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# GENERALIZED DERIVATIONS AND NORM EQUALITY IN NORMED IDEALS

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Abstract. We compare the norm of a generalized derivation on a Hilbert space with the norm of its restrictions to Schatten norm ideals.

**Introduction.** Let $H$ be a complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on $H$. For two bounded operators $A,\ B \in B(H)$,the left and right multiplications $L_A, R_B \in B(H)$ are defined by $L_A(X) = AX$ and $R_B(X) = XB$ respectively. The generalized derivation induced by $A$ and $B$ is the operator

$$\delta_{A,B} : B(H) \to B(H),\ X \mapsto (L_A - R_B)(X) = AX - XB.$$

The bimultiplication $M_{A,B}$ is the operator defined by

$$M_{A,B}(X) = (R_A \circ L_B)(X) = AXB.$$

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Let $(J, \|.\|_J)$ be a norm ideal in $B(H)$ in the sense of [13] and let $A, B \in B(H)$. If $X \in J$, then $\|AX - XB\|_J = \|(A-\lambda)X - X(B-\lambda)\|_J \leq (\|A-\lambda\| + \|B-\lambda\|)\|X\|_J$ for all $\lambda \in C$. Hence $\|AX - XB\|_J \leq \inf_{\lambda \in C}(\|A-\lambda\| + \|B-\lambda\|)\|X\|_J$. Since

$$\|\delta_{A,B}\| = \inf_{\lambda \in C}(\|A-\lambda\| + \|B-\lambda\|) \tag{1}$$

see [15], we conclude that $\|AX - XB\|_J \leq \|\delta_{A,B}\|\|X\|_J$. Thus the restriction $\delta_{J,A,B}$ of $\delta_{A,B}$ to $J$ defines a bounded linear operator on $(J, \|\cdot\|_J)$ and $\|\delta_{J,A,B}\| \leq \|\delta_{A,B}\|$ for each norm ideal $J$ in $B(H)$.

Let $J_p$ denote the Schatten $p$-ideal, $1 \leq p \leq \infty$; see for instance [9] or [13]. The space $J_p$ consists of compact operators $K$ such that $\sum_j s_j^p(K) < \infty$, where $\{s_j(K)\}_j$ denotes the sequence of the singular values of $K$. For $K \in J_p \quad (1 \leq p \leq \infty)$, we set $\|K\|_p = \left(\sum_j s_j^p(K)\right)^{1/p}$, where, by convention, $\|K\|_\infty = s_1(K)$ is the usual operator norm of $K$. Then $(J_p, \|\cdot\|_p)$ is a norm ideal. Moreover, $(J_2, \|\cdot\|_2)$ is a Hilbert space with inner product defined by $\langle X, Y\rangle = tr(Y^*X) \quad (X, Y \in J_2)$, where $tr$ denotes the usual trace functional.

For simplicity of notation, we write $\delta_{p,A,B}$ (respectively $M_{p,A,B}$) instead of $\delta_{J_p,A,B}$ (respectively $M_{J_p,A,B}$) and $\delta_{p,A}$ instead of $\delta_{J_p,A,A}$.

Stampfli [15] has given the elegant formula (1) for the norm of a genearlized derivation on $B(H)$. Fialkow [7] has given an example of an operator $A \in B(H)$ such that $\|\delta_{2,A}\| < \|\delta_A\|$. This leads us to search for the relation between the norms of $\|\delta_{A,B}\|$ and $\|\delta_{J,A,B}\|$. It is true that for certain norm ideals such as the compact operators or the trace class operators we do have equality, use duality or see [8].

In order to state our results in detail, we first recall some notations and results from the literature. Let $E$ be a complex Banach space. For $A \in B(E)$, let $\sigma(A)$, $\sigma_{ap}(A)$ and $r(A)$ denote respectively the spectrum, approximate point spectrum and spectral radius of $A$. Recall that a complex number $\lambda \in \sigma_{ap}(A)$ if there exists a unit sequence $\{x_n\}_n \subseteq E$ such that $\lim_n \|(A-\lambda)x_n\| = 0$. Since the boundary of $\sigma(A)$ is contained in $\sigma_{ap}(A)$, then $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \sigma_{ap}(A)$.

