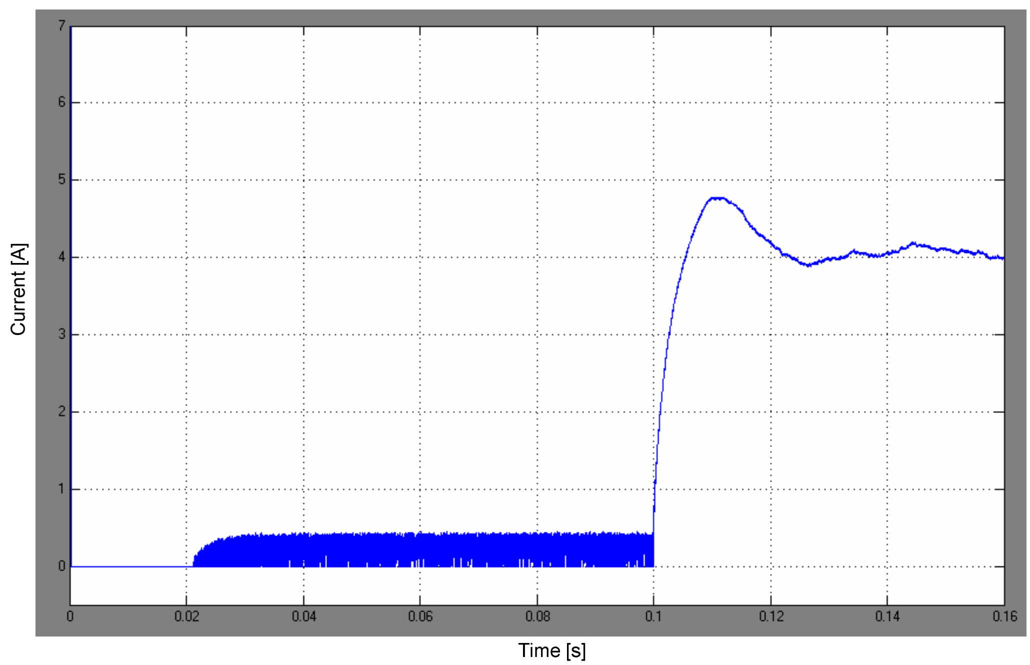


Figure 4.27: Current of upper magnet by overloaded system with 200 times heavier load at 0.1 s

