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Serdica Math. J. **37** (2011), 227–236

Serdica
Mathematical Journal

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REMARKS ON BLOW UP TIME FOR SOLUTIONS OF A NONLINEAR DIFFUSION SYSTEM WITH TIME DEPENDENT COEFFICIENTS

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Communicated by T. Gramchev

Abstract. We investigate the blow-up of the solutions to a nonlinear parabolic system with Robin boundary conditions and time dependent coefficients. We derive sufficient conditions on the nonlinearities and the initial data in order to obtain explicit lower and upper bounds for the blow up time t^*.

1. Introduction. The blow-up phenomena for solutions to linear and nonlinear parabolic equations and systems have been widely studied and different methods have been introduced in order to find bounds for blow-up time t^*. We refer to the books of Straughan [11] and Quittner-Souplet [10], and the papers of Vazquez [12] and Weissler [13], [14]. Recently Payne, Philippin, Schaefer in [5]–[6] and Payne, Philippin and Vernier-Piro in [7]–[8] (see also references therein) have considered nonlinear boundary value problems under different boundary

2010 *Mathematics Subject Classification:* 35K55, 35K60.
Key words: Parabolic problems, blow-up.

conditions, deriving lower and upper bounds for the blow-up time t^* as well as sufficient conditions for global existence of the solutions.

In this paper we investigate the blow-up phenomena of the classical solution $(u(x,t), v(x,t))$ of the following Robin system

$$\begin{cases} \Delta u + K_1(t) f_1(v) = u_t \quad \text{in } \Omega \times (0, t^*), \\ \Delta v + K_2(t) f_2(u) = v_t \quad \text{in } \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu}(x,t) = -\alpha u, \quad \text{on } \partial\Omega \times (0, t^*), \\ \dfrac{\partial v}{\partial \nu}(x,t) = -\beta v, \quad \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) \geq 0, \quad \text{on } \Omega, \\ v(x,0) = v_0(x) \geq 0, \quad \text{on } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$ with $\partial\Omega$ sufficiently smooth, the coefficients $K_1(t)$, $K_2(t)$ and f_1, f_2 positive functions and $\dfrac{\partial u}{\partial \nu}, \dfrac{\partial v}{\partial \nu}$ are the outward normal derivative of the vector–valued solution (u, v) on the boundary $\partial\Omega$, α and β are two positive constants satisfying

$$0 \leq \alpha \leq \beta. \tag{1.2}$$

The initial data $u_0(x), v_0(x)$ are nonnegative smooth functions, such that $\dfrac{\partial u_0}{\partial \nu} = -\alpha u_0$, $\dfrac{\partial v_0}{\partial \nu} = -\beta v_0$, $x \in \partial\Omega$, while t^* stands for the blow-up time if blow up occurs, otherwise $t^* = \infty$. It is well known that the solution can fail to exist only by blowing up at finite time ([1], [2]) and the geometry of the domain Ω, the nonlinearities, the boundary data and the initial conditions greatly affect the evolution in time of the solution.

We note that u and v are non negative in x and $t \in (0, t^*)$ by the parabolic maximum principle applied to the the system (1.1).

The main aim of this paper is to derive explicit upper and lower bounds for the blow-up time for the solution of (1.1). We remark that if K_1, K_2 are constant, bounds for the blow-up time are derived in [3], where on $\partial\Omega \times (0, t^*)$ $\dfrac{\partial u}{\partial \nu} = g_1(u)$, $\dfrac{\partial v}{\partial \nu} = g_2(v)$, g_1, g_2 positive functions.

For systems with Dirichlet and Neumann boundary conditions we refer to [3] and [9].

We consider another class of parabolic systems with Robin type boundary conditions in [4], where sufficient conditions are introduced in order to obtain global existence of the solution as well as upper and lower bounds for the blow-up, if blow-up occurs.

The paper is organized as follows. In Section 2 we derive a lower bound for the blow up time t^* under suitable conditions on data and for convex domain in $\mathbb{R}^3$. In Section 3 we remove the restriction on the domain and under alternative conditions on data an upper bound is obtained.

Throughout the paper the notations $u_i = \dfrac{\partial u}{\partial x_i}$, $i = 1, \ldots, N$ and $(\cdot)'$ for the derivative of the coefficients will be used and the summation convention over repeated indexes will be assumed.

