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Oscillatory difference equations and moment problems

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Abstract

In this paper, we first consider some new oscillatory results with respect to the discrete Hermite polynomials of type I, respectively, type II and the Heim-Lorek polynomials. In the second part, we investigate the oscillatory and boundedness properties of the related orthogonality measures and the functions representing them. The polynomials considered so far in this article are closely related to the concept of the Wess-Ruffing discretization.

1 Introduction

In recent years, interesting connections between orthogonal function systems and oscillation theory for ordinary difference equations have been established. They allow one to relate different methods, namely those from functional analysis with those from oscillatory function systems to each other. The benefit is new insight into one and the same object from two different directions. This helps also to give answers for problems arising in oscillation theory from the viewpoint of orthogonal polynomials and *vice versa*. To get started, we consider in the sequel first the classical Hermite polynomials in Section 2 and generalize then their oscillatory results to discrete versions of the polynomials in Section 4, having revised some principal concepts of a consistent quantum mechanical discretization method, the Wess-Ruffing discretization, in Section 3. In Section 5 the fascinating Heim-Lorek polynomials are looked at. Finally, in Section 6, there is a change of methods, namely using for the discrete Hermite polynomials classical results of moment problems to understand the possibly oscillatory behavior of their orthogonality measures. The concept of the Wess-Ruffing discretization, which has been developed through the last 20 years, is related to the mathematical structures outlined in the articles [1–17].

2 Hermite polynomials

Let us consider the recursion relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \in \mathbb{N}_0, \tag{1}$$

with initial conditions

$$H_{-1} = 0 \quad \text{and} \quad H_0 = 1,$$

which yield polynomial functions on $x \in \mathbb{R}$.

For a $x \in \mathbb{R}$ fixed we rewrite (1) as

$$h(n+1) = 2xh(n) - 2nh(n-1), \quad n \in \mathbb{N}_0, \tag{2}$$

under the initial conditions

$$h(-1) = 0 \quad \text{and} \quad h(0) = 1.$$

Here in this section we complement the oscillatory properties of $h(n)$ obtained in [18] with the study of the amplitude with respect to the real line of its oscillations. We recall that if $h(n)$ has two consecutive or alternate zeros then it is eventually null, a trivial situation which will be excluded here. In this way, for every integer and positive k, for $n > k$ the lowest number of either positive or negative consecutive terms of $h(n)$ is larger or equal to two and the largest number of either positive or negative consecutive terms of $h(n)$ does not exceed three (see [18, Theorems 3 and 4]).

Lemma 1 *Let $x \in \mathbb{R}^+$ and $n > \max\{x, 2x^2 - \frac{1}{2}, \frac{1}{4x}\}$. If*

$$h(n-1) > 0 \quad \textit{and} \quad h(n) > 0$$

then either $h(n+1) > 0$ and

$$\max\{-h(n+2), -h(n+3)\} > \max\{h(n-1), h(n), h(n+1)\}$$

or $h(n+1) < 0$ and

$$\max\{-h(n+1), -h(n+2)\} > \max\{h(n-1), h(n)\}.$$

Proof Let $h(n-1) > 0$, $h(n) > 0$ and $h(n+1) > 0$. Then by (2),

$$h(n) > \frac{n}{x}h(n-1) > h(n-1)$$

for every $n > x$. Noticing that by [18, Theorem 4], $h(n+2) < 0$ and $h(n+3) < 0$, we will prove that $-h(n+2) > h(n)$ and $-h(n+3) > h(n+1)$. In fact, using (2), we have

$$\begin{aligned} h(n+2) &= 2xh(n+1) - 2(n+1)h(n) \\ &= -2(n+1-2x^2)h(n) - 4xnh(n-1) \\ &< -2(n+1-2x^2)h(n), \end{aligned}$$

and for $n > 2x^2 - 1/2$ we obtain

$$h(n+2) < -h(n).$$

The same arguments enable us to conclude that

$$h(n+3) < -2(n+2-2x^2)h(n+1) < -h(n+1).$$

Let now $h(n-1) > 0$, $h(n) > 0$ and $h(n+1) < 0$. Notice that

$$h(n+2) < -2(n+1)h(n) < -h(n)$$

and, for every $n > \frac{1}{4x}$,

$$\begin{aligned} h(n+2) &= -2\bigl(n+1-2x^2\bigr)h(n) - 4xnh(n-1) \\ &< -4xnh(n-1) \\ &< -h(n-1), \end{aligned}$$

which proves the lemma. □

Observe that $u(n) = -h(n)$ is also a solution of (2) satisfying now the initial conditions $u(-1) = 0$ and $u(0) = -1$, which of course has the same oscillatory characteristics as $h(n)$. This fact enables Lemma 1 to be used in the following situation.