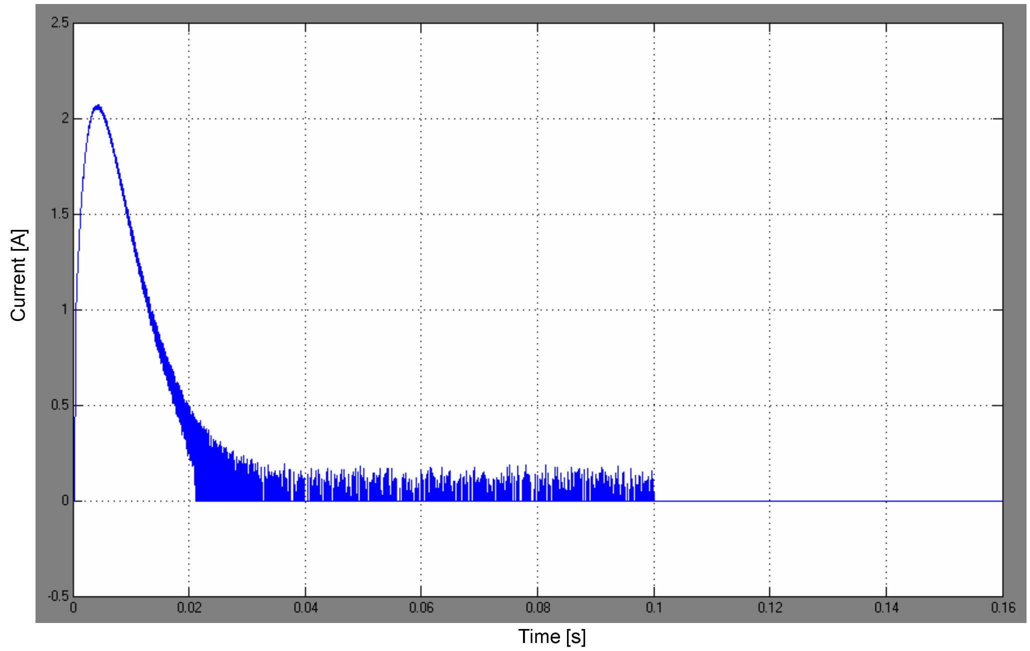


Figure 4.28: Current of lower magnet by overloaded system with 200 times heavier load at 0.1 s

4.3. Nonlinear Reduced-Order Observer Design

The representation of the nonlinear magnetic levitation system denoted in (4.10) can be applied to estimate all of the states including disturbance that contains all unknown disturbance and the gravitational acceleration with full-order state observer. Hence the full-order state vector is written as

$$\tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{d} \end{bmatrix} \quad \textbf{(4.34)}$$

In reduced-order observer design, there will be two groups of states of full-order states. The first group is measured and denoted as X_m and the other group consists of the states that are estimated from the measured states and denoted as X_e.

$$X = \begin{bmatrix} X_m \\ X_e \end{bmatrix} \quad \textbf{(4.35)}$$

where

$$X_m = x_1$$
$$X_e = \begin{bmatrix} x_2 \\ d \end{bmatrix} \quad \textbf{(4.36)}$$

Here, $y = x_1$ and can be rewritten as $y = g(x_1, u)$ in a general form. As explained in section 3.4.3, the dynamic equations are written as in (3.42) and (3.43).

$$\dot{X}_1 = f_1(X_1, X_2, u)$$
$$\dot{X}_2 = f_2(X_1, X_2, u)$$

Assuming the position represented as state variable x_1 is directly measured and the speed x_2 and the acceleration including unknown disturbance d can be estimated from x_1. Therefore, the equations regarding to the observation become

$$\begin{aligned} \tilde{x}_1 &= y = x_1 \\ \tilde{x}_2 &= z_1 + k_1 y \\ \tilde{d} &= z_2 + k_2 y \end{aligned} \quad \textbf{(4.37)}$$

Deriving the state variables in (4.37)

$$\begin{aligned} \dot{\tilde{x}}_1 &= \dot{x}_1 \\ \dot{\tilde{x}}_2 &= \dot{z}_1 + k_1\dot{x}_1 = \dot{z}_1 + k_1 x_2 \\ \dot{\tilde{d}} &= \dot{z}_2 + k_2\dot{x}_1 = \dot{z}_2 + k_2 x_2 \end{aligned} \quad \textbf{(4.38)}$$

z_1 and z_2 are the new state variables representing the reduced-order state dynamics where

$$\begin{aligned} \dot{z}_1 &= \phi_1 = -\frac{1}{2M}\frac{aL_0}{\left(1+\frac{\tilde{x}_1}{a}\right)^2}u^2 + \tilde{d} - k_1\tilde{x}_2 \\ \dot{z}_2 &= \phi_2 = -k_2\tilde{x}_2 \end{aligned} \quad \textbf{(4.39)}$$

for the system with one electromagnet placed above the levitated object. Using (4.39), the estimated state variables in (4.38) become

$$\begin{aligned} \dot{\tilde{x}}_2 &= -\frac{1}{2M}\frac{aL_0}{\left(1+\frac{\tilde{x}_1}{a}\right)^2}u^2 + \tilde{d} - k_1\tilde{x}_2 + k_1 x_2 \\ \dot{\tilde{d}} &= -k_2\tilde{x}_2 + k_2 x_2 \end{aligned} \quad \textbf{(4.40)}$$

According to the design approach to obtain the observer gain matrix, the error dynamic is derived where A_c is linearized at the origin $e = 0$

$$\dot{e} = A_c\big|_{e=0}\, e$$

and A_c in (3.49) applied for active magnetic bearing system is denoted as

$$A_c = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \tag{4.41}$$

The structure of the reduced-order nonlinear observer is given in figure 4.26.

