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On nonoscillation of mixed advanced-delay differential equations with positive and negative coefficients

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ABSTRACT

For a mixed (advanced-delay) differential equation with variable delays and coefficients

$$\dot{x}(t) \pm a(t)x(g(t)) \mp b(t)x(h(t)) = 0, \quad t \geq t_0$$

where

$$a(t) \geq 0, \qquad b(t) \geq 0, \qquad g(t) \leq t, \qquad h(t) \geq t$$

explicit nonoscillation conditions are obtained.

1. Introduction

Differential equations with delay in advanced arguments occur in many applied problems, see [1–6], especially in mathematical economics. There are natural delays in impact and processing in the economic models, like the policy lags [4]. A government's stabilization policy can be destabilizing due to delays in policy response [4] (if the delays are large enough). An advanced term may, for example, reflect the dependence on anticipated capital stock [5,6]. However, apart from the case of constant coefficients, there are only a few results on delayed-advanced (mixed) differential equations. The present paper partially fills this gap. It mainly deals with nonoscillation problems: we obtain criteria when a nonoscillatory solution with a prescribed tendency (either nonincreasing or nondecreasing) exists.

Recently, results on oscillation of delay differential equations (DDE) have taken the shape of a developed theory presented in monographs [7–11]. Most of the oscillation criteria for DDE can be extended to equations of advanced type (ADE) (see [7–12] and also the recent papers [13–19]). However, for mixed differential equations (MDE), i.e., equations with delay and advanced arguments, the theory is much less developed.

In this paper we consider a mixed differential equation

$$\dot{x}(t) + \delta_1 a(t)x(g(t)) + \delta_2 b(t)x(h(t)) = 0, \quad t \geq t_0 \tag{1}$$

with variable coefficients $a(t) \geq 0$, $b(t) \geq 0$, one delayed ($g(t) \leq t$) and one advanced ($h(t) \geq t$) arguments. To the best of our knowledge, oscillation of such equations has not been studied before, except partial cases of autonomous equations [20–24], equations of the second or higher order [25–27] and equations with constant delays [28]. In [29], nonoscillation only of Eq. (1) and higher order equations was considered, where δ_1 and δ_2 have the same sign. In [30] the author considers a differential equation with a deviating argument, without the assumption that it is either a delay or an advanced equation. Hence, the results of [30] can be applied to MDE (1). The results of the present paper and of [30] are independent.

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We consider Eq. (1) under the following conditions:

(a1) $a(t), b(t), g(t), h(t)$ are Lebesgue measurable locally essentially bounded functions, $a(t) \geq 0$, $b(t) \geq 0$;
(a2) $g(t) \leq t$, $h(t) \geq t$, $\lim_{t\to\infty} g(t) = \infty$.

For Eq. (1) we can consider the same initial value problem as for delay equations:

$$x(t) = \varphi(t), \quad t < t_0, x(t_0) = x_0. \tag{2}$$

Definition. An absolutely continuous (on each interval $[t_0, b]$) function $x : \mathbb{R} \to \mathbb{R}$ is called *a solution of problem* (1)–(2), if it satisfies Eq. (1) for almost all $t \in [t_0, \infty)$ and equalities (2) for $t \leq t_0$.

In the present paper, we will not discuss existence and uniqueness conditions for a solution of the problem (1)–(2). Instead, as mentioned before, we will only discuss asymptotic properties of the solutions.

Definition. Solution $x(t)$, $t_0 \leq t < \infty$, of a differential equation or inequality is called *nonoscillatory* if there exists T such that $x(t) \neq 0$ for any $t \geq T$ and *oscillatory* otherwise.

In [31] for Eq. (1), the cases when the delay and the advanced term have the same sign ($\delta_1 = \delta_2 = 1$ or $\delta_1 = \delta_2 = -1$) were investigated and the following main results were obtained.

Theorem A. *Suppose for the equation*

$$\dot{x}(t) + a(t)x(g(t)) + b(t)x(h(t)) = 0 \tag{3}$$

(a1)–(a2) *hold, functions* $a(t), b(t), g(t), h(t)$ *are equicontinuous on* $[0, \infty)$ *and*

$$\limsup_{t\to\infty}[t - g(t)] < \infty, \qquad \limsup_{t\to\infty}[h(t) - t] < \infty. \tag{4}$$

