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ON GROUPS WHOSE CONTRANORMAL SUBGROUPS ARE NORMALLY COMPLEMENTED

L. A. Kurdachenko, I. Ya. Subbotin

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Abstract. Groups in which every contranormal subgroup is normally complemented has been considered. The description of such groups G with the condition Max-n and such groups having an abelian nilpotent residual satisfying Min-G have been obtained.

J.S. Rose [8] has introduced the notion of a contranormal subgroup. A subgroup H *is contranormal in a group* G if its normal closure A^G coincides with G. Contranormal subgroups play a significant role in the investigations related to generalized nilpotency. Thus in finite groups, the absence of proper contranormal subgroups is equivalent to nilpotency, while all (finite and infinite) groups, in which every non-identity subgroup is contranormal are simple groups. Some useful criteria of generalized nilpotency involving the contranormality have been obtained for some classes of infinite groups (see, for example, [6]).

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Observe that *a contranormal subgroup supplements the derived subgroup in a group*. Recall, that a subgroup F is a *supplement* to a subgroup D in a group G if $G = FD$. In this case, the subgroup D is called *supplemented* in G. A partial case of a supplement subgroup is a *complement* subgroup. This is a subgroup H such that $G = DH$ and $D \cap H = \langle 1 \rangle$. Here the subgroup D is called *complemented in* G. Groups with some classes of complemented subgroups have been investigated by the authors. It was initiated by a remarkable theorem due to P. Hall [3] that characterizes finite soluble groups as the groups whose Sylow subgroups are complemented. The groups, in which all subgroups are complemented have been completely described by P. Hall [4] and N. V. Chernikova [1]. D. I. Zaitsev [11, 12] studied groups with the property that every normal subgroup is complemented. Many other important results in this area can be found in the book [2].

A group G is called *a C-group* if every subgroup of G either is normal or has a normal complement [10]. In [10], all finite C-groups are described. Since in every group a contranormal subgroup supplements the derived subgroup (i.e. the derived subgroup is *normally supplemented*), the questions about groups in which *all* contranormal *subgroups are normally complemented* looks as a logical next step. We will denote these groups by CNC-groups. Observe, that C-groups form a proper subclass of CNC-groups. All groups without contranormal subgroups (for example, all nilpotent groups) are CNC-groups.

There are many simple examples showing that the class of CNC-groups is diverse, large and contains many kinds of periodic and non-periodic groups. For example, let $G = Q \leftthreetimes B$, where Q is an abelian group having no B-invariant proper subgroups and B is an abelian group. Any proper contranormal subgroup C that contains B can be written as $C = (Q \cap C) \leftthreetimes B$ (see, for example, [2, Lemma 3.7]), and $Q \cap C$ is B-invariant, i.e. $Q \cap C = \langle 1 \rangle$ and $C = B$. Let $C \not\geq B$ be a contranormal in G subgroup. Since $QC = G$, $Q \lhd G$, and Q is abelian, we conclude that $Q \cap C = M \lhd G$ and M is B-invariant. Since Q is an abelian group having no B-invariant subgroups and $C \not\geq B$, $M = \langle 1 \rangle$, and $G = Q \leftthreetimes C$, i.e. C is complemented in G. So G is a CNC-group.

Since every subgroup of a nilpotent group is subnormal (see, for example, [5, Theorem 16.2.2]), there are no contranormal subgroups in a nilpotent group. Therefore all nilpotent groups are CNC-groups.

In the current article we consider some classes of CNC-groups. We obtain a description of such groups G satisfying the condition Max-n and CNC-groups group having an abelian nilpotent residual satisfying Min-G. Recall that a group G is called *a group satisfied the maximal condition for normal subgroups* (*the* *condition* Max-n) if every ascending series of its normal subgroups is finite. We say that a subgroup K of a group G *satisfies the minimal condition for G-normal subgroups (the* Min-G *condition)* if every descending series of G-normal subgroups of K is finite.

