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Oscillatory retarded functional systems

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Abstract

This note is concerned with the oscillatory behavior of a linear retarded system. Several criteria are obtained for having the system oscillatory. Conditions regarding the existence of nonoscillatory solutions are also given.

1. Introduction

The oscillation theory of delay equations has received a large amount of attention during the last two decades, as one can see through the textbooks [1–5] and references therein. However, excepting discrete difference systems and particular results which can be obtained through some studies regarding differential systems of neutral type, some gaps can be found in the literature with respect to functional retarded systems.

The aim of this note is to study the oscillatory behavior of the system

$$x(t) = \int_{-1}^{0} x\bigl(t - r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr], \tag{1}$$

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where $x(t) \in \mathbb{R}^n$, $r(\theta)$ is a real continuous function on $[-1,0]$, positive on $[-1,0[$, and $\nu(\theta)$ is a real n-by-n matrix valued function of bounded variation on $[-1,0]$, which in case of having $r(0)=0$ will be assumed atomic at zero, that is, such that

$$\lim_{\gamma\to 0^+} \int_{-\gamma}^{0} \left\| d\left[\eta(\theta)\right]\right\| = 0,$$

where for a given norm, $\|\cdot\|$, in the space $\mathbb{M}_n(\mathbb{R})$, of all real n-by-n matrices, by

$$\int_a^b \left\| d\left[\eta(\theta)\right]\right\|$$

we mean the total variation of ν on an interval $[a,b] \subset [-1,0]$. Notice that when $r(\theta)$ is positive on $[-1,0]$, no atomicity assumption on the function ν is necessary.

The system (1) for $r(\theta) = -r\theta$ $(r>0)$ and $\theta \in [-1,0]$, is the class of linear retarded functional systems

$$x(t) = \int_{-r}^{0} x(t+\theta)\, d\left[\eta(\theta)\right], \tag{2}$$

where $\eta(\theta) = \nu(\theta/r)$ is assumed to be atomic at zero. This is the most common general linear retarded functional system appearing in the literature (see [6]). Our preference on system (1) regards the possibility of understanding more clearly the role of the delays on the oscillatory behavior of functional retarded systems.

Considering the value $\|r\| = \max\{r(\theta)\colon -1 \leqslant \theta \leqslant 0\}$, a continuous function $x : [-\|r\|, +\infty[\to \mathbb{R}$, is said a solution of (1) if satisfies this equation for every $t \geqslant 0$. A solution of (1), $x(t) = [x_1(t),\dots,x_n(t)]^T$, is called *oscillatory* if every component, $x_j(t)$, $j=1,\dots,n$, has arbitrary large zeros; otherwise $x(t)$ is said a *nonoscillatory* solution. Whenever all solutions of (1) are oscillatory we will say that (1) is an *oscillatory* system.

Both systems (1) and (2) include the important class of the delay difference systems

$$x(t) = \sum_{j=1}^{p} A_j\, x(t-r_j), \tag{3}$$

where the A_j are n-by-n real matrices and the r_j are positive real numbers. This case corresponds to have in (1), ν as a step function of the form

$$\nu(\theta) = \sum_{j=1}^{p} H(\theta-\theta_j)A_j, \tag{4}$$

where H denotes the Heaviside function, $-1 < \theta_1 < \cdots < \theta_p < 0$, and $r(\theta)$ is any continuous and positive function on $[-1,0]$ such that $r(\theta_j) = r_j$, for $j=1,\dots,p$.

The oscillatory behavior of this class of systems is studied in [7]. A specific treatment for discrete difference systems is included in [3].

As is well known the systems (1), (2), and (3) can be looked, respectively, as particular cases of the differential systems of neutral type

$$\frac{d}{dt}\left(x(t) - \int_{-1}^{0} x\big(t - r(\theta)\big)\, d\big[\nu(\theta)\big]\right) = \int_{-1}^{0} x\big(t - r(\theta)\big)\, d\big[\eta(\theta)\big],$$

$$\frac{d}{dt}\left(x(t) - \int_{-r}^{0} x(t + \theta)\, d\big[\nu(\theta)\big]\right) = \int_{-r}^{0} x(t + \theta)\, d\big[\eta(\theta)\big], \tag{5}$$

$$\frac{d}{dt}\left(x(t) - \sum_{j=1}^{p} A_j x(t - r_j)\right) = \sum_{j=1}^{p} B_j\, x(t - r_j). \tag{6}$$

Several criteria for having (6) oscillatory can be found in [2–4], but in all of them, the matrices B_j cannot be null, which excludes necessarily the system (3). In [8], the Theorem 3.3 and the Corollaries 4.2 and 4.3, are oscillation criteria obtained in the regard of the systems (5) and (6), which can as well include the systems (2) and (3), respectively. However the results which will be presented here are of different kind.

According to [9], the analysis of the oscillatory behavior of solutions of the system (1), can be based upon the existence or absence of real zeros of the characteristic equation

$$\det\left[I - \int_{-1}^{0} \exp\big(-\lambda r(\theta)\big)\, d\big[\nu(\theta)\big]\right] = 0, \tag{7}$$

where by I we mean the n-by-n identity matrix. In fact, in this framework, one can conclude that (1) is oscillatory if and only if (7) has no real roots, that is, if and only if

$$1 \notin \sigma\left(\int_{-1}^{0} \exp\big(-\lambda r(\theta)\big)\, d\big[\nu(\theta)\big]\right) \tag{8}$$

for every $\lambda \in \mathbb{R}$; nonoscillatory solutions will exist, whenever (7) has at least a real root.

We will denote by BV_n the space of all functions of bounded variation, $\eta : [-1, 0] \to \mathbb{M}_n(\mathbb{R})$. The space BV_1, of all real functions of bounded variation on $[-1, 0]$, will be denoted simply by BV. For $\phi \in BV$ by

$$\int_{-1}^{0} \big|d\phi(\theta)\big|,$$

we will mean the total variation of ϕ on $[-1, 0]$. We notice that if $\eta \in BV_n$ and with $j, k = 1, \ldots, n$, $\eta(\theta) = [\eta_{jk}(\theta)]$, then each function $\eta_{jk} \in BV$. The matrix

$$|\eta| = \left[\int_{-1}^{0} \big|d\eta_{jk}(\theta)\big|\right].$$

will also be considered.

For any $\eta \in BV_n$ we can formulate the right and left hand limit matrices at any point $\theta \in [-1, 0]$, $\eta(\theta^+)$ and $\eta(\theta^-)$, as well as the right and left hand oscillation matrices

$$\Omega_\eta^+(\theta) = \eta(\theta^+) - \eta(\theta) \quad \text{and} \quad \Omega_\eta^-(\theta) = \eta(\theta) - \eta(\theta^-).$$

Nonnegative matrices enable us to consider monotonic n-by-n matrix valued functions. For this purpose we recall that a n-by-n real matrix $C = [c_{jk}]$ $(j, k = 1, \ldots, n)$ is said to be nonnegative (positive) whenever $c_{jk} \geqslant 0$, (respectively, $c_{jk} > 0$) for every $j, k = 1, \ldots, n$. These properties will be expressed as usual, through the notations $C \geqslant 0$ and $C > 0$, respectively. More generally given two n-by-n real matrices, C and D, we will say that $C \leqslant D$ ($C < D$) if $D - C \geqslant 0$ (respectively, $D - C > 0$).