If the delay equation

$$\dot{x}(t) + a(t)x(g(t)) + b(t)x(t) = 0$$

has a nonoscillatory solution then Eq. (3) *also has a nonoscillatory solution.*

In particular, if

$$\limsup_{t\to\infty} \int_{g(t)}^{t} a(s) \exp\left\{\int_{g(s)}^{s} b(\tau)\mathrm{d}\tau\right\} \mathrm{d}s < \frac{1}{\mathrm{e}}$$

then Eq. (3) *has a nonoscillatory solution.*

Theorem B. *Suppose for the equation*

$$\dot{x}(t) - a(t)x(g(t)) - b(t)x(h(t)) = 0 \tag{5}$$

(a1)–(a2) *hold, functions* $a(t), b(t), g(t), h(t)$ *are equicontinuous on* $[0, \infty)$ *and condition* (4) *holds. If the advanced equation*

$$\dot{x}(t) - a(t)x(t) - b(t)x(h(t)) = 0$$

has a nonoscillatory solution then Eq. (5) *also has a nonoscillatory solution.*

In particular, if

$$\lim_{t\to\infty}\sup \int_{t}^{h(t)} b(s) \exp\left\{\int_{s}^{h(s)} a(\tau)\mathrm{d}\tau\right\} \mathrm{d}s < \frac{1}{\mathrm{e}}$$

then Eq. (5) *has a nonoscillatory solution.*

In the present paper, we consider the two remaining cases where coefficients have different signs: $\delta_1 = 1$, $\delta_2 = -1$ and $\delta_1 = -1$, $\delta_2 = 1$. For these cases, we obtain some rather natural explicit nonoscillation conditions for Eq. (1): in Section 2 for the former case, in Section 3 for the latter one. Finally, Section 4 involves a discussion of the results and outlines some open problems on MDE.

It is important to emphasize that, in applications of ADE and MDE, for example in economics, it is interesting to obtain not only positive solutions but also positive monotone solutions, which retain the trend. Most of the theorems in this paper present sufficient conditions when solutions of this kind exist. Some of the results also present explicit estimates for positive solutions.

We note that the methods applied in the present paper are different from [31]. The criteria (like Theorem 2) and non-explicit results of the general form (Theorem 3) are supplemented by corollaries which provide easily verified sufficient conditions.

2. Positive delay term, negative advanced term

In this section, we consider the case $\delta_1 = 1, \delta_2 = -1$ in (1) which becomes

$$\dot{x}(t) + a(t)x(g(t)) - b(t)x(h(t)) = 0, \quad t \geq t_0. \tag{6}$$

Theorem 1. *Suppose* (a1)–(a2) *hold and* $a(t) \geq b(t)$. *Then the following conditions are equivalent:*

1. *Differential inequality*

$$\dot{x}(t) + a(t)x(g(t)) - b(t)x(h(t)) \leq 0, \quad t \geq t_0, \tag{7}$$

has an eventually nonincreasing positive solution.

2. *Integral inequality*

$$u(t) \geq a(t)\exp\left\{\int_{g(t)}^{t} u(s)\mathrm{d}s\right\} - b(t)\exp\left\{-\int_{t}^{h(t)} u(s)\mathrm{d}s\right\}, \quad t \geq t_1, \tag{8}$$

has a nonnegative locally integrable solution for some $t_1 \geq t_0$, *where we assume* $u(t) = 0$ *for* $t < t_1$.

3. *Differential equation* (6) *has an eventually positive nonincreasing solution.*

Proof. (1) $\Rightarrow$ (2). Let x be a solution of (7) such that $x(t) > 0, \dot{x}(t) \leq 0, t \geq t_0$. For some $t_1 \geq t_0$ we have $g(t) \geq t_0$ for $t \geq t_1$. Denote $u(t) = -\dot{x}(t)/x(t), t \geq t_1, u(t) = 0, t < t_1$. Then

$$x(t) = x(t_1)\exp\left\{-\int_{t_1}^{t} u(s)\mathrm{d}s\right\}, \quad t \geq t_1. \tag{9}$$

After substituting (9) into (7) and carrying the exponent out of the brackets, we obtain

$$-\exp\left\{-\int_{t_1}^{t} u(s)\mathrm{d}s\right\} x(t_1)\left[u(t) - a(t)\exp\left\{\int_{g(t)}^{t} u(s)\mathrm{d}s\right\} + b(t)\exp\left\{-\int_{t}^{h(t)} u(s)\mathrm{d}s\right\}\right] \leq 0.$$

Hence (8) holds.

(2) $\Rightarrow$ (3). Suppose $u_0(t) \geq 0, t \geq t_1$ is a solution of inequality (8). Consider the following sequence

$$u_{n+1}(t) = a(t)\exp\left\{\int_{g(t)}^{t} u_n(s)\mathrm{d}s\right\} - b(t)\exp\left\{-\int_{t}^{h(t)} u_n(s)\mathrm{d}s\right\}, \quad n \geq 0. \tag{10}$$

Since $u_n(t) \geq a(t) - b(t) \geq 0$ and $u_0 \geq u_1$ then by induction

$$0 \leq u_{n+1}(t) \leq u_n(t) \leq \cdots \leq u_0(t).$$

Hence, there exists a pointwise limit

$$u(t) = \lim_{n\to\infty} u_n(t).$$

The Lebesgue convergence theorem and (10) imply

$$u(t) = a(t)\exp\left\{\int_{g(t)}^{t} u(s)\mathrm{d}s\right\} - b(t)\exp\left\{-\int_{t}^{h(t)} u(s)\mathrm{d}s\right\}.$$

Then $x(t)$ denoted by (9) is a nonnegative nonincreasing solution of Eq. (6).

