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Oscillation Criteria For Even Order Nonlinear Neutral Differential Equations With Mixed Arguments

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ABSTRACT

This paper deals with the oscillation criteria for nth order nonlinear neutral mixed type differential equations of the form

$$\left((x(t)+ax(t-\tau_1)-bx(t+\tau_2))^{\alpha}\right)^{(n)} = q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2),$$

$$\left((x(t)-ax(t-\tau_1)+bx(t+\tau_2))^{\alpha}\right)^{(n)} = q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2),$$

and

$$\left((x(t)+ax(t-\tau_1)+bx(t+\tau_2))^{\alpha}\right)^{(n)} = q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2)$$

where n is an even positive integer, a and b are nonnegative constants, τ_1, τ_2, σ_1 and σ_2 are positive real constants, $q(t), p(t) \in C([t_0,\infty),(0,\infty))$ and α, β and γ are ratios of odd positive integers with $\beta, \gamma \geq 1$. Some examples are provided to illustrate the main results.

Keywords: Oscillation; Even order; Mixed arguments; Nonlinear neutral differential equations.

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1 INTRODUCTION

In this paper, we study the oscillatory behavior of all solutions of nth order nonlinear neutral differential equations with mixed arguments of the form

$$\left((x(t)+ax(t-\tau_1)-bx(t+\tau_2))^{\alpha}\right)^{(n)}=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2), \tag{1}$$

$$\left((x(t)-ax(t-\tau_1)+bx(t+\tau_2))^{\alpha}\right)^{(n)}=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2), \tag{2}$$

and

$$\left((x(t)+ax(t-\tau_1)+bx(t+\tau_2))^{\alpha}\right)^{(n)}=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2) \tag{3}$$

where n is an even positive integer, a and b are nonnegative constants, τ_1,τ_2,σ_1 and σ_2 are positive real constants, $q(t), p(t)\in C([t_0,\infty),(0,\infty))$ and α,β and γ are ratios of odd positive integers with $\beta,\gamma\geq 1$.

As is customary, a solution is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. Equations (1.1),(1.2) and (1.3) are called oscillatory if all its solutions are oscillatory.

Differential equations with delayed and advanced arguments (also called mixed differential equations or equations with mixed arguments) occur in many problems of economy, biology and physics (see for example [2,4,8,9,14]), because differential equations with mixed arguments are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena. The concept of delay is related to a memory of system, the past events are importance for the current behavior, and the concept of advance is related to a potential future events which can be known at the current time which could be useful for decision making. The study of various problems for differential equations with mixed arguments can be seen in [3,7,13,16,19,22]. It is well known that the solutions of these types of equations cannot be obtained in closed form. In the absence of closed form solutions a rewarding alternative is to resort to the qualitative study of the solutions of these types of differential equations. But it is not quite clear how to formulate an initial value problem for such equations and existence and uniqueness of solutions becomes a complicated issue. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equation on the half line.

The problem of asymptotic and oscillatory behavior of solutions of nth order delay and neutral type differential equations has received great attention in recent years see for example [1-27], and the references cited therein. However, there are few results regarding the oscillatory properties of neutral differential equations with mixed arguments.

In[10] the authors established some oscillation criteria for the following mixed neutral equations

$$\left(x(t)+cx(t-h)-c^{*}x(t+h^{*})\right)^{(n)}=qx(t-g)+px(t+g^{*}), \tag{4}$$

and

$$\left(x(t)+cx(t-h)+c^{*}x(t+h^{*})\right)^{(n)}=qx(t-g)+px(t+g^{*}), \tag{5}$$

where n is an even positive integer, c,c^{*},h,h^{*},p and q are real numbers and g and g^{*} are positive constants.

In[5] the author established some oscillation results for the following mixed neutral equation

$$\left(x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2)\right)=q_1x(t-\sigma_1)+q_2x(t+\sigma_2),\ t\geq t_0, \tag{6}$$

with q_1 and q_2 are nonnegative real valued functions.

In[25] the authors established some oscillation theorems for the following second order mixed neutral differential equation

$$\left((x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2))^{\alpha}\right)=q_1(t)x^{\beta}(t-\sigma_1)+q_2(t)x^{\gamma}(t+\sigma_2),\ t\geq t_0, \tag{7}$$

where α,β and γ are ratios of odd positive integers, p_i,τ_i and $\sigma_i, i=1,2$ are positive constants and $q_i\in C([t_0,\infty),[0,\infty)), i=1,2.$

In[26] the authors established some oscillation criteria for the following second order mixed neutral differential equation

$$\left((x(t)+ax(t-\tau_1)-bx(t+\tau_2))^{\alpha}\right)=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2),\ t\geq t_0, \tag{8}$$

and

$$\left((x(t)-ax(t-\tau_1)+bx(t+\tau_2))^{\alpha}\right)=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2),\ t\ge t_0, \qquad (9)$$

where α and β are ratios of odd positive integers with $\beta\ge1$, $a,b,\tau_1,\tau_2,\sigma_1,\sigma_2$ are positive constants and $q(t),\,p(t)\in C([t_0,\infty),[t_0,\infty)), i=1,2.$

Clearly equations (1.4) and (1.5) with $\alpha=\beta=\gamma=1$ and $q(t)=q,\,p(t)=p$ are special cases of equations (1.1) and (1.3) and equation (1.6) with $n=2$ and $\alpha=\beta=\gamma=1, q(t)=q_1, p(t)=q_2$ is a special case of equation (1.3) respectively. Moreover equations (1.7), (1.8) and (1.9) with $n=2$ are special cases of of equations (1.1), (1.2) and (1.3) respectively. Motivated by the above observation in this paper we study the oscillatory behavior of equations (1.1),(1.2) and (1.3) for different values of $\beta\ge1$ and $\gamma\ge1$. Therefore our results generalize and extend those of [5,10,25,26].

In Section 2, we present some sufficient conditions for the oscillation of all solutions of equations (1.1),(1.2) and (1.3). Examples are provided in Section 3 to illustrate the main results.