Therefore we will say that a function $\eta : [-1, 0] \to \mathbb{M}_n(\mathbb{R})$ is nondecreasing (nonincreasing) on a interval $J \subset [-1, 0]$, if for every $\theta_1, \theta_2 \in J$ such that $\theta_1 < \theta_2$ one has $\eta(\theta_1) \leqslant \eta(\theta_2)$ (respectively, $\eta(\theta_2) \leqslant \eta(\theta_1)$); η will be said increasing (decreasing) on J, if η is nondecreasing (respectively, nonincreasing) on J and there exist $\theta_1, \theta_2 \in J$ such that $\theta_1 < \theta_2$ and $\eta(\theta_1) < \eta(\theta_2)$ (respectively, $\eta(\theta_2) < \eta(\theta_1)$). If for every $\varepsilon > 0$, sufficiently small, η is increasing (decreasing) in $[\theta - \varepsilon, \theta + \varepsilon]$ ($[-\varepsilon, 0]$ if $\theta = 0$, $[-1, -1 + \varepsilon]$ if $\theta = -1$) we will say that θ is a point of increase of η (respectively, a point of decrease of η).

As is well known, any function $\phi \in BV$ can be decomposed as the difference of two nondecreasing functions α and β: $\phi = \alpha - \beta$. This decomposition is not unique and a particular decomposition of ϕ is given by

$$\phi = \varphi - \psi, \tag{9}$$

where by φ and ψ we denote, respectively, the positive and negative variation of ϕ, which are defined as follows. For each $\theta \in [-1, 0]$, let $\mathfrak{P}_\theta$ be the set of all partitions $P = \{-1 = \theta_0, \theta_1, \ldots, \theta_k = \theta\}$ of the interval $[-1, \theta]$ and to each $P \in \mathfrak{P}_\theta$ associate the sets

$$A(P) = \bigl\{j\colon\ \phi(\theta_j) - \phi(\theta_{j-1}) > 0\bigr\} \quad \text{and} \quad B(P) = \bigl\{j\colon\ \phi(\theta_j) - \phi(\theta_{j-1}) < 0\bigr\}.$$

Then φ and ψ are defined as

$$\varphi(\theta) = \sup\left\{ \sum_{j \in A(P)} \bigl(\phi(\theta_j) - \phi(\theta_{j-1})\bigr)\colon\ P \in \mathfrak{P}_\theta \right\},$$

$$\psi(\theta) = \sup\left\{ \sum_{j \in B(P)} \bigl|\phi(\theta_j) - \phi(\theta_{j-1})\bigr|\colon\ P \in \mathfrak{P}_\theta \right\}$$

(whenever $A(P)$ or $B(P)$ are empty, we make $\varphi(\theta) = 0$, $\psi(\theta) = 0$). One easily sees that both φ and ψ are nondecreasing functions such that $\phi(\theta) = \varphi(\theta) - \psi(\theta)$, for every $\theta \in [-1, 0]$.

These facts can be extended for functions $\eta \in BV_n$. In fact, since for each $\theta \in [-1, 0]$ we have $\eta(\theta) = [\eta_{jk}(\theta)]$ $(j, k = 1, \ldots, n)$ where $\eta_{jk} \in BV$, for every $j, k = 1, \ldots, n$, then decomposing each function η_{jk} $(j, k = 1, \ldots, n)$ as the difference of two nondecreasing functions α_{jk} and β_{jk}, the n-by-n matrix valued functions $A(\theta) = [\alpha_{jk}(\theta)]$, $B(\theta) = [\beta_{jk}(\theta)]$, are both nondecreasing functions in BV_n, and for every $\theta \in [-1, 0]$, we have

$$\eta(\theta) = A(\theta) - B(\theta). \tag{10}$$

If for every $j, k = 1, \ldots, n$, we decompose each function η_{jk} according to (9) then we obtain

$$\eta(\theta) = \Phi(\theta) - \Psi(\theta), \tag{11}$$

with $\Phi(\theta)=[\varphi_{jk}(\theta)]$ and $\Psi(\theta)=[\psi_{jk}(\theta)]$, where φ_{jk} and ψ_{jk} are, respectively, the positive and negative variation of η_{jk} $(j,k=1,\ldots,n)$.

2. Nonoscillatory solutions

Nonnegative matrices can play some role on the study of the oscillatory behavior of the system (1).

According to the Perron–Frobenius theorem, a nonnegative matrix $C\in\mathbb{M}_n(\mathbb{R})$ has several important spectral properties (see [10,11]). As a matter of fact, denoting by $\sigma(C)$ the spectrum of C and by $\rho(C)$ the spectral radius of C, one has that $\rho(C)\in\sigma(C)$. Moreover, $\rho(C)>0$ if $C>0$ and $0\leqslant C\leqslant D\Rightarrow\rho(C)\leqslant\rho(D)$.

For a matrix $C\in\mathbb{M}_n(\mathbb{R})$, considering the upper bound

$$s(C)=\max\bigl\{\operatorname{Re}z\colon\ z\in\sigma(C)\bigr\},$$

of the set $\operatorname{Re}\sigma(C)=\{\operatorname{Re}z\colon\ z\in\sigma(C)\}$, then $s(C)=\rho(C)\in\sigma(C)$ whenever $C\geqslant 0$. Through that same theorem, one can conclude that $s(C)\in\sigma(C)$, if $C=[c_{jk}]$ $(j,k=1,\ldots,n)$, is *essentially nonnegative*—that is, if the off-diagonal entries of C (c_{jk} for $j\neq k$) are nonnegative real numbers.

Therefore, if the matrix

$$\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr] \tag{12}$$

is essentially nonnegative, for every real λ, the spectral set

$$\sigma\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)$$

is dominated by the value

$$s(\lambda)=s\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right),$$

that is,

$$s(\lambda)\in\sigma\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)\quad\forall\lambda\in\mathbb{R}. \tag{13}$$

On this purpose, we notice that for every real λ, the matrix (12) is nonnegative when the function ν is nondecreasing on $[-1,0]$, and is essentially nonnegative when for $\nu(\theta)=[\nu_{jk}(\theta)]$ $(j,k=1,\ldots,n)$, the off-diagonal functions $\nu_{jk}(\theta)$ $(j\neq k)$ are nondecreasing on $[-1,0]$. Moreover the assumption (13) is also fulfilled when for each $\theta\in[-1,0]$, $\nu(\theta)$ is a symmetric real matrix.

Under (13), the continuous dependence of the spectrum with respect to parameters enable us to handle condition (8) in a more suitable manner.

Theorem 1. *Under assumption* (13), *the system* (1) *is oscillatory if and only if* $s(\lambda) < 1$, *for every real* λ.

Proof. Noticing that

$$\left|s(\lambda)\right| \leqslant \left\| \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \right\|,$$

we claim that $\lim_{\lambda\to+\infty} s(\lambda) = 0$.

In fact, if $r(\theta)$ is a positive function on $[-1, 0]$, then letting

$$m(r) = \min\bigl\{r(\theta):\ -1 \leqslant \theta \leqslant 0\bigr\},$$

one has for every real $\lambda \geqslant 0$

$$\left\| \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \right\| \leqslant \exp\bigl(-\lambda m(r)\bigr) \int_{-1}^{0} \bigl\|d\bigl[\nu(\theta)\bigr]\bigr\|,$$

and so $s(\lambda) \to 0$ as $\lambda \to +\infty$.

