Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 4, Number 1, pp. 107–121 (2009)
http://campus.mst.edu/adsa

Oscillations and Nonoscillations in Mixed Differential Equations with Monotonic Delays and Advances

Sandra Pinelas
Universidade dos Açores
Departamento de Matemática
R. Mãe de Deus
9500-321 Ponta Delgada, Portugal
spinelas@uac.pt

Abstract

In this work we study the oscillatory behavior of the delay differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x\left(t - r(\theta)\right) d\nu(\theta) + \int_{-1}^{0} x\left(t + \tau(\theta)\right) d\eta(\theta).$$

Some criteria are obtained in order to guarantee that all solutions are oscillatory. The existence of nonoscillatory solutions is also considered.

AMS Subject Classifications: 34K11.
Keywords: ODE, oscillation, nonoscillation, advanced, delayed.

1 Introduction

The aim of this work is to study the oscillatory behavior of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x\left(t - r(\theta)\right) d\nu(\theta) + \int_{-1}^{0} x\left(t + \tau(\theta)\right) d\eta(\theta), \tag{1.1}$$

where $x(t) \in \mathbb{R}$, $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. The advance $\tau(\theta)$ will be assumed such that

$$\tau(\theta_0) > \tau(\theta) \quad \forall \theta \neq \theta_0,$$

Received May 8, 2008; Accepted August 25, 2008
Communicated by Elena Braverman

where

$$\tau(\theta_0) = \max\{\tau(\theta) : \theta \in [-1, 0]\}.$$

Both $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$, and $\eta(\theta)$ is atomic at θ_0, that is, such that

$$\eta(\theta_0^+) - \eta(\theta_0^-) \neq 0. \tag{1.2}$$

The equation (1.1) represents a wide class of linear functional differential equations of mixed type. The same equation is considered by Krisztin [5] in basis of some mathematical applications appearing in the literature, such as in [1] and [6] (see also [2]). When $\eta(\theta)$ is the null function, one obtains a retarded functional differential equation whose oscillatory behavior is studied in [3, 4].

As usual, we will say that a solution x of (2.1) oscillates if it has arbitrary large zeros. Notice that for equations, this definition coincides with the nonexistence of an invariant cone as is considered an oscillatory solution in [5]. When all solutions oscillate, (2.1) will be called oscillatory.

By [5, Corollary 5], under the condition (1.2), the oscillatory behavior of (2.1) can be studied through the analysis of the real zeros of the characteristic equation

$$\lambda = \int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\nu(\theta) + \int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta). \tag{1.3}$$

Letting

$$M(\lambda) = \int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\nu(\theta) + \int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta),$$

the equation (1.1) is oscillatory if and only if $M(\lambda) \neq \lambda$ for every real λ. So, if either

$$M(\lambda) > \lambda, \quad \forall \lambda \in \mathbb{R} \tag{1.4}$$

or

$$M(\lambda) < \lambda, \quad \forall \lambda \in \mathbb{R} \tag{1.5}$$

then we can conclude that equation (1.1) is oscillatory. In case of having $M(\lambda) = \lambda$ for some real λ, there exists at least one nonoscillatory solution.

2 Oscillations and Nonoscillations independent of Delays and Advances

For $\nu(\theta)$ and $\eta(\theta)$ monotonous on $[-1, 0]$, some general criteria for oscillations can be obtained independently of either the delays or the advances.

Theorem 2.1. (*i*) *Let $\nu(\theta)$ and $\eta(\theta)$ be increasing functions on $[-1, 0]$. If*

$$\int_{-1}^{0} \tau(\theta) d\eta(\theta) > e^{-1}, \tag{2.1}$$

then equation (1.1) *is oscillatory independently of the delays.*

(*ii*) *If* $\nu(\theta)$ *and* $\eta(\theta)$ *are decreasing functions on* $[-1,0]$ *and*

$$\int_{-1}^{0} r(\theta)d\nu(\theta) < -e^{-1}, \tag{2.2}$$

then (1.1) *is oscillatory independently of the advances.*

Proof. We first prove (i). Since $M(\lambda) > 0$ for every $\lambda \in \mathbb{R}$, one has $M(\lambda) > \lambda$ for every $\lambda \leq 0$. For $\lambda > 0$ we have

$$\frac{M(\lambda)}{\lambda} = \int_{-1}^{0} \frac{\exp\left(-\lambda r(\theta)\right)}{\lambda r(\theta)} r(\theta)d\nu(\theta) + \int_{-1}^{0} \frac{\exp\left(\lambda \tau(\theta)\right)}{\lambda \tau(\theta)} \tau(\theta)d\eta(\theta),$$

and as $\dfrac{\exp(-u)}{u} > 0$ and $\dfrac{\exp u}{u} > e$ for every $u > 0$, we obtain

$$\frac{M(\lambda)}{\lambda} > e\int_{-1}^{0} \tau(\theta)d\eta(\theta) > 1.$$

Therefore (1.4) is verified.