Lemma 2 *Let $x \in \mathbb{R}^+$ and $n > \max\{x, 2x^2 - \frac{1}{2}, \frac{1}{4x}\}$. If*

$$h(n-1) < 0 \quad \textit{and} \quad h(n) < 0$$

then either $h(n+1) < 0$ and

$$\max\bigl\{h(n+2), h(n+3)\bigr\} > \max\bigl\{-h(n-1), -h(n), -h(n+1)\bigr\}$$

or $h(n+1) > 0$ and

$$\max\bigl\{h(n+1), h(n+2)\bigr\} > \max\bigl\{-h(n-1), -h(n)\bigr\}.$$

Proof If $h(n-1) < 0$, $h(n) < 0$ and $h(n+1) < 0$ then $-h(n-1) > 0$, $-h(n) > 0$ and $-h(n+1) > 0$. Therefore $-h(n+2) < 0$ and $-h(n+3) < 0$ and by Lemma 1, we have

$$\max\bigl\{h(n+2), h(n+3)\bigr\} > \max\bigl\{-h(n-1), -h(n), -h(n+1)\bigr\}.$$

By use of the same arguments one shows that

$$\max\bigl\{h(n+1), h(n+2)\bigr\} > \max\bigl\{-h(n-1), -h(n)\bigr\}.$$ □

From these lemmas one can conclude easily the following asymptotic result.

Theorem 3 *Let $x \in \mathbb{R}^+$. There are two increasing sequences $n_\ell, n_k \in \mathbb{N}$ such that*

$$h(n_\ell) \to +\infty \quad \textit{and} \quad h(n_k) \to -\infty$$

as $n_\ell, n_k \to +\infty$.

Proof Note first that by Lemmas 1 and 2, $h(n)$ is a divergent sequence.

Assume $h(n)$ bounded. Then if $K > 0$ is such that $|h(n)| \leq K$ for every n, from

$$2n\left|h(n-1)\right| \leq \left|h(n+1)\right| + 2x\left|h(n)\right|,$$

one obtains

$$\left|h(n-1)\right| \leq \frac{1+2x}{2n}K,$$

and consequently $h(n) \to 0$, as $n \to \infty$, which is contradictory.

Then $h(n)$ is unbounded and applying again Lemmas 5 and 6, the theorem follows. □

Remark 4 This theorem expresses a simple asymptotic behavior of the sequence $h(n) = H_n(x)$. It can also be obtained in a richer way as an immediate consequence of some asymptotic formulas for Hermite polynomials existing in the literature. This is the case of

$$H_n(x) \sim 2^{(n+1)/2} n^{n/2} e^{-n/2} e^{x^2/2} \cos\left(\sqrt{2n+1} - \frac{n\pi}{2}\right)$$

(see [19, Ex. 46, p.351]) or through the well-known asymptotic expansion theorem [1, Theorem 8.22.6].

$$\begin{aligned}\lambda_n^{-1} e^{-x^2/2} H_n(x) = {} & \cos\left((n+1)^{1/2}x - \frac{n\pi}{2}\right) \sum_{k=0}^{p-1} u_k(x)(n+1)^{-k/2} \\ & + (n+1)^{-1/2} \sin\left((n+1)^{1/2}x - \frac{n\pi}{2}\right) \sum_{k=0}^{p-1} v_k(x)(n+1)^{-k/2} + O\left(n^{-p}\right),\end{aligned}$$

where, using the Γ-function, one has for n even

$$\lambda_n = \frac{\Gamma(n+1)}{\Gamma(\frac{n}{2}+1)},$$

and for n odd

$$\lambda_n = \frac{\Gamma(n+2)}{\Gamma(\frac{n}{2}+\frac{3}{2})}(n+1)^{-1/2},$$

and the coefficients $u_k(x)$ and $v_k(x)$ are polynomials depending upon k containing only even and odd powers of x, respectively.