If $r(\theta)$ is a positive function on $[-1, 0[$ and $r(0) = 0$, as then $\nu(\theta)$ is atomic at zero, for every $\varepsilon > 0$ there exists a real $\gamma > 0$ such that

$$\int_{-\gamma}^{0} \bigl\|d\bigl[\nu(\theta)\bigr]\bigr\| < \frac{\varepsilon}{2}.$$

Thus taking $m_0 = \min\{r(\theta):\ -1 \leqslant \theta \leqslant -\gamma\} > 0$, we have for every real $\lambda \geqslant 0$

$$\begin{aligned}
&\left\| \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \right\| \\
&\quad \leqslant \left\| \int_{-1}^{-\gamma} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \right\| + \left\| \int_{-\gamma}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \right\| \\
&\quad \leqslant \exp(-\lambda m_0) \int_{-1}^{-\gamma} \bigl\|d\bigl[\nu(\theta)\bigr]\bigr\| + \int_{-\gamma}^{0} \bigl\|d\bigl[\nu(\theta)\bigr]\bigr\|,
\end{aligned}$$

and by consequence, for λ arbitrarily large, we conclude that

$$\left\| \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \right\| < \varepsilon.$$

Hence, also in this case, $s(\lambda) \to 0$ as $\lambda \to +\infty$.

Therefore supposing that (1) is oscillatory and that for some real λ_0 it is $s(\lambda_0) \geqslant 1$, by continuity, there exists a real $\lambda_1 \geqslant \lambda_0$ such that $s(\lambda_1) = 1$, which, in view of (13), contradicts (8). $\square$

Remark 2. The proof of this theorem enable us to conclude that under the assumption (13), the system (1) has at least a nonoscillatory solution whenever $s(\lambda) \to +\infty$, as $\lambda \to -\infty$.

Taking a decomposition of $\nu \in BV_n$ according to (10),

$$\nu = A - B,$$

where $A, B \in BV_n$ are nondecreasing functions, the matrix (12) is decomposed into the difference

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] = \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[A(\theta)\bigr] - \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[B(\theta)\bigr].$$

The following theorem states the existence of, at least, a nonoscillatory solution.

Theorem 3. *Let $\theta_0 \in [-1, 0]$ be such that $r(\theta_0) = \|r\|$. If either*

$$\Omega_A^+(\theta_0) > |B| \quad \text{or} \quad \Omega_A^-(\theta_0) > |B|$$

then (1) *has at least a nonoscillatory solution.*

Proof. Let us assume, for example, that $\Omega_A^+(\theta_0) > |B|$.

For $\varepsilon > 0$ small enough, we have

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[A(\theta)\bigr] \geqslant \int_{\theta_0}^{\theta_0+\varepsilon} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[A(\theta)\bigr].$$

By application of a mean value property of the functions of bounded variation, we can obtain for every real $\lambda < 0$,

$$\int_{\theta_0}^{\theta_0+\varepsilon} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[A(\theta)\bigr] \geqslant \exp\bigl(-\lambda r(\theta_0 + \delta\varepsilon)\bigr)\bigl(A(\theta_0 + \varepsilon) - A(\theta_0)\bigr),$$

for some $\delta \in]0, 1[$, depending upon r, θ_0, ε and A. Then, making $\varepsilon \to 0^+$, we have

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[A(\theta)\bigr] \geqslant \exp\bigl(-\lambda \|r\|\bigr)\Omega_A^+(\theta_0).$$

On the other hand, for every real $\lambda < 0$, the following matrix inequality holds

$$0 \leqslant \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[B(\theta)\bigr] \leqslant \exp\bigl(-\lambda \|r\|\bigr)|B|,$$

and by consequence

$$-\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[B(\theta)\bigr] \geqslant -\exp\bigl(-\lambda\|r\|\bigr)|B|.$$

Therefore, for every real $\lambda < 0$, we have that

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \geqslant \exp\bigl(-\lambda\|r\|\bigr)\bigl(\Omega_A^{+}(\theta_0) - |B|\bigr). \tag{14}$$

As the matrix $\Omega_A^{+}(\theta_0) - |B|$ is positive, this means in particular that also

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] > 0,$$

for every $\lambda < 0$. Thus not only assumption (13) is fulfilled but also

$$s(\lambda) = \rho\left(\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]\right).$$

But from (14) we can conclude that

$$\rho\left(\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]\right) \geqslant \exp\bigl(-\lambda\|r\|\bigr)\rho\bigl(\Omega_A^{+}(\theta_0) - |B|\bigr)$$

and by consequence $s(\lambda) \to +\infty$, as $\lambda \to -\infty$. Hence by the Remark 2, (1) has at least a nonoscillatory solution. □

The behavior of the function ν at the point $\theta_0 \in [-1, 0]$ where is attained the absolute maximum of the delay function $r(\theta)$, has a specific relevance, as was already shown in [12] for the scalar case of (1).

Theorem 4. *Let $\theta_0 \in [-1, 0]$ be such that $r(\theta_0) = \|r\|$ and $r(\theta) < \|r\|$ for every $\theta \neq \theta_0$. If θ_0 is a point of increase of ν, then* (1) *has at least a nonoscillatory solution.*

Proof. For a matter of simplicity, let us assume that $\theta_0 = -1$.

Considering the decomposition of ν, given by (11), then for some $\varepsilon > 0$, the matrix $\Psi(\theta)$ is constant on $[-1, -1 + \varepsilon]$ and by consequence, on this interval

$$\Phi(\theta) = \nu(\theta) - C,$$

for some real n-by-n real matrix C. Therefore

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]$$

$$= \int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Phi(\theta)\bigr] - \int_{-1+\varepsilon}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Psi(\theta)\bigr]$$

$$\geqslant \int_{-1}^{-1+\varepsilon} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Phi(\theta)\bigr] - \int_{-1+\varepsilon}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Psi(\theta)\bigr].$$

Take $0 < \delta < \varepsilon$ such that

$$m = \min\bigl\{r(\theta)\colon\ \theta \in [-1, -1+\delta]\bigr\} > M = \max\bigl\{r(\theta)\colon\ \theta \in [-1+\varepsilon, 0]\bigr\}.$$

One easily can see that the following matrix relations hold, for every real $\lambda < 0$:

$$0 \leqslant \int_{-1+\varepsilon}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Psi(\theta)\bigr] \leqslant \exp(-\lambda M)|\Psi|,$$

$$\int_{-1}^{-1+\varepsilon} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Phi(\theta)\bigr] \geqslant \exp(-\lambda m)\bigl(\Phi(-1+\varepsilon) - \Phi(-1)\bigr).$$

Thus we obtain for every real $\lambda < 0$,

$$\int_{-1}^{-1+\varepsilon} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Phi(\theta)\bigr] \geqslant \exp(-\lambda m)\bigl(\nu(-1+\varepsilon) - \nu(-1)\bigr),$$

$$-\int_{-1+\varepsilon}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\Psi(\theta)\bigr] \geqslant -\exp(-\lambda M)|\Psi|,$$

which imply that

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]$$
$$\geqslant \exp(-\lambda m)\bigl[\bigl(\nu(-1+\varepsilon) - \nu(-1)\bigr) - \exp\bigl(\lambda(m-M)\bigr)|\Psi|\bigr]. \tag{15}$$

Since the nonnegative matrix $\exp(\lambda(m-M))|\Psi|$ tends to the null matrix, as $\lambda \to -\infty$, we can conclude that there exists a real number $\ell > 0$ sufficiently large, such that, for every $\lambda < -\ell$, $(\nu(-1+\varepsilon) - \nu(-1)) - \exp(\lambda(m-M))|\Psi|$ is a positive matrix. Thus, for every $\lambda < -\ell$, we have, in particular,

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] > 0,$$

and so assumption (13) is satisfied in a way that

$$s(\lambda) = \rho\left(\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]\right).$$

Moreover (15) implies that, for every $\lambda < -\ell$,

$$\rho\left(\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]\right)$$
$$\geqslant \exp(-\lambda m)\rho\Bigl(\bigl(\nu(-1+\varepsilon) - \nu(-1)\bigr) - \exp\bigl(\lambda(m-M)\bigr)|\Psi|\Bigr).$$

Since the right hand member of this inequality tends to $+\infty$, as $\lambda \to -\infty$, we can state, as in Theorem 1, that (1) has at least a nonoscillatory solution. □

Theorem 4 can be applied to the system (3) and the following corollary can easily be obtained.