Now we show (ii). We have $M(\lambda) < 0$ for every $\lambda \in \mathbb{R}$ and so $M(\lambda) < \lambda$ for every $\lambda \geq 0$. Through the same arguments one has

$$\frac{M(\lambda)}{\lambda} > -e\int_{-1}^{0} r(\theta)d\nu(\theta) > 1$$

for every $\lambda < 0$, which implies (1.5). □

Remark 2.2. Denoting

$$m_r = \min\left\{r(\theta):\ \theta \in [-1,0]\right\}, \quad m_\tau = \min\left\{\tau(\theta):\ \theta \in [-1,0]\right\}$$

and considering the differences $\Delta\nu = \nu(0) - \nu(-1)$, $\Delta\eta = \eta(0) - \eta(-1)$, in the case (i) of the Theorem 2.1, the assumption (2.1) is fulfilled whenever $m_\tau \Delta\eta > e^{-1}$. In the case (ii) of the same theorem, also condition (2.2) is satisfied if $m_r \Delta\nu < -e^{-1}$.

Example 2.3. By Theorem 2.1, for every delay function $r(\theta)$ and any increasing function $\nu(\theta)$, the equation

$$x'(t) = \int_{-1}^{0} x\left(t - r(\theta)\right) d\nu(\theta) + \int_{-1}^{0} x\left(t + (\theta + 2)\right) d\theta$$

is oscillatory since

$$\int_{-1}^{0} (\theta + 2)d\theta = 1.5 > e^{-1}.$$

Example 2.4. Analogously for every advance function $\tau(\theta)$, the equation

$$x'(t) = \int_{-1}^{0} x\left(t - (\theta + 3)\right) d(-\theta) + \int_{-1}^{0} x\left(t + \tau(\theta)\right) d\eta(\theta),$$

where $\eta(\theta)$ is a decreasing function, is oscillatory since

$$\int_{-1}^{0} (\theta + 3) d(-\theta) = -2.5 < -e^{-1}.$$

In case that the functions $\nu(\theta)$ and $\eta(\theta)$ have an opposite type of monotonicity, one obtains nonoscillations, as is stated in the following theorem.

Theorem 2.5. (*i*) *If $\nu(\theta)$ is decreasing and $\eta(\theta)$ is increasing, then* (1.1) *is nonoscillatory independently of the delays and advances.*

(*ii*) *If $\nu(\theta)$ is increasing and $\eta(\theta)$ is decreasing, then* (1.1) *is nonoscillatory independently of the delays and advances.*

Proof. We prove only (i) as (ii) can be obtained analogously. Let $\nu(\theta)$ be decreasing and $\eta(\theta)$ increasing on $[-1, 0]$. For $\lambda > 0$ one has

$$M(\lambda) \geq \int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\nu(\theta) + \exp\left(\lambda m_\tau\right) \Delta\eta$$

and

$$\left| \int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\nu(\theta) \right| \leq \exp\left(-\lambda m_r\right) \int_{-1}^{0} \left| d\nu(\theta) \right|. \tag{2.3}$$

Then $M(\lambda) - \lambda \to +\infty$ as $\lambda \to +\infty$. On the other hand, for $\lambda < 0$, we have

$$M(\lambda) \leq \exp\left(-\lambda m_r\right) \Delta\nu + \int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta)$$

and

$$\left| \int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta) \right| \leq \exp\left(\lambda m_\tau\right) \int_{-1}^{0} \left| d\eta(\theta) \right|, \tag{2.4}$$

which imply that $M(\lambda) - \lambda \to -\infty$ as $\lambda \to -\infty$. So there is at least one $\lambda_0 \in \mathbb{R}$ such that $M(\lambda_0) = \lambda_0$ and so (1.1) is nonoscillatory independently of the delays and advances. □

Example 2.6. The equation

$$x'(t) = \int_{-1}^{0} x\left(t - r(\theta)\right) d(1 - \theta) + \int_{-1}^{0} x\left(t + \tau(\theta)\right) d\theta$$

is nonoscillatory for every delays and advances.