2. Lower bound. Under suitable conditions on non linearities and Ω we get a differential inequality and we derive a lower bound for t^*.

To this end we will use a Sobolev-type inequality introduced in [3] which plays an important role in deriving the lower bound. However, it must be noted that the inequality holds only if we consider our domain $\Omega \subset \mathbb{R}^3$.

Let us suppose $K_1 \le K_2$ and we define the auxiliary function

$$\Phi(t) = K_2^2(t) \int_\Omega (u^2 + v^2)^{2p} dx, \quad p > 1 \tag{2.1}$$

We say that (u, v) blows up in Φ-measure when

$$\lim_{t \to t^*} \Phi(t) = \infty.$$

We now prove the following result

Theorem 2.1. *Let (u, v) be the solution of (1.1), in a convex domain $\Omega \subset R^3$, blowing up at time t^* in Φ-measure. Assume that there exists a positive constant C_0 such that*

$$u f_1(v) + v f_2(u) \le C_0 (u^2 + v^2)^{p+1}. \tag{2.2}$$

Moreover assume that $K_2(t)$ satisfies

$$\frac{K_2'(t)}{K_2(t)} \le \delta, \quad \delta > 0. \tag{2.3}$$

Then the solution blows up in the Φ-measure and

$$t^* \ge T := \int_{\Phi(0)}^{\infty} \frac{d\eta}{\phi(\eta)}, \tag{2.4}$$

where $\phi(\Phi) = \sum_{i=1}^{3} (\xi_i \Phi^{h_i})$ *with some if not all* h_i *greater than one.*

Proof. We compute

$$\Phi'(t) = 2\left(\frac{K_2'}{K_2}\right) K_2^2 \int_\Omega (u^2+v^2)^{2p} + 4pK_2^2 \int_\Omega (u^2+v^2)^{2p-1}(u\Delta u + v\Delta v)dx$$
$$+ 4pK_2^2 \int_\Omega (u^2+v^2)^{2p-1}[K_1 u f_1(v) + K_2 v f_2(u)]dx.$$

Now by (2.2), (2.3) and the assumption $K_1 \le K_2$ we obtain

$$\Phi'(t) \le 2\delta\Phi + 4pK_2^2 \int_\Omega \operatorname{div}\Big[(u^2+v^2)^{2p-1}(u\nabla u + v\nabla v)\Big]dx$$
$$- 8p(2p-1)K_2^2 \int_\Omega (u^2+v^2)^{2p-2}(u\nabla u + v\nabla v)(u\nabla u + v\nabla v)dx$$
$$- 4pK_2^2 \int_\Omega (u^2+v^2)^{2p-1}(|\nabla u|^2 + |\nabla v|^2)dx + 4pC_0K_2^3 \int_\Omega (u^2+v^2)^{3p}dx.$$

Now by divergence theorem and boundary conditions in (1.1) and assumption (1.2) we get

$$\Phi'(t) \le 2\delta\Phi - 4pK_2^2\alpha \int_{\partial\Omega} (u^2+v^2)^{2p}ds \tag{2.5}$$
$$- 8p(2p-1)K_2^2 \int_\Omega (u^2+v^2)^{2p-2}(u\nabla u + v\nabla v)(u\nabla u + v\nabla v)dx$$
$$- 4pK_2^2 \int_\Omega (u^2+v^2)^{2p-1}(|\nabla u|^2 + |\nabla v|^2)dx + 4pC_0K_2^3 \int_\Omega (u^2+v^2)^{3p}dx.$$

Now we neglect the second (negative) term in (2.5), and we use the inequality $(u^2+v^2)(|\nabla u|^2+|\nabla v|^2) \ge (u\nabla u + v\nabla v)(u\nabla u + v\nabla v)$ in the fourth term to have

$$\Phi'(t) \le 2\delta\Phi - 4p(4p-1)K_2^2 \int_\Omega (u^2+v^2)^{2p-2}(u\nabla u + v\nabla v)(u\nabla u + v\nabla v)dx \tag{2.6}$$
$$+ 4pC_0K_2^3 \int_\Omega (u^2+v^2)^{3p}dx.$$

For simplicity we define

$$V = (u^2+v^2)^p, \tag{2.7}$$

then

$$V_iV_i = 4p^2(u^2+v^2)^{2p-2}(u\nabla u + v\nabla v)(u\nabla u + v\nabla v). \tag{2.8}$$