Corollary 5. *If* $r_k = \max\{r_j\colon\ j = 1, \ldots, p\}$ *and* $A_k > 0$ *then* (3) *has at least a nonoscillatory solution.*

Proof. As a matter of fact, if $A_k > 0$ then θ_k is a point of increase of $\nu(\theta) = \sum_{j=1}^{p} H(\theta - \theta_j)A_j$. Therefore choosing $r(\theta)$ continuous and positive on $[-1, 0]$ in manner that $r(\theta_k) = \|r\|$ and $r(\theta) < \|r\|$ for every $\theta \neq \theta_k$, one can conclude by Theorem 4 that (3) has at least a nonoscillatory solution. □

Similar arguments enable us to state the following theorem.

Theorem 6. *If* ν *is increasing on* $[-1, 0]$ *and nonconstant on* $[-1, 0[$ *then* (1) *has at least a nonoscillatory solution.*

Proof. Since ν is increasing on $[-1, 0[$, there exist $\theta_1, \theta_2 \in [-1, 0[$ such that $\theta_1 < \theta_2$ and

$$\Delta = \nu(\theta_2) - \nu(\theta_1)$$

is a positive matrix. Then the following matrix relation holds for every real $\lambda \leqslant 0$:

$$\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \geqslant \int_{\theta_1}^{\theta_2} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr] \geqslant \exp(-\lambda m)\Delta,$$

where

$$m = \min\bigl\{r(\theta)\colon\ \theta \in [\theta_1, \theta_2]\bigr\} > 0.$$

Thus as before, assumption (13) is fulfilled, $s(\lambda) \to +\infty$, as $\lambda \to -\infty$, and (1) has at least a nonoscillatory solution. □

3. Explicit conditions for oscillations

As is well known matrix measures are a relevant tool on the oscillation theory of delay systems. For a matter of completeness we recall briefly, its definition and the properties which will be used in the sequel.

For each induced norm, $\|\cdot\|$, in $\mathbb{M}_n(\mathbb{R})$, we associate a matrix measure $\mu : \mathbb{M}_n(\mathbb{R}) \to \mathbb{R}$, which is defined for any $C \in \mathbb{M}_n(\mathbb{R})$ as

$$\mu(C) = \lim_{\gamma \to 0^+} \frac{\|I + \gamma C\| - 1}{\gamma},$$

where by I we mean the identity matrix.

Well known matrix measures of a matrix $C = [c_{jk}] \in \mathbb{M}_n(\mathbb{R})$, are

$$\mu_1(C) = \max\left\{ c_{kk} + \sum_{j \neq k} |c_{jk}|:\ k = 1, \ldots, n \right\},$$

$$\mu_\infty(C) = \max\left\{ c_{jj} + \sum_{k \neq j} |c_{jk}|:\ j = 1, \ldots, n \right\},$$

which correspond, respectively, to the induced norms in $\mathbb{M}_n(\mathbb{R})$ given by:

$$\|C\|_1 = \max\left\{ \sum_{j=1}^{n} |c_{jk}|:\ k = 1, \ldots, n \right\},$$

$$\|C\|_\infty = \max\left\{ \sum_{k=1}^{n} |c_{jk}|:\ j = 1, \ldots, n \right\}.$$

Independently of the considered induced norm in $\mathbb{M}_n(\mathbb{R})$, a matrix measure has always the following properties (see [13]):

(i) $s(C) \leqslant \mu(C) \leqslant \|C\|$.
(ii) $\mu(C_1) - \mu(-C_2) \leqslant \mu(C_1 + C_2) \leqslant \mu(C_1) + \mu(C_2)$ $(C_1, C_2 \in \mathbb{M}_n(\mathbb{R}))$.
(iii) $\mu(\gamma C) = \gamma \mu(C)$, for every $\gamma \geqslant 0$.

If $\eta \in BV_n$, the continuity of μ on $\mathbb{M}_n(\mathbb{R})$ implies that $\mu \circ \eta \in BV$; in consequence, with $[a, b] \subset [-1, 0]$, the following inequalities hold (see [8]):

(iv) If $\phi \in C([a, b]; \mathbb{R})$ is nonincreasing and positive, then

$$\mu\left(\int_a^b \phi(\theta)\, d[\eta(\theta)] \right) \leqslant \int_a^b \phi(\theta) d\big(\mu\big(\eta(\theta) - \eta(a)\big)\big).$$

(v) If $\phi \in C([a, b]; \mathbb{R})$ is nondecreasing and positive, then

$$\mu\left(\int_a^b \phi(\theta)\, d[\eta(\theta)] \right) \leqslant -\int_a^b \phi(\theta)\, d\big(\mu\big(\eta(b) - \eta(\theta)\big)\big).$$

By (i), if for some matrix measure μ,

$$\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)<1\quad\forall\lambda\in\mathbb{R},\tag{16}$$

we can conclude that system (1) is oscillatory.

For any $\eta\in BV_n$, the functions η_0 and η_1 of BV_n, given, respectively, by

$$\eta_0(\theta)=\eta(0)-\eta(\theta),\qquad \eta_1(\theta)=\eta(\theta)-\eta(-1)\quad(\theta\in[-1,0]),$$

will be considered in the sequel.

In the following theorem we obtain conditions for having (1) oscillatory with respect to the family of all monotonic delay functions.

Theorem 7. *If on* $[-1,0]$, $\mu\circ\nu_1$ *is nonincreasing and* $\mu\circ\nu_0$ *is nondecreasing then* (1) *is oscillatory for all monotonic delay functions* $r(\theta)$.

Proof. Assume that $r:[-1,0]\to\mathbb{R}$ is a monotonic function.

