On the Oscillation of Fourth Order Superlinear Dynamic Equations on Time Scales

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Abstract: Some oscillation criteria for the oscillatory behavior of fourth order superlinear dynamic equations on time scales are established. Criteria are proved that ensure that all solutions of superlinear and linear equations are oscillatory. Many of our results are new for corresponding fourth order superlinear differential equations and fourth order superlinear difference equations.

1 Introduction

This paper deals with the oscillatory behavior of the fourth order superlinear and/or linear dynamic equation

$$x^{\Delta_4}(t) + q(t)\, x^{\gamma}(\sigma(t)) = 0, \tag{1}$$

on an arbitrary time scale $\mathbb{T} \subseteq \mathbb{R}$ with $\sup \mathbb{T} = \infty$, where $q : \mathbb{T} \to (0, \infty)$ is rd-continuous function and γ is the ratio of positive odd integers.

We recall that a solution of equation (1) is said to be nonoscillatory if there exists a $t_0 \in \mathbb{T}$ such that $x(t)\, x(\sigma(t)) > 0$ for all $t \in [t_0, \infty) \cap \mathbb{T}$; otherwise, it is said to be oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

In the last decade, there has been an increasing interest in studying the oscillatory behavior of first and second order dynamic equations on time scales [1]-[7]. With respect to dynamic equations on time scales, it is fairly new topic, and for general basic ideas and background, we refer to [1] and [2]. To the best of our knowledge, there are no results for the oscillation of equation (1). Therefore the main purpose of this paper is to establish some new criteria for the oscillation of equation (1). Our results are new even for the cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

2 Main Results

In order to prove our main results, we shall use the formula

$$\left((x(t))^{\lambda}\right)^{\Delta} = \lambda \int_0^1 \left[h x^{\sigma}(t) + (1-h)\, x(t)\right]^{\lambda-1} x^{\Delta}(t)\, dh, \tag{2}$$

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where $x(t)$ is delta-differentiable and eventually positive or eventually negative, which is a simple consequence of Keller's chain rule (see [1, Theorem 1.90]).

The following lemmas are needed in the proof of our main results.

Lemma 1 *Assume that $x(t)$ is an eventually positive solution of equation (1). Then there exists a $t_0 \in \mathbb{T}$ such that one of the following two cases holds:*

$$(I) \quad x(t) > 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) > 0, \quad x^{\Delta_3}(t) > 0, \quad x^{\Delta_4}(t) < 0 \quad \textit{for all} \quad t \in [t_0, \infty) \cap \mathbb{T}, \tag{3}$$

$$(II) \quad x(t) > 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad x^{\Delta_3}(t) > 0, \quad x^{\Delta_4}(t) < 0 \quad \textit{for all} \quad t \in [t_0, \infty) \cap \mathbb{T}. \tag{4}$$

The proof is easy and hence omitted.

In [1, Sec. 1.6], the Taylor monomials $\{h_n(t,s)\}_{n=0}^{\infty}$ are defined recursively by

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(u,s)\,\Delta u, \quad t, s \in \mathbb{T} \cap [t_0, \infty), \quad n \geqslant 1.$$

Lemma 2 *[4]. Let $y(t)$ be an eventually positive solution of the equation*

$$y^{\Delta\Delta\Delta}(t) + \bar{q}(t)\, y^{\gamma}(t) = 0$$

where $\bar{q}(t) \in C_{rd}([t_0, \infty), (0, \infty))$ and γ is as in equation (1). If

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad y^{\Delta\Delta}(t) > 0 \quad \textit{and} \quad y^{\Delta\Delta\Delta}(t) \leqslant 0 \quad \textit{for} \quad t_1 \in [t_0, \infty) \cap \mathbb{T}, \tag{5}$$

then

$$\liminf_{t \to \infty} \frac{t y(t)}{h_2(t, t_0)\, y^{\Delta}(t)} \geqslant 1. \tag{6}$$

The following result is a straightforward extension of Lemma in [4] and hence we omit the proof.