Lemma 1. *Let G be a CNC-group having an abelian normal subgroup K satisfying* Min-G *and defining a nilpotent factor-group G/K. Then the following conditions hold:*

(*i*) *G satisfies the minimal condition for contranormal subgroups; in particular, every contranormal subgroup of G contains a minimal contranormal subgroup (a contranormal subgroup containing no proper contranormal subgroups).*

(*ii*) *$G = A \leftthreetimes B$ where A is a nilpotent residual of G, A is an abelian direct product of a finite number of minimal normal characteristically free subgroups; B is a nilpotent group.*

Proof. Since a normal subgroup K satisfies Min-G and defines a nilpotent factor group G/K, we can state that G has a nilpotent residual of G that is a normal subgroup A which defines a nilpotent factor-group and A is an intersection of all normal subgroups with this property. Since G/K is nilpotent, K contains this residual A. Since K is abelian, A is also abelian.

Let B be a contranormal subgroup of G. Since any nilpotent group contains no contranormal subgroups and A is a nilpotent residual of G, $G = AB$ and $G/A \cong B/(A \cap B)$ is nilpotent. Since G is abelian-by-nilpotent, B is also abelian-by-nilpotent. If B is non-nilpotent, it has its own nilpotent residual A_1. Indeed, $G = KB$, K is an abelian subgroup with Min-G. It is easy to see, that in this case, for any subgroup M of K the condition for M to be normal in G is equivalent M to be B-invariant. So for K to be Min-G is equivalent to be Min-B (i.e. the minimal condition on B-invariant subgroups). Since $G = AB$, $A \leq B$, $B/(A \cap B) \cong AB/A$ is a nilpotent group, and $A \cap B$ satisfies Min-B. By the arguments listed above, B being non-nilpotent has its own nilpotent residual $A_1 \leq A \cap B \leq A$. Since A is abelian, $G = AB$ and A_1 is normal in B, so A_1 is normal in G. Since B is not a minimal contranormal subgroup in G, it contains its own proper contranormal subgroup B_1. By the reasons listed above, $B = A_1B_1$. Since the contranormality is a transitive property, B_1 is contranormal in G. Therefore $G = AB_1$. Since B_1 is a proper subgroup of B, $A > A_1$. Indeed, otherwise $A = A_1$, $B = A_1B_1 = AB_1 = G$. Repeating this arguments we obtain that if B_1 contains a proper contranormal subgroup B_2, then this subgroup has a nilpotent residual A_2, $B_1 = A_2B_2$. Note that A is abelian and $A > A_1 \geq A_2$, and $A_2 \trianglelefteq B_1$. Since $G = AB_1$, $A_2 \trianglelefteq G$. Since contranormality is a transitive property, B_2 is contranormal in G. Therefore $G = AB_2$. Since B_2 is a proper subgroup of B_1, using the arguments above we can prove that $A > A_1 > A_2$. Since A satisfies Min-G, this process is finite and we can continue it up to the step when B_n will not have a proper contranormal subgroups. Without loss of generality, we can put $B = B_n$. The condition (i) is proved.

Now we assume that B has no proper contranormal subgroups. By Corollary A2 of [?], B is nilpotent.

Since G is a CNC-group, B is complemented by a normal subgroup D; that is, $G = D \leftthreetimes B$. So since $G/D \cong B$ is a nilpotent group, $D \geq A$. As a contranormal subgroup, B supplements the nilpotent residual A of G, i.e. $G = AB$. So $G = A \leftthreetimes B$.

Let now M be a proper G-normal subgroup of A. Since $A > M$,

$$G/M = (A \leftthreetimes B)/M = A/M \leftthreetimes BM/M \ncong BM/M.$$

It follows that $M \leftthreetimes B = B_1$ is a proper in G contranormal subgroup. Let M_1 be a proper in M G-normal subgroup. Consider the subgroup $B_2 = M_1 \leftthreetimes B$. Clearly, $B_2 < B_1$ and B_2 is contranormal in G. If M_2 is a proper G-normal subgroup of M_1 we can continue this process. Finally we can construct a descending chain of contranormal in G subgroups $B_1 > B_2 > \cdots > B_m$. By the proved above condition (i), this series will be terminated at a finite number m. It means that the non-identity normal subgroup M_m has no G-normal proper subgroups. Note, that M_m must to be a characteristically free abelian subgroup.