By (iv) and (v), if $r(\theta)$ is nonincreasing we have that

$$\begin{aligned}
&\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)\leqslant\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d(\mu\circ\nu_1)(\theta),\quad \text{if } \lambda\leqslant 0,\\
&\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)\leqslant-\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d(\mu\circ\nu_0)(\theta),\quad \text{if } \lambda\geqslant 0,
\end{aligned}\tag{17}$$

and if $r(\theta)$ is nondecreasing,

$$\begin{aligned}
&\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)\leqslant\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d(\mu\circ\nu_1)(\theta),\quad \text{if } \lambda\geqslant 0,\\
&\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)\leqslant-\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d(\mu\circ\nu_0)(\theta),\quad \text{if } \lambda\leqslant 0.
\end{aligned}\tag{18}$$

Then take, respectively, for $\lambda\in\mathbb{R}$, the functions

$$g(\lambda)=\begin{cases}\int_{-1}^{0}\exp(-\lambda r(\theta))\,d(\mu\circ\nu_1)(\theta), & \text{if } \lambda<0,\\ \mu(\Delta\nu), & \text{if } \lambda=0,\\ -\int_{-1}^{0}\exp(-\lambda r(\theta))\,d(\mu\circ\nu_0)(\theta), & \text{if } \lambda>0,\end{cases}$$

$$h(\lambda)=\begin{cases}-\int_{-1}^{0}\exp(-\lambda r(\theta))\,d(\mu\circ\nu_0)(\theta), & \text{if } \lambda<0,\\ \mu(\Delta\nu), & \text{if } \lambda=0,\\ \int_{-1}^{0}\exp(-\lambda r(\theta))\,d(\mu\circ\nu_1)(\theta), & \text{if } \lambda>0,\end{cases}$$

where $\Delta\nu = \nu(0) - \nu(-1)$. Noticing that

$$g(0^+) = h(0^-) = -\int_{-1}^{0} d(\mu \circ \nu_0)(\theta) = \mu\big(\nu_0(-1)\big) = \mu\big(\nu(0) - \nu(-1)\big)$$

and

$$g(0^-) = h(0^+) = \int_{-1}^{0} d(\mu \circ \nu_1)(\theta) = \mu\big(\nu_1(0)\big) = \mu\big(\nu(0) - \nu(-1)\big),$$

one can conclude that g and h are both continuous in $\mathbb{R}$.

Since on $[-1, 0]$, $\mu \circ \nu_1$ is nonincreasing and $\mu \circ \nu_0$ is nondecreasing, one has

$$g(\lambda) \leqslant 0 \quad \forall\lambda \in \mathbb{R},$$

if $r(\theta)$ is nonincreasing and

$$h(\lambda) \leqslant 0 \quad \forall\lambda \in \mathbb{R},$$

if $r(\theta)$ is nondecreasing. Hence, for $r(\theta)$ monotonic, by (17) and (18) one has (16) satisfied, which achieves the proof. □

Remark 8. The assumptions in the Theorem 7 of $\mu \circ \nu_1$ be nonincreasing and $\mu \circ \nu_0$ be nondecreasing are fulfilled, if one assumes as in [14], that for $\theta_1, \theta_2 \in [-1, 0]$,

$$\theta_1 < \theta_2 \quad \Longrightarrow \quad \mu\big(\nu(\theta_2) - \nu(\theta_1)\big) \leqslant 0. \tag{19}$$

As a matter of fact, if $\theta_1 < \theta_2$, one has by (ii),

$$\begin{aligned}
&(\mu \circ \nu_1)(\theta_2) - (\mu \circ \nu_1)(\theta_1) \\
&\quad = \mu\big(\nu(\theta_2) - \nu(-1)\big) - \mu\big(\nu(\theta_1) - \nu(-1)\big) \\
&\quad \leqslant \mu\big(\nu(\theta_2) - \nu(-1) - \nu(\theta_1) + \nu(-1)\big) = \mu\big(\nu(\theta_2) - \nu(\theta_1)\big) \leqslant 0,
\end{aligned}$$

and

$$\begin{aligned}
&(\mu \circ \nu_0)(\theta_1) - (\mu \circ \nu_0)(\theta_2) \\
&\quad = \mu\big(\nu(0) - \nu(\theta_1)\big) - \mu\big(\nu(0) - \nu(\theta_2)\big) \\
&\quad \leqslant \mu\big(\nu(0) - \nu(\theta_1) - \nu(0) + \nu(\theta_2)\big) = \mu\big(\nu(\theta_2) - \nu(\theta_1)\big) \leqslant 0.
\end{aligned}$$

Following [8] let us define

$$\alpha(A_1) = \mu(A_1), \qquad \alpha(A_j) = \mu\left(\sum_{k=1}^{j} A_k\right) - \mu\left(\sum_{k=1}^{j-1} A_k\right) \quad \text{for } j = 2, \ldots, p,$$

$$\beta(A_p) = \mu(A_p), \qquad \beta(A_j) = \mu\left(\sum_{k=j}^{p} A_k\right) - \mu\left(\sum_{k=j+1}^{p} A_k\right)$$

$$\text{for } j = 1, \ldots, p-1.$$

Therefore, with respect to the system (3), from the Theorem 7 we can state the following corollary.

Corollary 9. *If for* $j = 1, \dots, p$, $\alpha(A_j) \leqslant 0$ *and* $\beta(A_j) \leqslant 0$, *then system* (3) *is oscillatory for every family of delays* $(r_1, \dots, r_p) \in \mathbb{R}_+^p$.

In particular, as by (ii)

$$\alpha(A_j) \leqslant \mu(A_j), \qquad \beta(A_j) \leqslant \mu(A_j),$$

we obtain as in [7]:

Corollary 10. *If* $\mu(A_k) \leqslant 0$, *for every* $k = 1, \dots, p$, *then* (3) *is oscillatory for every family of delays* $(r_1, \dots, r_p) \in \mathbb{R}_+^p$.

In the following we will analyze what happens when $r(\theta)$ is not a monotonic function on $[-1, 0]$.

For that purpose we notice that by property (ii) of the matrix measures,

$$\begin{aligned}
&\mu\left(\int_{-1}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]\right) \\
&\quad \leqslant \mu\left(\int_{-1}^{\theta_0-\delta} \exp\bigl(-\lambda r(\theta)\bigr)\, d\bigl[\nu(\theta)\bigr]\right) + \mu\left(\int_{\theta_0-\delta}^{\theta_0+\delta} \exp\bigl(-\lambda r(\theta)\bigr)\, d\nu(\theta)\right) \\
&\qquad + \mu\left(\int_{\theta_0+\delta}^{0} \exp\bigl(-\lambda r(\theta)\bigr)\, d\nu(\theta)\right),
\end{aligned} \tag{20}$$

for every $\theta_0 \in [-1, 0]$ and $\delta \geqslant 0$, small enough.

Theorem 11. *Let* $r(\theta)$ *be differentiable and positive on* $[-1, 0]$, *increasing on* $[-1, \theta_0 - \delta]$, *constant on* $[\theta_0 - \delta, \theta_0 + \delta]$ *and decreasing on* $[\theta_0 + \delta, 0]$. *If*

$$\mu(\nu(\theta_0 + \delta) - \nu(\theta_0 - \delta)) \leqslant 0, \tag{21}$$

$$\mu\bigl(\nu(\theta_0 - \delta) - \nu(\theta)\bigr) \leqslant 0, \quad (\mu \circ \nu_1)(\theta) \geqslant 0, \quad \textit{for every } \theta \in [-1, \theta_0 - \delta], \tag{22}$$

$$\mu\bigl(\nu(\theta) - \nu(\theta_0 + \delta)\bigr) \leqslant 0, \quad (\mu \circ \nu_0)(\theta) \geqslant 0, \quad \textit{for every } \theta \in [\theta_0 + \delta, 0], \tag{23}$$

and

$$\int_{-1}^{\theta_0-\delta} (\mu \circ \nu_1)(\theta)\, d\bigl(\log r(\theta)\bigr) - \int_{\theta_0+\delta}^{0} (\mu \circ \nu_0)(\theta)\, d\bigl(\log r(\theta)\bigr) < e, \tag{24}$$

then (1) *is oscillatory.*

Proof. We will prove that (16) holds.