2 SOME PRELIMINARY LEMMAS

In this section we shall obtain some sufficient conditions for the oscillation of all solutions of the equations (1.1),(1.2) and (1.3). Before proving the main results we state the following lemmas which are essential in the proofs of our oscillation theorems.

Lemma 2.1 *Let* $A\ge0, B\ge0$ *and* $\gamma\ge1.$ *Then*

$$A^{\gamma}+B^{\gamma}\ge\frac{1}{2^{\gamma-1}}(A+B)^{\gamma}.$$

If $A\ge B,$ then

$$A^{\gamma}-B^{\gamma}\ge(A-B)^{\gamma}.$$

The proof may be found in [23].

Lemma 2.2 *([21]) Let* $u\in C^n([t_0,\infty),\mathbb{R}^+).$ *If* $u^{(n)}(t)$ *is eventually of one sign for all large* $t,$ *then there exists a* $t_x>t_1,$ *for some* $t_1>t_0$, *and an integer* l, $0\le l\le n,$ *with* $n+l$ *even for* $u^{(n)}(t)\ge0$ *or* $n+l$ *odd for* $u^{(n)}(t)\le0$ *such that* $l>0$ *implies that* $u^{(k)}(t)>0$ *for* $t>t_x,$ $k=0,1,...,l-1,$ *and* $l\le n-1$, *implies that* $(-1)^{l+k}u^{(k)}(t)>0$ *for* $t>t_x,$ $k=l,l+1,...,n-1.$

Lemma 2.3 *([1]) Let* u *be as in Lemma 2.2. Assume that* $u^{(n)}(t)$ *is not identically zero on any interval* $[t_0,\infty),$ *and there exists a* $t_1\ge t_0$ *such that* $u^{(n-1)}(t)u^{(n)}(t)\le0$ *for all* $t\ge t_1.$ *If* $\lim\limits_{t\to\infty}u(t)\ne0,$ *then for every* $\lambda,$ $0<\lambda<1,$ *there exists* $T\ge t_1,$ *such that for all* $t\ge T,$

$$u(t)\ge\frac{\lambda}{(n-1)!}t^{n-1}u^{(n-1)}(t).$$

Lemma 2.4 *([15])Suppose* $q:[t_0,\infty)\to\mathbb{R}$ *is a continuous and eventually nonnegative function, and* σ *is a positive real number. Then the following hold.*

(I) If

$$\limsup_{t\to\infty}\int_t^{t+\sigma}\frac{(s-t)^i(t-s+\sigma)^{n-i-1}}{i!(n-i-1)!}q(s)ds>1,$$

hold for some $i=0,1,...,n-1,$ then the inequality

$$y^{(n)}(t)\ge q(t)y(t+\sigma)$$

has no eventually positive solution $y(t)$ which satisfies $y^{(j)}(t)>0$ eventually, $j=0,1,...,n.$

(II) If

$$\limsup_{t\to\infty}\int_{t-\sigma}^{t}\frac{(t-s)^{i}(s-t+\sigma)^{n-i-1}}{i!(n-i-1)!}q(s)ds>1,$$

hold for some $i=0,1,\ldots,n-1,$ then the inequality

$$(-1)^{n}z^{(n)}(t)\geq q(t)z(t-\sigma)$$

has no eventually positive solution $z(t)$ which satisfies $(-1)^{j}z^{(j)}(t)>0$ eventually, $j=0,1,\ldots,n.$

Lemma 2.5 *([24]) Assume that for large* t

$$q(s)\neq 0 \quad \textit{for all} \quad s\in[t,t^{*}],$$

where t^{*} satisfies $\sigma(t^{*})=t.$ Then

$$x'(t)+q(t)[x(\sigma(t))]^{\alpha}=0,\ t\geq t_{0},$$

has an eventually positive solution if and only if the corresponding inequality

$$x'(t)+q(t)[x(\sigma(t))]^{\alpha}\leq 0,\ t\geq t_{0},$$

has an eventually positive solution.

In [6,12,18,27], the authors investigated the oscillatory behavior of the following equation

$$x'(t)+q(t)[x(\sigma(t))]^{\alpha}=0,\ t\geq t_{0}, \qquad (10)$$

where $q\in C([t_0,\infty),\mathrm{R}^{+})$, $\sigma\in C([t_0,\infty),\mathrm{R})$, $\sigma(t)<t$, $\lim_{t\to\infty}\sigma(t)=\infty$ and $\alpha\in(0,\infty)$ is a ratio of odd positive integers.

Let $\alpha=1$. Then equation (2.1) reduces to the linear delay differential equation

$$x'(t)+q(t)x(\sigma(t))=0,\ t\geq t_{0}, \qquad (11)$$

and it is shown that every solution of equation (2.2) oscillates if

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}q(s)ds>\frac{1}{e}. \qquad (12)$$

3 Oscillation Results

First we study the oscillation of all solutions of equation (1.1).

Theorem 3.1 *Let* $\sigma_i>\tau_i$ *for* $i=1,2,a,b\leq 1,1\leq\beta\leq\gamma,(1+a^{\beta}-\frac{b^{\gamma}}{2^{\gamma-1}})>0,$ *and* $q(t)$ *and* $p(t)$ *are nonincreasing functions for* $t\geq t_0$. *Assume that the differential inequalities*

$$y^{(n)}(t)+c_1q(t)y^{\beta/\alpha}(t-\sigma_1-\tau_2)+c_1p(t)y^{\gamma/\alpha}(t+\sigma_2-\tau_2)\leq 0, \qquad (13)$$

$$y^{(n)}(t)-\frac{q(t)}{2^{\beta-1}(1+a^{\beta})^{\beta/\alpha}}y^{\beta/\alpha}(t-\sigma_1+\tau_1)\geq 0, \qquad (14)$$

and

$$y^{(n)}(t)-\frac{p(t)}{2^{\gamma-1}(1+a^{\beta}-\frac{b^{\gamma}}{2^{\gamma-1}})^{\gamma/\alpha}}y^{\gamma/\alpha}(t+\sigma_2-\tau_2)\geq 0, \qquad (15)$$

where $c_1 = \min\{\frac{1}{b^{\beta}}, \frac{1}{b^{\gamma}}, \frac{1}{2^{\beta-1}}(\frac{2^{\gamma-1}}{b^{\gamma}})^{\beta/\alpha}, \frac{1}{2^{\gamma-1}}(\frac{2^{\gamma-1}}{b^{\gamma}})^{\gamma/\alpha}\}$, have no eventually positive solution, no eventually positive decreasing solution and no eventually positive increasing solution respectively. Then every solution of equation (1.1) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t)$ is eventually positive,i.e., there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set

$$z(t) = (x(t) + ax(t-\tau_1) - bx(t+\tau_2))^{\alpha}.$$

Then

$$z^{(n)}(t) = q(t)x^{\beta}(t-\sigma_1) + p(t)x^{\gamma}(t+\sigma_2) > 0 \quad \text{for all} \quad t \geq t_1 \geq t_0. \tag{16}$$