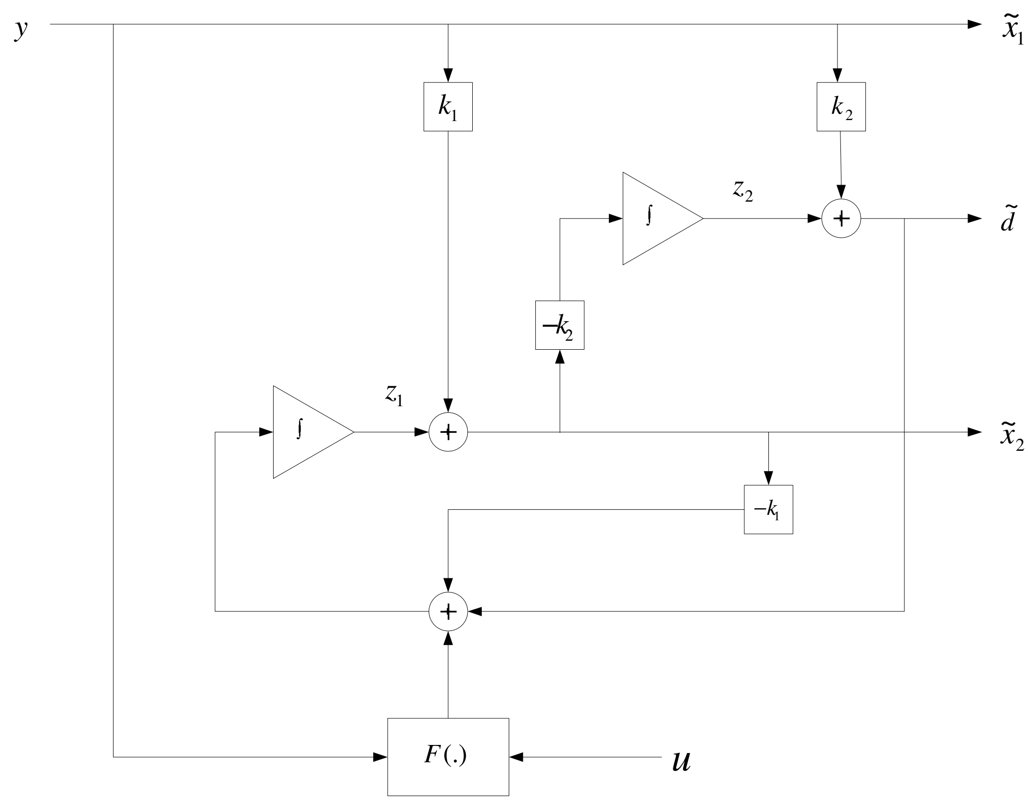


Figure 4.29: Schematic diagram of reduced-order nonlinear observer

In the figure 4.26, $F(.)$ represents the nonlinear term in the equation of observer and can include either one or two terms derived from the equation of magnetic force respectively for the systems with one or two electromagnets. Due to the difference in the system with two magnets, the equations (4.39) and (4.40) become

$$\dot{z}_1 = \phi_1 = \frac{1}{2M}\frac{aL_1}{\left(1+\frac{H-\tilde{x}_1}{a}\right)^2}u_1{}^2 - \frac{1}{2M}\frac{aL_2}{\left(1+\frac{\tilde{x}_1}{a}\right)^2}u_2{}^2 - \tilde{d} - k_1\tilde{x}_2$$
$$\dot{z}_2 = \phi_2 = -k_2\tilde{x}_2 \qquad \textbf{(4.42)}$$