For $\lambda = 0$, by (20), (21) and the first part of (22) and (23) we can conclude that

$$\begin{aligned}
&\mu\big(\nu(0) - \nu(-1)\big)\\
&\quad\leqslant \mu\big(\nu(0) - \nu(\theta_0+\delta)\big) + \mu\big(\nu(\theta_0+\delta) - \nu(\theta_0-\delta)\big)\\
&\qquad + \mu\big(\nu(\theta_0-\delta) - \nu(-1)\big) \leqslant 0.
\end{aligned}$$

Let now $\lambda < 0$. By (20) and the properties (iv) and (v) of the matrix measures we obtain

$$\begin{aligned}
&\mu\left(\int_{-1}^{0} \exp\big(-\lambda r(\theta)\big)\, d\big[\nu(\theta)\big]\right)\\
&\quad\leqslant -\int_{-1}^{\theta_0-\delta} \exp\big(-\lambda r(\theta)\big)\, d\mu\big(\nu(\theta_0-\delta) - \nu(\theta)\big)\\
&\qquad + \exp\big(-\lambda r(\theta_0)\big)\mu\big(\nu(\theta_0+\delta) - \nu(\theta_0-\delta)\big)\\
&\qquad + \int_{\theta_0+\delta}^{0} \exp\big(-\lambda r(\theta)\big)\, d\mu\big(\nu(\theta) - \nu(\theta_0+\delta)\big).
\end{aligned}$$

Therefore, integrating by parts, we have

$$\begin{aligned}
&\mu\left(\int_{-1}^{0} \exp\big(-\lambda r(\theta)\big)\, d\big[\nu(\theta)\big]\right)\\
&\quad\leqslant \exp\big(-\lambda r(-1)\big)\mu\big(\nu(\theta_0-\delta) - \nu(-1)\big)\\
&\qquad - \int_{-1}^{\theta_0-\delta} \lambda \exp\big(-\lambda r(\theta)\big)\mu\big(\nu(\theta_0-\delta) - \nu(\theta)\big)\, dr(\theta)\\
&\qquad + \exp\big(-\lambda r(\theta_0)\big)\mu\big(\nu(\theta_0+\delta) - \nu(\theta_0-\delta)\big) + \exp\big(-\lambda r(0)\big)\mu\big(\nu(0) - \nu(\theta_0+\delta)\big)\\
&\qquad + \int_{\theta_0+\delta}^{0} \lambda \exp\big(-\lambda r(\theta)\big)\mu\big(\nu(\theta) - \nu(\theta_0+\delta)\big)\, dr(\theta).
\end{aligned}$$

Then by (21) and the first part of (22) and (23) we state that, for every $\lambda < 0$,

$$\mu\left(\int_{-1}^{0} e^{-\lambda r(\theta)}\, d\big[\nu(\theta)\big]\right) \leqslant 0.$$

Consider now $\lambda > 0$. By the same arguments,

$$\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)$$
$$\leqslant \exp\bigl(-\lambda r(\theta_0-\delta)\bigr)\mu\bigl(\nu(\theta_0-\delta)-\nu(-1)\bigr)$$
$$+\int_{-1}^{\theta_0-\delta}\lambda\exp\bigl(-\lambda r(\theta)\bigr)(\mu\circ\nu_1)(\theta)\,dr(\theta)$$
$$+\exp\bigl(-\lambda r(\theta_0)\bigr)\mu\bigl(\nu(\theta_0+\delta)-\nu(\theta_0-\delta)\bigr)$$
$$+\exp\bigl(-\lambda r(\theta_0+\delta)\bigr)\mu\bigl(\nu(0)-\nu(\theta_0+\delta)\bigr)$$
$$-\int_{\theta_0+\delta}^{0}\lambda\exp\bigl(-\lambda r(\theta)\bigr)(\mu\circ\nu_0)(\theta)\,dr(\theta)$$
$$\leqslant \int_{-1}^{\theta_0-\delta}\lambda\exp\bigl(-\lambda r(\theta)\bigr)(\mu\circ\nu_1)(\theta)\,dr(\theta)-\int_{\theta_0+\delta}^{0}\lambda\exp\bigl(-\lambda r(\theta)\bigr)(\mu\circ\nu_0)(\theta)\,dr(\theta).$$

Noticing that the function $u\exp(-u)$ has an absolute maximum at $u=1$, we obtain for every $\theta\in[-1,0]$

$$\lambda\exp\bigl(-\lambda r(\theta)\bigr)\leqslant\frac{e^{-1}}{r(\theta)}.$$

As $r(\theta)$ is increasing in $[-1,\theta_0-\delta[$ and decreasing in $]\theta_0+\delta,0]$, by the second part of (22) and (23) we obtain

$$\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)$$
$$\leqslant\frac{1}{e}\left[\int_{-1}^{\theta_0-\delta}\frac{(\mu\circ\nu_1)(\theta)}{r(\theta)}\,dr(\theta)-\int_{\theta_0+\delta}^{0}\frac{(\mu\circ\nu_0)(\theta)}{r(\theta)}\,dr(\theta)\right],$$

and (24) implies

$$\mu\left(\int_{-1}^{0}\exp\bigl(-\lambda r(\theta)\bigr)\,d\bigl[\nu(\theta)\bigr]\right)<1,$$

for every $\lambda>0$.

This achieves the proof. □

Example 12. Let us consider the system (1) for

$$r(\theta)=\begin{cases}-\theta^2-\frac{6}{5}\theta, & \text{if } -1\leqslant\theta<-\frac{3}{5},\\ \frac{9}{25}, & \text{if } -\frac{3}{5}\leqslant\theta\leqslant-\frac{2}{5},\\ -\theta^2-\frac{4}{5}\theta+\frac{5}{25}, & \text{if } -\frac{2}{5}<\theta\leqslant 0,\end{cases}$$

and

$$\nu(\theta)=\begin{bmatrix} \theta\left(\theta+\frac{2}{5}\right) & 3 \\ 4 & -\left(\theta+\frac{3}{5}\right)(\theta+1) \end{bmatrix}.$$

With respect to the matrix measure μ_∞, (21) is satisfied since

$$\mu_\infty\left(\nu\left(-\frac{2}{5}\right)-\nu\left(-\frac{3}{5}\right)\right)=\mu_\infty\left(\begin{bmatrix} -\frac{3}{25} & 0 \\ 0 & -\frac{3}{25} \end{bmatrix}\right)<0.$$