Lemma 3 *Assume that $y(t)$ satisfies (5). If*

$$\int_{t_0}^{\infty} \bar{q}(\tau) \left(h_2(\tau, t_0)\right)^{\gamma} \Delta\tau = \infty, \tag{7}$$

then

$$y^{\Delta}(t) \geqslant t y^{\Delta\Delta}(t) \quad \textit{and} \quad y^{\Delta}(t)/t \quad \textit{is eventually nonincreasing.} \tag{8}$$

Next, we shall state some sufficient conditions for the oscillation of second order dynamic equation

$$y^{\Delta\Delta}(t) + Q(t)\, y^{\gamma}(\sigma(t)) = 0, \tag{9}$$

where $Q : \mathbb{T} \to (0, \infty)$ is rd-continuous, γ is as in equation (1), which are needed in the proof of our main results.

Theorem 4 *Equation (9) is oscillatory if one of the following conditions holds:*

(i)

$$\int_{t_0}^{\infty} Q(s)\,\Delta s = \infty \quad \textit{for all} \quad \gamma > 0; \tag{10}$$

(ii)

$$\limsup_{t\to\infty} \quad t\int_{t}^{\infty} Q(s)\,\Delta s > c, \quad c>0, \quad \textit{or} \quad \int_{t_0}^{\infty}\int_{s}^{\infty} Q(u)\,\Delta u\Delta s = \infty, \quad \textit{when } \gamma > 1; \tag{11}$$

(iii) There exists a positive nondecreasing delta differentiable function η such that for every $t_1 \in [t_0,\infty)\cap\mathbb{T}$

$$\begin{cases} (a_1) \quad \limsup\limits_{t\to\infty} \displaystyle\int_{t_1}^{t} \left[\eta(s)\,Q(s) - \frac{1}{s}\eta^{\Delta}(s)\right]\Delta s = \infty; \quad \textit{or} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{when } \gamma = 1. \\ (a_2) \quad \limsup\limits_{t\to\infty} \displaystyle\int_{t_1}^{t} \left[\eta(s)\,Q(s) - \frac{1}{4}\frac{\left(\eta^{\Delta}(s)\right)^2}{\eta(s)}\right]\Delta s = \infty; \end{cases} \tag{12}$$

The proof of Theorem 4 is given in [5] and [6].

For $t \geqslant t_0$, we let

$$Q(t) = \int_{t}^{\infty}\int_{s}^{\infty} q(u)\,\Delta u\Delta s.$$

$$\begin{cases} \text{We assume that there exists a rd-continuous function } g:\mathbb{T}\to\mathbb{T} \text{ such} \\ \text{that } g(t)<t,\ g(t) \text{ is non–decreasing for } t\geqslant t_0 \text{ and } \lim_{t\to\infty} g(t) = \infty. \end{cases} \tag{13}$$

We also let $\phi(t) = t - g(t)$ for $t \geq t_0$, and assume that

$$\int_{t_0}^{\infty} \left(\phi(s)h_2(g(s),t_0)\right)^{\gamma} q(u)\Delta u = \infty. \tag{14}$$

Now, we establish the following oscillation result for superlinear ($\gamma > 1$) as well as linear ($\gamma = 1$) equation (1).

Theorem 5 *Let $\gamma \geqslant 1$ and conditions (13) and (14), and condition (11) when $\gamma > 1$, and (12) when $\gamma = 1$ hold. Moreover, assume that there exists a positive function $\xi(t) \in C^1_{rd}([t_0,\infty),\mathbb{R})$ such that for every constant $k>0$, and $t \geqslant t_1 \in [t_0,\infty)\cap\mathbb{T}$*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\left(\frac{\phi(s)}{\sigma(s)}h_2(g(s),t_0)\right)^{\gamma} \xi^{\sigma}(s)\,q(s) - k\frac{\xi^{\Delta}(s)}{s}\right]\Delta s = \infty, \tag{15}$$

then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say, $x(t) > 0$ for $t \geqslant t_0 \in \mathbb{T}$. Then by Lemma 1, there are two cases to consider:

Assume that $x(t)$ satisfies Case (I). Then

$$x(t) = x(g(t)) + \int_{g(t)}^{t} x^{\Delta}(s)\,\Delta s$$

and since $x^{\Delta}(t)$ is an increasing function for $t \geqslant t_0$, we get

$$x(t) \geqslant (t-g(t))x^{\Delta}(g(t)) = \phi(t)x^{\Delta}(g(t)) \quad \text{for} \quad t \geqslant t_1 \geqslant t_0. \tag{16}$$