By plugging (2.7) and (2.8) in (2.6) we have

$$\Phi'(t) \leq 2\delta\Phi - (4-\frac{1}{p})K_2^2\int_\Omega V_iV_i + 4pC_0K_2^3\int_\Omega V^3dx. \tag{2.9}$$

In the third term in (2.9) we use the inequality

$$\int_\Omega V^3dx \leq \left\{\frac{3}{2\rho_0}\int_\Omega V^2dx + \left(1+\frac{d}{\rho_0}\right)\left(\int_\Omega V^2dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla V|^2dx\right)^{\frac{1}{2}}\right\}^{\frac{3}{2}}$$

with

$$\rho_0 = \min_{\partial\Omega}(x_i\cdot\nu_i), \qquad d^2 = \max_{\bar{\Omega}}(x_i\cdot x_i),$$

obtained by application of a Sobolev type inequality derived in [8] Lemma A.2, valid only in a convex domain $\subset R^3$ we have

$$\begin{aligned} K_2^3(t)\int_\Omega V^3dx \leq \Bigg\{&\frac{3}{2\rho_0}K_2^2\int_\Omega V^2dx \\ &+\left(1+\frac{d}{\rho_0}\right)\left(\frac{1}{\tau}K_2^2\int_\Omega V^2dx\right)^{\frac{1}{2}}\left(\tau K_2^2\int_\Omega |\nabla V|^2dx\right)^{\frac{1}{2}}\Bigg\}^{\frac{3}{2}} \end{aligned} \tag{2.10}$$

$\tau > 0$ a suitable constant to be chosen later on.

In (2.10) we use the basic inequality $(a+b)^{\frac{3}{2}} \leq \sqrt{2}\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}\right)$ and we obtain

$$\begin{aligned} K_2^3(t)\int_\Omega V^3dx \leq\; &\sqrt{2}\left(\frac{3}{2\rho_0}\right)^{\frac{3}{2}}\Phi^{\frac{3}{2}} \\ &+\sqrt{2}\left(1+\frac{d}{\rho_0}\right)^{\frac{3}{2}}\left(\frac{1}{\tau}\Phi\right)^{\frac{3}{4}}\left(\tau K_2^2\int_\Omega |\nabla V|^2dx\right)^{\frac{3}{4}}. \end{aligned} \tag{2.11}$$

Now we use the inequality $a^rb^s \leq ra+sb$, $r+s=1$, we have

$$\begin{aligned} K_2^3\int_\Omega V^3dx \leq\; &\frac{3^{\frac{3}{2}}}{2\rho_0^{\frac{3}{2}}}\Phi^{\frac{3}{2}} + \frac{1}{2^{\frac{3}{2}}\tau^3}\left(1+\frac{d}{\rho_0}\right)^{\frac{3}{2}}\Phi^3 \\ &+\left(\frac{3}{2^{\frac{3}{2}}}\right)\left(1+\frac{d}{\rho_0}\right)^{\frac{3}{2}}\tau K_2^2\int_\Omega V_iV_idx = C_1\Phi^{\frac{3}{2}} + C_2\Phi^3 + C_3\tau K_2^2\int_\Omega V_iV_idx \end{aligned} \tag{2.12}$$

with

$$C_1 = \frac{3^{\frac{3}{2}}}{2\rho_0^{\frac{3}{2}}}, \qquad C_2 = \frac{1}{2^{\frac{3}{2}}\tau^3}\left(1+\frac{d}{\rho_0}\right)^{\frac{3}{2}}, \qquad C_3 = \frac{3}{2^{\frac{3}{2}}}\left(1+\frac{d}{\rho_0}\right)^{\frac{3}{2}}. \tag{2.13}$$

Inserting (2.12) in (2.9) we obtain

$$\Phi'(t) \le 2\delta\Phi - \left(4 - \frac{1}{p} - 4pC_3C_0\tau\right)K_2^2\int_\Omega V_iV_i + 4pC_1C_0\Phi^{\frac{3}{2}} + 4pC_2C_0\Phi^3. \tag{2.14}$$

By choosing $\tau = \dfrac{4-\frac{1}{p}}{4pC_3C_0}$, we obtain

$$\Phi'(t) \le \xi_1\Phi + \xi_2\Phi^{\frac{3}{2}} + \xi_3\Phi^3 = \phi(\Phi), \tag{2.15}$$

with $\xi_1 = 2\delta$, $\xi_2 = 4pC_1C_0$, $\xi_3 = 4pC_2C_0$.