3 The concept of Wess-Ruffing discretization

The problem of isospectrality in the theory of linear operators is deep and challenging. Its connection in context of discretizations was looked at in [2], featuring the concept of Wess-Ruffing discretization. In this reference, it was shown how isospectrality occurs in context of supersymmetric difference ladder operator formalisms. The concept of superpotentials was first reviewed where naturally the concept of isospectrality appears. Difference versions of the quantum harmonic oscillator on an equidistant grid were looked at, revealing the same equidistant point spectrum which the continuum quantum harmonic oscillator has. Next self-similar supermodels were addressed, showing a point spectrum which consists of basic versions of the natural numbers. These supermodels were related to discrete versions of Schrödinger operators, which exhibit - at least partially - the same type of discrete point spectrum. The concept of strip discretizations was introduced on basic linear grids. This type of discretization showed the typical point spectrum, consisting of basic versions of the natural numbers. Precisely the same type of spectrum was finally also found in case of so-called basic multigrid discretizations. We therefore obtain a unified discrete model of some Schrödinger equations which allow both piecewise continuous solutions and sophisticated multigrid solutions. A mathematical tool used in a part of the Wess-Ruffing discretization deals with discrete versions of Hermite polynomials.

4 Discrete Hermite polynomials

Let us cite now remarks given in [3] on the connection of some different q-generalized Hermite polynomials: In literature, see for instance the Internet reference to the Koekoek-Swarttouw online report on orthogonal polynomials http://fa.its.tudelft.nl/~koekoek/askey/ there are listed two types of deformed discrete generalizations of the classical conventional Hermite polynomials, namely the discrete basic Hermite polynomials of type I and the discrete basic Hermite polynomials of type II. These polynomials appear in the mentioned Internet report under citations 3.28 and 3.29. Both types of polynomials, specified under the two respective citations by the symbol h_n while n is a non-negative integer, can be successively transformed (scaling the argument and renormalizing the coefficients) into the one and same form which is given by

$$H_{n+1}^q(x) - \alpha q^n x H_n^q(x) + \alpha \frac{q^n - 1}{q - 1} H_{n-1}^q(x) = 0, \quad n \in \mathbb{N}_0 \tag{3}$$

with initial conditions $H_0^q(x) = 1$, $H_1^q(x) = \alpha x$ for all $x \in \mathbb{R}$. Note that α is chosen as a fixed positive real number. Here, the number q may range in the set of all positive real numbers, without the number 1 - the case $q = 1$ being reserved for the classical conventional Hermite polynomials. Depending on the choice of q, the two different types of discrete basic Hermite polynomials can be found: The case $0 < q < 1$ corresponds to the discrete basic Hermite polynomials of type II, the case $q > 1$ corresponds to the discrete basic Hermite polynomials of type I.

Up to the late 1990s, the perception was that both type of discrete basic Hermite polynomials have only discrete orthogonality measures. This is certainly true in the case of $q > 1$ since the existence of such an orthogonality measure was shown explicitly and since the moment problem behind the discrete basic Hermite polynomials of type I is uniquely determined.

However, it could be shown that beside the known discrete orthogonality measure, specified in the above Internet report, the discrete basic Hermite polynomials of type II, hence being connected to (3) with $0 < q < 1$ allow also orthogonality measures with continuous support - and even going beyond this - characteristic supports which reveal particular strip structures, also measures with continuous and discrete parts and sophisticated mixtures of them.

We describe this phenomenon in some more detail:

We have the well-known conventional result that a symmetric orthogonality measure with discrete support for the polynomials (3), with $0 < q < 1$, yields moments being given by

$$\nu_{2m+2} = \frac{q^{-2m-1}-1}{\alpha(1-q)}\nu_{2m}, \qquad \nu_{2m+1} = 0, \quad m \in \mathbb{N}_0. \tag{4}$$

It was shown earlier - and here we refer again to the analytical background in [3] - that there exist continuous and piecewise continuous solutions to the difference equation

$$\psi(qx) = \left(1+\alpha(1-q)x^2\right)\psi(x), \quad x \in \mathbb{R} \tag{5}$$

leading to the same moments (4). Such a behavior of the discrete basic Hermite polynomials of type II, hence being related to the scenario (3) with $0 < q < 1$, was quite unexpected. *Vice versa*: Once moments ν_m with non-negative integer m of a given weight function are given through the relation (4), it can immediately be said that the weight function provides an orthogonality measure for the discrete basic Hermite polynomials of type II, related to scenario (3) with $0 < q < 1$.