3 Monotonic Differentiable Delays and Advances

In this section we analyze the oscillatory behavior of (1.1) for the relevant case where both delays and advances are monotonous. So in the remainder of this section, the delays and advances will be assumed monotonous differentiable functions on $[-1,0]$. For the sake of simplicity we will normalize the functions $\nu(\theta)$ and $\eta(\theta)$ in the following way:

a) if $r(\theta)$ is increasing, then $\nu(0)=0$,
b) if $r(\theta)$ is decreasing, then $\nu(-1)=0$,
c) if $\tau(\theta)$ is increasing, then $\eta(0)=0$,
d) if $\tau(\theta)$ is decreasing, then $\eta(-1)=0$.

With regard of condition (1.4) we obtain the following theorem.

Theorem 3.1. *Let*

$$\delta=\min\left\{-e\left(\frac{m_r\Delta\nu}{m_\tau\Delta\eta}\right)^{\frac{m_\tau}{m_r+m_\tau}}\left(\frac{m_\tau}{m_r}+1\right)\Delta\eta,\frac{e}{m_r}\left(1+\ln\left(-m_r\Delta\nu\right)\right)\right\}.$$

If

$$r'(\theta)\nu(\theta)\geq 0 \textit{ and } \tau'(\theta)\eta(\theta)\geq 0 \textit{ for every } \theta\in[-1,0] \tag{3.1}$$

and

$$\int_{-1}^{0}\nu(\theta)d\ln r(\theta)+\int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)<\delta, \tag{3.2}$$

then the equation (1.1) *is oscillatory.*

Proof. Assumptions a), b), c) and d) jointly with (3.1) imply that $\Delta\nu\leq 0$ and $\Delta\eta\leq 0$. However the value δ defined above has no meaning when either $\Delta\nu$ or $\Delta\eta$ are equal to zero. This way, implicitly we are assuming that $\Delta\nu<0$ and $\Delta\eta<0$. One easily notices that $M(0)=\Delta\nu+\Delta\eta$. So $M(0)<0$. Let $\lambda\neq 0$. Integrating by parts both integrals of (1.3) we have

$$\begin{aligned}M(\lambda) &= \exp\left(-\lambda r(0)\right)\nu(0)-\exp\left(-\lambda r(-1)\right)\nu(-1)\\ &\quad+\exp\left(\lambda\tau(0)\right)\eta(0)-\exp\left(\lambda\tau(-1)\right)\eta(-1)\\ &\quad+\int_{-1}^{0}\lambda\exp\left(-\lambda r(\theta)\right)\nu(\theta)dr(\theta)-\int_{-1}^{0}\lambda\exp\left(\lambda\tau(\theta)\right)\eta(\theta)d\tau(\theta).\end{aligned} \tag{3.3}$$

By (3.1) and the inequality $u\exp(-u)\leq e^{-1}$, we obtain

$$\begin{aligned}M(\lambda) &\leq \exp\left(-\lambda r(0)\right)\nu(0)-\exp\left(-\lambda r(-1)\right)\nu(-1)\\ &\quad+\exp\left(\lambda\tau(0)\right)\eta(0)-\exp\left(\lambda\tau(-1)\right)\eta(-1)\\ &\quad+e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta)+\int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right).\end{aligned}$$

For $r(\theta)$ increasing and $\tau(\theta)$ increasing, one has by a) and c)

$$\begin{aligned} M(\lambda) &\leq -\exp\left(-\lambda r(-1)\right)\nu(-1) - \exp\left(\lambda\tau(-1)\right)\eta(-1) \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right). \end{aligned}$$

For $r(\theta)$ increasing and $\tau(\theta)$ decreasing, by a) and d) we have

$$\begin{aligned} M(\lambda) &\leq -\exp\left(-\lambda r(-1)\right)\nu(-1) + \exp\left(\lambda\tau(0)\right)\eta(0) \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right). \end{aligned}$$

For $r(\theta)$ decreasing and $\tau(\theta)$ increasing, we have by b) and c)

$$\begin{aligned} M(\lambda) &\leq \exp\left(-\lambda r(0)\right)\nu(0) - \exp\left(\lambda\tau(-1)\right)\eta(-1) \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right). \end{aligned}$$

