# OSCILLATION CRITERIA FOR EVEN-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE 

## (COMMUNICATED BY IOANNIS P. STAVROULAKIS)

ETHIRAJU THANDAPANI, SANKARAPPAN PADMAVATHI AND SANDRA PINELAS


#### Abstract

This paper deals with oscillation criteria for even order nonlinear neutral mixed type differential equations of the form $\left(a(t)\left(x(t)+b x\left(t-\tau_{1}\right)+c x\left(t+\tau_{2}\right)\right)^{(n-1)}\right)^{\prime}+p(t) x^{\alpha}\left(t-\sigma_{1}\right)+q(t) x^{\beta}\left(t+\sigma_{2}\right)=0$, where $t \geq t_{0}$ and $n \geq 2$ is an even integer, $\alpha \geq 1$ and $\beta \geq 1$, are ratios of odd positive integers. The results are obtained both for the case $\int a^{-1}(t) d t=\infty$, and in case $\int^{\infty} a^{-1}(t) d t<\infty$. Some examples are given to illustrate our main results.


## 1. Introduction

In this paper, we study the oscillatory behavior of the following even order nonlinear neutral mixed type differential equation of the form

$$
\begin{equation*}
\left(a(t)\left(x(t)+b x\left(t-\tau_{1}\right)+c x\left(t+\tau_{2}\right)\right)^{(n-1)}\right)^{\prime}+p(t) x^{\alpha}\left(t-\sigma_{1}\right)+q(t) x^{\beta}\left(t+\sigma_{2}\right)=0, t \geq t_{0} \tag{1.1}
\end{equation*}
$$

where $n \geq 2$ is an even integer. We set $z(t)=x(t)+b x\left(t-\tau_{1}\right)+c x\left(t+\tau_{2}\right)$. Throughout this paper, we assume that
$\left(\mathrm{C}_{1}\right) a \in C\left(\left[t_{0}, \infty\right), \mathbb{R}\right), a(t)>0$ and $a^{\prime}(t)>0$ for all $t \geq t_{0} ;$
$\left(\mathrm{C}_{2}\right) p, q \in C\left(\left[t_{0}, \infty\right), \mathbb{R}\right), p(t)>0$ and $q(t)>0$ for all $t \geq t_{0}$;
$\left(\mathrm{C}_{3}\right) b$ and $c$ are positive constants, $\tau_{1}, \tau_{2}, \sigma_{1}, \sigma_{2}$ are nonnegative constants and $\alpha$ and $\beta$ are ratios of odd positive integers.
We shall consider the two cases:

$$
\begin{equation*}
\int_{t_{0}}^{\infty} \frac{1}{a(t)}=\infty \tag{1.2}
\end{equation*}
$$

[^0]and
\[

$$
\begin{equation*}
\int_{t_{0}}^{\infty} \frac{1}{a(t)}<\infty \tag{1.3}
\end{equation*}
$$

\]

Differential equations with delayed and advanced arguments (also called mixed differential equations or equations with mixed arguments) occur in many problems of economy, biology and physics (see for example [3, 6, 9, 10, 16]), because differential equations with mixed arguments are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena. The concept of delay is related to a memory of system, the past events are importance for the current behavior, and the concept of advance is related to a potential future events which can be known at the current time which could be useful for decision making. The study of various problems for differential equations with mixed arguments can be seen in [5, 8, 15, 17, 19, 24. It is well known that the solutions of these types of equations cannot be obtained in closed form. In the absence of closed form solutions a rewarding alternative is to resort to the qualitative study of the solutions of these types of differential equations. But it is not quite clear how to formulate an initial value problem for such equations and existence and uniqueness of solutions becomes a complicated issue. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equation on the half line.

In 20] the authors established some oscillation results for the nth order $(n>1)$ differential equations of mixed type

$$
\begin{equation*}
y^{(n)}(t)-\sum_{i=1}^{k} p_{i}^{n} y\left(t-n \tau_{i}\right)-\sum_{j=1}^{l} q_{j}^{n} y\left(t+n \sigma_{j}\right)=0 \tag{1.4}
\end{equation*}
$$

and

$$
\begin{equation*}
y^{(n)}(t)+\sum_{i=1}^{k} p_{i}^{n} y\left(t-n \tau_{i}\right)+\sum_{j=1}^{l} q_{j}^{n} y\left(t+n \sigma_{j}\right)=0 \tag{1.5}
\end{equation*}
$$

where $p_{i}, \tau_{i}, i=1,2, \ldots, k$ and $q_{j}, \sigma_{j}, j=1,2, \ldots, l$ are positive constants..
In 25 the author established some oscillation results for the solutions of the neutral equations of mixed type

$$
\begin{equation*}
\frac{d}{d t}(x(t)+c x(t-r))+\sum_{i=1}^{k} p_{i} x\left(t-\tau_{i}\right)+\sum_{j=1}^{l} q_{j} x\left(t+\sigma_{j}\right)=0 \tag{1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d}{d t}(x(t)+c x(t-r))-\sum_{i=1}^{k} p_{i} x\left(t+\tau_{i}\right)-\sum_{j=1}^{l} q_{j} x\left(t-\sigma_{j}=0\right. \tag{1.7}
\end{equation*}
$$

where $c \in \mathbb{R}, r \in(0, \infty), p_{i}, q_{j} \in(0, \infty)$ and $\tau_{i}, \sigma_{j} \in[0, \infty)$ for $i=1,2, \ldots, k, j=$ $1,2, \ldots, l$.

Grace [11] obtained some oscillation theorems for the odd order neutral differential equation

$$
\begin{equation*}
\left(x(t)+p_{1} x\left(t-\tau_{1}\right)+p_{2} x\left(t+\tau_{2}\right)\right)^{(n)}=q_{1} x\left(t-\sigma_{1}\right)+q_{2} x\left(t+\sigma_{2}\right), t \geq t_{0} \tag{1.8}
\end{equation*}
$$

where $n \geq 1$ is odd. In [13] the authors established some oscillation criteria for the following mixed neutral equation

$$
\begin{equation*}
\left(x(t)+p_{1} x\left(t-\tau_{1}\right)+p_{2} x\left(t+\tau_{2}\right)\right)^{\prime \prime}=q_{1} x\left(t-\sigma_{1}\right)+q_{2} x\left(t+\sigma_{2}\right), t \geq t_{0} \tag{1.9}
\end{equation*}
$$

with $q_{1}$ and $q_{2}$ are nonnegative real valued functions.
Zhang et al. 30 studied the even order nonlinear neutral functional equations

$$
\begin{equation*}
(x(t)+p(t) x(\tau(t)))^{(n)}+q(t) f(x(\sigma(t)))=0, t \geq t_{0} \tag{1.10}
\end{equation*}
$$

where $n$ is even, $0 \leq p(t)<1$. The authors established a comparison theorem for (1.10) and obtained results which improved and generalized some known results.