The discrete Hermite polynomials we are considering here are given through the recursive relation

$$H_{n+1}(x) = 2q^n x H_n(x) - 2\frac{q^n-1}{q-1}H_{n-1}(x), \quad n \in \mathbb{N}_0, \tag{6}$$

where $x \in \mathbb{R}$ and $q > 0$, $q \neq 1$, with the initial conditions

$$H_{-1} = 0 \quad \text{and} \quad H_0 = 1.$$

Fixing the real value x, equation (6) can be rewritten as

$$h(n+1) = 2q^n x h(n) - 2\frac{q^n-1}{q-1}h(n-1), \quad n \in \mathbb{N}_0 \tag{7}$$

with initial conditions

$$h(-1) = 0 \quad \text{and} \quad h(0) = 1.$$

The oscillatory properties of $h(n)$ are the same as in Section 1. For the amplitude with respect to the real axis of its oscillations the following results are obtained.

Lemma 5 *Let* $x \in \mathbb{R}^+$, $q \in]0,1[$ *and* $n > \max\{\log_q(\frac{q+1}{2}), -\log_q(2x)\}$. *If*

$$h(n-1) > 0 \quad \textit{and} \quad h(n) > 0$$

then either $h(n+1) > 0$ *and*

$$-h(n+2) > \max\left\{h(n-1), h(n), h(n+1)\right\} = h(n)$$

or $h(n+1) < 0$ *and*

$$\max\left\{-h(n+2), -h(n+1)\right\} > \max\left\{h(n-1), h(n)\right\} = h(n).$$

Proof Let $h(n-1) > 0$, $h(n) > 0$ and $h(n+1) > 0$. Using (7), we have, for every $n > \log_q(\frac{q+1}{2})$, that

$$\begin{aligned} h(n-1) &= q^n x \frac{q-1}{q^n-1} h(n) - \frac{q-1}{2(q^n-1)} h(n+1) \\ &< q^n x \frac{q-1}{q^n-1} h(n) \\ &< \frac{q-1}{2(q^n-1)} h(n) < h(n). \end{aligned}$$

On the other hand for every $n > -\log_q(2x)$, it is

$$h(n+1) < 2q^n x h(n) < h(n).$$

By [18, Theorem 10], one has $h(n+2) < 0$. We will show that $-h(n+2) > h(n)$. In fact, by (7), we obtain

$$\begin{aligned} h(n+2) &= 2q^{n+1} x h(n+1) - 2\frac{q^{n+1}-1}{q-1} h(n) \\ &< 2q^{n+1} x h(n) - 2\frac{q^{n+1}-1}{q-1} h(n) \\ &= 2\left(q^{n+1} x - \frac{q^{n+1}-1}{q-1}\right) h(n) \\ &< -2\left(q^n + q^{n-1} + \cdots + q + 1 - q^{n+1} x\right) h(n) \\ &< -2\left(q^n + q^{n-1} + \cdots + \frac{1}{2} q + 1\right) h(n) \end{aligned}$$

since $q^{n+1} x < \frac{1}{2} q$. Consequently

$$h(n+2) < -h(n).$$

Suppose now that $h(n-1) > 0$, $h(n) > 0$ and $h(n+1) < 0$. Notice that

$$h(n+2) < -2\frac{q^{n+1}-1}{q-1} h(n) < -h(n)$$

and since, by (7),

$$2\frac{q^n-1}{q-1} h(n-1) = 2q^n x h(n) - h(n+1) < 2q^n x h(n),$$

we obtain

$$h(n-1) < \frac{q^n-1}{q-1} h(n-1) < q^n x h(n) < h(n).$$

Thus

$$\max\{-h(n+2), -h(n+1)\} > \max\{h(n-1), h(n)\} = h(n). \qquad \square$$

Lemma 6 *Let* $x \in \mathbb{R}^+$, $q \in]0,1[$ *and* $n > \max\{\log_q(\frac{q+1}{2}), -\log_q(2x)\}$. *If*

$$h(n-1) < 0 \quad \textit{and} \quad h(n) < 0$$

then either $h(n+1) < 0$ *and*

$$h(n+2) > \max\{-h(n-1), -h(n), -h(n+1)\} = -h(n)$$

or $h(n+1) > 0$ *and*

$$\max\{h(n+1), h(n+2)\} > \max\{-h(n-1), -h(n)\} = -h(n).$$

Proof Let $n > -\log_q(2x)$. Since

$$-h(n-1) > 0, \qquad -h(n) > 0 \quad \text{and} \quad -h(n+1) > 0$$

by Lemma 5 we have

$$h(n+2) > \max\{-h(n-1), -h(n), -h(n+1)\} = -h(n),$$

and as

$$-h(n-1) > 0, \qquad -h(n) > 0 \quad \text{and} \quad -h(n+1) < 0,$$

one concludes that

$$\max\{h(n+1), h(n+2)\} > \max\{-h(n-1), -h(n)\} = -h(n).$$ □

As before the next theorem is a consequence of Lemmas 5 and 6.