For $r(\theta)$ decreasing and $\tau(\theta)$ decreasing, by b) and d) one has that

$$\begin{aligned} M(\lambda) &\leq \exp\left(-\lambda r(0)\right)\nu(0) + \exp\left(\lambda\tau(0)\right)\eta(0) \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right). \end{aligned}$$

So, for all possible cases one obtains

$$\begin{aligned} M(\lambda) - \lambda &\leq \exp\left(-\lambda m_r\right)\Delta\nu + \exp\left(\lambda m_\tau\right)\Delta\eta - \lambda \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right). \end{aligned}$$

Let $\lambda > 0$. Then

$$\begin{aligned} M(\lambda) - \lambda &< M(\lambda) \leq \exp\left(-\lambda m_r\right)\Delta\nu + \exp\left(\lambda m_\tau\right)\Delta\eta \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right). \end{aligned}$$

The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} f(\lambda) &= \exp\left(-\lambda m_r\right)\Delta\nu + \exp\left(\lambda m_\tau\right)\Delta\eta \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right) \end{aligned}$$

has an absolute maximum at

$$\lambda_0 = \frac{1}{m_r + m_\tau}\ln\frac{m_r\Delta\nu}{m_\tau\Delta\eta},$$

and in view of (3.2)

$$\begin{aligned} f(\lambda_0) &= \left(\frac{m_r \Delta\nu}{m_\tau \Delta\eta}\right)^{\frac{m_\tau}{m_r+m_\tau}} \left(\frac{m_\tau}{m_r}+1\right)\Delta\eta \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right) < 0. \end{aligned}$$

Therefore, for every $\lambda > 0$,

$$M(\lambda) - \lambda < 0.$$

For $\lambda < 0$, one has

$$\begin{aligned} M(\lambda) - \lambda &\leq \exp\left(-\lambda m_r\right)\Delta\nu - \lambda \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right). \end{aligned}$$

The function $g : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} g(\lambda) &= \exp\left(-\lambda m_r\right)\Delta\nu - \lambda \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right) \end{aligned}$$

attains its absolute maximum at

$$\lambda_1 = \frac{1}{m_r}\ln\left(-m_r\Delta\nu\right),$$

and by (3.2) it is

$$\begin{aligned} g(\lambda_1) &= -\frac{1}{m_r} - \frac{1}{m_r}\ln\left(-m_r\Delta\nu\right) \\ &\quad + e^{-1}\left(\int_{-1}^{0}\nu(\theta)d\ln r(\theta) + \int_{-1}^{0}\eta(\theta)d\ln\tau(\theta)\right) < 0. \end{aligned}$$

Thus, also

$$M(\lambda) - \lambda < 0$$

for every $\lambda < 0$. Hence (1.5) holds and (1.1) is oscillatory. □

Example 3.2. By Theorem 3.1 the equation

$$x'(t) = \int_{-1}^{0} x\left(t - (\theta+2)\right) d\left(-\frac{1}{2}\theta\right) + \int_{-1}^{0} x\left(t + (4-\theta)\right) d(-4\theta - 4)$$

is oscillatory. In fact, for $\theta \in [-1, 0]$, we have

$$r(\theta) \text{ increasing}, \quad r'(\theta)\nu(\theta) = -\frac{1}{2}\theta \geq 0, \text{ with } \nu(0) = 0$$

$$\tau(\theta) \text{ decreasing}, \quad \tau'(\theta)\eta(\theta) = 4\theta + 4 \geq 0, \text{ with } \eta(-1) = 0$$

and

$$\int_{-1}^{0} \nu(\theta) d \ln r(\theta) + \int_{-1}^{0} \eta(\theta) d \ln \tau(\theta) = \int_{-1}^{0} \left(-\frac{1}{2}\theta\right) d \ln(\theta+2)$$
$$+ \int_{-1}^{0} (-4\theta - 4) d \ln(4-\theta) \approx 0.65602 < \delta$$

since

$$-e \left(\frac{m_r \Delta\nu}{m_\tau \Delta\eta}\right)^{\frac{m_\tau}{m_r+m_\tau}} \left(\frac{m_\tau}{m_r} + 1\right) \Delta\eta = 4e \left(\frac{1}{2}\frac{1}{16}\right)^{\frac{4}{1+4}} (4+1) \approx 3.3979$$

and

$$\frac{e}{m_r}\left(1 + \ln\left(-m_r \Delta\nu\right)\right) \approx 0.83411.$$