The (algebraic) numerical range of $A$ is defined by

$$V(A) = \{\Phi(A) : \Phi \in B(E)^* \text{ and } \|\Phi\| = \Phi(I) = 1\},$$

and the numerical radius of $A$ is defined by $v(A) = \sup\{|\lambda| : \lambda \in V(A, B(E))\}$. Note that $V(A)$ is a compact convex subset of the plane and $\sigma(A) \subseteq V(A)$

[4]. If $E = H$ is a complex Hilbert space, then from [10], it turns out that the norm $\|A\|$ lies in $\overline{W(A)} = V(A)$ if and only if $\|A\|$ lies in $\sigma_{ap}(A)$. An operator $T \in B(E)$ is said to be of class $\sigma$ (respectively a normaloid operator) if $r(T) = \|T\|$ (respectively $v(T) = \|T\|$). For any operator $T \in B(H)$, $r(T) = \|T\|$ if and only if $v(T) = \|T\|$ see ([10]).

The numerical range of a generalized derivation on norm ideals in $B(E)$ was studied by several authors, see for instance [11] or [14]. In [14] S. Shaw considered generalized derivations $\delta_{J,A,B}$ acting on subspaces $(J, \| \cdot \|_J)$ of $B(E)$ ($E$ : Banach space) which satisfies axioms like those of norm ideals. He showed the following equality:

$$V(\delta_{J,A,B}) = V(A) - V(B). \tag{2}$$

Let $K$ be a nonempty bounded subset of the plane. The diameter of $K$ is defined by $\mathrm{diam}(K) = \sup_{\alpha,\beta \in K} |\alpha - \beta|$. For $A, B$ in $B(H)$, we see from above that $v(\delta_{J,A,B}) = \sup\{|\alpha - \beta| : \alpha \in V(A) \text{ and } \beta \in V(B)\}$. On the other hand, it turns out [6] that $\sigma(\delta_{J,A,B}) = \sigma(A) - \sigma(B)$. Hence we deduce that $r(\delta_{J,A,B}) = \sup\{|\alpha - \beta| : \alpha \in \sigma(A) \text{ and } \beta \in \sigma(B)\}$.

In the following we denote

$$d(A, B) = \inf\{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in C\}$$

This is the norm of $\delta_{A,B}$. Note that by a compactness argument there exists $\mu \in C$ such that $d(A, B) = \|A - \mu I\| + \|B - \mu I\|$.

**Definition 0.1.**

1. *An operator $A \in B(H)$ is $S$-universal if $\|\delta_{J,A}\| = d(A, A) = 2 \inf\{\|A - \lambda I\| : \lambda \in C\}$ for each norm ideal $J$.*
2. *A generalized derivation is said to be $S$-universal if $\|\delta_{J,A,B}\| = d(A, B)$ for each norm ideal $J$.*

The concept of a $S$-universal operator was introduced by L. Fialkow [7], who studied criteria for $S$-universality and posed several questions in this context. The $S-$universality of an operator was studied in [2]. The present paper studies the $S-$ universality of a generalized derivation. More precisely we shall be concerned with equality $\|\delta_{A,B}\| = \|\delta_{J,A,B}\|$ for all normed ideals $J$.

The main result of this paper is the following theorem:

**Theorem 0.2.** *Let $A, B \in B(H)$ be non-zero. Let $J$ be a norm ideal of $B(H)$. Then the following conditions are equivalent:*

1. $\|\delta_{2,A,B}\| = d(A, B)$;

2. *There exists* $\mu \in C$ *such that* $M_{2,(A-\mu I)^*,(B-\mu I)}$ *is a normaloid operator.*
3. $r(\delta_{J,A,B)} = d(A,B)$ *(*$\delta_{J,A,B}$ *is a* $\sigma$ *operator);*
4. $v(\delta_{J,A,B}) = d(A,B)$ *(*$\delta_{J,A,B}$ *is a normaloid operator);*
5. $\|\delta_{J,A,B}\| = d(A,B)$ *(*$\delta_{A,B}$ *is S-universal).*

*If any one of this conditions is satisfied then* $r(A-\mu I) = \|A-\mu I\|$ *and* $r(B-\mu I) = \|B-\mu I\|$ *where* $\mu \in C$ *such that* $d(A,B) = \|A-\mu I\| + \|B-\mu I\|$.