Implication (3) $\Rightarrow$ (1) is evident. □

For comparison, now consider the following MDE

$$\dot{x}(t) + a_1(t)x(g_1(t)) - b_1(t)x(h_1(t)) = 0, \quad t \geq t_0. \tag{11}$$

Corollary 1.1. *Suppose* (a1)–(a2) *hold for* $a, b, h, g, a_1, b_1, h_1, g_1$ *and*

$$b_1(t) \leq b(t) \leq a(t) \leq a_1(t), \qquad g(t) \geq g_1(t), \qquad h(t) \leq h_1(t). \tag{12}$$

If Eq. (11) *has an eventually positive solution with an eventually nonpositive derivative, then the same is valid for Eq.* (6).

Proof. Suppose (11) has an eventually positive solution with an eventually nonpositive derivative. By Theorem 1 the integral inequality

$$u(t) \geq a_1(t)\exp\left\{\int_{g_1(t)}^{t} u(s)\mathrm{d}s\right\} - b_1(t)\exp\left\{-\int_{t}^{h_1(t)} u(s)\mathrm{d}s\right\}, \quad t \geq t_1,$$

has a nonnegative locally integrable solution $u(t)$ for some t_1. Inequalities (12) imply that $u(t)$ also satisfies (8). Thus by Theorem 1 Eq. (6) has an eventually positive solution with an eventually nonpositive derivative. □

Corollary 1.2. *Suppose* (a1)–(a2) *hold, for t sufficiently large $a(t) \geq b(t)$ and*

$$b(t) \geq a(t)\left[\exp\left\{\int_{g(t)}^{t} a(s)\mathrm{d}s\right\} - 1\right]\exp\left\{\int_{t}^{h(t)} a(s)\mathrm{d}s\right\}.$$

Then there exists an eventually positive solution with an eventually nonpositive derivative of Eq. (6).

Proof. It is easy to see that $u(t) = a(t)$ is a nonnegative solution of inequality (8). □

Corollary 1.3. *Suppose* (a1)–(a2) *hold and there exist $a > 0, b > 0, \tau > 0, \sigma > 0$ such that*

$$b \leq b(t) \leq a(t) \leq a, \qquad g(t) \geq t - \tau, \qquad h(t) \leq t + \sigma.$$

If there exists a solution $\lambda > 0$ of the algebraic equation

$$-\lambda + a\mathrm{e}^{\lambda\tau} - b\mathrm{e}^{-\lambda\sigma} = 0 \tag{13}$$

then Eq. (6) *has an eventually positive solution with an eventually nonpositive derivative.*

Proof. The function $x(t) = \mathrm{e}^{-\lambda t}$ is a positive solution of the equation

$$\dot{x}(t) + ax(t-\tau) - bx(t+\sigma) = 0, \quad t \geq 0. \tag{14}$$

By Corollary 1.1 Eq. (6) has a nonoscillatory solution. □

Corollary 1.4. *Suppose* (a1)–(a2) *hold, $a(t) \geq b(t)$ and there exists a nonoscillatory solution of the delay equation*

$$\dot{x}(t) + a(t)x(g(t)) = 0. \tag{15}$$

Then there exists an eventually positive solution with an eventually nonpositive derivative of Eq. (6).

Proof. Theorem 1 in [32] implies that there exists a nonnegative solution $u(t)$ of the inequality

$$u(t) \geq a(t)\exp\left\{\int_{g(t)}^{t} u(s)\mathrm{d}s\right\}.$$

Hence $u(t)$ is also a nonnegative solution of inequality (8), then by Theorem 1 Eq. (6) has a nonoscillatory solution. □

Remark. Eq. (15) has a nonoscillatory solution [8] if for t sufficiently large

$$\int_{g(t)}^{t} a(s)\,\mathrm{d}s \leq \frac{1}{\mathrm{e}}.$$

Corollary 1.5. *Suppose* (a1)–(a2) *hold, $a(t) \geq b(t)$, the integral inequality* (8) *has a nonnegative solution for $t \geq t_1$ and $\int_0^\infty [a(s) - b(s)]\mathrm{d}s = \infty$. Then there exists an eventually positive solution $x(t)$ with an eventually nonpositive derivative of Eq.* (6) *such that* $\lim_{t\to\infty} x(t) = 0$.