Using (16) in equation (1) and setting $y(t) = x^{\Delta}(t)$ in the resulting inequality, we have

$$y^{\Delta\Delta\Delta}(t) + (\phi(t))^{\gamma} q(t) y^{\gamma}(g(t)) \leqslant 0 \quad \text{for} \quad t \geqslant t_1. \tag{17}$$

Define

$$W(t) = \xi(t) \frac{y^{\Delta\Delta}(t)}{(y^{\Delta}(t))^{\gamma}} \quad \text{for} \quad t \geqslant t_1. \tag{18}$$

Then $W(t) > 0$ for $t \geqslant t_1$ and by using the product rule, we find

$$\begin{aligned} W^{\Delta}(t) &= \xi^{\Delta}(t) \frac{y^{\Delta\Delta}(t)}{(y^{\Delta}(t))^{\gamma}} + \xi^{\sigma}(t) \left(\frac{y^{\Delta\Delta\Delta}(t)\left(y^{\Delta}(t)\right)^{\gamma} - y^{\Delta\Delta}(t)\left(\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}}{(y^{\Delta}(t))^{\gamma}(y^{\Delta}(\sigma(t)))^{\gamma}} \right) \\ &= \xi^{\Delta}(t) \frac{y^{\Delta\Delta}(t)}{(y^{\Delta}(t))^{\gamma}} + \xi^{\sigma}(t) \frac{y^{\Delta\Delta\Delta}(t)}{(y^{\Delta}(\sigma(t)))^{\gamma}} - \xi^{\sigma}(t) \frac{\left(\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}}{(y^{\Delta}(t))^{\gamma}(y^{\Delta}(\sigma(t)))^{\gamma}} \quad \text{for} \quad t \geqslant t_1. \end{aligned} \tag{19}$$

Using (17) – (19), we get

$$W^{\Delta}(t) \leqslant -\phi^{\gamma}(t)\xi^{\sigma}(t) q(t) \left(\frac{y(g(t))}{y^{\Delta}(\sigma(t))} \right)^{\gamma} + \xi^{\Delta}(t) \frac{y^{\Delta\Delta}(t)}{(y^{\Delta}(t))^{\gamma}} - \xi^{\sigma}(t) \frac{\left(\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}}{(y^{\Delta}(t))^{\gamma}(y^{\Delta}(\sigma(t)))^{\gamma}} \quad \text{for} \quad t \geqslant t_1. \tag{20}$$

Thus,

$$W^{\Delta}(t) \leqslant -\phi^{\gamma}(t)\xi^{\sigma}(t) q(t) \left(\frac{y(g(t))}{y^{\Delta}(\sigma(t))} \right)^{\gamma} + \xi^{\Delta}(t) \frac{y^{\Delta\Delta}(t)}{(y^{\Delta}(t))^{\gamma}} \quad \text{for} \quad t \geqslant t_1. \tag{21}$$

From (6) and (8), for any constant c, $0 < c < 1$, we obtain

$$\left(\frac{y(g(t))}{y^{\Delta}(\sigma(t))} \right)^{\gamma} = \left(\frac{y(g(t))}{y^{\Delta}(g(t))} \right)^{\gamma} \left(\frac{y^{\Delta}(g(t))}{y^{\Delta}(\sigma(t))} \right)^{\gamma} \geqslant \left(c \frac{h_2(g(t), t_0)}{g(t)} \right)^{\gamma} \left(\frac{g(t)}{\sigma(t)} \right)^{\gamma} \quad \text{for} \quad t \geqslant t_1. \tag{22}$$

Also from (8), there exists a $t_2 \geqslant t_1 \in [t_0, \infty) \cap \mathbb{T}$ such that

$$y^{\Delta}(t) \geqslant t y^{\Delta\Delta}(t) \quad \text{for} \quad t \geqslant t_2. \tag{23}$$

Using (22) and (23) in (21), we get

$$W^{\Delta}(t) \leqslant -c^{\gamma} \left(\frac{\phi(t)}{\sigma(t)} h_2(g(t), t_0) \right)^{\gamma} \xi^{\sigma}(t) q(t) + \xi^{\Delta}(t) \frac{1}{t} \left(y^{\Delta}(t) \right)^{1-\gamma} \quad \text{for} \quad t \geqslant t_2. \tag{24}$$