Theorem 3.3. *If*

$$r'(\theta)\nu(\theta) \geq 0 \textit{ and } \tau'(\theta)\eta(\theta) \geq 0 \textit{ for every } \theta \in [-1, 0] \tag{3.4}$$

and

$$1 + e m_r \Delta\nu < \int_{-1}^{0} \nu(\theta) dr(\theta) - \int_{-1}^{0} \eta(\theta) d\tau(\theta) < 1 - e m_\tau \Delta\eta, \tag{3.5}$$

then the equation (1.1) *is oscillatory.*

Proof. As in Theorem 3.1 one has $\Delta\nu \leq 0$ and $\Delta\eta \leq 0$. For $\lambda = 0$, we have

$$M(0) = \Delta\nu + \Delta\eta < 0$$

since by (3.5) one cannot have both $\Delta\nu$ and $\Delta\eta$ equal to zero. Notice that for every $u > 0$ we have

$$\exp(-u) < 1,\ \exp(u) > 1,\ \frac{\exp(-u)}{u} > 0 \text{ and } \frac{\exp u}{u} > e. \tag{3.6}$$

Let $\lambda > 0$. By (3.3) and (3.6) we have, for $\tau(\theta)$ increasing

$$\frac{M(\lambda)}{\lambda} < -e\tau(-1)\eta(-1) + \int_{-1}^{0} \nu(\theta) dr(\theta) - \int_{-1}^{0} \eta(\theta) d\tau(\theta),$$

and for $\tau(\theta)$ decreasing

$$\frac{M(\lambda)}{\lambda} < e\tau(0)\eta(0) + \int_{-1}^{0} \nu(\theta) dr(\theta) - \int_{-1}^{0} \eta(\theta) d\tau(\theta).$$

Therefore

$$\frac{M(\lambda)}{\lambda} < e m_\tau \Delta\eta + \int_{-1}^{0} \nu(\theta) dr(\theta) - \int_{-1}^{0} \eta(\theta) d\tau(\theta) < 1$$

and consequently

$$M(\lambda) < \lambda \quad \text{for every} \quad \lambda > 0.$$

If $\lambda < 0$, then by (3.3) and (3.6) we have for $r(\theta)$ increasing

$$\frac{M(\lambda)}{\lambda} > er(-1)\nu(-1) + \int_{-1}^{0} \nu(\theta)dr(\theta) - \int_{-1}^{0} \eta(\theta)d\tau(\theta)$$

and

$$\frac{M(\lambda)}{\lambda} > -er(0)\nu(0) + \int_{-1}^{0} \nu(\theta)dr(\theta) - \int_{-1}^{0} \eta(\theta)d\tau(\theta)$$

for $r(\theta)$ decreasing. Thus

$$\frac{M(\lambda)}{\lambda} > -em_r\Delta\nu + \int_{-1}^{0} \nu(\theta)dr(\theta) - \int_{-1}^{0} \eta(\theta)d\tau(\theta) > 1$$

and also

$$M(\lambda) < \lambda \quad \text{for every} \quad \lambda < 0.$$

Then (1.5) is verified and (1.1) is oscillatory. □

We illustrate this theorem through the following example.

Example 3.4. Through Theorem 3.3 one can show that the equation

$$x'(t) = \int_{-1}^{0} x\left(t - (7 - 2\theta)\right) d(-\theta - 1) + \int_{-1}^{0} x\left(t + (-\theta + 1)\right) d(-3\theta - 3)$$

is oscillatory. As a matter of fact, for $\theta \in [-1, 0]$ one has

$$r'(\theta) = -2, \quad \nu(-1) = 0,$$
$$\tau'(\theta) = -1, \quad \eta(-1) = 0$$

and

$$\begin{aligned} 1 - em_\tau\Delta\eta &\approx 9.1548, \\ 1 + em_r\Delta\nu &\approx -18.028, \end{aligned}$$

while

$$\int_{-1}^{0} (-\theta - 1)\, d(7 - 2\theta) - \int_{-1}^{0} (-3\theta - 3)d(-\theta + 1) = -0.5.$$

Remark 3.5. Notice that Theorem 3.1 cannot be applied to the equation of Example 3.4 since

$$\begin{aligned} \int_{-1}^{0} \nu(\theta)d\ln r(\theta) + \int_{-1}^{0} \eta(\theta)d\ln \tau(\theta) &\approx 1.2898 \\ &\not< \frac{e}{m_r}\left(1 + \ln\left(-m_r\Delta\nu\right)\right) \approx 1.144. \end{aligned}$$

On the other hand, Theorem 3.3 cannot be used in the equation of Example 3.2 since

$$\int_{-1}^{0} \nu(\theta) dr(\theta) - \int_{-1}^{0} \eta(\theta) d\tau(\theta) \approx -1.75 \not> 1 + e m_r \Delta\nu \approx -0.35914.$$

Similar results can be obtained through the use of condition (1.4), as is stated in the two following theorems.