Proof. By the assumption of the theorem, there exists a nonnegative solution $u(t)$ of Eq. (8), which obviously satisfies $u(t) \geq a(t) - b(t)$. Then the function $x(t)$ defined by (9) is a solution of (6). For this solution we have $0 < x(t) \leq x(t_1) \exp\left\{-\int_0^t [a(s) - b(s)]\mathrm{d}s\right\}$. Hence $\lim_{t\to\infty} x(t) = 0$. □

Corollaries 1.4 and 1.5 imply the following statement.

Corollary 1.6. *Suppose* (a1)–(a2) *hold, $a(t) \geq b(t)$, for t sufficiently large $\int_{g(t)}^t a(s)\,\mathrm{d}s \leq \frac{1}{\mathrm{e}}$ and $\int_0^\infty [a(s) - b(s)]\,\mathrm{d}s = \infty$. Then the equation*

$$\dot{x}(t) + a(t)x(t) - b(t)x(h(t)) = 0$$

has an eventually positive solution $x(t)$ with an eventually nonpositive derivative such that $\lim_{t\to\infty} x(t) = 0$.

Example 1. Consider the equation

$$\dot{x}(t) + 1.4x(t-0.3) - 1.3x(t+0.3) = 0. \tag{16}$$

Then $u(t) \equiv 1$ is a solution of the relevant inequality (8) since $1.4\mathrm{e}^{0.3} - 1.3\mathrm{e}^{-0.3} \approx 0.9267 < 1$. We remark that since $1.3 \cdot 0.3 = 0.39 > \frac{1}{\mathrm{e}} \approx 0.368$, then all solution of both equations

$$\dot{x}(t) + 1.4x(t-0.3) = 0 \tag{17}$$

and

$$\dot{x}(t) - 1.3x(t+0.3) = 0 \tag{18}$$

are oscillatory [9]. The characteristic equation

$$-\lambda + 1.4e^{0.3\lambda} - 1.3e^{-0.3\lambda} = 0 \tag{19}$$

has three real roots: $\lambda_1 \approx -4.2282$, $\lambda_2 \approx 0.5436$ and $\lambda_3 \approx 3.3541$, thus $e^{-\lambda_1 t}$, $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$ are three nonoscillatory solutions of (16), the first one is unbounded on $[0, \infty)$ and the other two are bounded and have a negative derivative.

Example 2. Consider the equation

$$\dot{x}(t) + (1.375 + 0.025\sin t)x(t-0.3) - (1.325 + 0.025\cos t)x(t+0.3) = 0. \tag{20}$$

Since $1.3 \le 1.325 + 0.025\cos t \le 1.35 \le 1.375 + 0.025\sin t \le 1.4$ then applying Corollary 1.1 and the results of Example 1 we obtain that (20) has an eventually positive solution $x(t)$ with an eventually nonpositive derivative. Moreover, since the integral of a continuous nonnegative periodic function $\int_0^\infty (0.05 + 0.025\sin t - 0.025\cos t)\,dt$ diverges, then by Corollary 1.5 this solution satisfies $\lim_{t\to\infty} x(t) = 0$.

Theorem 2. *Suppose* (a1)–(a2) *hold and* $b(t) \ge a(t)$. *Then the following conditions are equivalent:*

1. *Differential inequality*

$$\dot{x}(t) + a(t)x(g(t)) - b(t)x(h(t)) \ge 0, \quad t \ge t_0, \tag{21}$$

has an eventually positive solution with an eventually nonnegative derivative.

2. *Integral inequality*

$$u(t) \ge b(t)\exp\left\{\int_t^{h(t)} u(s)ds\right\} - a(t)\exp\left\{-\int_{g(t)}^t u(s)ds\right\}, \quad t \ge t_1, \tag{22}$$

$u(t) = 0$, $t < t_1$, *has a nonnegative locally integrable solution for some* $t_1 \ge t_0$.

3. *Differential equation* (6) *has an eventually positive solution with an eventually nonnegative derivative.*

Proof. (1) $\Rightarrow$ (2). Let x be a solution of (21) such that $x(t) > 0$, $\dot{x}(t) \ge 0$, $t \ge t_0$. For some $t_1 \ge t_0$ we have $g(t) \ge t_0$ for $t \ge t_1$. Denote $u(t) = \dot{x}(t)/x(t)$, $t \ge t_1$. Then

$$x(t) = x(t_1)\exp\left\{\int_{t_1}^t u(s)ds\right\}, \quad t \ge t_1. \tag{23}$$

After substituting (23) into (21) and carrying the exponent out of the brackets we obtain

$$\exp\left\{\int_{t_1}^t u(s)ds\right\} x(t_1)\left[u(t) - b(t)\exp\left\{\int_t^{h(t)} u(s)ds\right\} + a(t)\exp\left\{-\int_{g(t)}^t u(s)ds\right\}\right] \ge 0.$$

Hence (22) holds.