Thus $z^{(i)}(t), i = 0,1,...,n,$ are of one sign on $[t_2, \infty);\ t_2 \geq t_1$. As a result we have two cases:(a) $z(t) < 0$ for $t \geq t_2$, (b) $z(t) > 0$ for $t \geq t_2$.

Case(a): $z(t) < 0$ for $t \geq t_2$. In this case, we let

$$0 < u(t) = -z(t) = (bx(t+\tau_2) - x(t) - ax(t-\tau_1))^{\alpha} \leq b^{\alpha}x^{\alpha}(t+\tau_2).$$

Then

$$x(t) \geq \frac{1}{b}u^{1/\alpha}(t-\tau_2),\ t \geq t_2.$$

Using above inequality in equation (1.1), we have

$$0 = u^{(n)}(t) + q(t)x^{\beta}(t-\sigma_1) + p(t)x^{\gamma}(t+\sigma_2)$$

$$\geq u^{(n)}(t) + \frac{q(t)}{b^{\beta}}u^{\beta/\alpha}(t-\sigma_1-\tau_2) + \frac{p(t)}{b^{\gamma}}u^{\gamma/\alpha}(t+\sigma_2-\tau_2),$$

or

$$u^{(n)}(t) + c_1 q(t)u^{\beta/\alpha}(t-\sigma_1-\tau_2) + c_1 p(t)u^{\gamma/\alpha}(t+\sigma_2-\tau_2) \leq 0$$

has a positive solution, which is a contradiction.

Case(b): $z(t) > 0$ for $t \geq t_2$. Now we set

$$y(t) = z(t) + a^{\beta}z(t-\tau_1) - \frac{b^{\gamma}}{2^{\gamma-1}}z(t+\tau_2),\ t \geq t_2. \tag{17}$$

Then

$$y^{(n)}(t) = z^{(n)}(t) + a^{\beta}z^{(n)}(t-\tau_1) - \frac{b^{\gamma}}{2^{\gamma-1}}z^{(n)}(t+\tau_2)$$

$$= q(t)x^{\beta}(t-\sigma_1) + p(t)x^{\gamma}(t+\sigma_2)$$

$$+ a^{\beta}\left(q(t-\tau_1)x^{\beta}(t-\sigma_1-\tau_1) + p(t-\tau_1)x^{\gamma}(t+\sigma_2-\tau_1)\right)$$

$$- \frac{b^{\gamma}}{2^{\gamma-1}}\left(q(t+\tau_2)x^{\beta}(t-\sigma_1+\tau_2) + p(t+\tau_2)x^{\gamma}(t+\sigma_2+\tau_2)\right).$$

Using the monotonicity of $q(t)$ and $p(t)$, $a, b \leq 1, 1 \leq \beta \leq \gamma$ and Lemma 2.1 in the above inequality, we get

$$y^{(n)}(t) \geq \frac{q(t)}{2^{\beta-1}}(x(t-\sigma_1) + ax(t-\sigma_1-\tau_1) - bx(t-\sigma_1+\tau_2))^{\beta}$$

$$+\frac{p(t)}{2^{\gamma-1}}\left(x(t+\sigma_2)+ax(t+\sigma_2-\tau_1)-bx(t+\sigma_2+\tau_2)\right)^{\gamma}.$$

Now using $z(t)>0$ for $t\geq t_2$ in the above inequality, we obtain

$$y^{(n)}(t)\geq\frac{q(t)}{2^{\beta-1}}z^{\beta/\alpha}(t-\sigma_1)+\frac{p(t)}{2^{\gamma-1}}z^{\gamma/\alpha}(t+\sigma_2)>0,\ t\geq t_2, \quad (18)$$

which implies that the function $y^{(i)}(t),\ i=0,1,...,n$ are of one sign. We shall prove that $y(t)>0$ eventually. If not, then

$$0<v(t)=-y(t)=-z(t)-a^{\beta}z(t-\tau_1)+\frac{b^{\gamma}}{2^{\gamma-1}}z(t+\tau_2)\leq\frac{b^{\gamma}}{2^{\gamma-1}}z(t+\tau_2).$$

Hence

$$z(t)\geq\frac{2^{\gamma-1}}{b^{\gamma}}v(t-\tau_2).$$

Using the last inequality in (3.6), we obtain

$$v^{(n)}(t)+\frac{q(t)}{2^{\beta-1}}\left(\frac{2^{\gamma-1}}{b^{\gamma}}\right)^{\beta/\alpha}v^{\beta/\alpha}(t-\sigma_1-\tau_2)+\frac{p(t)}{2^{\gamma-1}}\left(\frac{2^{\gamma-1}}{b^{\gamma}}\right)^{\gamma/\alpha}v^{\gamma/\alpha}(t+\sigma_2-\tau_2)\leq 0.$$

or

$$v^{(n)}(t)+c_1q(t)v^{\beta/\alpha}(t-\sigma_1-\tau_2)+c_1p(t)v^{\gamma/\alpha}(t+\sigma_2-\tau_2)\leq 0,\ t\geq t_2.$$

Using the procedure of case(a), $v(t)$ is a positive solution of (3.1), a contradiction. Thus we consider two possible cases:(i): $z'(t)<0$ eventually,(ii) $z'(t)>0$ eventually.