In 2011, Zhang et al. [29] studied the oscillatory behavior of the following higher order half quasilinear delay differential equation

$$
\begin{equation*}
\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)^{\prime}+q(t) x^{\beta}(\tau(t))=0, t \geq t_{0} \tag{1.11}
\end{equation*}
$$

under the condition $\int_{t_{0}}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}} d t<\infty$. The authors obtained some sufficient conditions, which guarantee that every solution of (1.11) is oscillatory or tends to zero.

In 2012, Y.B.Sun, Z.L.Han, S.R.Sun, Ch.Zhang [26] studied the oscillation criteria for even order nonlinear neutral differential equations

$$
\begin{equation*}
\left(r(t)(x(t)+p(t) x(\tau(t)))^{(n-1)}\right)^{\prime}+q(t) f(x(\sigma(t)))=0, t \geq t_{0} \tag{1.12}
\end{equation*}
$$

where $\int_{t_{0}}^{\infty} r^{-1}(t) d t=\infty, \int_{t_{0}}^{\infty} r^{-1}(t) d t<\infty, \tau(t) \leq t, \sigma(t) \leq t, 0 \leq p(t) \leq p_{0}<\infty$. The authors obtained some oscillation theorems, which guarantee that every solution of equation (1.12) is oscillatory. For the particular case when $n=2$, equation (1.1) reduces to the following equation

$$
\begin{equation*}
\left(r(t)(x(t)+p(t) x(\tau(t)))^{\prime}\right)^{\prime}+q(t) f(x(\sigma(t)))=0, t \geq t_{0} \tag{1.13}
\end{equation*}
$$

Han et al. [14] established the oscillation criteria for the solutions of (1.13), where $\int_{t_{0}}^{\infty} r^{-1}(t) d t=\infty, \tau(t) \leq t, \sigma(t) \leq t, 0 \leq p(t) \leq p_{0}<\infty$.

In [28] the authors obtained several sufficient conditions for the oscillation of solutions of second order neutral differential equation of the form
$\left(a(t)\left(\left[x(t)+b(t) x\left(t-\sigma_{1}\right)+c(t) x\left(t+\sigma_{2}\right)\right]^{\alpha}\right)^{\prime}\right)^{\prime}+q(t) x^{\beta}\left(t-\tau_{1}\right)+p(t) x^{\gamma}\left(t+\tau_{2}\right)=0, t \geq t_{0}$
where $\int_{t_{0}}^{\infty} a^{-1}(t) d t=\infty, 0 \leq b(t) \leq b, 0 \leq c(t) \leq c$ and $p$ and $q$ are nonnegative continuous real valued functions.

Motivated by the above observations, in this paper we establish some sufficient conditions for the oscillation of all solutions of equation (1.1) when the condition (1.2) or (1.3) is satisfied.

In Section 2, we establish some preliminary lemmas and in Section 3, we present sufficient conditions for the oscillation all solutions of equation (1.1). Examples are provided to illustrate the main results.

## 2. Some preliminary lemmas

In this section, we present some useful lemmas, which will be used in the proofs of our main results.

Lemma 2.1. [23]] Let $u \in C^{n}\left(\left[t_{0}, \infty\right), \mathbb{R}^{+}\right)$. If $u^{(n)}(t)$ is eventually of one sign for all large $t$, then there exists a $t_{x}>t_{1}$, for some $t_{1}>t_{0}$, and an integer $l$, $0 \leq l \leq n$, with $n+l$ even for $u^{(n)}(t) \geq 0$ or $n+l$ odd for $u^{(n)}(t) \leq 0$ such that $l>0$ implies that $u^{(k)}(t)>0$ for $t>t_{x}, k=0,1, \ldots, l-1$, and $l \leq n-1$, implies that $(-1)^{l+k} u^{(k)}(t)>0$ for $t>t_{x}, k=l, l+1, \ldots, n-1$.

Lemma 2.2. [1] Let $u$ be as in Lemma 2.1. Assume that $u^{(n)}(t)$ is not identically zero on any interval $\left[t_{0}, \infty\right)$, and there exists a $t_{1} \geq t_{0}$ such that $u^{(n-1)}(t) u^{(n)}(t) \leq 0$ for all $t \geq t_{1}$. If $\lim _{t \rightarrow \infty} u(t) \neq 0$, then for every $\lambda, 0<\lambda<1$, there exists $T \geq t_{1}$, such that for all $t \geq T$,

$$
u(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} u^{(n-1)}(t)
$$

Lemma 2.3. Assume that condition (1.2) holds. Furthermore, assume that $x$ is an eventually positive solution of equation (1.1). Then there exists $t_{1} \geq t_{0}$, such that

$$
z(t)>0, z^{\prime}(t)>0, z^{(n-1)}(t)>0 \text { and } z^{(n)}(t) \leq 0, \text { for all } t \geq t_{1}
$$

The proof is similar to that of Meng and Xu [22],Lemma 2.3] and so omitted.
Lemma 2.4. [21]] Assume that $\alpha \in(0, \infty)$ and $c \geq 0$ and $d \geq 0$. Then

$$
c^{\alpha}+d^{\alpha} \geq(c+d)^{\alpha} \text { if } 0<\alpha<1
$$

and

$$
c^{\alpha}+d^{\alpha} \geq \frac{1}{2^{\alpha-1}}(c+d)^{\alpha} \text { if } \alpha \geq 1
$$

Lemma 2.5. [[27]] Assume that for large $t$

$$
q(s) \neq 0 \text { for all } s \in\left[t, t^{*}\right]
$$

where $t^{*}$ satisfies $\sigma\left(t^{*}\right)=t$. Then

$$
x^{\prime}(t)+q(t)[x(\sigma(t))]^{\alpha}=0, t \geq t_{0}
$$

has an eventually positive solution if and only if the corresponding inequality

$$
x^{\prime}(t)+q(t)[x(\sigma(t))]^{\alpha} \leq 0, t \geq t_{0}
$$

has an eventually positive solution.
In [7, 12, 18, 30, the authors investigated the oscillatory behavior of the following equation

$$
\begin{equation*}
x^{\prime}(t)+q(t)[x(\sigma(t))]^{\alpha}=0, t \geq t_{0} \tag{2.1}
\end{equation*}
$$

where $q \in C\left(\left[t_{0}, \infty\right), \mathbb{R}^{+}\right), \sigma \in C\left(\left[t_{0}, \infty\right), \mathbb{R}\right), \sigma(t)<t, \lim _{t \rightarrow \infty} \sigma(t)=\infty$ and $\alpha \in(0, \infty)$ is a ratio of odd positive integers.