## 1. Proof of the main result.

P r o o f. 1) $\Rightarrow$ 2) Assume that $\|\delta_{2,A,B}\| = d(A,B) = \|A-\mu\| + \|B-\mu\|$. Since $\delta_{2,A,B} = \delta_{2,A-\mu,B-\mu} = L_{2,A-\mu} - R_{2,B-\mu}$, it follows that

$$\|L_{2,A-\mu} - R_{2,B-\mu}\| = \|A-\mu\| + \|B-\mu\|.$$

On the other hand, $\|L_{2,A-\mu}\| = \|A-\mu\|$ and $\|R_{2,B-\mu}\| = \|B-\mu\|$. Hence

$$\|L_{2,A-\mu} - R_{2,B-\mu}\| = \|L_{2,A-\mu}\| + \|R_{2,B-\mu}\|.$$

Without loss of generality we may assume that $\mu = 0$, and then $\|L_{2,A} - R_{2,B}\| = \|L_{2,A}\| + \|R_{2,B}\|$. By theorem 1 of [2], this is equivalent to $\|L_{2,A}\|\|R_{2,B}\| \in \overline{W(-L_{2,A^*}R_{2,B})}$. So

$$\|L_{2,A}\|\|R_{2,B}\| \le v(-L_{2,A^*}R_{2,B}) \le \|L_{2,A}R_{2,B}\| \le \|L_{2,A}\|\|R_{2,B}\|.$$

Thus $-L_{2,A^*}R_{2,B}$ is a normaloid operator.

2) $\Rightarrow$ 3) We know that $J_2$ is a Hilbert space. In this case the condition 2) implies that $\|L_{2,A}\|\|R_{2,B}\| \in \sigma(-L_{2,A^*}R_{2,B})$. But $\sigma(-L_{2,A^*}R_{2,B}) = -\sigma(A^*)\sigma(B)$ see [5] and $\|L_{2,A}\|\|R_{2,B}\| = \|A\|\|B\|$. So there exist $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$ such that $\|A\|\|B\| = -\overline{\alpha}\beta$ ($\overline{\alpha}$: complex conjugate of $\alpha$). Since $|\alpha| \le \|A\|$ and $|\beta| \le \|B|$, then one can find $\theta \in R$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|B\|e^{i\theta}$. So

$$r(\delta_{2,A,B}) = \sup\{|\lambda - \mu| : \lambda \in \sigma(A), \mu \in \sigma(B)\} \ge |\alpha - \beta| = \|A\| + \|B\| = d(A,B).$$

But $\sigma(\delta_{2,A,B}) = \sigma(\delta_{J,A,B})$ see [7]. So $r(\delta_{J,A,B}) = d(A,B)$.

3) $\Rightarrow$ 4) This is obvious.

4) $\Rightarrow$ 5) By the inequality $v(\delta_{J,A,B}) \le \|\delta_{J,A,B}\| \le \|\delta_{A,B}\|$ we see that $v(\delta_{J,A,B}) = d(A,B)$ imply that $\|\delta_{J,A,B}\| = d(A,B)$.

5) $\Rightarrow$ 1) Just take $J = J_2$ The Hilbert Schmidt class $\|\delta_{2,A,B}\| = \|\delta_{A,B}\|$.

From the proof of 1) imply 2), we see that if $d(A,B) = \|A - \mu I\| + \|B - \mu I\|$ then $r(A - \mu I) = \|A - \mu I\|$ and $r(B - \mu I) = \|B - \mu I\|$. $\square$

## 2. Examples and remarks.

**Remark 2.1.** The ideal $J_2$ in the condition 2) can't be remplaced by $B(H)$. The following example shows that $M_{A,B}$ can be normaloid and no other condition in the theorem is satisfied.