Case(i): Assume that $z'(t)<0$ for all $t\geq t_3^*\geq t_2$. Then from (3.4) we have $(-1)^i z^{(i)}(t)>0$ for $t\geq t_4\geq t_3^*$ and $i=0,1,...,n-1$. We claim that $y'(t)<0$ for $t\geq t_4$. To prove it, assume $y'(t)>0$ for $t\geq t_4$. Then differentiate (3.5), we have

$$0<y'(t)=z'(t)+a^{\beta}z'(t-\tau_1)-\frac{b^{\gamma}}{2^{\gamma-1}}z'(t+\tau_2)$$

or

$$0>-y'(t)=w(t)+a^{\beta}w(t-\tau_1)-\frac{b^{\gamma}}{2^{\gamma-1}}w(t+\tau_2)$$

where $w=-z'>0$ on $[t_4,\infty)$. Since the function w is decreasing on $[t_4,\infty)$, we have

$$0>-y'(t)\geq(1+a^{\beta}-\frac{b^{\gamma}}{2^{\gamma-1}})w(t+\tau_2)\geq 0,\quad for\ \ t\geq t_4,$$

a contradiction. Thus $y'(t)<0$ for $t\geq t_4$ and from (3.6), we have

$$(-1)^i y^{(i)}(t)>0\quad for\ \ t\geq t_4\ \ and\ \ i=0,1,...,n. \quad (19)$$

Now, using the monotonicity of $z(t)$, we obtain

$$y(t)=z(t)+a^{\beta}z(t-\tau_1)-\frac{b^{\gamma}}{2^{\gamma-1}}z(t+\tau_2)\leq(1+a^{\beta})z(t-\tau_1),\ t\geq t_4.$$

Then from the above inequality and (3.6), we have

$$y^{(n)}(t)-\frac{q(t)}{2^{\beta-1}(1+a^{\beta})^{\beta/\alpha}}y^{\beta/\alpha}(t-\sigma_1+\tau_1)\geq 0,\ t\geq t_4, \tag{20}$$

has a positive decreasing solution, a contradiction.

Case(ii): Assume that $z'(t)>0$ for all $t\geq t_3\geq t_2$. Now we consider the following two subcases:

Subcase(i): Assume that $y'(t)<0$ for all $t\geq t_3$. Proceeding as in Case(i) and using the monotonicity of $z(t)$, we obtain

$$y(t-\sigma_1)\leq(1+a^{\beta})z(t-\sigma_1).$$

Using the last inequality in (3.6) and the monotonicity of $y(t)$, we obtain

$$\begin{aligned} y^{(n)}(t) &\geq \frac{q(t)}{2^{\beta-1}}z^{\beta/\alpha}(t-\sigma_1) \\ &\geq \frac{q(t)}{2^{\beta-1}(1+a^{\beta})^{\beta/\alpha}}y^{\beta/\alpha}(t-\sigma_1) \\ &\geq \frac{q(t)}{2^{\beta-1}(1+a^{\beta})^{\beta/\alpha}}y^{\beta/\alpha}(t-\sigma_1+\tau_1). \end{aligned}$$

Thus once again $y(t)$ is a positive decreasing solution of the equation (3.2), which is a contradiction.

Subcase(ii): Assume that $y'(t)>0$ for all $t\geq t_3$. Then we have

$$y(t)=z(t)+a^{\beta}z(t-\tau_1)-\frac{b^{\gamma}}{2^{\gamma-1}}z(t+\tau_2)\leq(1+a^{\beta}-\frac{b^{\gamma}}{2^{\gamma-1}})z(t+\tau_2)$$

and this with (3.6) implies

$$y^{(n)}(t)-\frac{p(t)}{2^{\gamma-1}(1+a^{\beta}-\frac{b^{\gamma}}{2^{\gamma-1}})^{\gamma/\alpha}}y^{\gamma/\alpha}(t+\sigma_2-\tau_2)\geq 0,\ t\geq t_3, \tag{21}$$

has a positive increasing solution which satisfies

$$y^{(i)}(t)>0 \quad for \quad i=1,2,...,n, \quad and \quad t\geq t_3, \tag{22}$$

a contradiction. This completes the proof.

Corollary 3.1 *Let* $\sigma_i>\tau_i$ *for* $i=1,2,a,b\leq 1,(1+a^{\alpha}-\frac{b^{\alpha}}{2^{\alpha-1}})>0$ *and* $\alpha=\beta=\gamma\geq 1$ *and* $q(t)$ *and* $p(t)$ *are nonincreasing functions for* $t\geq t_0$. *If*

$$\liminf_{t\to\infty}\int_{t-\sigma_1-\tau_2}^{t}(s-\sigma_1-\tau_2)^{n-1}(q(s)+p(s))ds>\frac{b^{\alpha}(n-1)!}{e\lambda},\ 0<\lambda<1 \tag{23}$$

$$\limsup_{t\to\infty}\int_{t-\sigma_1+\tau_1}^{t}\frac{(t-s)^{i}(s-t+\sigma_1-\tau_1)^{n-i-1}}{i!(n-i-1)!}q(s)ds>2^{\alpha-1}(1+a^{\alpha}),i=0,1,...,n-1 \tag{24}$$

and

$$\limsup_{t\to\infty}\int_t^{t+\sigma_2-\tau_2}\frac{(s-t)^i(t-s+\sigma_2-\tau_2)^{n-i-1}}{i!(n-i-1)!}p(s)ds > 2^{\alpha-1}(1+a^\alpha-\frac{b^\alpha}{2^{\alpha-1}}), \tag{25}$$

where $i=0,1,...,n-1,$ then every solution of equation (1.1) is oscillatory.