Let $\alpha \in(0,1)$. Then it is shown that every solution of the sublinear equation (2.1) oscillates if and only if

$$
\begin{equation*}
\int_{t_{0}}^{\infty} q(s) d s=\infty \tag{2.2}
\end{equation*}
$$

Let $\alpha=1$. Then equation (2.1) reduces to the linear delay differential equation

$$
\begin{equation*}
x^{\prime}(t)+q(t) x(\sigma(t))=0, t \geq t_{0} \tag{2.3}
\end{equation*}
$$

and it is shown that every solution of equation (2.3) oscillates if

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{\sigma(t)}^{t} q(s) d s>\frac{1}{e} \tag{2.4}
\end{equation*}
$$

Let $\alpha \in(1, \infty)$ and $\sigma(t)=t-\sigma$. Then equation (2.3) reduces to

$$
\begin{equation*}
x^{\prime}(t)+q(t) x^{\alpha}(t-\sigma)=0, t \geq t_{0} \tag{2.5}
\end{equation*}
$$

for which the following results was obtained: If there exists $\lambda \in\left(\sigma^{-1} \ln \alpha, \infty\right)$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} q(t) \exp \left(e^{-\lambda t}\right)>0 \tag{2.6}
\end{equation*}
$$

then every solution of equation (2.5) oscillates.

## 3. Oscillation Results

In this section, we state and prove our main results. Define for all $t \geq t_{0}$,

$$
R(t)=P(t)+Q(t)
$$

where

$$
P(t)=\min \left\{p(t), p\left(t-\tau_{1}\right), p\left(t+\tau_{2}\right)\right\}
$$

and

$$
\begin{equation*}
Q(t)=\min \left\{q(t), q\left(t-\tau_{1}\right), q\left(t+\tau_{2}\right)\right\} \tag{3.1}
\end{equation*}
$$

Theorem 3.1. Assume that condition (1.2) holds and $1 \leq \alpha \leq \beta$. If

$$
\begin{equation*}
\int_{t_{0}}^{\infty} R(t) d t=\infty \tag{3.2}
\end{equation*}
$$

then every solution of equation (1.1) is oscillatory.
Proof. Suppose, on the contrary, $x$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a constant $t_{1} \geq t_{0}$, such that $x(t)>0$, for all $t \geq t_{1}$. From the definition of $z$, we have $z(t)>0$ for all $t \geq t_{1}$. From the equation (1.1), we obtain

$$
\left(a(t) z^{(n-1)}(t)\right)^{\prime}=-\left(p(t) x^{\alpha}\left(t-\sigma_{1}\right)+q(t) x^{\beta}\left(t+\sigma_{2}\right)\right)<0, t \geq t_{1}
$$

Therefore, by Lemma $2.3 a(t) z^{(n-1)}(t)$ is a positive decreasing function. Furthermore, we have

$$
\begin{gather*}
\left(a(t) z^{(n-1)}(t)\right)^{\prime}+p(t) x^{\alpha}\left(t-\sigma_{1}\right)+q(t) x^{\beta}\left(t+\sigma_{2}\right)=0,  \tag{3.3}\\
b^{\alpha}\left(a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right)^{\prime}+b^{\alpha} p\left(t-\tau_{1}\right) x^{\alpha}\left(t-\tau_{1}-\sigma_{1}\right)+b^{\alpha} q\left(t-\tau_{1}\right) x^{\beta}\left(t-\tau_{1}+\sigma_{2}\right)=0, \tag{3.4}
\end{gather*}
$$

and

$$
\begin{equation*}
c^{\alpha}\left(a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right)^{\prime}+c^{\alpha} p\left(t+\tau_{2}\right) x^{\alpha}\left(t+\tau_{2}-\sigma_{1}\right)+c^{\alpha} q\left(t+\tau_{2}\right) x^{\beta}\left(t+\tau_{2}+\sigma_{2}\right)=0 \tag{3.5}
\end{equation*}
$$

Combining (3.3), (3.4), (3.5) and using Lemma 2.4 and (3.1) we obtain for $t \geq t_{1}$,

$$
\begin{align*}
& \left(a(t) z^{(n-1)}(t)\right)^{\prime}+b^{\alpha}\left(a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right)^{\prime}+c^{\alpha}\left(a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right)^{\prime} \\
& +P(t) \frac{1}{4^{\alpha-1}} z^{\alpha}\left(t-\sigma_{1}\right)+Q(t) \frac{1}{4^{\alpha-1}} z^{\alpha}\left(t+\sigma_{2}\right) \leq 0, t \geq t_{1} \tag{3.6}
\end{align*}
$$

But $z(t)>0$ and increasing, we have

$$
\begin{align*}
& \left(a(t) z^{(n-1)}(t)\right)^{\prime}+b^{\alpha}\left(a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right)^{\prime}+c^{\alpha}\left(a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right)^{\prime} \\
& +\frac{1}{4^{\alpha-1}} R(t) z^{\alpha}\left(t-\sigma_{1}\right) \leq 0, t \geq t_{1} \tag{3.7}
\end{align*}
$$

Integrating (3.7) from $t_{1}$ to $t$, we have

$$
\begin{aligned}
& \int_{t_{1}}^{t}\left(a(s) z^{(n-1)}(s)\right)^{\prime} d s+\int_{t_{1}}^{t} b^{\alpha}\left(a\left(s-\tau_{1}\right) z^{(n-1)}\left(s-\tau_{1}\right)\right)^{\prime} d s \\
& +\int_{t_{1}}^{t} c^{\alpha}\left(a\left(s+\tau_{2}\right) z^{(n-1)}\left(s+\tau_{2}\right)\right)^{\prime} d s+\int_{t_{1}}^{t} \frac{1}{4^{\alpha-1}} R(s) z^{\alpha}\left(s-\sigma_{1}\right) d s \leq 0, t \geq t_{1}
\end{aligned}
$$

again we get

$$
\begin{align*}
& \frac{1}{4^{\alpha-1}} \int_{t_{1}}^{t} R(s) z^{\alpha}\left(s-\sigma_{1}\right) d s \leq-\int_{t_{1}}^{t}\left(a(s) z^{(n-1)}(s)\right)^{\prime} d s \\
& -b^{\alpha} \int_{t_{1}}^{t}\left(a\left(s-\tau_{1}\right) z^{(n-1)}\left(s-\tau_{1}\right)\right)^{\prime} d s-c^{\alpha} \int_{t_{1}}^{t}\left(a\left(s+\tau_{2}\right) z^{(n-1)}\left(s+\tau_{2}\right)\right)^{\prime} d s \\
& \leq a\left(t_{1}\right) z^{(n-1)}\left(t_{1}\right)-a(t) z^{(n-1)}(t) \\
& +b^{\alpha}\left(a\left(t_{1}-\tau_{1}\right) z^{(n-1)}\left(t_{1}-\tau_{1}\right)-a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right) \\
& +c^{\alpha}\left(a\left(t_{1}+\tau_{2}\right) z^{(n-1)}\left(t_{1}+\tau_{2}\right)-a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right) \tag{3.8}
\end{align*}
$$

Since $z^{\prime}(t)>0$ for $t \geq t_{1}$, we can find a constant $M>0$ such that $z\left(t-\sigma_{1}\right) \geq M$,for all $t \geq t_{1}$. Then from (3.8) and the fact that $a(t) z^{(n-1)}(t)$ is positive, we obtain

$$
\int_{t_{1}}^{\infty} R(s) d s<\infty
$$

which is in contradiction with (3.2). This completes the proof.
Theorem 3.2. Assume that condition (1.2) holds. Further assume that $\alpha=1$ and $\sigma_{1}>\tau_{1}$. If either