Theorem 3.6. *Let*

$$\varepsilon = \max\left\{ -e\left(\frac{m_\tau \Delta\eta}{m_r \Delta\nu}\right)^{\frac{m_r}{m_r+m_\tau}} \left(\frac{m_r}{m_\tau} + 1\right)\Delta\nu, -\frac{e}{m_\tau}\left(1 + \ln\left(m_\tau \Delta\eta\right)\right)\right\}.$$

If

$$r'(\theta)\nu(\theta) \le 0 \text{ and } \tau'(\theta)\eta(\theta) \le 0 \text{ for every } \theta \in [-1, 0] \tag{3.7}$$

and

$$\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta) > \varepsilon, \tag{3.8}$$

then the equation (1.1) *is oscillatory.*

Proof. We will follow closely the proof of Theorem 3.1. Here the assumptions a), b), c), d) and (3.7) imply that $\Delta\nu \ge 0$ and $\Delta\eta \ge 0$. As the value ε above has no meaning when either $\Delta\nu = 0$ or $\Delta\eta = 0$, we implicitly are assuming that $\Delta\nu$ and $\Delta\eta$ are both positive real numbers. For $\lambda = 0$ one has then

$$M(0) = \Delta\nu + \Delta\eta > 0.$$

Let $\lambda \neq 0$. Integrating again by parts both integrals which define the function $M(\lambda)$, the assumptions (3.7) and the inequality $u \exp(-u) \le 1$ imply that

$$\begin{aligned} M(\lambda) \ \ge \ & \exp\left(-\lambda r(0)\right)\nu(0) - \exp\left(-\lambda r(-1)\right)\nu(-1) \\ & + \exp\left(\lambda \tau(0)\right)\eta(0) - \exp\left(\lambda \tau(-1)\right)\eta(-1) \\ & + e^{-1}\left(\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta)\right). \end{aligned}$$

The arguments used in proof of Theorem 3.1 enable us to conclude that

$$\begin{aligned} M(\lambda) \ \ge \ & \exp\left(-\lambda m_r\right)\Delta\nu + \exp\left(\lambda m_\tau\right)\Delta\eta \\ & + e^{-1}\left(\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta)\right) \end{aligned}$$

and

$$\begin{aligned} M(\lambda) - \lambda \ \ge \ & \exp\left(-\lambda m_r\right)\Delta\nu + \exp\left(\lambda m_\tau\right)\Delta\eta - \lambda \\ & + e^{-1}\left(\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta)\right). \end{aligned}$$

Letting $\lambda < 0$, analogously to the proof of Theorem 3.1 we obtain

$$\begin{aligned} M(\lambda)-\lambda &> M(\lambda) \geq \exp\left(-\lambda m_r\right)\Delta\nu + \exp\left(\lambda m_\tau\right)\Delta\eta \\ &\quad + e^{-1}\left(\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta)\right) \\ &\geq \left(\frac{m_\tau \Delta\eta}{m_r \Delta\nu}\right)^{\frac{m_r}{m_r+m_\tau}} \left(\frac{m_r}{m_\tau}+1\right)\Delta\nu \\ &\quad + e^{-1}\left(\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta)\right) > 0. \end{aligned}$$

For $\lambda > 0$, we have in the same way

$$\begin{aligned} M(\lambda)-\lambda &\geq \exp\left(\lambda m_\tau\right)\Delta\eta - \lambda \\ &\quad + e^{-1}\left(\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta)\right) \\ &\geq \frac{1}{m_\tau} + \frac{1}{m_\tau}\ln\left(m_\tau \Delta\eta\right) \\ &\quad + e^{-1}\left(\int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta)\right) > 0. \end{aligned}$$