Since $y^{\Delta}(t)$ is an increasing function for $t \geqslant t_1$, there exist a constant $c_2 > 0$ such that

$$y^{\Delta}(t) \geqslant c_1 \quad \text{for} \quad t \geqslant t_2. \tag{25}$$

Using (25) in (24), we get

$$c^{-\gamma} W^{\Delta}(t) \leqslant -\left(\frac{\phi(t)}{\sigma(t)} h_2(g(t), t_0) \right)^{\gamma} \xi^{\sigma}(t) q(t) + c^{-\gamma} c_1^{1-\gamma} \frac{\xi^{\Delta}(t)}{t} \quad \text{for} \quad t \geqslant t_2.$$

Integrating the above inequality from t_2 to $t \geqslant t_2$, we have

$$-c^{-\gamma} W(t_2) \leqslant -\int_{t_2}^{t} \left[\left(\frac{\phi(s)}{\sigma(s)} h_2(g(s), t_0) \right)^{\gamma} \xi^{\sigma}(s) q(s) - C \left(\frac{\xi^{\Delta}(s)}{s} \right) \right] \Delta s,$$

where $C = c^{-\gamma} c_1^{1-\gamma}$, which yields

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\left(\frac{\phi(s)}{\sigma(s)} h_2(g(s), t_0) \right)^{\gamma} \xi^{\sigma}(s) q(s) - C \left(\frac{\xi^{\Delta}(s)}{s} \right) \right] \Delta s \leqslant c^{-\gamma} W(t_2) < \infty \quad \text{for all} \quad t \geqslant t_2,$$

which contradicts (15).

Assume that $x(t)$ satisfies Case (II). Integrating equation (1) from $t \geqslant t_0$ to $u \geqslant t$ and letting $u \to \infty$, we get

$$x^{\Delta\Delta\Delta}(t) \geqslant \left(\int_t^{\infty} q(s) \Delta s \right) x^{\gamma}(\sigma(t)) \quad \text{for} \quad t \geqslant t_0. \tag{26}$$

Integrating (26) from $t \geqslant t_0$ to $u \geqslant t$ and letting $u \to \infty$, we have

$$-x^{\Delta\Delta}(t) \geqslant \left(\int_t^{\infty} \int_s^{\infty} q(\tau) \Delta\tau \Delta s \right) x^{\gamma}(\sigma(t)) \quad \text{for} \quad t \geqslant t_0,$$

or

$$x^{\Delta\Delta}(t) + Q(t) x^{\gamma}(\sigma(t)) \leqslant 0 \quad \text{for} \quad t \geqslant t_0. \tag{27}$$

By a comparison result (see [7]), the equation

$$y^{\Delta\Delta}(t) + Q(t) y^{\gamma}(\sigma(t)) = 0 \tag{28}$$

has a positive solution, while condition (11) or (12) implies the oscillation of equation (28), a contradiction. This completes the proof. ■

The following corollary is immediate.

Corollary 6 *In Theorem 5, let the condition (15) be replaced by*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{\phi(s)}{\sigma(s)} h_2(g(s), t_0) \right)^{\gamma} \xi^{\sigma}(s) q(s) \Delta s = \infty, \tag{29}$$

and

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{\xi^{\Delta}(s)}{s} \Delta s < \infty, \tag{30}$$

then the conclusion of Theorem 5 holds.

Next, we present the following result.

Theorem 7 *Let $\gamma > 1$, conditions (11) and (14) hold and assume that there exists a function $\xi(t) \in C_{rd}^2([t_0, \infty), \mathbb{R})$ such that*

$$\xi(t) > 0, \quad \xi^{\Delta}(t) \geqslant 0 \quad \text{and} \quad \xi^{\Delta\Delta}(t) \leqslant 0 \quad \text{for} \quad t \geqslant t_0. \tag{31}$$

If for $t \geqslant t_1 \in [t_0, \infty) \cap \mathbb{T}$

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{\phi(s)}{\sigma(s)} h_2\left(g(s), t_0\right) \right)^{\gamma} \xi^{\sigma}(s)\, q(s)\, \Delta s = \infty, \tag{32}$$

then equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say, $x(t) > 0$ for $t \geqslant t_0$. Then by Lemma 1, there are two cases to consider. The proof of Case (II) is similar to that of Theorem 5 - Case (II) and hence omitted. Now, we consider Case (I). As in the proof of Theorem 5 we obtain the inequality (17). Next, we define W by (18) and apply the product rule to

$$W(t) = \left(\xi(t)\, y^{\Delta\Delta}(t)\right) \left(y^{\Delta}(t)\right)^{-\gamma} \quad \text{for} \quad t \geqslant t_1 \in [t_0, \infty) \cap \mathbb{T}$$

to find

$$W^{\Delta}(t) = \left[\xi^{\Delta}(t)\, y^{\Delta\Delta}(t) + \xi^{\sigma}(t)\, y^{\Delta\Delta\Delta}(t)\right] \left(y^{\Delta}(\sigma(t))\right)^{-\gamma} + \xi(t)\, y^{\Delta\Delta}(t) \left(\left(y^{\Delta}(t)\right)^{-\gamma}\right)^{\Delta}.$$