Theorem 7 *Let* $x \in \mathbb{R}^+$, $q \in]0,1[$. *There are two increasing sequences* $n_\ell, n_k \in \mathbb{N}$ *such that*

$$h(n_\ell) \to +\infty \quad \textit{and} \quad h(n_k) \to -\infty$$

as $n_\ell, n_k \to +\infty$.

5 Heim-Lorek polynomials

With the same initial conditions as we used them in the last section, namely

$$H_{-1} = 0 \quad \text{and} \quad H_0 = 1,$$

for $q > 0$, $q \neq 1$, let us now consider the recurrence relation

$$H_{n+1}(x) = 2q^n x H_n(x) - 2\sqrt{n}\sqrt{\frac{q^n - 1}{q - 1}} H_{n-1}(x), \quad n \in \mathbb{N}_0, \tag{8}$$

yielding a polynomial function on $x \in \mathbb{R}$.

These are the Heim-Lorek polynomials which are interpolating structures between the type of q-generalized Hermite polynomials we considered in the last section and the classical Hermite polynomials.

With x fixed, we obtain a sequence $h(n)$ defined through

$$h(n+1) = 2q^n x h(n) - 2\sqrt{n}\sqrt{\frac{q^n-1}{q-1}}h(n-1), \quad n \in \mathbb{N}_0, \tag{9}$$

for the initial conditions

$$h(-1) = 0 \quad \text{and} \quad h(0) = 1,$$

which has the same oscillatory characteristics as before. With respect to the amplitude of its oscillations, similar results are obtained in the sequel.

Lemma 8 *Let* $x \in \mathbb{R}^+$, $q \in]0,1[$, $N = \min\{n \in \mathbb{N} : \frac{q^n x}{\sqrt{n}}\sqrt{\frac{q-1}{q^n-1}} < 1\}$ *and* $n > \max\{N, -\log_q(2x)\}$. *If*

$$h(n-1) > 0 \quad \textit{and} \quad h(n) > 0$$

then either $h(n+1) > 0$ *and*

$$-h(n+2) > \max\{h(n-1), h(n), h(n+1)\} = h(n)$$

or $h(n+1) < 0$ *and*

$$-h(n+2) > \max\{h(n-1), h(n)\}.$$

Proof Let us assume first that $h(n-1) > 0$, $h(n) > 0$ and $h(n+1) > 0$. By (9), for $n > N$, we have

$$\begin{aligned} h(n-1) &= \left(2\sqrt{n}\sqrt{\frac{q^n-1}{q-1}}\right)^{-1}\left(2q^n x h(n) - h(n+1)\right) \\ &< \frac{q^n x}{\sqrt{n}}\sqrt{\frac{q-1}{q^n-1}}h(n) < h(n), \end{aligned}$$

and, for $n > -\log_q(2x)$,

$$h(n+1) < 2q^n x h(n) < h(n).$$

On the other hand by [18, Theorem 18] one has $h(n+2) < 0$ and, using (9), one obtains

$$\begin{aligned} h(n+2) &= 2q^{n+1} x h(n+1) - 2\sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}}h(n) \\ &< 2q^{n+1} x h(n) - 2\sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}}h(n) \end{aligned}$$

$$\begin{aligned}&= -2\left(\sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}} - q^{n+1}x\right)h(n)\\&< -h(n),\end{aligned}$$

since, for every $n > N$,

$$\sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}} - q^{n+1}x > \sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}} > \frac{1}{q^n x} > 1.$$

Then $-h(n+2) > h(n)$.

Assume now that $h(n-1) > 0$, $h(n) > 0$ and $h(n+1) < 0$. Notice that

$$\begin{aligned}h(n+2) &= 2q^{n+1}xh(n+1) - 2\sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}}h(n)\\&< -2\sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}}h(n) < -h(n)\end{aligned}$$

and

$$\begin{aligned}h(n+2) &= \left(4q^{2n+1}x^2 - 2\sqrt{n+1}\sqrt{\frac{q^{n+1}-1}{q-1}}\right)h(n) - 4q^{n+1}x\sqrt{n}\sqrt{\frac{q^n-1}{q-1}}h(n-1)\\&= \left(4q^{2n+1}x^2 - 2\frac{q^{n+1}-1}{q-1}\right)h(n) - 4q^{n+1}x\sqrt{n}\sqrt{\frac{q^n-1}{q-1}}h(n-1)\\&< -4q^{n+1}x\sqrt{n}\sqrt{\frac{q^n-1}{q-1}}h(n-1),\end{aligned}$$

since

$$\sqrt{n}\sqrt{\frac{q^n-1}{q-1}} > \frac{1}{q^n x} \quad\Leftrightarrow\quad 4q^{n+1}x\sqrt{n}\sqrt{\frac{q^n-1}{q-1}} > 4q.$$

Hence

$$h(n+2) < -h(n-1)$$

for $n > \max\{N, -\log_q(2x)\}$ and the lemma is proved. □

Analogously one obtains the following result.

