Forward-Backward Differential Equations: Approximation of Small Solutions

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Abstract

In the context of physics, economic dynamics, finance, optimal control, biology and other applied sciences, many mathematical models contain mixed type functional differential equations (MTFDEs), equations with both delayed and advanced arguments. Knowing from the analysis of delay differential equations (DDEs) that the evaluation of small solutions (that decay faster than any exponential) often leads to computational problems (degeneracy), we investigate this subject in the case of MTFDEs. Some computations have been carried out and are presented here, concerning the linear nonautonomous case. We continue this work and extend the investigation to other problems.

Key words: Mixed-type functional differential equation, method of steps, small solutions, collocation, least squares, finite element method.
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1 Introduction

In the context of physics, economic dynamics, finance, optimal control, biology and other applied sciences, many mathematical models contain forward-backward differential equations, usually denominated mixed type functional differential equations (MTFDEs):

$$x'(t) = F(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)), \tag{1}$$

where the shifts τ_i may take negative or positive values. The equation (1) is a multi-delay-advance differential equation form.

For the problems studied in this work, we focus our attention on computation of solutions. We are particularly interested in obtaining the numerical solution of the equations:

$$x'(t) = F(t, x(t), x(t-\tau), x(t+\tau)), \quad \tau > 0 \tag{2}$$

Interest in this type of equation is motivated by applications in optimal control [20] [21], economic dynamics [22], nerve conduction [6] and travelling waves in a spatial lattice [1], [19].

Important contributions in such fields have appeared in the literature in the two last decades of past century and it remains an actual subject of research until now.

Concerning the application of MTFDEs to optimal control and economic dynamics, a fundamental contribuition was introduced by Rustichini in 1989 [21]. However, the basis for the application of MTFDEs to optimal control was given earlier by Mischenko and his co-authors [20].

Nowadays, MTFDEs continue to play an important role in analysis and modeling of economic growth. The have been used, for example, by Boucekkine et al. in [5] to study the relation between demographic and economic variables in the vintage capital problem.

In 2009 [2] a system of functional differential equations having leads and lags has been applied by Albis et al. to model the dynamic behaviour of the capital growth in an overlapping-generations model with continous trading and finitely lived agents.

Differential equations with advanced and delayed time arguments may also arise in simple growth models with delays, such as models with investment generations lags proposed by Collard et al [7] in 2004. In this article, the authors proposed and implemented a numerical scheme which combine a Runge-Kutta and a shooting method to solve the short run dynamics of a neoclassical growth model with a simple time to build lag.

Another application of MTFDEs is investigated in [3, 4, 6, 16] where a nonlinear functional differential equation with delay and advanced arguments is used to model conduction in a myelinated nerve axon.

Special attention has been paid by many authors to mixed-type functional differential equations of the form

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-1) + \gamma(t)x(t+1), \tag{3}$$

where x is the unknown function, α, β and γ are known functions.

In 2005, Iakovleva and Vanegas [13] studied equation (3) in a particular case, presenting existence and uniqueness results. A similar approach has been followed by the authors of [8]. Inspired on their earlier work about delay differential equations [9, 17] and based on the existing insights on the qualitative behaviour of MTFDEs, they have developed a new approach to the analysis of these equations in the autonomous case. More precisely, they analysed MTFDEs as boundary value problems, that is, for a MTFDEs of the form (3), but with constant coefficients α, β and γ, they considered the problem of finding a differentiable solution on a certain real interval $[-1, k]$, $k \in \mathbb{N}$, given its values on the intervals $[-1, 0]$ and $(k-1, k]$. They concluded that in general the specification of such boundary functions is not sufficient to ensure that a solution can be found. With arbitrary boundary conditions, the problem turns out to be ill-posed. They provided sufficient analytical background on which the numerical scheme could be built. For the case where such a solution exists they introduced a numerical algorithm based on the θ-method to compute it. The same authors of [8] and their colaborators extended this work in [11]. In 2011, Da Silva and Escalante [23] presented a numerical algorithm based on the method of steps and tau method, for the approximation of solutions of BVP equation (3) in the autonomous case.

2 Numerical Experiments

We say that $x(t)$ is a small solution of equation (3) iff $t \to +\infty \lim x(t)e^{\lambda t} = 0$, $\forall \lambda \in \mathbb{R}^+$.

It is known that the detection and computation of small solutions are important problems when studying DDEs. Such problems have been discussed in [9, 10, 12, 17]. Therefore, we decided to investigate how our method for MTFDEs work in the case of a small solution.

We are concerned with the numerical solution of the linear homogeneous nonautonomous equation (3) in the case of a small solution.

The numerical algorithms applied to the solution of this problem, based on the collocation method and on the method of steps, were introduced in [24, 14, 25]. In [26, 15] new algorithms were introduced, based on the finite elements and on the least squares methods.

In the present work, the BVP under consideration is defined in Example 2.

The choosen boundary value problem (BVP) is defined by equations (4)-(6).

$$x'(t) = -2t\ x(t) - e^{-2t}x(t-1) + e^{2t}x(t+1), \tag{4}$$

with boundary conditions

$$x(t) = e^{-t^2}, \qquad t \in [-1, 0]; \tag{5}$$

$$x(t) = e^{-t^2}, \qquad t \in [k-1, k]. \tag{6}$$

The exact solution is $x(t) = e^{-t^2}$.

The numerical results of BVP (4)-(6) are presented in Table 1. There is a good agreement with the convergence order of the collocation method ($k = 2$). Similar results are obtained when the least squares and finite element methods are applied. We are still working with larger intervals of estimation ($k = 3, 4$).

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ϵ and estimates for the order p		
Step	$k=2$	
h_i	ϵ	p
1/8	$1.18101e-3$	
1/16	$2.94768e-4$	2.0024
1/32	$7.26304e-5$	2.0209
1/64	$1.79561e-5$	2.0161
1/128	$4.45928e-6$	2.0096

Table 1: Collocation method. Estimates of error ϵ and convergence order p for the approximate solution $\tilde{x}$ in the case $k=2$. $\epsilon = \|x-\tilde{x}\|_\infty$ (error on $[0,1]$).

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