Let $\langle 1 \rangle \neq N \neq M_m$ be another G-normal subgroup of A. It is obvious that $N \cap M_m = \langle 1 \rangle$. Using the above procedure, we can find a minimal non-identity G-normal subgroup $N_n \leq N$.

Continuing this process, we can identify all minimal non-identity G-normal subgroups in A. Since A satisfies Min-G, there is a finite number of them. It is clear that we can consider a direct product (with a finite amount of factors) C of these subgroups.

Since C is a G-normal subgroup, we can consider a contranormal subgroup $C \leftthreetimes B$. This subgroup is normally complemented in G with the help of some G-normal subgroup, let say F, i.e. $G = F \leftthreetimes (C \leftthreetimes B)$. Consider $F \cap A = R$. If $R \neq \langle 1 \rangle$, then R contains a minimal G-normal subgroup R_r belonging to the product of all such subgroups C. Since $F \cap (C \leftthreetimes B) = \langle 1 \rangle$, $F \cap C = \langle 1 \rangle$. Therefore $R_r = \langle 1 \rangle$, and $F \cap A = R = \langle 1 \rangle$. Then $G = F \leftthreetimes (C \leftthreetimes B) = (F \times C) \leftthreetimes B$. Indeed, if $(F \times C) \cap B = X$ and $x \in X$, then $x = fc = b$ where $f \in F$, $c \in C$, $b \in B$. It follows that $f = c^{-1}b = 1$.

Since $G/(F \times C) \cong B$ is a nilpotent group, $F \times C$ contains a nilpotent residual A. This subgroup A is complemented in $F \times C$ by the subgroup $B \cap$

$(F \times C)$. Since $(F \times C) \cap B = \langle 1 \rangle$, $F \times C = A$ which is a contradiction. Therefore $A = F \times C$, and hence $F = \langle 1 \rangle$ and $A = C$. □

Lemma 2. *The conditions of Lemma* 1 *are sufficient.*

Proof. Let by (ii) $G = A \leftthreetimes B$, where A is a nilpotent residual of G, A is an abelian direct product of a finite number of minimal normal characteristically free subgroups, B is a nilpotent group. If $A = \langle 1 \rangle$, then G is nilpotent and everything is clear. So we assume that $A \neq 1$. Note, that B is a contranormal in G subgroup. Indeed if R is a proper normal subgroup of G and $R \geq B$, $G = AR$, and $F = A \cap R \ngeq A$. Then the group G/F is a direct product of an abelian subgroup A/F and a nilpotent subgroup R/F. Then G/F is nilpotent and $F > A$. This is a contradiction. Since B is nilpotent, it does not contain contranormal in B (and therefore in G) proper contranormal subgroups. So B is a minimal contranormal in G subgroup.

If Q is any contranormal subgroup of G, then by (i) without loss of generality with respect to the notations we can assume that $Q \geq B$. Then $Q = (A \cap Q) \leftthreetimes B$. Since A is a completely-factorized group (see [1]), $A \cap Q$ is complemented in A with the help of some subgroup T. If $A = \prod_{i=1}^{n} A_i$, where A_i is a minimal G-normal subgroup, then A_i either entirely belongs to $A \cap Q$, or $A_i \cap Q = \langle 1 \rangle$. Since we have a finite number n of factors A_i, we can conclude that $A = (A \cap Q) \times L$, $L = \prod_{j=1}^{k} A_j$, where each $A_j \cap Q = \langle 1 \rangle$. Now it is clear, that $G = L \leftthreetimes Q$. □

The following theorem is a direct corollary of Lemmas 1 and 2.

Theorem 1. *Let G be a group having an abelian normal subgroup K satisfying* Min-G *and defining a nilpotent factor-group G/K. Then $G$is a CNC-group if and only if the following conditions hold:*

(i) *G satisfies the minimal condition for contranormal subgroups; in particular, every contranormal subgroup of G contains a minimal contranormal subgroup (that is a contranormal subgroup containing no proper contranormal subgroups).*

(ii) *$G = A \leftthreetimes B$, where A is a nilpotent residual of G, A is a direct product of finite number of minimal normal characteristically free subgroups; B is a nilpotent group.*

We proved that the group G from Theorem 1 satisfied the minimal condition for contranormal subgroups. In this setting it will be interesting to mention the following assertion.