$$\dot{\tilde{x}}_2 = \frac{1}{2M}\frac{aL_1}{\left(1+\frac{H-\tilde{x}_1}{a}\right)^2}u^2 - \frac{1}{2M}\frac{aL_2}{\left(1+\frac{\tilde{x}_1}{a}\right)^2}u^2 - \tilde{d} - k_1\tilde{x}_2 + k_1x_2$$
$$\dot{\tilde{d}} = -k_2\tilde{x}_2 + k_2x_2 \qquad \textbf{(4.43)}$$

4.4. Lyapunov Stability Analysis

To investigate the stability, a candidate Lyapunov function is chosen according to the Lyapunov's Second Method and denoted by

$$V(E) = E^T PE \qquad \textbf{(4.44)}$$

satisfying that

$$PA + A^T P = -Q \qquad \textbf{(4.45)}$$

where P and Q are positive definite matrices with $P = P^T$ and $Q = Q^T > 0$

Considering (4.21) and differentiating $V(E)$ by time

$$\dot{V}(E) = \dot{E}^T PE + E^T P\dot{E}$$
$$\dot{V}(E) = E^T(A^T P + PÅ) + U^T(x)B^T PE + E^T PBU(x) \qquad \textbf{(4.46)}$$

Denoting (4.31) by

$$\dot{V}(E) = -E^T QE + \phi(U(x)) \qquad \textbf{(4.47)}$$

It is necessary to find out an $\phi(U(x)) = 0$ for $t > 0$ so that

$$\dot{V}(E) = -E^T QE < 0 \text{ for every } e(t) \neq 0. \qquad \textbf{(4.48)}$$

4.5. Constructing Control System

As applied in Svoboda's study [7], the disturbance will be applied to the control signal equation which consists of it as well in its expression. Differently from his study, there is the variable structure control based on a sliding surface and a disturbance to provide the unknown terms to the control equation in this control scheme without having any learning mechanism. The proposed control scheme is iluustrated in figure 4.27.

On the other hand, similarly to the study of Gurleyen, Bahadir and Tekin [9], the estimated disturbance output of the nonlinear disturbance can be applied to the dynamic behavioural equation of the model reference based control, since the model still consists of an uncertain term disturbance (see figure 4.28).