Since $y^{\Delta\Delta}(t) > 0$ and $\left(\left(y^{\Delta}(t)\right)^{-\gamma}\right)^{\Delta} \leqslant 0$ for $t \geqslant t_1$, we see that

$$W^{\Delta}(t) \leqslant \xi^{\Delta}(t)\, y^{\Delta\Delta}(t) \left(y^{\Delta}(\sigma(t))\right)^{-\gamma} - \phi^{\gamma}(t) \xi^{\sigma}(t)\, q(t) \frac{y^{\gamma}(g(t))}{\left(y^{\Delta}(\sigma(t))\right)^{\gamma}} \quad \text{for} \quad t \geqslant t_1. \tag{33}$$

As in the proof of Theorem 5, we obtain (22) and hence (33) becomes

$$W^{\Delta}(t) \leqslant \xi^{\Delta}(t)\, y^{\Delta\Delta}(t) \left(y^{\Delta}(\sigma(t))\right)^{-\gamma} - \xi^{\sigma}(t)\, \phi^{\gamma}(t) q(t) \left(\frac{c h_2(g(t), t_0)}{g(t)}\right)^{\gamma} \left(\frac{g(t)}{\sigma(t)}\right)^{\gamma} \quad \text{for} \quad t \geqslant t_1. \tag{34}$$

By applying (2), we have

$$\begin{aligned} \left(\left(y^{\Delta}(t)\right)^{1-\gamma}\right)^{\Delta} &= (1-\gamma) \int_0^1 \left[h y^{\Delta}(\sigma(t)) + (1-h)\, y^{\Delta}(t)\right]^{-\gamma} y^{\Delta\Delta}(t)\, dh \\ &\leqslant (1-\gamma) \int_0^1 \left[h y^{\Delta}(\sigma(t)) + (1-h)\, y^{\Delta}(\sigma(t))\right]^{-\gamma} y^{\Delta\Delta}(t)\, dh \\ &= (1-\gamma) \left(y^{\Delta}(\sigma(t))\right)^{-\gamma} y^{\Delta\Delta}(t). \end{aligned} \tag{35}$$

Using (35) in (34), we get

$$W^{\Delta}(t) \leqslant \frac{1}{1-\gamma} \xi^{\Delta}(t) \left(\left(y^{\Delta}(t)\right)^{1-\gamma}\right)^{\Delta} - c^{\gamma} \left(\frac{\phi(t)}{\sigma(t)} h_2(g(t), t_0)\right)^{\gamma} \xi^{\sigma}(t)\, q(t) \quad \text{for} \quad t \geqslant t_1.$$

Integrating this inequality from t_1 to t, we obtain

$$\begin{aligned} -W(t_1) \leqslant\ & W(t) - W(t_1) \leqslant \frac{1}{1-\gamma} \left[\xi^{\Delta}(t) \left(y^{\Delta}(t)\right)^{1-\gamma} - \xi^{\Delta}(t_1) \left(y^{\Delta}(t_1)\right)^{1-\gamma}\right] \\ & - \frac{1}{1-\gamma} \int_{t_1}^{t} \xi^{\Delta\Delta}(s) \left(y^{\Delta}(s)\right)^{1-\gamma} \Delta s - c^{\gamma} \int_{t_1}^{t} \left(\frac{\phi(s)}{\sigma(s)} h_2(g(s), t_0)\right)^{\gamma} \xi^{\sigma}(s)\, q(s)\, \Delta s. \end{aligned}$$

Using condition (31) in the above inequality, we get

$$\int_{t_1}^{t} \left(\frac{\phi(s)}{\sigma(s)} h_2(g(s), t_0)\right)^{\gamma} \xi^{\sigma}(s)\, q(s)\, \Delta s \leqslant W(t_1) < \infty.$$

Taking lim sup of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (32). This completes the proof. $\blacksquare$

The following corollary is immediate.