(2) $\Rightarrow$ (3). Let $u_0(t) \ge 0$ be a solution of inequality (22). Consider the sequence

$$u_{n+1}(t) = b(t)\exp\left\{\int_t^{h(t)} u_n(s)ds\right\} - a(t)\exp\left\{-\int_{g(t)}^t u_n(s)ds\right\}, \quad n \ge 0. \tag{24}$$

Inequalities $u_n(t) \ge b(t) - a(t)$ and $u_0 \ge u_1$ imply

$$0 \le u_{n+1}(t) \le u_n(t) \le \cdots \le u_0(t).$$

Hence, there exists a pointwise limit

$$u(t) = \lim_{n\to\infty} u_n(t).$$

By the Lebesgue convergence theorem and (24), we have

$$u(t) = b(t)\exp\left\{\int_t^{h(t)} u(s)ds\right\} - a(t)\exp\left\{-\int_{g(t)}^t u(s)ds\right\}. \tag{25}$$

Then $x(t)$ defined by (23) is a positive solution of Eq. (6) with a nonnegative derivative.

Implication (3) $\Rightarrow$ (1) is evident. □

The proofs of the following corollaries can be given, and are very similar to those given for Theorem 1, hence we omit most of them.

Corollary 2.1. *Suppose* (a1)–(a2) *hold for Eqs.* (6) *and* (11),

$$a_1(t) \le a(t) \le b(t) \le b_1(t), \qquad g(t) \ge g_1(t), \qquad h(t) \le h_1(t).$$

If Eq. (11) *has an eventually positive solution with an eventually nonnegative derivative then so does Eq.* (6).

Corollary 2.2. *Suppose* (a1)–(a2) *hold, for t sufficiently large $b(t) \ge a(t)$ and*

$$a(t) \ge b(t)\left[\exp\left\{\int_t^{h(t)} b(s)\mathrm{d}s\right\} - 1\right]\exp\left\{\int_{g(t)}^t b(s)\mathrm{d}s\right\}.$$

Then there exists an eventually positive solution with an eventually nonnegative derivative of Eq. (6).

Proof. It is easy to see that $u(t) = b(t)$ is a nonnegative solution of inequality (22). □

Corollary 2.3. *Suppose* (a1)–(a2) *hold and there exist $a > 0$, $b > 0$, $\tau > 0$, $\sigma > 0$ such that*

$$a \le a(t) \le b(t) \le b, \qquad g(t) \ge t - \tau, \qquad h(t) \le t + \sigma.$$

If there exists a positive solution $\lambda > 0$ of the algebraic equation

$$\lambda + a\mathrm{e}^{-\lambda\tau} - b\mathrm{e}^{\lambda\sigma} = 0 \tag{26}$$

then there exists an eventually positive solution with an eventually nonnegative derivative of Eq. (6).

Corollary 2.4. *Suppose* (a1)–(a2) *hold, $b(t) \ge a(t)$ and there exists a nonoscillatory solution of the advanced equation*

$$\dot{x}(t) - b(t)x(h(t)) = 0. \tag{27}$$

Then there exists an eventually positive solution with an eventually nonnegative derivative of Eq. (6).

Remark. If for t sufficiently large

$$\int_t^{h(t)} b(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}},$$

then [9] there exists a nonoscillatory solution of (27).

Corollary 2.5. *Suppose* (a1)–(a2) *hold, $b(t) \ge a(t)$, the integral inequality* (22) *has a nonnegative solution for $t \ge t_1$ and $\int_0^\infty [b(s) - a(s)]\,\mathrm{d}s = \infty$. Then there exists an eventually positive solution with an eventually nonnegative derivative $x(t)$ of* (6) *such that $\lim_{t\to\infty} x(t) = \infty$.*

3. Negative delay term, positive advanced term

In this section we consider the following scalar MDE (1) with $\delta_1 = -1$, $\delta_2 = 1$:

$$\dot{x}(t) - a(t)x(g(t)) + b(t)x(h(t)) = 0, \quad t \ge t_0, \tag{28}$$

Theorem 3. *Suppose that $a(t)$, $b(t)$ are continuous bounded on $[0, \infty)$ functions, $g(t)$ and $h(t)$ are equicontinuous on $[0, \infty)$ functions, there exist positive numbers a_1, a_2, b_1, b_2, τ, σ, t_1 such that*

$$a_1 \le a(t) \le a_2, \qquad b_1 \le b(t) \le b_2, \qquad t - g(t) \le \tau, \qquad h(t) - t \le \sigma, t \ge t_1, \tag{29}$$

and the following algebraic system

$$\begin{cases} a_2\mathrm{e}^{y\tau} - b_1\mathrm{e}^{-y\sigma} \le x, \\ b_2\mathrm{e}^{x\sigma} - a_1\mathrm{e}^{-x\tau} \le y \end{cases} \tag{30}$$

has a solution $x = d_1 > 0$, $y = d_2 > 0$.