Hence

$$M(\lambda) - \lambda > 0 \quad \text{for every} \quad \lambda \in \mathbb{R}$$

and (1.1) is oscillatory. □

Example 3.7. By Theorem 3.6 the equation

$$x'(t) = \int_{-1}^{0} x\left(t-(2\theta+4)\right) d\left(-\theta^2\right) + \int_{-1}^{0} x(t+\theta+3) d\theta$$

is oscillatory. In fact, for every $\theta \in [-1, 0]$ we have

$$\begin{gathered} r(\theta) \text{ increasing}, \quad r'(\theta)\nu(\theta) = -2\theta^2 \leq 0, \text{ with } \nu(0)=0, \\ \tau(\theta) \text{ increasing}, \quad \tau'(\theta)\eta(\theta) = \theta \leq 0, \text{ with } \eta(0)=0. \end{gathered}$$

Moreover

$$\begin{aligned} \int_{-1}^{0} \nu(\theta) d\ln r(\theta) + \int_{-1}^{0} \eta(\theta) d\ln \tau(\theta) &= \int_{-1}^{0} \left(-\theta^2\right) d\ln(2\theta+4) \\ &\quad + \int_{-1}^{0} \theta d\ln(\theta+3) \approx -0.48898 > \varepsilon \end{aligned}$$

since

$$-e\left(\frac{m_\tau\Delta\eta}{m_r\Delta\nu}\right)^{\frac{m_r}{m_r+m_\tau}}\left(\frac{m_r}{m_\tau}+1\right)\Delta\nu = -e\left(\frac{2}{2}\right)^{\frac{2}{4}}\left(\frac{2}{2}+1\right)\approx -5.4366$$

and

$$-\frac{e}{m_\tau}\left(1+\ln\left(m_\tau\Delta\eta\right)\right) = -\frac{e}{2}(1+\ln 2)\approx -2.3012.$$

Theorem 3.8. *If*

$$r'(\theta)\nu(\theta)\le 0 \textit{ and } \tau'(\theta)\eta(\theta)\le 0, \textit{ for every } \theta\in[-1,0] \tag{3.9}$$

and

$$1-em_\tau\Delta\eta < \int_{-1}^{0}\nu(\theta)dr(\theta)-\int_{-1}^{0}\eta(\theta)d\tau(\theta) < 1+em_r\Delta\nu, \tag{3.10}$$

then the equation (1.1) *is oscillatory.*

Proof. As before, one has $\Delta\nu\ge 0$ and $\Delta\eta\ge 0$, and (3.10) implies that either $\Delta\nu>0$ or $\Delta\eta>0$. Therefore

$$M(0)=\Delta\nu+\Delta\eta>0.$$

Let $\lambda>0$. By (3.3) and (3.6) we have, for $\tau(\theta)$ increasing

$$\frac{M(\lambda)}{\lambda} > -e\tau(-1)\eta(-1)+\int_{-1}^{0}\nu(\theta)dr(\theta)-\int_{-1}^{0}\eta(\theta)d\tau(\theta)$$

and

$$\frac{M(\lambda)}{\lambda} > e\tau(0)\eta(0)+\int_{-1}^{0}\nu(\theta)dr(\theta)-\int_{-1}^{0}\eta(\theta)d\tau(\theta)$$

for $\tau(\theta)$ decreasing. So, for every real $\lambda>0$,

$$\frac{M(\lambda)}{\lambda} > em_\tau\Delta\eta+\int_{-1}^{0}\nu(\theta)dr(\theta)-\int_{-1}^{0}\eta(\theta)d\tau(\theta) > 1$$

and consequently

$$M(\lambda)>\lambda.$$

If $\lambda<0$, by (3.3) and (3.6) we have

$$\frac{M(\lambda)}{\lambda} < er(-1)\nu(-1)+\int_{-1}^{0}\nu(\theta)dr(\theta)-\int_{-1}^{0}\eta(\theta)d\tau(\theta)$$

when $r(\theta)$ is increasing, and

$$\frac{M(\lambda)}{\lambda} < -er(0)\nu(0)+\int_{-1}^{0}\nu(\theta)dr(\theta)-\int_{-1}^{0}\eta(\theta)d\tau(\theta)$$

for $r(\theta)$ decreasing. So, for every real $\lambda < 0$,

$$\frac{M(\lambda)}{\lambda} < -em_r\Delta\nu + \int_{-1}^{0} \nu(\theta)dr(\theta) - \int_{-1}^{0} \eta(\theta)d\tau(\theta) < 1$$

and consequently $M(\lambda) > \lambda$. □

Example 3.9. Let

$$x'(t) = \int_{-1}^{0} x\left(t - (2-\theta)\right) d(\theta+1) + \int_{-1}^{0} x\left(t + (2\theta+3)\right) d\theta.$$

By Theorem 3.8 the equation is oscillatory since

$$\int_{-1}^{0} (\theta+1)d(2-\theta) - \int_{-1}^{0} \theta d(2\theta+3) = 0.5$$

and

$$\begin{aligned} 1 - em_\tau\Delta\eta = 1 - e &\approx -1.7183, \\ 1 + em_r\Delta\nu = 1 + 2e &\approx 6.4366. \end{aligned}$$

Nonoscillations appear in situations as given in the following theorem.