Remark 1. Let $G = A \leftthreetimes B$ be a group with Min-cn (i.e. G satisfies the minimal condition for contranormal subgroups), B is a contranormal subgroup of G. Then A is a group satisfying the minimal condition for B-normal subgroups (Min-B-n).

Indeed, let

$$A = A_0 > A_1 > A_2 > \cdot > A_n > \cdot$$

be a descending series of B-normal subgroups from A. For any $A_i, 0 \leq i$, there is a contranormal subgroup $A_i \leftthreetimes B$ of G. We will show that $A_i \leftthreetimes B > A_{i+1} \leftthreetimes B$. Let $A_i \leftthreetimes B = A_{i+1} \leftthreetimes B$. Let $a \in A_i \backslash A_{i+1}$. Then $a \in A_{i+1} \leftthreetimes B$, and $a = db$ for some $d \in A_{i+1}, b \in B$. So $b = d^{-1}a \in B$. But $d^{-1}a \in A_i$ and $A_i \cap B = \langle 1 \rangle$. Therefore $d^{-1}a = 1$ and $d = a$. This is a contradiction. Thus $A_i \leftthreetimes B > A_{i+1} \leftthreetimes B$ for all $i \geq 0$. Since G is a group with Min-cn, the above series is finite.

Theorem 2. *Let G be a Chernikov group. Then G is a CNC-group if and only if $G = A \leftthreetimes B$, where A is a finite group and the nilpotent residual A of G is a direct product of a finite number of minimal normal characteristically free subgroups, B is a nilpotent group.*

P r o o f. The proof of the theorem is very similar to the proof of Theorem 1. Note that since contranormality is a transitive property, every contranormal in a Chernikov group G subgroup H contains a minimal contranormal in G subgroup B. Note that B has no contranormal subgroups, and therefore in our case by Corollary A2 of [7], B is nilpotent. If G is a CNC-group, B is complemented by a normal subgroup A, that is, $G = A \leftthreetimes B$.

Since G is a Chernikov group, G has a nilpotent residual R. In particular, A contains this residual R. As a contranormal subgroup, B supplements this residual. So A is a nilpotent residual of G and $G = A \leftthreetimes B$.

Repeating step by step the proof of Theorem 1 and taking into account that being a Chernikov group A cannot contains an infinite characteristically free subgroup, we come to the theorem statement. □

Lemma 3. *If G is a CNC-group with the maximal condition on normal subgroups (*Max-n*), then G satisfies the minimal condition on contranormal subgroups* (Min-cn).

P r o o f. If G has no proper contranormal subgroups then everything is clear.

Let B_1 be a proper contranormal subgroup in G. Let B_2 be a proper contranormal subgroup of B_1. Since the contranormality is a transitive relation, B_2 is contranormal in G. The CNC-group G contains G-normal subgroups X_1 and X_2 such that the following decompositions hold: $G = X_1 \leftthreetimes B_1$ and $G =$

$X_2 \leftthreetimes B_2$. Since B_2 is complemented in G with the help of X_2, and B_2 is a proper subgroup in B_1, then by a well known result (see, for example, [2, Lemma 3.7]), $B_1 = (X_2 \cap B_1) \leftthreetimes B_2$. Consider the subgroup $Y = X_1 \leftthreetimes (X_2 \cap B_1)$. Since $G = X_1 \leftthreetimes B_1 = X_1 \leftthreetimes ((X_2 \cap B_1) \leftthreetimes B_2)$, the subgroup $Y = X_1 \leftthreetimes (X_2 \cap B_1)$ is normal in $G = YB_2$.