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t-\left(\sigma_{1}-\tau_{1}\right)}^{t} \frac{R(s)\left(s-\sigma_{1}\right)^{n-1}}{a\left(s-\sigma_{1}\right)} d s>\frac{(1+b+c)(n-1)!}{\lambda_{0} e} \tag{3.9}
\end{equation*}
$$

or

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{t-\left(\sigma_{1}-\tau_{1}\right)}^{t} \frac{R(s)\left(s-\sigma_{1}\right)^{n-1}}{a\left(s-\sigma_{1}\right)} d s>\frac{(1+b+c)(n-1)!}{\lambda_{0}} \tag{3.10}
\end{equation*}
$$

for some $\lambda_{0} \in(0,1)$, then every solution of equation (1.1) is oscillatory.
Proof. Suppose, on the contrary, $x$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a constant $t_{1} \geq t_{0}$, such that $x(t)>0$, for all $t \geq t_{1}$. Proceeding as in the proof of Theorem 3.1 we have (3.7). By Lemma 2.2 and (3.7), for every $\lambda, 0<\lambda<1$, we obtain

$$
\begin{aligned}
& \left(a(t) z^{(n-1)}(t)\right)^{\prime}+b^{\alpha}\left(a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right)^{\prime}+c^{\alpha}\left(a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right)^{\prime} \\
& +\frac{R(t)}{4^{\alpha-1}}\left(\frac{\lambda}{(n-1)!}\left(t-\sigma_{1}\right)^{n-1} z^{(n-1)}\left(t-\sigma_{1}\right)\right)^{\alpha} \leq 0, t \geq t_{1}
\end{aligned}
$$

Let $y(t)=a(t) z^{(n-1)}(t)>0$. Then for all $t$ large enough, we have

$$
\begin{align*}
& \left(y(t)+b^{\alpha} y\left(t-\tau_{1}\right)+c^{\alpha} y\left(t+\tau_{2}\right)\right)^{\prime} \\
& +\frac{R(t)}{4^{\alpha-1} a^{\alpha}\left(t-\sigma_{1}\right)}\left(\frac{\lambda}{(n-1)!}\left(t-\sigma_{1}\right)^{n-1}\right)^{\alpha} y^{\alpha}\left(t-\sigma_{1}\right) \leq 0, t \geq t_{1} . \tag{3.11}
\end{align*}
$$

Next, set

$$
w(t)=y(t)+b^{\alpha} y\left(t-\tau_{1}\right)+c^{\alpha} y\left(t+\tau_{2}\right)
$$

Since $y$ is decreasing, it follows that

$$
\begin{equation*}
w(t) \leq\left(1+b^{\alpha}+c^{\alpha}\right) y\left(t-\tau_{1}\right), t \geq t_{1} \tag{3.12}
\end{equation*}
$$

Combining (3.11) and (3.12), we get
$w^{\prime}(t)+\frac{R(t)}{4^{\alpha-1}\left(1+b^{\alpha}+c^{\alpha}\right)^{\alpha} a^{\alpha}\left(t-\sigma_{1}\right)}\left(\frac{\lambda}{(n-1)!}\left(t-\sigma_{1}\right)^{n-1}\right)^{\alpha} w^{\alpha}\left(t-\sigma_{1}+\tau_{1}\right) \leq 0$.
Hence for $\alpha=1$, we have

$$
\begin{equation*}
w^{\prime}(t)+\frac{R(t)}{(1+b+c) a\left(t-\sigma_{1}\right)}\left(\frac{\lambda}{(n-1)!}\left(t-\sigma_{1}\right)^{n-1}\right) w\left(t-\sigma_{1}+\tau_{1}\right) \leq 0 \tag{3.14}
\end{equation*}
$$

Therefore, $w$ is a positive solution of (3.14). Now, we consider the following two cases, depending on whether (3.9) or (3.10) holds.
Case (i):It is easy to see that if (3.9) holds, then we can choose a constant $0<$ $\lambda_{0}<1$, such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t-\left(\sigma_{1}-\tau_{1}\right)}^{t} \frac{R(s)\left(s-\sigma_{1}\right)^{n-1} \lambda}{a\left(s-\sigma_{1}\right)(n-1)!(1+b+c)} d s>\frac{1}{e} \tag{3.15}
\end{equation*}
$$

But according to the Lemma 2.5, (3.15) guarantees that (3.14) has no positive solution, which is a contradiction.
Case (ii): Using the definition of $w$ and (3.7), we obtain

$$
\begin{align*}
& w^{\prime}(t)=y^{\prime}(t)+b^{\alpha} y^{\prime}\left(t-\tau_{1}\right)+c^{\alpha} y^{\prime}\left(t+\tau_{2}\right) \\
& \leq\left(a(t) z^{(n-1)}(t)\right)^{\prime}+b^{\alpha}\left(a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right)^{\prime}+c^{\alpha}\left(a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right)^{\prime} \\
& \leq-\frac{1}{4^{\alpha-1}} R(t) z^{\alpha}\left(t-\sigma_{1}\right) \leq 0, t \geq t_{1} \tag{3.16}
\end{align*}
$$

Noting that $\alpha=1$ and $\sigma_{1} \geq \tau_{1}$, there exists $t_{2} \geq t_{1}$, such that

$$
\begin{equation*}
w\left(t-\sigma_{1}+\tau_{1}\right) \geq w(t), t \geq t_{2} \tag{3.17}
\end{equation*}
$$

Integrating (3.14) from $t-\sigma_{1}+\tau_{1}$ to $t$, we have
$w(t)-w\left(t-\sigma_{1}+\tau_{1}\right)+\frac{\lambda}{(1+b+c)(n-1)!} \int_{t-\sigma_{1}+\tau_{1}}^{t} \frac{\left(s-\sigma_{1}\right)^{n-1} R(s)}{a\left(s-\sigma_{1}\right)} w\left(s-\sigma_{1}+\tau_{1}\right) d s \leq 0$, wheret $\geq t_{2}$.Thus
$w(t)-w\left(t-\sigma_{1}+\tau_{1}\right)+\frac{\lambda}{(1+b+c)(n-1)!} w\left(t-\sigma_{1}+\tau_{1}\right) \int_{t-\sigma_{1}+\tau_{1}}^{t} \frac{\left(s-\sigma_{1}\right)^{n-1} R(s)}{a\left(s-\sigma_{1}\right)} d s \leq 0$, where $t \geq t_{2}$. From the above inequality, we obtain

$$
\frac{w(t)}{w\left(t-\sigma_{1}+\tau_{1}\right)}-1+\frac{\lambda}{(1+b+c)(n-1)!} \int_{t-\sigma_{1}+\tau_{1}}^{t} \frac{\left(s-\sigma_{1}\right)^{n-1} R(s)}{a\left(s-\sigma_{1}\right)} d s \leq 0, t \geq t_{2}
$$

Hence from (3.17), we have

$$
\begin{equation*}
\frac{\lambda}{(1+b+c)(n-1)!} \int_{t-\sigma_{1}+\tau_{1}}^{t} \frac{\left(s-\sigma_{1}\right)^{n-1} R(s)}{a\left(s-\sigma_{1}\right)} d s \leq 1, t \geq t_{2} \tag{3.18}
\end{equation*}
$$