Proof: Let $y(t)$ be a positive solution of (3.1), for $t\ge t_1\ge t_0$. Then we have $y^{(n)}(t)\le 0$ for all $t\ge t_1$. Further $y^{(n-1)}(t)>0$ for all $t\ge t_1$, otherwise $y(t)\to-\infty$ as $t\to-\infty$. Hence we have $y(t)>0, y^{(n-1)}(t)>0$ and $y^{(n)}(t)\le 0,$ for $t\ge t_1$. Then by Lemma 2.2 and Lemma 2.3 we obtain

$$y(t)\ge\frac{\lambda}{(n-1)!}t^{n-1}y^{(n-1)}(t),\ \lambda\in(0,1),\ t\ge t_2\ge t_1.$$

From (3.1) and the monotonicity of $y(t),$ we have

$$y^{(n)}(t)+\frac{q(t)+p(t)}{b^\alpha}y(t-\sigma_1-\tau_2)\le 0,\ t\ge t_2.$$

Combining the last two inequalities, we obtain

$$y^{(n)}(t)+(q(t)+p(t))\left(\frac{\lambda}{b^\alpha(n-1)!}(t-\sigma_1-\tau_2)^{n-1}y^{(n-1)}(t-\sigma_1-\tau_2)\right)\le 0,\ t\ge t_2.$$

Let $\omega(t)=y^{(n-1)}(t)$. Then we see that $\omega(t)$ is a positive solution of equation

$$\omega'(t)+\frac{(q(t)+p(t))\lambda}{b^\alpha(n-1)!}(t-\sigma_1-\tau_2)^{n-1}\omega(t-\sigma_1-\tau_2)\le 0,\ t\ge t_2 \tag{26}$$

But according to the Lemma 2.5 and the condition (2.3), condition (3.11) guarantees that inequality (3.14) has no positive solution, which is a contradiction. Hence (3.1) has no eventually positive solution. Moreover in view of Lemma 2.4 (II) and the condition (3.12), inequality (3.8) has no eventually positive solution which satisfies (3.7), which is a contradiction. Hence (3.2) has no eventually positive decreasing solution. Also in view of Lemma 2.4 (I) and the condition (3.13), inequality (3.9) has no eventually positive decreasing solution which satisfies (3.10), which is a contradiction. Hence (3.3) has no eventually positive increasing solution.

Next we consider the equation (1.2) and present sufficient conditions for the oscillation of all solutions.

Theorem 3.2 *Let* $\sigma_i>\tau_i$ *for* $i=1,2,(1-\frac{a^\beta}{2^{\beta-1}}+b^\gamma)>0,a,b\le 1,$ $1\le\gamma\le\beta,$ *and* $q(t)$ *and* $p(t)$ *are nondecreasing functions for* $t\ge t_0$. *Assume that the differential inequalities*

$$y^{(n)}(t)+c_2q(t)y^{\beta/\alpha}(t-\sigma_1+\tau_1)+c_2p(t)y^{\gamma/\alpha}(t+\sigma_2+\tau_1)\le 0, \tag{27}$$

$$y^{(n)}(t)-\frac{q(t)}{2^{\beta-1}(1+b^\gamma)^{\beta/\alpha}}y^{\beta/\alpha}(t-\sigma_1+\tau_1)\ge 0, \tag{28}$$

and

$$y^{(n)}(t)-\frac{p(t)}{2^{\gamma-1}(1+b^\gamma)^{\gamma/\alpha}}y^{\gamma/\alpha}(t+\sigma_2-\tau_2)\ge 0, \tag{29}$$

where $c_2=\min\{\frac{1}{a^\beta},\frac{1}{a^\gamma},\frac{1}{2^{\beta-1}}(\frac{2^{\gamma-1}}{a^\gamma})^{\beta/\alpha},\frac{1}{2^{\gamma-1}}(\frac{2^{\gamma-1}}{a^\gamma})^{\gamma/\alpha}\},$ have no eventually positive solution, no eventually positive decreasing solution and no eventually positive increasing solution respectively. Then every solution of equation (1.2) is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of equation (1.2). Without loss of generality we may assume that $x(t)$ is eventually positive,i.e., there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set

$$z_1(t) = (x(t) - ax(t-\tau_1) + bx(t+\tau_2))^\alpha$$

and proceeding as in the proof of Theorem 3.1, we see that the function $z_1^{(i)}(t), i = 0,1,...,n,$ are of one sign on $[t_2, \infty);\ t_2 \geq t_1$. As a result we have two cases:(a) $z_1(t) < 0$ for $t \geq t_2$, (b) $z_1(t) > 0$ for $t \geq t_2$.

Case(a): $z_1(t) < 0$ for $t \geq t_2$. In this case, we let

$$0 < u_1(t) = -z_1(t) = (-x(t) + ax(t-\tau_1) - bx(t+\tau_2))^\alpha \leq a^\alpha x^\alpha (t-\tau_1).$$

Then

$$x(t) \geq \frac{1}{a} u_1^{1/\alpha}(t+\tau_1),\ t \geq t_2.$$

Using above inequality in equation (1.2), we have

$$0 = u_1^{(n)}(t) + q(t)x^\beta (t-\sigma_1) + p(t)x^\gamma (t+\sigma_2)$$

$$\geq u_1^{(n)}(t) + \frac{q(t)}{a^\beta} u_1^{\beta/\alpha}(t-\sigma_1+\tau_1) + \frac{p(t)}{a^\gamma} u_1^{\gamma/\alpha}(t+\sigma_2+\tau_1),$$

or

$$u_1^{(n)}(t) + c_2 q(t) u_1^{\beta/\alpha}(t-\sigma_1+\tau_1) + c_2 p(t) u_1^{\gamma/\alpha}(t+\sigma_2+\tau_1) \leq 0$$

has a positive solution, which is a contradiction.