Let $b \in Y \cap B_2$. Since $Y = X_1 \leftthreetimes (X_2 \cap B_1)$, we can write $b = xb_1$ where $x \in X_1, b_1 \in X_2 \cap B_1, b \in Y \cap B_2 \leq B_1$. So, $x = b_1^{-1}b$. Since $x \in X_1$, $b_1^{-1}b \in B_1$, and $X_1 \cap B_1 = \langle 1 \rangle$, we can conclude that $x = b_1^{-1}b = 1$ and $b = b_1 \in X_2 \cap B_1$. Hence, $Y \cap B_2 \leq X_2 \cap B_1$. The decomposition $B_1 = (X_2 \cap B_1) \leftthreetimes B_2$ implies that $(X_2 \cap B_1) \cap B_2 = \langle 1 \rangle$, and therefore $Y \cap B_2 = \langle 1 \rangle$.

It follows from the arguments above that $G = Y \leftthreetimes B_2$, and $Y > X_1$. Without lost of generality we can assume that $Y = X_2$ and $G = X_2 \leftthreetimes B_2$, $X_2 > X_1$. If B_2 has no proper contranormal subgroups, then everything is clear. If B_2 contains a proper contranormal subgroup B_3, then repeating word by word the above arguments, we come to the existence of a normal subgroup X_3 such that $G = X_3 \leftthreetimes B_3$, and $X_3 > X_2 > X_1$. Since G satisfies Max-n, this process must stop at a finite number of steps n for which B_n does not have any contranormal subgroup. □

The following theorem gives a general description of CNC-groups satisfying Max-n.

Theorem 3. *Let a group G satisfying* Max-n *be a CNC-group. Then every contranormal subgroup of G contains a minimal contranormal subgroup B of G, and for each such subgroup B, there is a decomposition $G = A \leftthreetimes B$, where A is a direct product of a finite number of minimal B-normal subgroups.*

Proof. In fact by Lemma 3, every contranormal in G subgroup H contains a minimal contranormal in G subgroup B. Since G is a CNC-group, B is complemented by a normal subgroup A, that is, $G = A \leftthreetimes B$. Let M be a proper in A G-normal subgroup. Since $A > M$,

$$G/M = (A \leftthreetimes B)/M = A/M \leftthreetimes BM/M \ncong BM/M.$$

It follows that $M \leftthreetimes B = B_1$ is a proper in G contranormal subgroup. Let M_1 be a proper in M G-normal subgroup. Consider the subgroup $B_2 = M_1 \leftthreetimes B$. Clearly, $B_2 < B_1$ and B_2 is contranormal in G. If M_2 is a proper G-normal subgroup of M_1 we can continue this process and construct a descending chain of contranormal in G subgroups $B_1 > B_2 > \cdots > B_f$ that by Proposition 1, will be terminated at a finite number m. It means that the normal subgroup M_m has no G-normal proper subgroups. Since the subgroup $B_m = M_m \leftthreetimes B$ is contranormal in G, it is complemented in G by some normal subgroup N. Repeating the arguments above and operating with the contranormal subgroup B_m instead of B, we will come to the contranormal subgroup $B_n = N_n \leftthreetimes B_m = N_n \leftthreetimes (M_m \leftthreetimes B)$. Note, that since N_n and M_m are G-normal subgroups, we can write $B_n = (N_n \times M_m) \leftthreetimes B$. Continuing this process, we will construct an ascending chain of G-normal subgroups $M_m < N_n \times M_m < \cdots < K_k \times \cdots \times N_n \times M_m < \cdots$. Since G satisfies Max-n, this process will be terminated after finite numbers of steps, and we will have $G = (K_k \times \cdots \times N_n \times M_m) \leftthreetimes B$ where $M_m, N_n, \ldots, K_k$ are minimal G-normal subgroups.

Assume that one of these minimal G-normal subgroups $M_m, N_n, \ldots, K_k$, let say M_m, is not a minimal B-normal subgroup. Let $F < M_m$ be a B-normal subgroup. Consider the subgroup $R = F \leftthreetimes B$. Since G is a CNC-group, R is normally complemented in G. It means that there is a G-normal subgroup L, such that $G = L \leftthreetimes (F \leftthreetimes B)$. It follows that the subgroup R is normally complemented in $M_m \leftthreetimes B$, and $M_m \leftthreetimes B = (L \cap M_m) \leftthreetimes R$. Since both M_m and L are G-normal, $1 \neq L \cap M_m \lhd G$ and $M_m \neq L \cap M_m$. This ontradiction completes the proof. □