Corollary 8 *In Theorem 7, let conditions (31) and (32) be replaced by*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{\phi(s)}{\sigma(s)} h_2 \left(g(s), t_0\right) \right)^{\gamma} \sigma\left(s\right) q\left(s\right) \Delta s = \infty,$$

then the conclusion of Theorem 7 holds.

Proof. The proof is similar to that of Theorem 7 by setting $\xi\left(t\right) = t$. $\blacksquare$

Finally, we establish the following result.

Theorem 9 *In Theorem 5, let condition (15) be replaced by: for every constant $\beta > 0$*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\left(\frac{\phi(s)}{\sigma(s)} h_2 \left(g(s), t_0\right) \right)^{\gamma} \xi^{\sigma}\left(s\right) q\left(s\right) - \beta \left(\frac{\sigma\left(s\right)}{s} \right)^{\gamma} \frac{\left(\xi^{\Delta}\left(s\right)\right)^2}{\xi^{\sigma}\left(s\right)} \right] \Delta s = \infty, \tag{36}$$

then the conclusion of Theorem 5 holds.

Proof. Let $x\left(t\right)$ be a nonoscillatory solution of equation (1), say, $x\left(t\right) > 0$ for $t \geqslant t_0$. Then by Lemma 1, there are two cases to consider and the proof of Case (II) is similar to that of Theorem 5 - Case (II) and hence omitted. Now, we consider Case (I). Proceeding as in the proof of Theorem 5 we obtain (17) and by defining W as in (18), we obtain (19) and (22), that is,

$$W^{\Delta}\left(t\right) \leqslant -c^{\gamma}\xi^{\sigma}\left(t\right) q\left(t\right) \left(\frac{\phi(t)}{\sigma(t)} h_2 \left(g(t), t_0\right) \right)^{\gamma} + \frac{\xi^{\Delta}\left(t\right)}{\xi\left(t\right)} W\left(t\right) - \xi^{\sigma}\left(t\right) \frac{y^{\Delta\Delta}\left(t\right) \left(\left(y^{\Delta}\left(t\right)\right)^{\gamma}\right)^{\Delta}}{\left(y^{\Delta}\left(t\right)\right)^{\gamma} \left(y^{\Delta}\left(\sigma\left(t\right)\right)\right)^{\gamma}} \quad \text{for} \quad t \geqslant t_1. \tag{37}$$

From (2), $\gamma > 1$, we have

$$\begin{aligned} \left(\left(y^{\Delta}\left(t\right)\right)^{\gamma}\right)^{\Delta} &= \gamma \int_0^1 \left[h y^{\Delta}\left(\sigma\left(t\right)\right) + \left(1-h\right) y^{\Delta}\left(t\right)\right]^{\gamma-1} y^{\Delta\Delta}\left(t\right) dh \\ &\geqslant \gamma \left(y^{\Delta}\left(t\right)\right)^{\gamma-1} y^{\Delta\Delta}\left(t\right) \geqslant \gamma \left(y^{\Delta}\left(t_1\right)\right)^{\gamma-1} y^{\Delta\Delta}\left(t\right) := C y^{\Delta\Delta}\left(t\right) \quad \text{for} \quad t \geqslant t_1, \end{aligned}$$

where $C = \gamma \left(y^{\Delta}\left(t_1\right)\right)^{\gamma-1}$. Thus, (37) takes the form

$$W^{\Delta}\left(t\right) \leqslant -c^{\gamma}\xi^{\sigma}\left(t\right) q\left(t\right) \left(\frac{\phi(t)}{\sigma(t)} h_2 \left(g(t), t_0\right) \right)^{\gamma} + \frac{\xi^{\Delta}\left(t\right)}{\xi\left(t\right)} W\left(t\right) - C\xi^{\sigma}\left(t\right) \frac{\left(y^{\Delta\Delta}\left(t\right)\right)^2}{\left(y^{\Delta}\left(t\right)\right)^{\gamma} \left(y^{\Delta}\left(\sigma\left(t\right)\right)\right)^{\gamma}} \quad \text{for} \quad t \geqslant t_1. \tag{38}$$