Case(b): $z_1(t) > 0$ for $t \geq t_2$. Now we set

$$y_1(t) = z_1(t) - \frac{a^\beta}{2^{\beta-1}} z_1(t-\tau_1) + b^\gamma z_1(t+\tau_2),\ t \geq t_2. \tag{30}$$

Then

$$y_1^{(n)}(t) = z_1^{(n)}(t) - \frac{a^\beta}{2^{\beta-1}} z_1^{(n)}(t-\tau_1) + b^\gamma z_1^{(n)}(t+\tau_2)$$

$$= q(t)x^\beta(t-\sigma_1) + p(t)x^\gamma(t+\sigma_2)$$

$$-\frac{a^\beta}{2^{\beta-1}}\left(q(t-\tau_1)x^\beta(t-\sigma_1-\tau_1) + p(t-\tau_1)x^\gamma(t+\sigma_2-\tau_1)\right)$$

$$+b^\gamma\left(q(t+\tau_2)x^\beta(t-\sigma_1+\tau_2) + p(t+\tau_2)x^\gamma(t+\sigma_2+\tau_2)\right).$$

Using the monotonicity of $q(t)$ and $p(t), a, b \leq 1, 1 \leq \gamma \leq \beta$ and Lemma 2.1 in the above inequality, we get

$$y_1^{(n)}(t) \geq \frac{q(t)}{2^{\beta-1}}\left(x(t-\sigma_1) - ax(t-\sigma_1-\tau_1) + bx(t-\sigma_1+\tau_2)\right)^\beta$$

$$+\frac{p(t)}{2^{\gamma-1}}\left(x(t+\sigma_2) - ax(t+\sigma_2-\tau_1) + bx(t+\sigma_2+\tau_2)\right)^\gamma.$$

Now using $z_1(t) > 0$ for $t \geq t_2$ in the above inequality, we obtain

$$y_1^{(n)}(t) \geq \frac{q(t)}{2^{\beta-1}} z_1^{\beta/\alpha}(t-\sigma_1) + \frac{p(t)}{2^{\gamma-1}} z_1^{\gamma/\alpha}(t+\sigma_2) > 0,\ t \geq t_2. \tag{31}$$

As in the proof of Theorem 3.1, case (2), we can easily see that $y_1(t) > 0$ for $t \geq t_2$. Next we consider two possible cases:(i): $z_1'(t) < 0$ eventually,(ii) $z_1'(t) > 0$ eventually.

Case(i): Assume that $z_1'(t) < 0$ for all $t \geq t_3^* \geq t_2$. Then we have $(-1)^i z_1^{(i)}(t) > 0$ for $t \geq t_4 \geq t_3^*$ and $i = 0,1,...,n-1$. We claim that $y_1'(t) < 0$ for $t \geq t_4$. To prove it, assume $y_1'(t) > 0$ for $t \geq t_4$. Then differentiate (3.18), we have

$$0 < y_1'(t) = z_1'(t) - \frac{a^\beta}{2^{\beta-1}} z_1'(t-\tau_1) + b^\gamma z_1'(t+\tau_2)$$

or

$$0 > -y_1'(t) = w_1(t) - \frac{a^\beta}{2^{\beta-1}} w_1(t-\tau_1) + b^\gamma w_1(t+\tau_2)$$

where $w_1 = -z_1' > 0$ on $[t_4, \infty)$. Since the function w_1 is decreasing on $[t_4, \infty)$, we have

$$0 > -y_1'(t) \geq (1 - \frac{a^\beta}{2^{\beta-1}} + b^\gamma) w_1(t+\tau_2) > 0, \quad for \quad t \geq t_4,$$

a contradiction. Thus $y_1'(t) < 0$ for $t \geq t_4$ and from (3.19), we have

$$(-1)^i y_1^{(i)}(t) > 0 \quad for \quad t \geq t_4 \quad and \quad i = 0,1,...,n. \tag{32}$$

Now, using the monotonicity of $z(t)$, we obtain

$$y_1(t) = z_1(t) - \frac{a^\beta}{2^{\beta-1}} z_1(t-\tau_1) + b^\gamma z_1(t+\tau_2) \leq (1+b^\gamma) z_1(t), \; t \geq t_4.$$

Then from the above inequality and (3.19), we have

$$y_1^{(n)}(t) \geq \frac{q(t)}{2^{\beta-1}} z_1^{\beta/\alpha}(t-\sigma_1)$$

$$\geq \frac{q(t)}{2^{\beta-1}(1+b^\gamma)^{\beta/\alpha}} y_1^{\beta/\alpha}(t-\sigma_1)$$

$$\geq \frac{q(t)}{2^{\beta-1}(1+b^\gamma)^{\beta/\alpha}} y_1^{\beta/\alpha}(t-\sigma_1+\tau_1), \; t \geq t_4.$$

has a positive decreasing solution, a contradiction.

Case(ii): Assume that $z_1'(t) > 0$ for all $t \geq t_3 \geq t_2$. Now we consider the following two subcases:

Subcase(i): Assume that $y_1'(t) < 0$ for all $t \geq t_3$. Proceeding as in Case(i) and using the monotonicity of $z_1(t)$, we obtain

$$y_1(t) \leq (1+b^\gamma) z_1(t+\tau_2).$$

Using the last inequality in (3.19) and the monotonicity of $y_1(t)$, we obtain

$$y_1^{(n)}(t) \geq \frac{q(t)}{2^{\beta-1}} z_1^{\beta/\alpha}(t-\sigma_1)$$

$$\geq \frac{q(t)}{2^{\beta-1}(1+b^\gamma)^{\beta/\alpha}} y_1^{\beta/\alpha}(t-\sigma_1-\tau_2)$$

$$\geq \frac{q(t)}{2^{\beta-1}(1+b^{\gamma})^{\beta/\alpha}} y_1^{\beta/\alpha}(t-\sigma_1+\tau_1).$$

Thus once again $y_1(t)$ is a positive decreasing solution of the inequality (3.16), which is a contradiction.

Subcase(ii): Assume that $y_1'(t)>0$ for all $t \geq t_3$. Then we have

$$y_1(t)=z_1(t)-\frac{a^{\beta}}{2^{\beta-1}}z_1(t-\tau_1)+b^{\gamma}z_1(t+\tau_2) \leq (1+b^{\gamma})z_1(t+\tau_2)$$

and this with (3.19) implies

$$y_1^{(n)}(t)-\frac{p(t)}{2^{\gamma-1}(1+b^{\gamma})^{\gamma/\alpha}} y_1^{\gamma/\alpha}(t+\sigma_2-\tau_2) \geq 0, \ t \geq t_3, \tag{33}$$

has a positive increasing solution, which satisfies

$$y_1^{(i)}(t)>0 \quad for \quad i=1,2,...,n, \ and \ \ t \geq t_3, \tag{34}$$

a contradiction. This completes the proof.