**Example 2.2.** Let A=$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and B=$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then AB=$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Easy computation gives $W(AB) \subset V(M_{A,B})$ and so

$$\| M_{J,A,B} \| = \| A \| \| B \| = 1 = v(M_{A,B}).$$

But $\delta_{J,A,B}$ is nilpotent for any ideal $J$, hence $r(\delta_{J,A,B}) = 0$. Note that $v(A) = v(B) = \frac{1}{2}$ and $v(\delta_{J,A,B}) = 1$. It is also easy to show that $\|\delta_{A,B}\| = 2$ and $\|\delta_{2,A,B}\| = \sqrt{2}$.

The following corollary summarizes some results from the third section of [2].

**Corollary 2.3.** *Let $A \in B(H)$ non-zero. The Following conditions are equivalent:*

(1) $\|\delta_{2,A}\| = \|\delta_A\|$;

(2) *There exists $\mu \in C$ such that $M_{2,(A-\mu I)^*,(A-\mu I)}$ is a normaloid operator.*

(3) $\operatorname{diam}(\sigma(A)) = r(\delta_{J,A}) = \|\delta_A\|$; ($\delta_{J,A}$ *is a $\sigma$ operator.*)

(4) $\operatorname{diam} W(A) = v(\delta_{J,A}) = \|\delta_A\|$; ($\delta_{J,A}$ *is a normaloid operator.*)

(5) $\|\delta_{J,A}\| = \|\delta_A\|$ *( A is a S-universal operator).*

Recently Timoney in [16] and [17], obtained a couple of general formulas for the norm of an elementary operators. But it seems that this formulas throw no light on the norms of restrictions of such operators.

## REFERENCES


[1] Y. A. Abramovich, C. D. Aliprantis, O. Burkinshaw. The Daugavet equation in uniformly convex Banach spaces. *J. Func. Anal.* **97** (1991), 215–230.

[2] M. Barraa, M. Boumazgour. Inner Derivations and Norm Equality. *Proc. Amer. Math. Soc.* **130** (2001), 471–476.

[3] M. Boumazgour. An estimate for the norm of a derivation on a norm ideal. *Linear and Multilear Algebra* **54**, *5* (2006), 321–327.

[4] F. F. Bonsall, J. Duncan. Numerical ranges I,II. Cambridge Univ. Press, 1973.

[5] A. Brown, C. Pearcy. Spectra of tensor products of operators. *Proc. Amer. Math. Soc.* **17** (1966), 162–166.

[6] A. Brown, C. Pearcy. On the spectra of derivations on norm ideals. Preprint.

[7] L. Fialkow. A note on norm ideal and the operator $X \longrightarrow AX - XB$. Israel. J. Math. **32** (1979), 331–348.

[8] L. Fialkow. Elementary operators & Applications. Proceeding of the international workshop, 1991.

[9] I. C. Goheberg, M. G. Krein. Introduction to the theory of linear nonselfadjoint operators. Trans. Math. Monographs vol. **18**, Amer. Math. Soc., Providence, RI, 1969.

[10] P. R. Halmos. A Hilbert space Problem Book. Van Nostrand, Princeton, 1970.

[11] J. Kyle. Numerical ranges of derivations. *Proc. Edinburgh. Math. Soc.* **21** (1978), 33–39.

[12] C. L. Lin. The unilateral shift and norm equality for bounded linear operators. *Proc. Amer. Math. Soc.* **127** (1999), 1693–1696.

[13] R. Schatten. Norm ideals of completely continuous operators. Springer-Verlag, Berlin, 1960.

[14] S. Y. Shaw. On numerical ranges of generalized derivations and related properties. *J. Austral. Math. Soc. Ser. A* **36** (1984), 134–142.

[15] J. Stampfli. The norm of a derivation, *Pacific J. Math.* **33** (1970), 737–747.

[16] R. M. Timoney. Computing the norm of elementary operators. *Illinois. J. Math.* **47** (2003), 1207–1226.

[17] R. M. Timoney. Some formulae for the norm of elementary operators. *J. Operator Theory* **57**, *1* (2007), 121–145

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