Theorem 3.10. (*i*) *If for every* $\theta \in [-1, 0]$

$$r'(\theta)\nu(\theta) \le 0 \quad \textit{and} \quad \tau'(\theta)\eta(\theta) \ge 0,$$

then (1.1) *is nonoscillatory.*

(*ii*) *If for every* $\theta \in [-1, 0]$

$$r'(\theta)\nu(\theta) \ge 0 \quad \textit{and} \quad \tau'(\theta)\eta(\theta) \le 0,$$

then (1.1) *is nonoscillatory.*

Proof. We will prove only (i) as analogous arguments enable to obtain (ii). Let then

$$r'(\theta)\nu(\theta) \le 0 \quad \text{and} \quad \tau'(\theta)\eta(\theta) \ge 0$$

for every $\theta \in [-1, 0]$. Integrating by parts the first integral in (1.3) we have

$$\int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\nu(\theta) = -\exp\left(-\lambda r(-1)\right)\nu(-1) + \lambda \int_{-1}^{0} \exp\left(-\lambda r(\theta)\right)\nu(\theta)dr(\theta).$$

Therefore with $\lambda < 0$ we obtain for $r(\theta)$ increasing

$$\int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\nu(\theta) \ge -\exp\left(-\lambda r(-1)\right)\left(\nu(-1) - \lambda \int_{-1}^{0} \nu(\theta)r'(\theta)d\theta\right)$$

and for $r(\theta)$ decreasing

$$\int_{-1}^{0} \exp\left(-\lambda r(\theta)\right) d\nu(\theta) \geq \exp\left(-\lambda r(0)\right) \left(\nu(0) + \lambda \int_{-1}^{0} \nu(\theta) r(\theta) d\theta\right).$$

As both right-hand parts of these inequalities tend to $+\infty$ as $\lambda \to -\infty$, by (2.4) we have $\lim_{\lambda \to -\infty} M(\lambda) - \lambda = +\infty$ for every monotonous delay function $r(\theta)$. Analogously, integrating by parts the second integral in (1.3)

$$\int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta) = -\exp\left(\lambda \tau(-1)\right) \eta(-1) - \lambda \int_{-1}^{0} \eta(\theta) \exp\left(\lambda \tau(\theta)\right) d\tau(\theta)$$

with $\lambda > 0$, we have for $\tau(\theta)$ increasing

$$\int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta) \leq -\exp\left(\lambda \tau(-1)\right) \left(\eta(-1) + \lambda \int_{-1}^{0} \eta(\theta) \tau'(\theta) d\theta\right)$$

and for $\tau(\theta)$ decreasing

$$\int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta) = \exp\left(\lambda \tau(0)\right) \left(\eta(0) - \lambda \int_{-1}^{0} \eta(\theta) \tau'(\theta) d\theta\right).$$

This enables to conclude that, as $\lambda \to +\infty$,

$$\int_{-1}^{0} \exp\left(\lambda \tau(\theta)\right) d\eta(\theta) \to -\infty$$

exponentially, for every monotonous advance function $\tau(\theta)$. In view of (2.3), the same conclusion holds for the function $M(\lambda) - \lambda$. Thus there exists a $\lambda_0 \in \mathbb{R}$ such that $M(\lambda_0) = \lambda_0$ and consequently (1.1) is nonoscillatory. $\square$

Example 3.11. Through (i) of Theorem 3.10 one easily sees that the equation

$$x'(t) = \int_{-1}^{0} x\left(t - (2 - \theta)\right) d\left((-\theta - 1)(2\theta - 1)\right) + \int_{-1}^{0} x\left(t + (\theta + 3)\right) d\left(\theta(-\theta - 1)\right)$$

is nonoscillatory. Notice that in this example we cannot use Theorem 2.5 since $\nu(\theta)$ and $\eta(\theta)$ are not monotonic functions on $[-1, 0]$.

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