Then if Φ blows up at time t^*, there exists a time t_0 such that $\Phi(t) > \Phi(0)$ and by integration from 0 to t^*, we obtain the following lower bound for t^*

$$\int_{\Phi(0)}^{\infty}\frac{d\eta}{\phi(\eta)} \le t^* - t_0 \le t^*$$

which is the desired lower bound (2.4).

3. Upper bound. We seek in this section an upper bound for the blow up time t^* by defining the following auxiliary function

$$\chi(t) := \chi_1(t) + \chi_2(t) = K_1(t)^{\frac{1}{q-1}}\int_\Omega u\varphi_1 dx + K_2(t)^{\frac{1}{q-1}}\int_\Omega v\varphi_1 dx, \tag{3.1}$$

where φ_1 is the first eigenfunction of the following membrane problem

$$\begin{cases} \Delta\varphi_1 + \lambda_1\varphi_1 = 0, \quad \varphi_1 > 0, \quad \text{in } \Omega, \\ \dfrac{\partial\varphi_1}{\partial\nu} = -\beta\varphi_1, \quad \text{on } \partial\Omega \end{cases} \tag{3.2}$$

with

$$\int_\Omega \varphi_1 dx = 1 \tag{3.3}$$

with β in (1.1) satisfying (1.2). We prove the following

Theorem 3.1. *Let (u, v) be the solution of (1.1), in a bounded domain Ω in $\mathbb{R}^N$. Let $K(t) = \min\{K_1(t), K_2(t)\}$. Assume that*

$$f_i(s) \geq s^q \quad q > 1, \quad i = 1, 2, \tag{3.4}$$

and

$$\frac{K_i'}{K_i} \geq \tilde{\gamma}, \tag{3.5}$$

with $\tilde{\gamma}$ a positive constant. Assume moreover that there exists a constant $\bar{\gamma}$ with $0 < \bar{\gamma} \leq 1$ such that

$$\frac{K}{K_i} \geq \bar{\gamma}. \tag{3.6}$$

If (u,v) becomes unbounded in χ measure at some finite time t^*, then t^* is bounded from above by T with

$$T = \int_{\chi(0)}^{\infty} \frac{d\eta}{\gamma_2 \eta^q - \gamma_1 \eta}, \tag{3.7}$$

where

$$\gamma_1 = \lambda_1 - \frac{\tilde{\gamma}}{q-1}, \quad \gamma_2 = 2^{1-q} \bar{\gamma}^{\frac{q}{q-1}},$$

provided

$$\gamma_2 \chi^{q-1}(0) > \gamma_1. \tag{3.8}$$

P r o o f. By derivative of (3.1) we have

$$\begin{aligned} \chi'(t) = {} & \frac{1}{q-1} \left(\frac{K_1'}{K_1} \right) K_1^{\frac{1}{q-1}} \int_\Omega u \varphi_1 dx + \frac{1}{q-1} \left(\frac{K_2'}{K_2} \right) K_2^{\frac{1}{q-1}} \int_\Omega v \varphi_1 dx \\ & + K_1^{\frac{1}{q-1}} \int_\Omega u_t \varphi_1 dx + K_2^{\frac{1}{q-1}} \int_\Omega v_t \varphi_1 dx. \end{aligned} \tag{3.9}$$

By applying in (3.9) the condition (3.5) we obtain

$$\begin{aligned} \chi'(t) \geq {} & \frac{\tilde{\gamma}}{q-1} \chi + K_1^{\frac{1}{q-1}} \int_\Omega \Delta u \varphi_1 dx + K_2^{\frac{1}{q-1}} \int_\Omega \Delta v \varphi_1 dx \\ & + K_1^{\frac{q}{q-1}} \int_\Omega f_1 \varphi_1 dx + K_2^{\frac{q}{q-1}} \int_\Omega f_2 \varphi_1 dx, \end{aligned} \tag{3.10}$$

We compute

$$\int_\Omega \Delta u\varphi_1 dx = \int_\Omega div[\nabla(u\varphi_1)]dx - \int_\Omega u\Delta\varphi_1 dx - 2\int_\Omega \nabla u\nabla\varphi_1 dx.$$