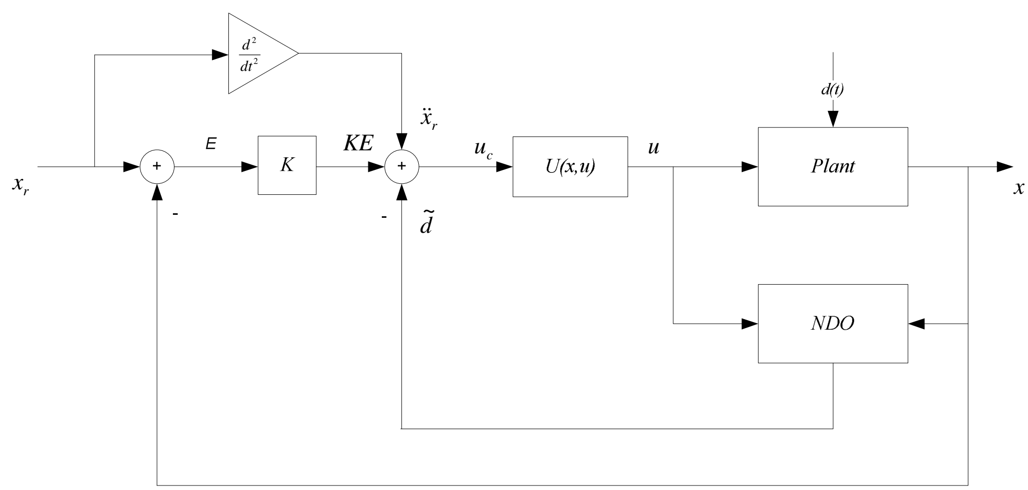


Figure 4.30: The control system with design approach in 4.2.1 and the plant

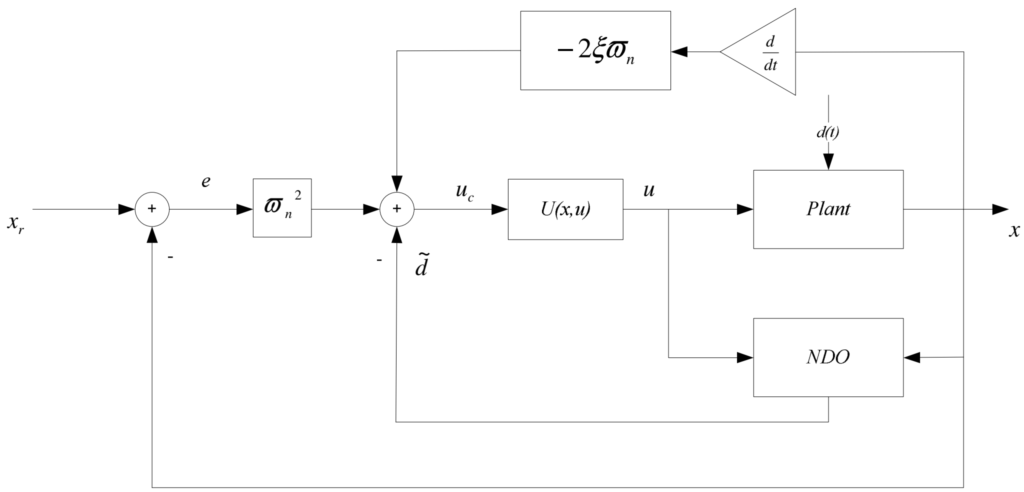


Figure 4.31: The control system with design approach in 4.2.2 and the plant

5. CONCLUSION

Two design approaches for sliding mode control are proposed to control the model of an active magnetic bearing system which consists of two electromagnets. The simulation results are observed to distinguish the control schemas by advantages over each other.

Basically both of the approaches are similar in choosing a sliding surface for error dynamics where they give pretty close results. Since the design of the first approach is based on finding coefficients of the terms e and $\dot{e}$ in sliding surface, the design turns out to find the coefficients of a PD controller where and Integral controller term can be inserted to improve the steady state error.

Linearization in second approach provides rather fast rising time where it can cause an overshoot and steady state error for the same heavier loads. This can be improved by using PID controller to make the system asymptotically and this costs only adding Integral and Derivative factors since there is already a proportional controller in the system.

In both systems, the disturbance is an issue which can disaffect the performance of the system. Therefore an observation of the system and estimation of the disturbance definitely has a positive effect on the performance. To decrease the computation in the observation system, the reduced order disturbance observer is very simplified to design.

Nevertheless, the disturbance is only estimated and still remains difficult to obtain precisely. By going one step further, a fuzzy neural network can be adapted to estimate the error in estimation of disturbance as it is already applied in control of other types of nonlinear systems.

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ÖZGEÇMİŞ

Rüstem Tolga BÜYÜKBAŞ, 15.04.1980 İstanbul doğumludur. Orta okul ve lise eğitimini 1997 yılına kadar Bahçelievler Anadolu Lisesi'nde devam etmiş; ardından 1998 yılında Bahçeşehir Koleji'nde tamalamıştır. Ocak 2004'te İstanbul Teknik Üniversitesi Elektrik-Elektronik Fakültesi, Elektronik ve Haberleşme Mühendisliği Bölümünden mezun olmuştur. Aynı yıl İ.T.Ü. Elektrik-Elektronik Fakültesi, Kontrol ve Otomasyon Mühendisliği Programında yüksek lisans eğitimine başlamıştır.