Theorem 4. *If G is a group, in which every contranormal subgroup contains a minimal contranormal in G subgroup, and for each such subgroup B there is a decomposition $G = A \leftthreetimes B$, where A is a direct product of a finite number of minimal G-normal subgroups, then G is a CNC-group.*

Proof. If K is a contranormal subgroup in G, K includes a minimal contranormal subgroup B such that $G = A \leftthreetimes B = (X_1 \times X_2 \times \cdots \times X_n) \leftthreetimes B$, $A = X_1 \times X_2 \times \cdots \times X_n$, all X_i, $i = 1, 2, 3, \ldots, n$, are minimal G-normal subgroups. Then B is complemented in K by $R = A \cap K \lhd K$, so that, R is G-normal and $R \leq A$. Since every X_i is a minimal B-normal subgroup, the intersection $X_i \cap R$ is equal either to X_i itself or to $\langle 1 \rangle$ for any $i = 1, 2, 3, \ldots, n$. Let for some i_1 $X_{i_1} \cap R = \langle 1 \rangle$. Consider the subgroup $R_1 = X_{i_1} \times R$. If $R_1 = A$, then $G = A \leftthreetimes B = (X_{i_1} \times R) \leftthreetimes B = X_{i_1} \leftthreetimes (R \leftthreetimes B)$, and everything is clear.

Let now $R_1 \neq A$. This means that there is $X_{i_2} \in \{X_n, X_{n-1}, \ldots, X_1\}$ such that $X_{i_2} \nleq R_1$. Since X_{i_2} is minimal G-normal, $X_{i_2} \cap R_1 = \langle 1 \rangle$. Consider the subgroup $R_2 = X_{i_2} \times R_1$. If $R_2 = A$, then

$$G = A \leftthreetimes B = (X_{i_2} \times R_1) \leftthreetimes B = (X_{i_2} \times X_{i_1} \times R) \leftthreetimes B$$

$$= (X_{i_2} \times X_{i_1}) \leftthreetimes (R \leftthreetimes B) = (X_{i_2} \times X_{i_1}) \leftthreetimes K,$$

and everything is clear.

If $R_2 \neq A$, we can chose $X_{i_3} \in \{X_n, X_{n-1}, \ldots, X_1\}$ such that $X_{i_3} \nleq R_2$, $X_{i_3} \cap R_2 = \langle 1 \rangle$, and consider the subgroup $R_3 = X_{i_3} \times R_2$.

Continuing this process we will come to a number $k \leq n$, such that $R_k = A$ and $G = A \rightthreetimes B = (X_{i_2} \times X_{i_1} \times \cdots \times X_{i_k}) \rightthreetimes K$. $\square$

Observe, that by the MacLane's theorem (see, for example, [9, 12.1.7]), any locally nilpotent group with Max-n is nilpotent and finitely generated. So, we come to the following simple result.

Proposition 1. *All locally nilpotent groups with* Max-n *are* CNC*-groups.*

However, a locally nilpotent group and even a hypercentral group could have a proper non-identity contranormal subgroup. The following simple example supports this statement: $G = P \rightthreetimes \langle b \rangle$, where P is a Pruefer 2-group, $b^2 = 1$, and $x^b = x^{-1}$. Observe, that every subgroup of G containing $\langle b \rangle$ is contranormal.

Note, that a periodic soluble group G with Max-n is finite. The following proposition is a direct corollary from Theorems 1 and 2. It provides us with a descriptions of the finite CNC-groups. This description is a generalization of some of the main results from [10].

Corollary 1. *A finite group G is a CNC-group if and only if for every nilpotent contranormal subgroup B there is a decomposition $G = A \rightthreetimes B$, where A is a direct product of minimal G-normal proper subgroups.*

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L. A. Kurdachenko
Department of Algebra
National Dnipropetrovsk University
Vul. Naukova 13
Dnipropetrovsk 50, Ukraine 49050
e-mail: lkurdachenko@hotmail.com

I. Ya. Subbotin
Mathematics Department
National University
5245 Pacific Concourse Drive
Los Angeles, CA 90045-6904, USA
e-mail: isubboti@nu.edu

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