By the equation in (3.2), the divergence theorem and boundary condition in (3.2) we obtain

$$\int_\Omega \Delta u\varphi_1 dx = (\beta-\alpha)\int_{\partial\Omega} u\varphi_1 ds - \lambda_1\int_\Omega u\varphi_1 dx \geq -\lambda_1\int_\Omega u\varphi_1 dx \tag{3.11}$$

analogously, we obtain

$$\int_\Omega \Delta v\varphi_1 dx = -\lambda_1\int_\Omega v\varphi_1 dx. \tag{3.12}$$

By inserting (3.11) and (3.12) in (3.10) we have

$$\begin{aligned}\chi'(t) &\geq \frac{\tilde\gamma}{q-1}\chi - \lambda_1\left[K_1^{\frac{1}{q-1}}\int_\Omega u\varphi_1 dx + K_2^{\frac{1}{q-1}}\int_\Omega v\varphi_1 dx\right] \\ &\quad + K_1^{\frac{q}{q-1}}\int_\Omega f_1\varphi_1 dx + K_2^{\frac{q}{q-1}}\int_\Omega f_2\varphi_1 dx \\ &= -\left(\lambda_1 - \frac{\tilde\gamma}{q-1}\right)\chi + K_1^{\frac{q}{q-1}}\int_\Omega v^q\varphi_1 dx + K_2^{\frac{q}{q-1}}\int_\Omega u^q\varphi_1 dx.\end{aligned} \tag{3.13}$$

In the last two terms of (3.13) we used (3.4), and by expression of $K(t)$ we obtain

$$\chi'(t) \geq -\gamma_1\chi + K^{\frac{q}{q-1}}\int_\Omega (u^q+v^q)\varphi_1 dx, \tag{3.14}$$

with $\gamma_1 = \lambda_1 - \frac{\tilde\gamma}{q-1}$.

To estimate $\int_\Omega (u^q+v^q)\varphi_1 dx$ we use Hölder inequality and (3.3)

$$\begin{cases} \displaystyle\int_\Omega u^q\varphi_1 dx \geq \left(\int_\Omega u\varphi_1 dx\right)^q \\ \displaystyle\int_\Omega v^q\varphi_1 dx \geq \left(\int_\Omega v\varphi_1 dx\right)^q \end{cases} \tag{3.15}$$

By substitution of (3.15) in (3.14) we find

$$\chi'(t) \geq -\gamma_1\chi + K^{\frac{q}{q-1}}\left\{\left(\int_\Omega u\varphi_1 dx\right)^q + \left(\int_\Omega v\varphi_1 dx\right)^q\right\} \tag{3.16}$$

$$= -\gamma_1 \chi + \left(\frac{K}{K_1}\right)^{\frac{q}{q-1}} \chi_1^q + \left(\frac{K}{K_2}\right)^{\frac{q}{q-1}} \chi_2^q.$$

By inserting in the last term of (3.16) the assumption (3.6), we obtain

$$\chi'(t) \geq -\gamma_1 \chi + \bar{\gamma}^{\frac{q}{q-1}} (\chi_1^q + \chi_2^q). \tag{3.17}$$

Using in (3.17) the inequality $\chi_1^q + \chi_2^q \geq 2^{1-q}(\chi_1 + \chi_2)^q$ we obtain

$$\chi'(t) \geq -\gamma_1 \chi + \gamma_2 \chi^q. \tag{3.18}$$

with $\gamma_2 = 2^{1-q} \bar{\gamma}^{\frac{q}{q-1}}$.

Since (3.8) holds we conclude that $\chi(t)$ is an increasing function of $t \in (0, t^*)$ and that by integration of (3.18) from 0 to t we find

$$\int_{\chi(0)}^{\chi(t)} \frac{d\eta}{\gamma_2 \eta^q - \gamma_1 \eta} \geq t. \tag{3.19}$$

and we obtain the upper bound for t^*:

$$t^* \leq \int_{\chi(0)}^{\infty} \frac{d\eta}{\eta(\gamma_2 \eta^{q-1} - \gamma_1)}.$$

Acknowledgments. The author thanks G.Philippin and S.Vernier Piro for useful discussions on the subject.

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Received July 4, 2011