Lemma 9 *Let* $x \in \mathbb{R}^+$, $q \in]0,1[$, $N = \min\{n \in \mathbb{N} : \frac{q^n x}{\sqrt{n}}\sqrt{\frac{q-1}{q^n-1}} < 1\}$ *and* $n > \max\{N, -\log_q(2x)\}$. *If*

$$h(n-1) < 0 \quad \textit{and} \quad h(n) < 0$$

then either $h(n+1) < 0$ *and*

$$h(n+2) > \max\bigl\{-h(n-1), -h(n), -h(n+1)\bigr\} = -h(n)$$

or $h(n+1) > 0$ *and*

$$h(n+2) > \max\{-h(n-1), -h(n)\}.$$

We conclude this section with the next theorem.

Theorem 10 *Let* $x \in \mathbb{R}^+$, $q \in]0,1[$. *There are two increasing sequences* $n_\ell, n_k \in \mathbb{N}$ *such that*

$$h(n_\ell) \to +\infty \quad \text{and} \quad h(n_k) \to -\infty$$

as $n_\ell, n_k \to +\infty$.

6 Related moment problems

Let $\Omega \subseteq \mathbb{R}$ such that for a fixed $0 < q < 1$ we have

$$x \in \Omega \quad \Leftrightarrow \quad qx \in \Omega, \qquad x \in \Omega. \tag{10}$$

For $f : \Omega \to \mathbb{R}$ we define

$$(Rf)(x) = f(qx), \qquad (Lf)(x) = f\left(q^{-1}x\right)$$

for all $x \in \Omega$.

We use the abbreviations R^n ($n \in \mathbb{N}$) where for instance $R^{-1} = L$. With this, we obtain the operator expressions

$$F = \sum_{(m,n)\in\mathbb{Z}^2} a_{mn} R^m X^n,$$

with only finitely many $a_{mn} \neq 0$, and the numbers $(a_{mn})_{(m,n)\in\mathbb{Z}^2}$ being real or even complex. We can in a natural way introduce addition and multiplication of objects of type F and see that they constitute a noncommutative $\mathbb{C}$-algebra.

Example 11

$$\begin{aligned}
&F = RX^2 + R^2X, \qquad G = RX + R^3X^4, \\
&F + G = RX + RX^2 + R^2X + R^3X^4, \\
&FG = \left(RX^2 + R^2X\right)\left(RX + R^3X^4\right) = q^{-2}R^2X^3 + q^{-6}R^4X^6 + q^{-1}R^3X^2 + q^{-3}R^5X^5.
\end{aligned}$$

We denote the noncommutative $\mathbb{C}$-algebra specified as above by $\mathcal{A}$ and feel ourselves attracted to the following problem.

Problem 12 Given $F \in \mathcal{A}$. Look for $f : \Omega \to \mathbb{R}$ with suitable $\Omega \in \mathbb{R}$ such that

$$F(f) = 0.$$

What can we say about classical analytic properties of f? In case that f exists:

1. Is f bounded?
2. Is f oscillatory?
3. Is f in $\mathcal{L}^1(\Omega)$?
4. Is f holomorphic?

So far it is clear that we meet here a very rich structure of objects and answers to the questions 1-4 may be expected in a large variety.

Let us look at the objects

$$F = R + \alpha I + \beta X,$$
$$G = L + \alpha I + \beta X,$$

I denoting the identity, α and β being real numbers. It turns out that the case study of these objects is not purely academic but is related to mathematical modeling in quantum optics, hence exhibiting interesting physical structures.

To get started, let us look at the problem

$$F(f) = 0.$$

This may be rewritten in terms of adaptive difference equations as

$$f(qx) + \alpha f(x) + \beta x f(x) = 0. \tag{11}$$

Already now it is crucial to say some words about the support for this type of difference equations.