Taking the sup limit as $t \rightarrow \infty$ in (3.18), we get

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{t-\left(\sigma_{1}-\tau_{1}\right)}^{t} \frac{R(s)\left(s-\sigma_{1}\right)^{n-1}}{a\left(s-\sigma_{1}\right)} d s \leq \frac{(1+b+c)(n-1)!}{\lambda} \tag{3.19}
\end{equation*}
$$

If (3.10) holds, we can choose a constant $0<\lambda_{0}<1$, such that

$$
\limsup _{t \rightarrow \infty} \int_{t-\left(\sigma_{1}-\tau_{1}\right)}^{t} \frac{R(s)\left(s-\sigma_{1}\right)^{n-1}}{a\left(s-\sigma_{1}\right)} d s>\frac{(1+b+c)(n-1)!}{\lambda}
$$

which is in contradiction with (3.18). This completes the proof.
Theorem 3.3. Assume that condition (1.2) holds and $1 \leq \beta \leq \alpha$. If

$$
\begin{equation*}
\int_{t_{0}}^{\infty} R(t) d t=\infty \tag{3.20}
\end{equation*}
$$

then every solution of equation (1.1) is oscillatory.
Proof. The proof is similar to that of Theorem3.1 and hence the details are omitted.

Theorem 3.4. Assume that condition (1.2) holds. Further assume that $\beta=1$ and $\sigma_{1}>\tau_{1}$. If either

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \int_{t-\left(\sigma_{1}-\tau_{1}\right)}^{t} \frac{R(s)\left(s-\sigma_{1}\right)^{n-1}}{a\left(s-\sigma_{1}\right)} d s>\frac{(1+b+c)(n-1)!}{\lambda_{0} e} \tag{3.21}
\end{equation*}
$$

or

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{t-\left(\sigma_{1}-\tau_{1}\right)}^{t} \frac{R(s)\left(s-\sigma_{1}\right)^{n-1}}{a\left(s-\sigma_{1}\right)} d s>\frac{(1+b+c)(n-1)!}{\lambda_{0}} \tag{3.22}
\end{equation*}
$$

for some $\lambda_{0} \in(0,1)$, then every solution of equation (1.1) is oscillatory.
Proof. The proof is similar to that of Theorem 3.2 and hence the details are omitted.

Corollary 3.5. Assume that condition (1.2) holds, $\sigma_{1}-\tau_{1}>0$ and $\alpha \in(1, \infty)$. If there exists $\mu \in\left(\left(\sigma_{1}-\tau_{1}\right)^{-1} \ln \alpha, \infty\right)$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} R(t)\left(\frac{\left(t-\sigma_{1}\right)^{n-1}}{a\left(t-\sigma_{1}\right)(n-1)!}\right)^{\alpha} \exp \left(e^{-\mu t}\right)>0 \tag{3.23}
\end{equation*}
$$

then every solution of equation (1.1) is oscillatory.
Proof. According to Lemma 2.5, the condition (3.23) guarantees that (3.13) with $\alpha>1$ has no positive solution. Hence by Theorem 3.2, every solution of equation (1.1) is oscillatory. This completes the proof.

Corollary 3.6. Assume that condition (1.2) holds, $\sigma_{1}-\tau_{1}>0$ and $\beta \in(1, \infty)$. If there exists $\nu \in\left(\left(\sigma_{1}-\tau_{1}\right)^{-1} \ln \beta, \infty\right)$ such that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} R(t)\left(\frac{\left(t-\sigma_{1}\right)^{n-1}}{a\left(t-\sigma_{1}\right)(n-1)!}\right)^{\beta} \exp \left(e^{-\nu t}\right)>0 \tag{3.24}
\end{equation*}
$$

then every solution of equation (1.1) is oscillatory.
Proof. According to Lemma 2.5, the condition (3.24) guarantees that (3.13) with $\beta>1$ has no positive solution. Hence by Theorem 3.2, every solution of equation (1.1) is oscillatory. This completes the proof.

Theorem 3.7. Assume that condition (1.3) holds and $\sigma_{1}-\tau_{1}>0$. Suppose, further that the first order differential equation
$w^{\prime}(t)+\frac{R(t)}{4^{\alpha-1}\left(1+b^{\alpha}+c^{\alpha}\right)^{\alpha} a^{\alpha}\left(t-\sigma_{1}\right)}\left(\frac{\lambda_{0}}{(n-1)!}\left(t-\sigma_{1}\right)^{n-1}\right)^{\alpha} w^{\alpha}\left(t-\sigma_{1}+\tau_{1}\right)=0$
is oscillatory for some constant $\lambda_{0} \in(0,1)$. If

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \int_{t_{1}}^{t}\left[k_{1} R(s)\left(\frac{1}{(n-2)!}\left(s-\sigma_{1}\right)^{n-2}\right)^{\alpha} \delta(s)-\frac{\left(1+b^{\alpha}+c^{\alpha}\right)}{4 a\left(s+\tau_{2}\right) \delta(s)}\right] d s=\infty \tag{3.26}
\end{equation*}
$$

for all constants $k_{1}>0$, then equation (1.1) is almost oscillatory.
Proof. Suppose, on the contrary, $x$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a constant $t_{1} \geq t_{0}$, such that $x(t)>0$, for all $t \geq t_{1}$. It follows from the equation (1.1) and Kiguradze's Lemma 2.1 that there exist three possible cases:
(i) $z(t)>0, z^{\prime}(t)>0, z^{(n-1)}(t)>0, z^{(n)}(t) \leq 0$;
(ii) $z(t)>0, z^{\prime}(t)>0, z^{(n-2)}(t)>0, z^{(n-1)}(t)<0$;
(iii) $z(t)>0, z^{\prime}(t)<0, z^{(n-2)}(t)>0, z^{(n-1)}(t)<0 ;$
for $t \geq t_{2} \geq t_{1}, t_{1}$ is sufficiently large. Assume that case(i) holds. From the proof of Theorem 3.2 we get
$w^{\prime}(t)+\frac{R(t)}{4^{\alpha-1}\left(1+b^{\alpha}+c^{\alpha}\right)^{\alpha} a^{\alpha}\left(t-\sigma_{1}\right)}\left(\frac{\lambda_{0}}{(n-1)!}\left(t-\sigma_{1}\right)^{n-1}\right)^{\alpha} w^{\alpha}\left(t-\sigma_{1}+\tau_{1}\right) \leq 0$.