Then Eq. (28) *has a nonoscillatory solution.*

Proof. Define the following operator in the space $\mathbf{C}[t_1, \infty)$ of all bounded continuous on $[t_1, \infty)$ functions with the usual sup-norm

$$(Au)(t) = a(t)\exp\left\{-\int_{g(t)}^t u(s)\mathrm{d}s\right\} - b(t)\exp\left\{\int_t^{h(t)} u(s)\mathrm{d}s\right\}, \quad t \ge t_1,$$

$u(t) = 0, t \le t_1$. Let $x = d_1, y = d_2$ be a positive solution of system (30). Then the inequality $-d_2 \le u(t) \le d_1$ implies $-d_2 \le (Au)(t) \le d_1$. It means that $A(S) \subset S$, where

$$S = \{u(t) : -d_2 \le u(t) \le d_1\}.$$

Now we will prove that $A(S)$ is a compact set in the space $\mathbf{C}[t_1, \infty)$. Denote the integral operators

$$(Hu)(t) := \int_{g(t)}^{t} u(s)\mathrm{d}s, \qquad (Ru)(t) := \int_{t}^{h(t)} u(s)\mathrm{d}s.$$

We have for $u \in S$

$$|(Hu)(t)| \le \max\{d_1, d_2\}\tau, \qquad |(Ru)(t)| \le \max\{d_1, d_2\}\sigma.$$

Hence the sets $H(S)$ and $R(S)$ are bounded in the space $\mathbf{C}[t_1, \infty)$.

Let $u \in S$. Then

$$\begin{aligned}|(Hu)(\tau_2) - (Hu)(\tau_1)| &\le \left|\int_{g(\tau_1)}^{g(\tau_2)} |u(s)|\mathrm{d}s\right| + \left|\int_{\tau_1}^{\tau_2} |u(s)|\mathrm{d}s\right| \\ &\le \max\{d_1, d_2\}\left(|g(\tau_2) - g(\tau_1)| + |\tau_2 - \tau_1|\right)\end{aligned}$$

and, similarly,

$$|(Ru)(\tau_2) - (Ru)(\tau_1)| \le \max\{d_1, d_2\}(|h(\tau_2) - h(\tau_1)| + |\tau_2 - \tau_1|).$$

Since g, h are equicontinuous in $[t_1, \infty)$, then functions in $H(S)$ and $R(S)$ are also equicontinuous. Then the sets $H(S)$ and $R(S)$ are compact. Consequently, $A(S)$ is also a compact set.

Schauder's fix point theorem implies that there exists a solution $u \in S$ of operator equation $u = Au$. Therefore the function $x(t) = x(t_1) \exp\left\{\int_{t_1}^{t} u(s)\mathrm{d}s\right\}$, $t \ge t_1$, is a positive solution of Eq. (28). $\square$

Corollary 3.1. *Suppose that $a(t), b(t)$ are continuous bounded on $[0, \infty)$ functions, $g(t)$ and $h(t)$ are equicontinuous on $[0, \infty)$ and there exist positive numbers $a_1, a_2, b_1, b_2, \tau, \sigma$ such that (29) is satisfied and at least one of the following conditions holds:*

(1) $b_2 < a_1, 0 < a_2 - b_1 < \frac{1}{\tau+\sigma} \ln \frac{a_1}{b_2}$,

(2) $a_2 < b_1, 0 < b_2 - a_1 < \frac{1}{\tau+\sigma} \ln \frac{b_1}{a_2}$.

Then Eq. (28) has a nonoscillatory solution.

Proof. It suffices to prove that system (30) has a positive solution. Suppose condition (1) holds. For the second condition, the proof is similar.

Let us define the functions $f(x) = b_2 e^{x\sigma} - a_1 e^{-x\tau}$ and $g(y) = a_2 e^{y\tau} - b_1 e^{-y\sigma}$ which are both monotone increasing. We have $f(0) < 0, f(x_1) = 0$, where

$$x_1 = \frac{1}{\tau + \sigma} \ln \frac{a_1}{b_2}.$$

Since $g(y)$ is a monotone function then there exists the monotone increasing inverse function $g^{-1}(x)$, for which we have $g^{-1}(x_2) = 0$, where $x_2 = a_2 - b_1 > 0$.