The difference equation of type (11) can be considered:

(1) on the real axis,
(2) on discrete supports of type $\{cq^n | n \in \mathbb{Z}\}$ where $c \in \mathbb{R}\backslash\{0\}$ or unions of the support of this type,
(3) on strip structures.

Suppose that we have a solution of (11) on a non-empty interval (a, b). With the help of (11) we then can extend the solution to the interval (qa, qb) provided $(qa, qb) \cap (a, b) = \varnothing$.

This procedure can be iterated successively. As a result we obtain a sequence of intervals on which (11) holds: this is a strip structure.

Let us now try to distillate some elementary properties of (11) first.

Lemma 13 *Let $\alpha < 0$ and $\beta < 0$. Let $\Omega \subseteq \mathbb{R}^+$ such that*

$$x \in \Omega \quad \Leftrightarrow \quad qx \in \Omega.$$

Any solution $f : \Omega \to \mathbb{R}$ of $F(f) = 0$ with $F = R + \alpha I + \beta X$ has the following properties:

(i) *f is nonoscillatory.*
(ii) *f is bounded $\Leftrightarrow \alpha \geq -1$.*

Let us denote furthermore the Lebesgue measure of Ω by μ. If $\mu(\Omega) > 0$ then

(iii) *the sequence of numbers* $(\mu_n^{\Omega})_{n\in\mathbb{N}_0}$ *exist, given by*

$$\mu_n^{\Omega} := \int_{\mathbb{R}^+} x^n f(x)\mathcal{X}_{\Omega}(x)\,dx, \quad n \in \mathbb{N}_0$$

provided $\alpha \geq -1$.

(iv) *Let* $\alpha \geq -1$ *and* $\Omega_1, \Omega_2 \subseteq \mathbb{R}^+$ *such that* $\mu(\Omega_1)\mu(\Omega_2) > 0$. *Then the following identity holds*:

$$\mu_{n+1}^{\Omega_1}\mu_n^{\Omega_2} = \mu_{n+1}^{\Omega_2}\mu_n^{\Omega_1}, \quad n \in \mathbb{N}_0.$$

(v) *Provided* $\varphi : \mathbb{C} \to \mathbb{C}$ *such that* $\varphi|_{\mathbb{R}^+} = f$, φ *cannot be a holomorphic solution to* $F(\varphi) = 0$.

Remark 14 $\mathcal{X}_{\Omega}$ in (iii) denotes the characteristic function of Ω.

Remark 15 It is not necessary to have $\Omega_1 \cap \Omega_2 = \varnothing$ in (iv).

Remark 16 Spectacular is the fact (iv) since it shows that F generates the same orthogonal polynomials on different $\Omega_1, \Omega_2 \subseteq \mathbb{R}^+$, which can - from the viewpoint of possible physical applications - be interpreted as two different phases of one and the same object.

We now want to compare the respective properties of the objects

$$F = R + \alpha I + \beta X \quad \text{and} \quad G = L + \alpha I + \beta X.$$

Lemma 17 *Let* $\alpha < 0$ *and* $\beta < 0$. *Let* $\Omega \subseteq \mathbb{R}^+$ *such that* $x \in \Omega \Leftrightarrow qx \in \Omega$. *Any solution* $g : \Omega \to \mathbb{R}$ *of* $G(g) = 0$ *with* $G = L + \alpha I + \beta X$ *has the following properties*:

(i) g *is nonoscillatory.*
(ii) *In no case* g *is bounded.*
(iii) *In no case* g *is in* $\mathcal{L}^1(\Omega)$ *if* $\mu(\Omega) > 0$.
(iv) *There exist - up to a multiplicative constant - precisely one holomorphic* $\varphi : \mathbb{C} \to \mathbb{C}$ *with* $F(\varphi) = 0$.

We are now going to prove Lemma 13 and Lemma 17.

Proof of Lemma 13 We translate first the requirement $F(f) = 0$ into (11)

$$f(qx) + \alpha f(x) + \beta x f(x) = 0.$$

It follows that

$$f(qx) = (-\alpha - \beta x)f(x)$$

where $x \in \Omega \subseteq \mathbb{R}^+$ as specified and $\alpha, \beta < 0$. Immediately we see that there is no sign change between $f(qx)$ and $f(x)$. Hence statement (i) is correct.