By [[12],Corollary 3.2.2], $w$ is a positive solution of
$w^{\prime}(t)+\frac{R(t)}{4^{\alpha-1}\left(1+b^{\alpha}+c^{\alpha}\right)^{\alpha} a^{\alpha}\left(t-\sigma_{1}\right)}\left(\frac{\lambda_{0}}{(n-1)!}\left(t-\sigma_{1}\right)^{n-1}\right)^{\alpha} w^{\alpha}\left(t-\sigma_{1}+\tau_{1}\right)=0$
for every $\lambda \in(0,1)$, which contradicts the fact that (3.25) is oscillatory.
Assume that case (ii) holds. Define the function $u$ by

$$
\begin{equation*}
u(t)=\frac{a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)}{z^{(n-2)}(t)}, t \geq t_{2} \tag{3.27}
\end{equation*}
$$

Clearly, $u(t)<0$ for $t \geq t_{2}$. Noting that $a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)$ is decreasing, we obtain

$$
\begin{equation*}
a\left(s+\tau_{2}\right) z^{(n-1)}\left(s+\tau_{2}\right) \leq a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right), s \geq t \geq t_{2} \tag{3.28}
\end{equation*}
$$

Dividing (3.28) by $a\left(s+\tau_{2}\right)$ and integrating it from $t$ to $l(l \geq t)$, we have

$$
z^{(n-2)}\left(l+\tau_{2}\right) \leq z^{(n-2)}\left(t+\tau_{2}\right)+a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right) \int_{t}^{l} \frac{d u}{a(u)}
$$

Letting $l \rightarrow \infty$, we get

$$
0 \leq z^{(n-2)}(t)+a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right) \delta(t)
$$

that is

$$
-1 \leq \frac{a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right) \delta(t)}{z^{(n-2)}(t)}
$$

Therefore, from (3.28), we obtain

$$
\begin{equation*}
-1 \leq u(t) \delta(t) \leq 0, t \geq t_{2} \tag{3.29}
\end{equation*}
$$

Next, we define the function $w$ as

$$
\begin{equation*}
w(t)=\frac{a(t) z^{(n-1)}(t)}{z^{(n-2)}(t)}, t \geq t_{2} \tag{3.30}
\end{equation*}
$$

Clearly, $w(t)<0$ for $t \geq t_{2}$. Noting that $a(t) z^{(n-1)}(t)$ is decreasing, we have

$$
a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right) \leq a(t) z^{(n-1)}(t)
$$

then $u(t) \leq w(t)$. Thus, by (3.30), we get

$$
\begin{equation*}
-1 \leq w(t) \delta(t) \leq 0, t \geq t_{2} \tag{3.31}
\end{equation*}
$$

Next, define the function $v$ as

$$
\begin{equation*}
v(t)=\frac{a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)}{z^{(n-2)}(t)}, t \geq t_{2} \tag{3.32}
\end{equation*}
$$

Clearly, $v(t)<0$ for $t \geq t_{2}$. Noting that $a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)$ is decreasing, we have

$$
a(t) z^{(n-1)}(t) \leq a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)
$$

then $u(t) \leq w(t) \leq v(t)$. Thus, by (3.32), we get

$$
\begin{equation*}
-1 \leq v(t) \delta(t) \leq 0, t \geq t_{2} \tag{3.33}
\end{equation*}
$$

Differentiating (3.27), we obtain

$$
\begin{equation*}
u^{\prime}(t) \leq \frac{\left(a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right)^{\prime}}{z^{(n-2)}(t)}-\frac{u^{2}(t)}{a\left(t+\tau_{2}\right)} \tag{3.34}
\end{equation*}
$$

Differentiating (3.30) and from (3.28), we obtain

$$
\begin{equation*}
w^{\prime}(t) \leq \frac{\left(a(t) z^{(n-1)}(t)\right)^{\prime}}{z^{(n-2)}(t)}-\frac{w^{2}(t)}{a\left(t+\tau_{2}\right)} \tag{3.35}
\end{equation*}
$$

Differentiating (3.32) and from (3.28), we obtain

$$
\begin{equation*}
v^{\prime}(t) \leq \frac{\left(a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right)^{\prime}}{z^{(n-2)}(t)}-\frac{v^{2}(t)}{a\left(t+\tau_{2}\right)} \tag{3.36}
\end{equation*}
$$

Combining (3.34), (3.35) and (3.36), we get

$$
\begin{align*}
& w^{\prime}(t)+b^{\alpha} v^{\prime}(t)+c^{\alpha} u^{\prime}(t) \leq \frac{1}{z^{(n-2)}(t)}\left(a(t) z^{(n-1)}(t)\right)^{\prime}+b^{\alpha}\left(a\left(t-\tau_{1}\right) z^{(n-1)}\left(t-\tau_{1}\right)\right)^{\prime} \\
& +c^{\alpha}\left(a\left(t+\tau_{2}\right) z^{(n-1)}\left(t+\tau_{2}\right)\right)^{\prime}-\frac{1}{a\left(t+\tau_{2}\right)}\left(w^{2}(t)+b^{\alpha} v^{2}(t)+c^{\alpha} u^{2}(t)\right) \tag{3.37}
\end{align*}
$$

Therefore, by (3.7) and (3.37), we obtain

$$
\begin{align*}
& w^{\prime}(t)+b^{\alpha} v^{\prime}(t)+c^{\alpha} u^{\prime}(t) \leq-\frac{z^{\alpha}\left(t-\sigma_{1}\right)}{z^{(n-2)}(t) 4^{\alpha-1}} R(t) \\
& -\frac{1}{a\left(t+\tau_{2}\right)}\left(w^{2}(t)+b^{\alpha} v^{2}(t)+c^{\alpha} u^{2}(t)\right) . \tag{3.38}
\end{align*}
$$

On the other hand, by Lemma 2.2, we get

$$
\begin{equation*}
z(t) \geq \frac{\lambda}{(n-2)!} t^{n-2} z^{(n-2)}(t) \tag{3.39}
\end{equation*}
$$

for every $\lambda \in(0,1)$ and for all sufficiently large $t$. Then there exists a constant $M>0$ such that

$$
\begin{aligned}
& w^{\prime}(t)+b^{\alpha} v^{\prime}(t)+c^{\alpha} u^{\prime}(t) \leq-\frac{R(t)}{4^{\alpha-1}} \frac{z^{\alpha}\left(t-\sigma_{1}\right)}{\left(z^{(n-2)}\left(t-\sigma_{1}\right)\right)^{\alpha}}\left(z^{(n-2)}\left(t-\sigma_{1}\right)\right)^{\alpha-1} \frac{z^{(n-2)}\left(t-\sigma_{1}\right)}{z^{(n-2)}(t)} \\
& -\frac{1}{a\left(t+\tau_{2}\right)}\left(w^{2}(t)+b^{\alpha} v^{2}(t)+c^{\alpha} u^{2}(t)\right) \\
& \leq-\left(\frac{M}{4}\right)^{\alpha-1} R(t)\left(\frac{\lambda}{(n-2)!}\left(t-\sigma_{1}\right)^{n-2}\right)^{\alpha}-\frac{1}{a\left(t+\tau_{2}\right)}\left(w^{2}(t)+b^{\alpha} v^{2}(t)+c^{\alpha} u^{2}(t)\right)
\end{aligned}
$$