Corollary 3.2 *Let* $\sigma_i > \tau_i$ *for* $i=1,2,a,b \leq 1,(1-\frac{a^{\alpha}}{2^{\alpha-1}}+b^{\alpha})>0$ *and* $\alpha=\beta=\gamma \geq 1$ *and* $q(t)$ *and* $p(t)$ *are nonincreasing functions for* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} (s-\sigma_1+\tau_1)^{n-1}(q(s)+p(s)ds > \frac{a^{\alpha}(n-1)!}{e\lambda_1}, \ 0<\lambda_1<1, \tag{35}$$

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} \frac{(t-s)^i(s-t+\sigma_1-\tau_1)^{n-i-1}}{i!(n-i-1)!} q(s)ds > 2^{\alpha-1}(1+b^{\alpha}), \ i=0,1,...,n-1 \tag{36}$$

and

$$\limsup_{t\to\infty} \int_{t}^{t+\sigma_2-\tau_2} \frac{(s-t)^i(t-s+\sigma_2-\tau_2)^{n-i-1}}{i!(n-i-1)!} p(s)ds > 2^{\alpha-1}(1+b^{\alpha}), \ i=0,1,...,n-1, \tag{37}$$

then every solution of equation (1.2) is oscillatory.

Proof: The proof is similar to that of Corollary 3.1 and hence the details are omitted.

Next we consider the equation (1.3) and present sufficient conditions for the oscillation of all solutions.

Theorem 3.3 *Let* $\sigma_i > \tau_i$ *for* $i=1,2,a \leq 1,b \geq 1$ *and* $1 \leq \beta \leq \gamma$ *, and*

$$Q(t)=\min\{q(t),q(t-\tau_1),q(t+\tau_2)\},$$

$$P(t)=\min\{p(t),p(t-\tau_1),p(t+\tau_2)\},$$

are positive functions for $t \geq t_0$. Assume that the differential inequalities

$$y^{(n)}(t)-\frac{Q(t)}{4^{\beta-1}(1+a^{\beta}+b^{\gamma})^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1) \geq 0, \tag{38}$$

and

$$y^{(n)}(t)-\frac{P(t)}{4^{\gamma-1}(1+a^{\beta}+b^{\gamma})^{\gamma/\alpha}}\,y^{\gamma/\alpha}(t+\sigma_2-\tau_2)\geq 0, \tag{39}$$

have no eventually positive decreasing solution and no eventually positive increasing solution. Then every solution of equation (1.3) is oscillatory.

Proof: Let $x(t)$ be an eventually positive solution of equation (1.3), then there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set

$$z_2(t)=(x(t)+ax(t-\tau_1)+bx(t+\tau_2))^{\alpha},\ t\geq t_1.$$

and proceeding as in the proof of Theorem 3.1, we see that the function $z_2^{(i)}(t), i=0,1,...,n$ is of one sign on $[t_2,\infty)$, for some $t_2 \geq t_1$. Now we define

$$y_2(t)=z_2(t)+a^{\beta}z_2(t-\tau_1)+b^{\gamma}z_2(t+\tau_2),\ t\geq t_2. \tag{40}$$

Then $y_2(t)>0$ for $t\geq t_2$ and then

$$\begin{aligned} y_2^{(n)}(t) &= z_2^{(n)}(t)+a^{\beta}z_2^{(n)}(t-\tau_1)+b^{\gamma}z_2^{(n)}(t+\tau_2)\\ &= q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\gamma}(t+\sigma_2)\\ &+a^{\beta}\left(q(t-\tau_1)x^{\beta}(t-\sigma_1-\tau_1)+p(t-\tau_1)x^{\gamma}(t+\sigma_2-\tau_1)\right)\\ &+b^{\gamma}\left(q(t+\tau_2)x^{\beta}(t-\sigma_1+\tau_2)+p(t+\tau_2)x^{\gamma}(t+\sigma_2+\tau_2)\right).\end{aligned}$$

Using the monotonicity of $q(t)$ and $p(t), a\leq 1, b\geq 1, 1\leq\beta\leq\gamma$ and Lemma 2.1 in the above inequality, we get

$$\begin{aligned} y_2^{(n)}(t) &\geq \frac{Q(t)}{4^{\beta-1}}\left(x(t-\sigma_1)+ax(t-\sigma_1-\tau_1)+bx(t-\sigma_1+\tau_2)\right)^{\beta}\\ &+\frac{P(t)}{4^{\gamma-1}}\left(x(t+\sigma_2)-ax(t+\sigma_2-\tau_1)+bx(t+\sigma_2+\tau_2)\right)^{\gamma}.\end{aligned}$$

Now using $z_2(t)>0$ for $t\geq t_2$ in the above inequality, we obtain

$$y_2^{(n)}(t)\geq \frac{Q(t)}{4^{\beta-1}}z_2^{\beta/\alpha}(t-\sigma_1)+\frac{P(t)}{4^{\gamma-1}}z_2^{\gamma/\alpha}(t+\sigma_2)>0,\ t\geq t_2, \tag{41}$$

which implies that the function $y_2^{(i)}(t), i=0,1,...,n$ are of one sign. Next we consider two possible cases:(i): $z_2'(t)<0$ eventually,(ii) $z_2'(t)>0$ eventually.

Case(i): Assume that $z_2'(t)<0$ for all $t\geq t_3^*\geq t_2$. Then there exists a $t_3^*\geq t_2$ so that $(-1)^i y_2^{(i)}(t)>0$ for $i=0,1,...,n-1$ and $t\geq t_4\geq t_3^*$. Using the fact that the function $y_2^{(n)}$ is decreasing on $[t_4,\infty)$ in the equation (3.28) we have

$$y_2(t)=z_2(t)+a^{\beta}z_2(t-\tau_1)+b^{\gamma}z_2(t+\tau_2)\leq(1+a^{\beta}+b^{\gamma})z_2(t-\tau_1),$$

and then we have $z_2(t-\tau_1)\geq \dfrac{y_2(t)}{1+a^{\beta}+b^{\gamma}}$. Using the last inequality in (3.29), we obtain

$$y_2^{(n)}(t)-\frac{Q(t)}{4^{\beta-1}(1+a^{\beta}+b^{\gamma})^{\beta/\alpha}}\,y_2^{\beta/\alpha}(t-\sigma_1+\tau_1)\geq 0, \tag{42}$$

has a positive decreasing solution, a contradiction.