Denote $h(x) = g^{-1}(x) - f(x)$. Condition (1) implies $x_2 < x_1$, then $f(x_2) < 0$ and thus $h(x_2) > 0$.

From the equality $g(y) = a_2 e^{y\tau} - b_1 e^{-y\sigma} = x$ we have $a_2 e^{y\tau} \le x + b_1$ for $y \ge 0$. Then

$$g^{-1}(x) \le \frac{1}{\tau} \ln \frac{x + b_1}{a_2} \quad \text{and} \quad h(x) \le \frac{1}{\tau} \ln \frac{x + b_1}{a_2} - b_2 e^{x\sigma} + a_1$$

for x large enough. Hence $\lim_{x\to\infty} h(x) = -\infty$.

Since h is continuous, then there exists $x_0 > x_2 > 0$ such that $h(x_0) = 0$. It means that $f(x_0) = g^{-1}(x_0)$. Therefore $x_0, y_0 = f(x_0)$ is a solution of the system (30). Hence Eq. (28) has a nonoscillatory solution. $\square$

If the conditions of Corollary 3.1 do not hold, we can apply numerical methods to prove that system (30) has a positive solution.

Example 3. Consider the equation

$$\dot{x}(t) - (1.3 + 0.1 \sin t)x(t - 0.1 - 0.1 \cos t) + (1.7 + 0.1 \cos t)x(t + 0.2 + 0.1 \sin t) = 0, \quad t \ge 0. \tag{31}$$

Then $a_1 = 1.2, a_2 = 1.4, b_1 = 1.6, b_2 = 1.8, \tau = 0.2, \sigma = 0.3$ and (30) has a positive solution $x = 2, y = 3$, since

$$a_2 e^{y\tau} - b_1 e^{-y\sigma} = 1.4e^{0.6} - 1.6e^{-0.9} \approx 1.9 < x = 2,$$
$$b_2 e^{x\sigma} - a_1 e^{-x\tau} \le y = 1.8e^{0.6} - 1.2e^{-0.4} \approx 2.48 < y = 3.$$

Hence Eq. (31) has a nonoscillatory solution.

Fig. 1 illustrates the domain of values (x, y) satisfying the system of inequalities (30) for Eq. (31) which is between the two curves.

Example 4. Consider the equation with constant coefficients and variable advance and delay

$$\dot{x}(t) - ax(g(t)) + bx(h(t)) = 0, \quad t \geq 0, \tag{32}$$

where $t \geq g(t) \geq t - 0.2$, $t \leq h(t) \leq t + 0.3$, as in Example 3. Thus, in (30) we have $\tau = 0.2$, $\delta = 0.3$. All values below the curve in Fig. 2 are such that the system of inequalities (30) has a positive solution and hence Eq. (32) has a nonoscillatory solution.

For comparison, we also included the line

$$0.2a + 0.3b < \frac{1}{\mathrm{e}}. \tag{33}$$

Remark. The autonomous equation

$$\dot{x}(t) - ax(t - \tau) + bx(t + \sigma) = 0, \quad a > 0, b > 0, \tau > 0, \sigma > 0, \tag{34}$$

always has a positive solution $\mathrm{e}^{\lambda t}$, where λ is a solution of the characteristic equation

$$f(\lambda) = \lambda - a\mathrm{e}^{-\tau\lambda} + b\mathrm{e}^{\sigma\lambda} = 0. \tag{35}$$

Since $\lim_{\lambda \to \pm\infty} f(\lambda) = \pm\infty$, then there is always a real λ satisfying (35). There is a positive solution satisfying $\lim_{t\to\infty} x(t) = \infty$ if $b < a$, and a positive solution satisfying $\lim_{t\to\infty} x(t) = 0$ if $b > a$.

4. Discussion and open problems

In this paper, we presented results for equations with variable arguments and coefficients, one delay and one advanced term in the case when coefficients have any of four possible sign combinations; the results for coefficients of different signs are new. If the delayed term is positive, we not only claim the existence of a positive solution but present sufficient conditions under which its asymptotics can be deduced (i.e., a nonincreasing positive solutions which tends to zero or a nondecreasing solution which tends to infinity).

Below, we present some open problems and topics for research and discussion.

1. If we consider the characteristic equation of the autonomous equation (14), it is easy to prove that Eq. (14) has a positive solution for any positive coefficients $a > 0$, $b > 0$.

   Prove or disprove:

   If conditions (a1)–(a2) hold,

   $$a_1 \leq a(t) \leq a_2, \qquad b_1 \leq b(t) \leq b_2, \qquad t - g(t) \leq \tau, \qquad h(t) - t \leq \sigma,$$

   then Eq. (6) has an eventually positive solution.