Multiplying the above inequality by $\delta(t)$ and integrating from $t_{2}$ to $t$, we obtain

$$
\begin{align*}
& \delta(t) w(t)-\delta\left(t_{2}\right) w\left(t_{2}\right)+\int_{t_{2}}^{t} \frac{w(s)}{a\left(s+\tau_{2}\right)} d s+\int_{t_{2}}^{t} \frac{w^{2}(s) \delta(s)}{a\left(s+\tau_{2}\right)} d s \\
& +b^{\alpha}\left(\delta(t) v(t)-\delta\left(t_{2}\right) v\left(t_{2}\right)+\int_{t_{2}}^{t} \frac{v(s)}{a\left(s+\tau_{2}\right)} d s+\int_{t_{2}}^{t} \frac{v^{2}(s) \delta(s)}{a\left(s+\tau_{2}\right)} d s\right) \\
& +c^{\alpha}\left(\delta(t) u(t)-\delta\left(t_{2}\right) u\left(t_{2}\right)+\int_{t_{2}}^{t} \frac{u(s)}{a\left(s+\tau_{2}\right)} d s+\int_{t_{2}}^{t} \frac{u^{2}(s) \delta(s)}{a\left(s+\tau_{2}\right)} d s\right) \\
& +\left(\frac{M}{4}\right)^{\alpha-1} \int_{t_{2}}^{t} R(s)\left(\frac{\lambda}{(n-2)!}\left(s-\sigma_{1}\right)^{n-2}\right)^{\alpha} \delta(s) d s \leq 0 . \tag{3.40}
\end{align*}
$$

It follows from (3.40), taking into account that $-1 \leq w(t) \delta(t) \leq 0,-1 \leq v(t) \delta(t) \leq$ 0 and $-1 \leq u(t) \delta(t) \leq 0$,

$$
\begin{aligned}
& \delta(t) w(t)-\delta\left(t_{2}\right) w\left(t_{2}\right)+b^{\alpha}\left(\delta(t) v(t)-\delta\left(t_{2}\right) v\left(t_{2}\right)\right) \\
& +c^{\alpha}\left(\delta(t) u(t)-\delta\left(t_{2}\right) u\left(t_{2}\right)\right)+\left(\frac{M}{4}\right)^{\alpha-1}\left(\frac{\lambda}{(n-2)!}\right)^{\alpha} \int_{t_{2}}^{t} \delta(s) R(s)\left(s-\sigma_{1}\right)^{\alpha(n-2)} d s \\
& -\frac{1+b^{\alpha}+c^{\alpha}}{4} \int_{t_{2}}^{t} \frac{1}{a\left(s+\tau_{2}\right) \delta(s)} \leq 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \delta(t) w(t)+b^{\alpha} \delta(t) v(t)+c^{\alpha} \delta(t) u(t) \\
& +\int_{t_{2}}^{t}\left[k_{1}\left(\frac{1}{(n-2)!}\left(s-\sigma_{1}\right)^{n-2}\right)^{\alpha} R(s) \delta(s)-\frac{1+b^{\alpha}+c^{\alpha}}{4} \frac{1}{a\left(s+\tau_{2}\right) \delta(s)}\right] d s \\
& \leq \delta\left(t_{2}\right) w\left(t_{2}\right)+b^{\alpha} \delta\left(t_{2}\right) v\left(t_{2}\right)+c^{\alpha} \delta\left(t_{2}\right) u\left(t_{2}\right)
\end{aligned}
$$

From (3.26) and the above inequality, we get a contradiction to (3.29), (3.31) and (3.33).

Assume that case(iii) holds. Similar to the proof of that of [4],Lemma2], there exists a constant $k>0$ such that

$$
\begin{equation*}
x(t) \geq k z(t) \tag{3.41}
\end{equation*}
$$

The conclusion of the proof is similar to that of case (ii) and we can obtain the contradiction to (3.26), and so is omitted. This completes the proof.

## 4. Examples

In this section we present some examples to illustrate the main results.
Example 4.1. Consider the even order nonlinear mixed type differential equation

$$
\begin{equation*}
(x(t)+x(t-\pi)+4 x(t+2 \pi))^{(i v)}+2 x(t-3 \pi)+2 x(t+\pi)=0, t \geq 0 \tag{4.1}
\end{equation*}
$$

Here $a(t)=1, p(t)=q(t)=2, b=1, c=4, \tau_{1}=\pi, \tau_{2}=2 \pi, \sigma_{1}=3 \pi, \sigma_{2}=\pi$ and $\alpha=\beta=1$. It satisfies all the conditions of the Theorem 3.2. Hence, every solution of equation (4.1) oscillates. For, example, $x(t)=\sin t$ is an oscillatory solution of equation 4.1.
Example 4.2. Consider the even order nonlinear mixed type differential equation

$$
\begin{equation*}
\left(2(x(t)+x(t-\pi)+x(t+\pi))^{\prime \prime \prime}\right)^{\prime}+x(t-2 \pi)+x(t+\pi)=0, t \geq 0 \tag{4.2}
\end{equation*}
$$

Here $a(t)=2, p(t)=q(t)=1, b=1, c=1, \tau_{1}=\pi, \tau_{2}=\pi, \sigma_{1}=2 \pi, \sigma_{2}=2 \pi$ and $\alpha=\beta=1$. It satisfies all the conditions of the Theorem 3.4. Hence, every solution of equation (4.2) oscillates. For, example, $x(t)=\sin t$ is an oscillatory solution of equation (4.2).

Example 4.3. Consider the fourth-order differential equation

$$
\begin{align*}
& \left(\frac{1}{e^{2 t}}(x(t)+x(t-1)+x(t+1))^{\prime \prime \prime}\right)^{\prime}+\frac{1}{e^{4 t}}\left(e^{6}+e^{7}\right) x^{3}(t-2) \\
& +\frac{1}{e^{4 t+10}} x^{3}(t+3)=0 \tag{4.3}
\end{align*}
$$

where $t \geq 0$. Here $a(t)=1 / e^{2 t}, p(t)=\frac{1}{e^{4 t}}, q(t)=\frac{1}{e^{4 t+10}}, b=c=1, \tau_{1}=\tau_{2}=1, \sigma_{1}=$ $2, \sigma_{2}=3$ and $\alpha=\beta=3$. Then one can see that all conditions of Theorem 3.4 are satisfied except the condition (3.2). Therefore all the solutions of equation (4.3) not necessarily oscillatory. In fact $x(t)=e^{t}$ is an oscillatory solution of equation (4.3).