If $\alpha < -1$, we introduce the sequence $(x_n)_{n\in\mathbb{N}}$, given by

$$x_1 := 1, \qquad x_{n+1} := qx_n, \quad n \in \mathbb{N}.$$

It then follows that

$$\lim_{n\to\infty}\left|\frac{f(x_{n+1})}{f(x_n)}\right| = \alpha > 1.$$

Hence $(f(x_n))_{n\in\mathbb{N}}$ and therefore also f is unbounded. By pointwise argumentation, one can also show that $-1 \leq \alpha < 0$ implies the boundedness of f. Therefore (ii) holds.

If the Lebesgue measure fulfills $\mu(\Omega) = 0$, then the statement is trivial. If $\mu(\Omega) > 0$ and $\alpha \geq -1$, then the boundedness of f together with (11) implies that

$$\int_0^\infty f(x)\,dx < \infty,$$

provided f was chosen as positive. By induction follows for $n \in \mathbb{N}_0$

$$\begin{aligned}\mu_{n+1}^{\Omega} &= \int_0^\infty x^{n+1} f(x)\,dx = -\frac{1}{\beta}\int_0^\infty \bigl[f(qx)x^n + \alpha f(x)x^n\bigr]\,dx \\ &= -\frac{1}{\beta}q^{-n}\mu_n^{\Omega} - \frac{\alpha}{\beta}\mu_n^{\Omega}.\end{aligned}$$

Thus (iii) holds.

The last equation in (iii) reads

$$\mu_{n+1}^{\Omega} = \left(-\frac{1}{\beta}q^{-n} - \frac{\alpha}{\beta}\right)\mu_n^{\Omega}, \quad n \in \mathbb{N}_0$$

and since Ω is arbitrary with $\mu(\Omega) > 0$ and we have the properties of Ω specified in Lemma 13, statement (iv) is true.

Suppose $\varphi : \mathbb{C} \to \mathbb{C}$ is holomorphic, such that $\varphi|_{\mathbb{R}^+} = f$. Then

$$\varphi(z) = \sum_{n=0}^\infty c_n z^n \quad \text{and} \quad \varphi(qz) = \sum_{n=0}^\infty c_n q^n z^n.$$

The equation

$$\varphi(qz) = (-\alpha - \beta z)\varphi(z)$$

implies

$$c_n q^n = -\alpha c_n - \beta c_{n-1}, \quad n \in \mathbb{N}_0.$$

Looking at the expression

$$\left|\frac{c_n z^n}{c_{n-1}z^{n-1}}\right| = \left|\frac{(q^n + \alpha)^{-1} z^n}{-\beta^{-1} z^{n-1}}\right|$$

we see that there is a contradiction to analyticity for $|z| \to \infty$. Hence (v) holds. □

Proof of Lemma 17 The equation

$$G(g) = 0 \quad \text{for } G = L + \alpha I + \beta X$$

yields

$$g(q^{-1}x) + \alpha g(x) + \beta x g(x) = 0.$$

Since we have $\Omega \subseteq \mathbb{R}^+$ and by assumption $\alpha, \beta < 0$, we are through,

$$g(q^{-1}x) = -\alpha g(x) - \beta x g(x),$$

automatically leading to the fact that the sequence

$$\bigl(g(q^{-n}x)\bigr)_{n\in\mathbb{N}_0}$$

is unbounded; hence (ii) holds.

The fact that g is nonoscillatory is obvious from the choice $\alpha, \beta < 0$ and $x \in \Omega \subseteq \mathbb{R}^+$. So (i) is true.

The unboundedness of g from (ii) also implies statement (iii); note that $\mu(\Omega) > 0$.

We show now the existence of a holomorphic $g : \mathbb{C} \to \mathbb{C}$ such that

$$g(q^{-1}z) = -\alpha g(z) - \beta g(z) z.$$

Inserting

$$g(z) = \sum_{n=0}^{\infty} c_n z^n,$$

we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n q^{-n} z^n = -\alpha \sum_{n=0}^{\infty} c_n z^n - \beta \sum_{n=0}^{\infty} c_n z^{n+1} \\
&\quad\Rightarrow \quad c_n(q^{-n} + \alpha) = -\beta c_{n-1}, \quad n \in \mathbb{N} \\
&\quad\Rightarrow \quad \left|\frac{c_n}{c_{n-1}}\right| = \frac{-\beta}{|q^{-n} + \alpha|} \underset{n\to\infty}{\rightarrow} 0.
\end{aligned}$$

This implies in any case the analyticity statement from (iv). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JMF and SP study oscillatory behavior of the Hermit polynomials, discrete Hermit polynomials and Heim-Lorek polynomials; AR introduce the concept of Wess-Ruffing discretization and study the related moments.

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