The same holds to assumptions (22) and (23), since for every $\theta\in[-1,-3/5]$,

$$\begin{aligned}\mu_\infty\left(\nu\left(-\frac{3}{5}\right)-\nu(\theta)\right)&=\mu_\infty\left(\begin{bmatrix} \frac{3}{25}-\theta\left(\theta+\frac{2}{5}\right) & 0 \\ 0 & \left(\theta+\frac{3}{5}\right)(\theta+1) \end{bmatrix}\right)\\ &=\left(\theta+\frac{3}{5}\right)(\theta+1)\leqslant 0,\end{aligned}$$

$$\begin{aligned}\mu_\infty\left(\nu(\theta)-\nu(-1)\right)&=\mu_\infty\left(\begin{bmatrix} \theta\left(\theta+\frac{2}{5}\right)-\frac{3}{5} & 0 \\ 0 & -\left(\theta+\frac{3}{5}\right)(\theta+1) \end{bmatrix}\right)\\ &=\theta\left(\theta+\frac{2}{5}\right)-\frac{3}{5}\geqslant 0,\end{aligned}$$

and, for every $\theta\in[-2/5,0]$,

$$\begin{aligned}\mu_\infty\left(\nu(\theta)-\nu\left(-\frac{2}{5}\right)\right)&=\mu_\infty\left(\begin{bmatrix} \theta\left(\theta+\frac{2}{5}\right) & 0 \\ 0 & -\left(\theta+\frac{3}{5}\right)(\theta+1)+\frac{3}{25} \end{bmatrix}\right)\\ &=\theta\left(\theta+\frac{2}{5}\right)\leqslant 0,\end{aligned}$$

$$\begin{aligned}\mu_\infty\left(\nu(0)-\nu(\theta)\right)&=\mu_\infty\left(\begin{bmatrix} -\theta\left(\theta+\frac{2}{5}\right) & 0 \\ 0 & -\frac{3}{5}+\left(\theta+\frac{3}{5}\right)(\theta+1) \end{bmatrix}\right)\\ &=-\frac{3}{5}+\left(\theta+\frac{3}{5}\right)(\theta+1)\geqslant 0.\end{aligned}$$

Moreover, as

$$\begin{aligned}&\int_{-1}^{-3/5}\frac{\mu(\nu(\theta)-\nu(-1))}{r(\theta)}\,dr(\theta)-\int_{-2/5}^{0}\frac{\mu(\nu(0)-\nu(\theta))}{r(\theta)}\,dr(\theta)\\ &\quad=-\int_{-1}^{-3/5}\frac{\left(\theta+\frac{3}{5}\right)(\theta+1)\left(2\theta+\frac{6}{5}\right)}{\theta\left(\theta+\frac{6}{5}\right)}\,d\theta+\int_{-2/5}^{0}\frac{\theta\left(\theta+\frac{2}{5}\right)\left(2\theta+\frac{4}{5}\right)}{(\theta+1)\left(\theta-\frac{1}{5}\right)}\,d\theta\\ &\quad=-\frac{8}{25}-\frac{36}{25}\ln 3+\frac{6}{5}\ln 5<e,\end{aligned}$$

condition (24) is also verified and so the corresponding system (1) is oscillatory.

Making in the Theorem 11, $\delta = 0$, we obtain the following corollary.

Corollary 13. *Let* $r : [-1, 0] \to \mathbb{R}^+$ *be differentiable on* $[-1, 0]$, *increasing on* $[-1, \theta_0]$ *and decreasing on* $[\theta_0, 0]$. *If for every* $\theta \in [-1, \theta_0]$,

$$\mu\big(\nu(\theta_0) - \nu(\theta)\big) \leqslant 0, \qquad (\mu \circ \nu_1)(\theta) \geqslant 0, \tag{25}$$

for $\theta \in [\theta_0, 0]$

$$\mu\big(\nu(\theta) - \nu(\theta_0)\big) \leqslant 0, \qquad (\mu \circ \nu_0)(\theta) \geqslant 0, \tag{26}$$

and

$$\int_{-1}^{\theta_0} (\mu \circ \nu_1)(\theta)\, d\big(\log r(\theta)\big) - \int_{\theta_0}^{0} (\mu \circ \nu_0)(\theta)\, d\big(\log r(\theta)\big) < e, \tag{27}$$

then (1) *is oscillatory.*

This corollary is illustrated in the following example.

Example 14. Consider (1) with $r(\theta) = -\theta(\theta + 1)$ and

$$\nu(\theta) = \begin{bmatrix} \theta(\theta+1)\big(\theta + \frac{1}{2}\big) & 5 \\ 0 & -6\theta - 7 \end{bmatrix}.$$

For the matrix measure μ_1, we have for every $\theta \in [-1, -1/2]$

$$\mu_1\left(\nu\left(-\frac{1}{2}\right) - \nu(\theta)\right) = \mu_1\left(\begin{bmatrix} -\theta(\theta+1)\big(\theta + \frac{1}{2}\big) & 0 \\ 0 & 3 + 6\theta \end{bmatrix}\right)$$
$$= -\theta(\theta+1)\left(\theta + \frac{1}{2}\right) \leqslant 0,$$

$$\mu_1(\nu(\theta) - \nu(-1)) = \mu_1\left(\begin{bmatrix} \theta(\theta+1)\big(\theta + \frac{1}{2}\big) & 0 \\ 0 & -6\theta - 6 \end{bmatrix}\right)$$
$$= \theta(\theta+1)\left(\theta + \frac{1}{2}\right) \geqslant 0,$$

and for every $\theta \in [-1/2, 0]$,

$$\mu_1\left(\nu(\theta) - \nu\left(-\frac{1}{2}\right)\right) = \mu_1\left(\begin{bmatrix} \theta(\theta+1)\big(\theta + \frac{1}{2}\big) & 0 \\ 0 & -3 - 6\theta \end{bmatrix}\right)$$
$$= \theta(\theta+1)\left(\theta + \frac{1}{2}\right) \leqslant 0,$$

$$\mu_1\big(\nu(0) - \nu(\theta)\big) = \mu_1\left(\begin{bmatrix} -\theta(\theta+1)\big(\theta + \frac{1}{2}\big) & 0 \\ 0 & 6\theta \end{bmatrix}\right)$$
$$= -\theta(\theta+1)\left(\theta + \frac{1}{2}\right) \geqslant 0.$$

Therefore conditions (25) and (26) of the Corollary 13 are fulfilled. The same holds to assumption (27), since

$$\int_{-1}^{-1/2} \frac{\theta(\theta+1)\big(\theta+\frac{1}{2}\big)}{-\theta(\theta+1)}\, d\big(-\theta(\theta+1)\big) - \int_{-1/2}^{0} \frac{-\theta(\theta+1)\big(\theta+\frac{1}{2}\big)}{-\theta(\theta+1)}\, d\big(-\theta(\theta+1)\big)$$
$$= \int_{-1}^{-1/2} \left(\theta+\frac{1}{2}\right)(2\theta+1)\, d\theta + \int_{-1/2}^{0} \left(\theta+\frac{1}{2}\right)(2\theta+1)\, d\theta = \frac{1}{6} < e.$$

Hence the corresponding system (1) is oscillatory.

Now, assuming that $\delta = 0$, as before, and further that $\theta_0 = -1$, the following corollary can be stated.

Corollary 15. *Let on* $[-1, 0]$*, be* $(\mu \circ \nu_0)(\theta) \geqslant 0$*,* $(\mu \circ \nu_1)(\theta) \leqslant 0$ *and* $r(\theta)$ *positive, differentiable and decreasing. Then system* (1) *is oscillatory if*

$$\int_{-1}^{0} (\mu \circ \nu_0)(\theta)\, d\big(\log r(\theta)\big) > -e. \tag{28}$$

Proof. Just notice that, in this case, (25) is fulfilled, since $\mu(0) = 0$. On the other hand, assumption (26) becomes equivalent to $(\mu \circ \nu_0)(\theta) \geqslant 0$ and $(\mu \circ \nu_1)(\theta) \leqslant 0$, for every $\theta \in [-1, 0]$, while (27) gives rise to (28). □

Analogously, for $\delta = 0$ and $\theta_0 = 0$, we obtain the following corollary.