Case(ii): Assume that $z_2'(t) > 0$ for all $t \geq t_3 \geq t_2$. so that $y_2'(t) > 0$ for all $t \geq t_3$. Using the fact that the function $y_2^{(n)}$ is increasing on $[t_3, \infty)$, in (3.29), we have

$$y_2^{(n)}(t) - \frac{P(t)}{4^{\gamma-1}(1+a^{\beta}+b^{\gamma})^{\gamma/\alpha}} y_2^{\gamma/\alpha}(t+\sigma_2-\tau_2) \geq 0, \quad (43)$$

has a positive increasing solution, which satisfies

$$y_2^{(i)}(t) > 0 \;\; for \;\; i = 1,2,...,n, \;\; and \;\; t \geq t_3, \quad (44)$$

a contradiction. The proof is now complete.

Corollary 3.3 *Let* $\sigma_i > \tau_i$ *for* $i = 1,2, a \leq 1, b \geq 1$ *and* $\alpha = \beta = \gamma \geq 1$. *If*

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} \frac{(t-s)^i (s-t+\sigma_1-\tau_1)^{n-i-1}}{i!(n-i-1)!} Q(s)ds > 4^{\alpha-1}(1+a^{\alpha}+b^{\alpha}), \quad (45)$$

where $i = 0,1,...,n-1$, and

$$\limsup_{t\to\infty} \int_{t}^{t+\sigma_2-\tau_2} \frac{(s-t)^i (t-s+\sigma_2-\tau_2)^{n-i-1}}{i!(n-i-1)!} P(s)ds > 4^{\alpha-1}(1+a^{\alpha}+b^{\alpha}), \quad (46)$$

where $i = 0,1,...,n-1$, then every solution of equation (1.3) is oscillatory.

Proof: The proof is similar to that of Corollary 3.1 and hence the details are omitted.

4 Examples

In this section we present some examples to illustrate the main results.

Example 4.1 *Consider the differential equation*

$$\left((x(t)+\frac{1}{2}x(t-\pi)-\frac{1}{2}x(t+\pi))^3\right)^{(iv)} = \frac{1}{2}x^3(t-2\pi)+\frac{1}{2}x^3(t+2\pi), \; t \geq 0. \quad (47)$$

Here $a = b = \frac{1}{2}, \alpha = \beta = \gamma = 3, \tau_1 = \tau_2 = \pi, \sigma_1 = \sigma_2 = 2\pi, q(t) = p(t) = \frac{1}{2}$. Then one can see that all conditions of Corollary 3.1 are satisfied. Therefore all the solutions of equation (4.1) are oscillatory. In fact $x(t) = \sin^{1/3} t$ is one such oscillatory solution of equation (4.1).

Example 4.2 *Consider the differential equation*

$$\left((x(t)-\frac{1}{9}e^{\pi/3}x(t-\pi)+\frac{1}{3e^{\pi/3}}x(t+\pi))^3\right)^{(iv)} = \frac{1372}{729}e^{2\pi}x^3(t-2\pi)+\frac{2744}{729e^{3\pi}}x^3(t+3\pi), \quad (48)$$

where $t \geq 0$.

Here

$a = \frac{1}{9}e^{\pi/3}, b = \frac{1}{3e^{\pi/3}}, \alpha = \beta = \gamma = 3, \tau_1 = \tau_2 = \pi, \sigma_1 = 2\pi, \sigma_2 = 3\pi/2, q(t) = \frac{1372}{729}e^{2\pi}, p(t) = \frac{2744}{729e^{3\pi}}$.

Then one can see that all conditions of Corollary 3.2 are satisfied. Therefore all the solutions of equation (4.2) are oscillatory. In fact $x(t) = e^{t/3}\sin^{1/3} t$ is one such oscillatory solution of equation (4.2).

Example 4.3 *Consider the differential equation*

$$\left(x(t)+\frac{1}{2}x(t-\pi)+2x(t+\pi)\right)^{(iv)} = \frac{6}{t-3\pi/2}x(t-3\pi/2)+\frac{3(t+\pi)}{2(t+3\pi)}x(t+3\pi), \quad (49)$$

where $t \geq 3\pi/2.$

Here $a = 1/2, b = 2, \alpha = \beta = \gamma = 1, \tau_1 = \tau_2 = \pi, \sigma_1 = 3\pi/2, \sigma_2 = 3\pi, q(t) = \dfrac{6}{t - 3\pi/2}, p(t) = \dfrac{3(t+\pi)}{2(t+3\pi)}$, and we can see that all the conditions of Corollary 3.3 are satisfied. Therefore all the solutions of equation (4.3) are oscillatory. In fact $x(t) = t\sin t$ is one such oscillatory solution of equation (4.3).

Example 4.4 *Consider the differential equation*

$$\left((x(t) + \frac{1}{e}x(t-1) + ex(t+1))^3\right)^{(iv)} = \frac{2187}{2e^6}x^3(t-2) + \frac{2187}{2}e^{12}x^3(t+4),\ t \geq 1. \qquad (50)$$

Here $a = \frac{1}{e}, b = e, \alpha = \beta = \gamma = 3, \tau_1 = \tau_2 = 1, \sigma_1 = 2, \sigma_2 = 4, q(t) = \dfrac{2187}{2e^6}, p(t) = \dfrac{2187}{2}e^{12}.$ Then one can see that all conditions of Corollary 3.3 are satisfied except condition (3.33). Therefore all the solutions of equation (4.4) not necessarily oscillatory. In fact $x(t) = e^{-t}$ is one such nonoscillatory solution, since it satisfies equation (4.4).

We conclude this paper with the following remark.

Remark 1 *It would be interesting to obtain oscillation results for the equations (1.1) to (1.4) when* $0 < \beta, \gamma < 1$ *or* $0 < \beta < 1$ *and* $\gamma > 1$ *or* $\beta > 1$ *and* $0 < \gamma < 1.$

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