   Consider the same problem for Eq. (28).
2. Prove or disprove:

   If $a(t) \geq b(t) \geq 0$ and (6) has an eventually positive solution with an eventually nonpositive derivative which tends to zero, then all solutions of the ordinary differential equation

   $$\dot{x}(t) + [a(t) - b(t)]x(t) = 0 \tag{36}$$

   tend to zero as $t \to \infty$. If $b(t) \geq a(t) \geq 0$ and (6) has an eventually positive solution with an eventually nonnegative derivative which tends to infinity as $t \to \infty$, then all solutions of (36) tend to infinity.
3. For Eq. (6), obtain sufficient conditions under which there exists a positive solution which tends to some $d \neq 0$. Is convergence of $\int_0^\infty a(s)\,\mathrm{d}s$ and $\int_0^\infty b(s)\,\mathrm{d}s$ sufficient, together with the other conditions of Theorem 1? Consider the same problem for (28). We remark that, for first order delay neutral equations with positive and negative coefficients in [33,34], sufficient conditions were found, under which all solutions either oscillate or tend to zero, and for the second order neutral delay equation such conditions can be found in the recent paper [35]. It would be interesting to find conditions which guarantee the existence of a positive solution (which tends to zero) for these equations. For some sufficient conditions for linear models without a neutral part, see [36].
4. Find sufficient conditions when Eq. (28) has a positive nonincreasing solution $x(t)$ such that $\lim_{t\to\infty} x(t) = 0$ or find sufficient conditions when Eq. (28) has a positive solution $x(t)$ with a nonnegative derivative such that $\lim_{t\to\infty} x(t) = \infty$.
5. Obtain sufficient conditions when the equation with one term which can be both advanced and delayed, and an oscillating coefficient

   $$\dot{x}(t) + a(t)x(g(t)) = 0 \tag{37}$$

   has a positive solution. For instance, if $a(t)[t - g(t)]$ is either positive or negative for any t, then (37) can be rewritten in the form (6) or (28), thus some conditions can be deduced from the results of the present paper.

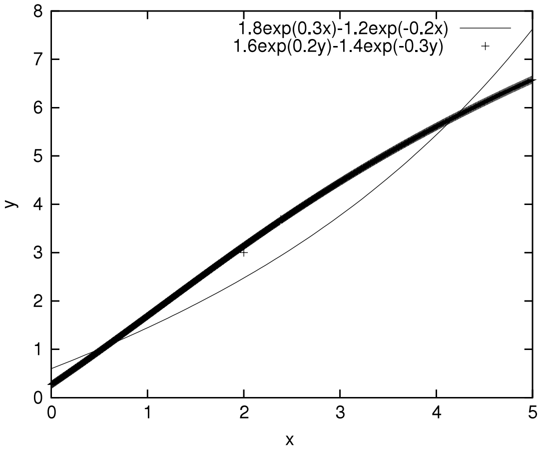

Fig. 1. The domain of values (x, y) satisfying the system of inequalities (30) for Eq. (31) is between the curves. The chosen value $x = 2, y = 3$ inside the domain is also marked on the graph.

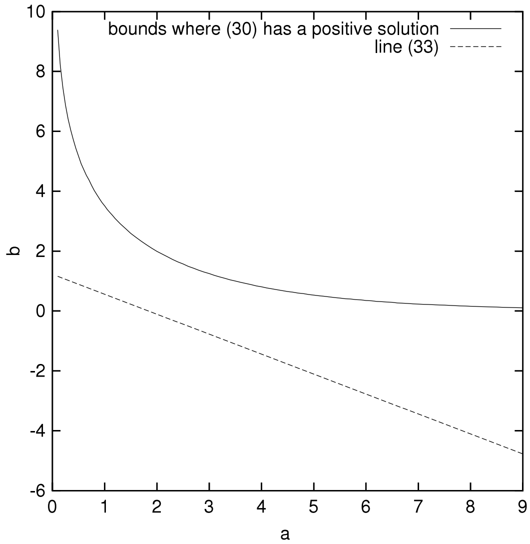

Fig. 2. The domain of values a, b such that the system of inequalities (30) for Eq. (32) has a positive solution is under the curve.

6. Prove or disprove:
   Suppose $a(t) \geq 0, b(t) \geq 0, \int_0^\infty [a(t) + b(t)]dt = \infty$.
   If Eq. (3) has a positive solution then this equation is asymptotically stable.
   If Eq. (5) has a positive solution then the absolute value of any nontrivial solution tends to infinity.
7. Instead of initial condition (2) for MDE we can formulate the "final" conditions

$$x(t) = \varphi(t), \quad t > t_0, \qquad x(t_0) = x_0, \tag{38}$$

studying nonoscillation for $t < T \leq t_0$ and asymptotics for $t \to -\infty$. How would the results of the present paper change if obtained for (6), (38) or (28), (38)?

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