Corollary 16. *Let on* $[-1, 0]$*, be* $(\mu \circ \nu_0)(\theta) \leqslant 0$*,* $(\mu \circ \nu_1)(\theta) \geqslant 0$ *and* $r(\theta)$ *positive, differentiable and increasing. Then system* (1) *is oscillatory if*

$$\int_{-1}^{0} (\mu \circ \nu_1)(\theta)\, d\big(\log r(\theta)\big) < e.$$

The following example illustrates the Corollary 15.

Example 17. Let us consider (1) for

$$\nu(\theta) = \begin{bmatrix} \theta^2 & -\theta \\ \theta & -2\theta \end{bmatrix} \quad \text{and} \quad r(\theta) = 1 - \theta.$$

We have, for the matrix measure μ_∞,

$$(\mu_\infty \circ \nu_0)(\theta) = \mu_\infty \left(\begin{bmatrix} -\theta^2 & \theta \\ -\theta & 2\theta \end{bmatrix} \right)$$
$$= \max\{-\theta^2 - \theta, \theta\} = -\theta(\theta+1) \geqslant 0,$$

for every $\theta \in [-1, 0]$, and

$$\begin{aligned}\big(\mu_\infty \circ \nu_1(\theta)\big) &= \mu_\infty\left(\begin{bmatrix} \theta^2 - 1 & -\theta - 1 \\ \theta + 1 & -2\theta - 2 \end{bmatrix}\right)\\ &= \max\{\theta^2 + \theta, -\theta - 1\} = \theta(\theta + 1) \leqslant 0,\end{aligned}$$

for every $\theta \in [-1, 0]$. As

$$\begin{aligned}\int_{-1}^{0} \frac{\theta(\theta+1)}{1-\theta}\,d\theta &= \int_{-1}^{0}\left(-\theta - 2 + \frac{2}{1-\theta}\right)d\theta = \left[-\frac{\theta^2}{2} - 2\theta - \ln(1-\theta)^2\right]_{-1}^{0}\\ &= \frac{1}{2} - 2 + \ln 4 \approx -0,2 > -e,\end{aligned}$$

the corresponding system (1) is then oscillatory.

Considering in (1), $\nu(\theta)$ given by (4), for $-1 < \theta_1 < \cdots < \theta_p < 0$, and $r(\theta)$ differentiable, decreasing and positive on $[-1, 0]$, one obtains the system (3) with $r_j = r(\theta_j)$ for $j = 1, \ldots, p$, such that $r_1 > \cdots > r_p$. In this situation we can apply the Corollary 15 and consequently obtain the following, corollary.

Corollary 18. *Let*

$$\mu\left(\sum_{k=j}^{p} A_k\right) \geqslant 0 \quad \textit{and} \quad \mu\left(\sum_{k=1}^{j} A_k\right) \leqslant 0, \tag{29}$$

for every $j = 1, \ldots, p$. *Then system* (3) *is oscillatory if*

$$\sum_{j=2}^{p} \mu\left(\sum_{k=j}^{p} A_k\right) \log \frac{r_j}{r_{j-1}} > -e. \tag{30}$$

Proof. The conditions corresponding to $(\mu \circ \nu_0)(\theta) \geqslant 0$ and $(\mu \circ \nu_1)(\theta) \leqslant 0$ in the Corollary 15, are in this case, respectively, $\mu(\sum_{k=j}^{p} A_k) \geqslant 0$ and $\mu(\sum_{k=1}^{j} A_k) \leqslant 0$, for every $j = 1, \ldots, p$.

On the other hand, as then $\mu(\sum_{k=1}^{p} A_k) = 0$, (28) becomes,

$$\begin{aligned}&\int_{-1}^{0} (\mu \circ \nu_0)(\theta)\, d\big(\log r(\theta)\big)\\ &\quad = \int_{\theta_1}^{\theta_2} \mu\left(\sum_{k=2}^{p} A_k\right) d\big(\log r(\theta)\big) + \cdots + \int_{\theta_{p-1}}^{\theta_p} \mu(A_p)\, d\big(\log r(\theta)\big)\\ &\qquad + \int_{\theta_p}^{0} \mu(0)\, d\big(\log r(\theta)\big)\end{aligned}$$

$$= \mu\left(\sum_{k=2}^{p} A_k\right)\log\frac{r_2}{r_1} + \cdots + \mu\left(\sum_{k=p-1}^{p} A_k\right)\log\frac{r_{p-1}}{r_{p-2}} + \mu(A_p)\log\frac{r_p}{r_{p-1}}$$
$$= \sum_{j=2}^{p}\mu\left(\sum_{k=j}^{p} A_k\right)\log\frac{r_j}{r_{j-1}} > -e,$$

which proves the corollary. □

An analogous result can be obtained for (3) through the Corollary 16, by considering the system (1) with $\nu(\theta)$ given by (4), for $-1 < \theta_1 < \cdots < \theta_p < 0$, and $r(\theta)$ differentiable and increasing on $[-1, 0]$. In fact, now one obtains the system (3) with $r_j = r(\theta_j)$ for $j = 1, \ldots, p$, such that $r_1 < \cdots < r_p$. Hence the following corollary can be stated.

Corollary 19. *Let*

$$\mu\left(\sum_{k=j}^{p} A_k\right) \leqslant 0 \quad \textit{and} \quad \mu\left(\sum_{k=1}^{j} A_k\right) \geqslant 0,$$

for every $j = 1, \ldots, p$. *Then system* (3) *is oscillatory if*

$$\sum_{j=1}^{p-1}\mu\left(\sum_{k=1}^{j} A_k\right)\log\frac{r_{j+1}}{r_j} < e.$$

The following example illustrates the Corollary 18.

Example 20. Let us consider the system

$$x(t) = A_1 x\left(t - \frac{7}{4}\right) + A_2 x\left(t - \frac{3}{2}\right) + A_3 x\left(t - \frac{5}{4}\right), \tag{31}$$

where

$$A_1 = \begin{bmatrix} -3 & -5 \\ 1 & -9 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}.$$

We have

$$\mu_1(A_1) = \max\{-2, -4\} = -2 \leqslant 0,$$
$$\mu_1(A_1 + A_2) = \mu_1\left(\begin{bmatrix} -2 & -3 \\ -1 & -4 \end{bmatrix}\right) = \max\{-1, -1\} = -1 \leqslant 0,$$
$$\mu_1(A_1 + A_2 + A_3) = \mu_1\left(\begin{bmatrix} -3 & -2 \\ 3 & -7 \end{bmatrix}\right) = \max\{0, -5\} = 0,$$
$$\mu_1(A_2 + A_3) = \mu_1\left(\begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix}\right) = \max\{2, 5\} = 5 \geqslant 0,$$
$$\mu_1(A_3) = \max\{3, -2\} = 3 \geqslant 0.$$

So, (29) is satisfied. The same holds to (30), since

$$\mu\left(\sum_{k=2}^{3} A_k\right) \ln \frac{r_2}{r_1} + \mu\left(\sum_{k=3}^{3} A_k\right) \ln \frac{r_3}{r_2}$$
$$= 5 \ln \frac{3/2}{7/4} + 3 \ln \frac{5/4}{3/2} = 5 \ln \frac{6}{7} + 3 \ln \frac{5}{6} \approx -1,32 > -e.$$

Hence the system (31) is oscillatory.

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