Example 4.4. Consider the fourth-order differential equation

$$
\begin{equation*}
\left(e^{t} z^{\prime \prime \prime}(t)\right)^{\prime}+\left(\frac{e^{t-1 / 2}+e^{t-1}}{16}\right) x(t-2)+\frac{e^{t}}{16} x(t+1)=0, t \geq 2 \tag{4.4}
\end{equation*}
$$

where $z(t)=x(t)+x(t-1)+x(t+1)$. We can see that all conditions of Theorem 3.2 satisfied except the condition (1.2). Therefore all the solutions of equation (4.4) not necessarily oscillatory. In fact $x(t)=e^{-t / 2}$ is one such nonoscillatory solution, since it satisfies the equation (4.4).
Example 4.5. Consider the fourth-order differential equation

$$
\begin{equation*}
\left(\frac{1}{e^{2 t}}\left(x(t)+\frac{1}{3} x(t-1)+\frac{1}{3} x(t+1)\right)^{\prime \prime \prime}\right)^{\prime}+\frac{1}{e^{2 t}}\left(e^{2}+\frac{1}{3} e\right) x(t-2)+\frac{1}{3 e^{2 t}} x(t+1)=0, t \geq 0 \tag{4.5}
\end{equation*}
$$

Here $a(t)=1 / e^{2 t}, p(t)=\frac{1}{e^{2 t}}\left(e^{2}+\frac{1}{3} e\right), q(t)=\frac{1}{3 e^{2 t}}, b=c=1 / 3, \tau_{1}=1, \tau_{2}=1, \sigma_{1}=$ $2, \sigma_{2}=1$ and $\alpha=\beta=1$. Then one can see that all conditions of Theorem 3.4 are satisfied except the condition (3.2). Therefore all the solutions of equation (4.5) not necessarily oscillatory. In fact $x(t)=e^{t}$ is one such nonoscillatory solution, since it satisfies the equation (4.5).

## Acknowledgements

The authors thank the referees for his/her suggestions and corrections which improved the content of the paper.

## References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic, Dordrecht, 2000.
[2] R. P. Agarwal, S. R. Grace, D.O'Regan, Oscillation criteria for certain nth order differential equations with deviating arguments, J. Math. Anal. Appl. 262 (2001), 601-622.
[3] T. Asaela, H. YoshidaH, Stability, instability and complex behavior in macrodynamic models with policy lag, Discrete Dynamics in Nature and Society, 5, (2001), 281-295.
[4] B. Baculíková, J. Džurina, Oscillation of third order neutral differential equations, Math.Comput. Modelling, 52, (2010), 215-226.
[5] L. Berenzansky, E. Braverman, Some oscillation problems for a second order linear delay differential equations, J. Math. Anal. Appl. 220,(1998), 719-740.
[6] D. M. DuboisX, Extension of the Kaldor-Kalecki models of business cycle with a computational anticipated capital stock, Journal of Organisational Transformation and Social Change, 1, (2004), 63-80.
[7] L. H. Erbe, Qingkai Kong, B.G.Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.
[8] J. M. Ferreira, S. Pinelas, Oscillatory mixed difference systems, Hindawi publishing corporation, Advanced in Difference Equations ID (2006), 1-18.
[9] R. Frish and H. Holme, The Characteristic solutions of mixed difference and differential equation occuring in economic dynamics, Econometrica, 3, (1935), 219-225.
[10] G. Gandolfo, Economic dynamics, Third Edition, Berlin Springer-verlag, 1996.
[11] S.R.Grace, On the oscillations of mixed neutral equations, J. Math. Anal. Appl.,194, (1995),377-388.
[12] I. Győri and G. Ladas, Oscillation Theory of Delay Differential Equations, Clarendon Press, New York, 1991.
[13] Z. L. Han, T. X. Li, S. R. Sun, W. S. Chen, On the oscillation of second order neutral delay differential equations, Adv. Diff. Eqn. 2010, (2010), 1-8.
[14] Z. L. Han, T. X. Li, S. R. Sun, Y. B. Sun, Remarks on the paper [Appl. Math. Comput. 207 (2009) 388-396], Appl. Math. Comput. 215, (2010), 3998-4007.
[15] V. Iakoveleva and C. J. Vanegas, On the oscillation of differential equations with delayed and advanced arguements, Elec. J. Diff. Equation, 13, (2005), 57-63.
[16] R. W. James and M. H. Belz, The significance of the characteristic solutions of mixed difference and differential equations, Econometrica, 6, (1938), 326-343.
[17] T. Kristin, Non oscillation for functional differential equations of mixed type, J.Math.Anal.Appl., 245, (2000), 326-345.
[18] G. S. Ladde, V. Lakshmikantham and B. G. Zhang, Oscillatory Theory of Differential Equations with Deviation Arguments Marcel Dekker, NewYork, 1987.
[19] G. Ladas and I. P. Stavroulakis, Oscillation caused by several retarded and advanced arguments, Journal of Differential Equation, 44, (1982), 134-152.
[20] G. Ladas and I. P. Stavroulakis, Oscillations of differential equations of mixed type, J. Math. Phys. Sci., 18, (1984), 245-262.
[21] T. Li and E. Thandapani, Oscillation of solutions to odd order nonlinear neutral functional differential equations, Elec.J. Diff. Eqns., 23, (2011), 1-12.
[22] F. W. Meng, R. Xu, Oscillation criteria for certain even order quasilinear neutral differential equations with deviating arguments, Appl. Math. Comput. 190, (2007), 458-464.
[23] Ch. G. Philos, A new criteria for the oscillatory and asymptotic behavior of delay differential equations, Bull. Polish Acad. Sci. Sér. Sci. Math., 39, (1981), 61-64.
[24] Y. V. Rogovchenko, Oscillation criteria for certain nonlinear differential equations, J.Math.Anal.Appl., 229, (1999), 399-416.
[25] I. P. Stavroulakis, Oscillations of mixed neutral equations, Hiroshima Math. J., 19, (1989), 441-456.
[26] Y. B. Sun, Z. L. Han, S. R. Sun and Ch. Zhang, Oscillation criteria for even order nonlinear neutral differential equations, Elec. J. Qual. Diff. Eqn. 30, (2012), 1-12.
[27] X. H. Tang, Oscillation for first order superlinear delay differential equations, J.London Math.Soc.(2), $\mathbf{6 5 ( 1 ) , ( 2 0 0 2 ) , 1 1 5 - 1 2 2 .}$
[28] E. Thandapani and R. Rama, Comparison and oscillation theorems for second order nonlinear neutral differential equations of mixed type, Seridica Math.J., 39, (2013), 1-16.
[29] C. H. Zhang, T. X. Li, B. Sun and E. Thandapani, On the oscillation of higher order half linear delay differential equations, Appl. Math. Lett. 24, (2011), 1618-1621.
[30] Q. X. Zhang, J. R. Yan and L. Gao, Oscillation behavior of even order nonlinear neutral delay differential equations with variable coefficients, Comput. Math. Appl. 59, (2010), 426430.

Ethiraju Thandapani
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai- 600 005, India.
E-mail address: ethandapani@yahoo.co.in.
Sankarappan Padmavathi
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai- 600 005, India.
E-mail address: padmavathi.sankarappan@gmail.com.
Sandra Pinelas
Academia Militar, Departamento de Ciências Exactas e Naturais, Av.
Conde Castro Guimarães, 2720-113 Amadora, Portugal.
E-mail address: sandra.pinelas@gmail.com.


[^0]:    2010 Mathematics Subject Classification. 34C15.
    Key words and phrases. Oscillation, Even order, Mixed type,Nonlinear neutral differential equations.
    © 2014 Universiteti i Prishtinës, Prishtinë, Kosovë.
    Submitted June 10, 2013. Published January 5, 2014.