By (8), we see that $y^{\Delta}\left(t\right)/t$ is nonincreasing, and hence

$$y^{\Delta}\left(t\right) \geqslant \left(\frac{t}{\sigma\left(t\right)} \right) y^{\Delta}\left(\sigma\left(t\right)\right) \quad \text{for} \quad t \geqslant t_1. \tag{39}$$

Using (39) in (38), we have

$$W^{\Delta}\left(t\right) \leqslant -c^{\gamma}\xi^{\sigma}\left(t\right) q\left(t\right) \left(\frac{\phi(t)}{\sigma(t)} h_2 \left(g(t), t_0\right) \right)^{\gamma} + \frac{\xi^{\Delta}\left(t\right)}{\xi\left(t\right)} W\left(t\right) - C \left(\frac{t}{\sigma\left(t\right)} \right)^{\gamma} \frac{\xi^{\Delta}\left(t\right)}{\xi^2\left(t\right)} W^2\left(t\right) \quad \text{for} \quad t \geqslant t_1. \tag{40}$$

By completing the square on the right-hand side of (40), we find

$$c^{-\gamma}W^{\Delta}(t) \leqslant -\xi^{\sigma}(t)\,q(t)\left(\frac{\phi(t)}{\sigma(t)}h_2(g(t),t_0)\right)^{\gamma} + \frac{1}{4c^{\gamma}C}\left(\frac{\sigma(t)}{t}\right)^{\gamma}\frac{\left(\xi^{\Delta}(t)\right)^2}{\xi^{\sigma}(t)} \quad \text{for} \quad t \geqslant t_1.$$

Integrating this inequality from t_1 to t, we have

$$\begin{aligned} -c^{-\gamma}W(t_1) &\leqslant c^{-\gamma}(W(t)-W(t_1)) \\ &\leqslant -\int_{t_1}^{t}\left[\xi^{\sigma}(s)\,q(s)\left(\frac{\phi(s)}{\sigma(s)}h_2(g(s),t_0)\right)^{\gamma} - a\left(\frac{\sigma(s)}{s}\right)^{\gamma}\frac{\left(\xi^{\Delta}(s)\right)^2}{\xi^{\sigma}(s)}\right]\Delta s, \end{aligned}$$

which yields

$$\int_{t_1}^{t}\left[\xi^{\sigma}(s)\,q(s)\left(\frac{\phi(s)}{\sigma(s)}h_2(g(s),t_0)\right)^{\gamma} - a\left(\frac{\sigma(s)}{s}\right)^{\gamma}\frac{\left(\xi^{\Delta}(s)\right)^2}{\xi^{\sigma}(s)}\right]\Delta s \leqslant c^{-\gamma}W(t_1) < \infty,$$

where $a = 1/4c^{\gamma}C$, which contradicts (36). This completes the proof. ■

As an example, we let $\xi(t) = 1$ or t in Theorem 9 and obtain the following immediate result.

Corollary 10 *In Theorem 9, let condition (36) be replaced by: for every constant $\beta > 0$*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\sigma(s)\,q(s)\left(\frac{\phi(s)}{\sigma(s)}h_2(g(s),t_0)\right)^{\gamma} - \frac{\beta}{\sigma(s)}\left(\frac{\sigma(s)}{s}\right)^{\gamma}\right]\Delta s = \infty, \tag{41}$$

then the conclusion of Theorem 9 holds.

Proof. Set $\xi(t) = t$ in the proof of Theorem 9. ■

Corollary 11 *In Theorem 9, let condition (36) be replaced by:*

$$\limsup_{t\to\infty}\int_{t_1}^{t} q(s)\left(\frac{\phi(s)}{\sigma(s)}h_2(g(s),t_0)\right)^{\gamma}\Delta s = \infty, \tag{42}$$

then the conclusion of Theorem 9 holds.

Proof. Set $\xi(t) = 1$ in the proof of Theorem 9. ■

Next, let $\mathbb{T} = \mathbb{R}$. In this case equation (1) takes the form

$$x^{(4)}(t) + q(t)\,x^{\gamma}(\sigma(t)) = 0. \tag{43}$$

Now Theorem 9 when applied to equation (43) becomes:

Theorem 12 *Let $\gamma \geqslant 1$,and condition (13) with $\mathbb{T} = \mathbb{R}$ hold,*

$$\int_{t_0}^{\infty}\left((s-g(s))g^2(s)\right)^{\gamma} q(s)\,ds = \infty,$$

$$\int_{t_0}^{\infty}\int_{s}^{\infty} Q(u)\,duds = \infty, \qquad when \qquad \lambda > 1$$

and assume that there exist two nondecreasing functions $\eta(t), \xi(t) \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\eta(s)Q(s) - \frac{1}{4}\frac{(\eta'(s))^2}{\eta(s)}\right]ds = \infty, \; when \; \gamma = 1,$$

where

$$Q(t) = \int_{t}^{\infty}\int_{s}^{\infty} q(u)\,duds.$$

If for every constant $\beta > 0$, $t \geqslant t_1 \in [t_0,\infty)\cap\mathbb{T}$

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[s^{\lambda}\xi(s)q(s) - \beta\frac{(\xi'(s))^2}{\xi(s)}\right]ds = \infty,$$

then equation (43) is oscillatory.

When $\mathbb{T} = \mathbb{Z}$. In this discrete case equation (1) becomes

$$\Delta^4 x(t) + q(t)x^{\lambda}(t+1) = 0. \tag{44}$$

Now, Theorem 5 when applied to equation (44) takes the form:

Theorem 13 *Let* $\gamma \geqslant 1$,*and condition (13) with* $\mathbb{T} = \mathbb{Z}$ *hold,*

$$\sum_{s=t_0}^{\infty}\left((s-g(s))\,g^2(s)\right)q^{\gamma}(s) = \infty,$$

$$\sum_{s=t_0}^{\infty}\sum_{u=s}^{\infty} Q(u) = \infty, \qquad when \qquad \lambda > 1$$

and there exist two positive nondecreasing sequences $\{\eta(t)\}$ *and* $\{\xi(t)\}$ *such that*

$$\limsup_{t\to\infty}\sum_{s=t_1\geqslant t_0}^{t-1}\left[\eta(s)Q(s) - \frac{(\Delta\eta(s))^2}{4\eta(s)}\right] = \infty \quad when \quad \gamma = 1,$$

where

$$Q(t) = \sum_{s=t}^{\infty}\sum_{u=s}^{\infty} q(u).$$

If for every constant $k > 0$, *and* $t \geqslant t_1$,

$$\limsup_{t\to\infty}\sum_{s=t_0}^{t}\left[\left(\frac{s^3}{s+1}\right)^{\gamma}\xi(s+1)q(s) - k\frac{\Delta\xi(s)}{s}\right] = \infty,$$

then equation (44) is oscillatory.

Remark 14 *The results of this paper are presented in a form which is essentially new even for the corresponding differential equation (43) and difference equation (44). The obtained results are also extendable to delay dynamic equations of the form*

$$x^{\Delta_4}(t) + q(t)\left(x^{\sigma}(\tau(t))\right)^{\lambda} = 0,$$

where $\tau : \mathbb{T} \to \mathbb{T}$ *satisfies* $\tau(t) \leqslant t$ *for* $t \in \mathbb{T}$, $\tau(t)$ *is nondecreasing and* $\lim_{t\to\infty}\tau(t) = \infty$.

Remark 15 *The literature is filled with many criteria for the oscillation of the second order dynamic equations of type (9), and so, one may apply those results rather than presented here.*

Remark 16 *We may employ other types of time scales, e.g.,* $\mathbb{T} = h\mathbb{Z}$ *with* $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ *with* $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$ *etc., see [1] and [2]. The details are left to the reader.*

References

[1] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[2] M. Bohner and A. Peterson, Advances on Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[3] L. Erbe, A. Peterson and S. Saker, Kamenev type oscillation criteria for second order linear delay dynamic equations, Dynamic System Appl., 15 (2006), 65-78.

[4] L. Erbe, A. Peterson and S. Saker. Hille and Nehari type criteria for third order dynamic equations, J. Math. Anal. Appl., 333 (2007), 505-522.

[5] S. R. Grace, M. Bohner and R. P. Agarwal, On the oscillation of second order half-linear dynamic equations, J. Difference Equations Appl., 15 (2009), 451-460.

[6] S. R. Grace, R. P. Agarwal, M. Bohner and D. O'Regan, Oscillation of second order strongly superlinear and strongly sublinear dynamic equations, Communications in Nonlinear Sciences and Numerical Simulations, 14 (2009), 3463-3471.

[7] S. Saker, R.P. Agarwal and D. O'Regan, Oscillation results for second order neutral dynamic equations on time scales, Appl. Appl., 86(2007), 1–17.