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BAYESIAN AND FREQUENTIST TWO-SAMPLE PREDICTIONS OF THE INVERSE WEIBULL MODEL BASED ON GENERALIZED ORDER STATISTICS

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Abstract. This paper is concerned with the problem of deriving Bayesian prediction bounds for the future observations (two-sample prediction) from the inverse Weibull distribution based on generalized order statistics (GOS). Study the two side interval Bayesian prediction, point prediction under symmetric and asymmetric loss functions and the maximum likelihood (ML) prediction using "plug-in" procedure for future observations from the inverse Weibull distribution based on GOS. Study the problem of predicting future records based on observed progressive type II censored data and observed order statistics from the inverse Weibull distribution. Finally, a numerical example using real data are used to illustrate the procedure.

1. Introduction. A concept of GOS was introduced by Kamps [20]. Ordinary order statistics (OS) (David [15], Castillo [12], and Arnold, Balakrishnan and Nagaraja [10]), record values, K^{th} record values and Pfeifer's records

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(Ahsanullah [6]), sequential order statistics (SOS) (Cramer and Kamps [14]), progressive Type-II censoring order statistics (PCOS) (Soliman [29] and Sarhan, Ammar and Abuammoh [27]) and censoring schemes can be discussed as they are special cases of the GOS, [for a survey of the models contained and of the results obtained in the GOS, see Kamps [20].

Bayesian prediction intervals under GOS and record values have discussed by many authors, Ahmadi, Doostparast and Parsian [5], Calabria and Pulcini [13], AL-Hussaini and Jaheen [8], Nigm and Abd AL-wahab [25], Abd Ellah [1, 2, 3], Soliman and Abd-Ellah [30], Escobar and Meeker [17], Soliman and Al-Ohaly [31] present Bayes 2-sample prediction for the Pareto distribution and Raqab and Balakrishnan [26].

The rest of the paper is as a follows. In Section 2, we present some preliminaries as the model, the loss function , priors and the posterior and progressively type-II censored data. In Section 3, Bayesian predictive distribution for the future GOS in the two-sample prediction case based on past number of GOS, the ML prediction both point and interval prediction using "plug-in" procedure are derived and also, point and interval prediction for the lower record values based on progressively type-II censored data and ordinary order statistics are obtained as a special case of GOS. Finally, a practical example using real data set was used for illustration are presented in Section 4.

In this paper we consider the problem of two side interval Bayesian prediction, point prediction under symmetric and asymmetric loss functions and maximum likelihood (ML) prediction for future observations from the inverse Weibull distribution, based on (GOS), we will consider the two sample prediction techniques, we will consider the past observations are order statistics and progressive type II censored and the future observations are record values as a special cases of GOS.

2. Preliminaries.

2.1. The model and the concept of the GOS. The IWD plays an important role in many applications, including the dynamic components of diesel engines and several data set such as the times to breakdown of an insulating fluid subject to the action of a constant tension; see Nelson [24]. Calabria and Pulcini [13] provide an interpretation of the IWD in the context of the load-strength relationship for a component. Recently, Maswadah [22] has fitted the IWD to the flood data reported in Dumonceaux and Antle [16]. For more details on the IWD, see, for example Johnson et al. [18] and Murthy et al. [23]. The two parameter IWD has probability density function (*pdf*), cumulative distribution function (*cdf*) and reliability function $S(t)$, which are given respectively as

$$f(x) = \theta\beta x^{-\beta-1}\exp(-\theta x^{-\beta}), \qquad x \geq 0,\ \theta, \beta > 0, \tag{1}$$

$$F(x) = \exp(-\theta x^{-\beta}), \qquad x \geq 0,\ \theta, \beta > 0, \tag{2}$$

and the reliability function at time t is

$$S(t) = 1 - \exp(-\theta t^{-\beta}), \qquad t \geq 0,\ \theta, \beta > 0, \tag{3}$$

where θ and β are scale and shape parameters respectively.

We recall the concept of GOS (cf. Kamps [17]).

Let $n \in N$, $n \geq 2$ and $\widetilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, then the random variables $X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k)$ are called the GOS if their joint pdf is given by

$$f_{X_{(1,n,\widetilde{m},k)},\ldots,X_{(n,n,\widetilde{m},k)}}(x_1, \ldots, x_n) = c_{n-1}\left[\prod^{n-1} i = 1[\overline{F}(x_i]^{m_i} f(x_i)\right][\overline{F}(x_n)]^{k-1} f(x_n), \tag{4}$$

For $F^{-1}(0) < x_1 \leq \cdots \leq x_n < F^{-1}(1)$, where

$$c_{n-1} = \prod_{i=1}^{n}\gamma_i = k\prod_{i=1}^{n-1}\gamma_i, \quad \gamma_j = k + n - j + \sum_{i=j}^{n-1} m_i \quad \text{and} \quad \overline{F}(x) = 1 - F(x). \tag{5}$$

Let $\underline{x} = X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k)$ are n GOS drawn from inverse Weibull distribution whose pdf is given by (1), the likelihood function (L.F), By substituting (1), (2) in (3) we obtain

$$\begin{aligned} L(\theta, \beta \mid \underline{x}) &= c_{n-1}\theta^n\beta^n\left[\prod_{i=1}^{n} x_i^{-\beta-1}\right]\left[\exp(-\theta\sum_{i=1}^{n} x_i^{-\beta})\right]\left[\prod_{i=1}^{n-1}(1 - \exp(-\theta x_i^{-\beta}))^{m_i}\right] \\ &\quad [1 - \exp(-\theta x_n^{-\beta})]^{k-1}. \end{aligned} \tag{6}$$

The log-likelihood function is given by

$$\ell(\theta,\beta \mid \underline{x}) = \ln c_{r-1} + r\ln\theta + r\ln\beta - (\beta+1)\sum_{i=1}^{r}\ln x_i - \theta\sum_{i=1}^{r} x_i^{-\beta} + \sum_{i=1}^{r-1} m_i \ln(1-\exp(-\theta x_i^{-\beta})) + (\gamma_r - 1)\ln(1-\exp(-\theta x_r^{-\beta})). \tag{7}$$

If both of the parameters θ and β are unknown, and from the log-likelihood function given by (11) the MLEs, $\widehat{\theta}_{ML} = \widehat{\theta}$ and $\widehat{\beta}_{ML} = \widehat{\beta}$ can be obtained by the numerical solution of the following eq.s

$$\frac{\partial\ell(\theta,\beta \mid \underline{x})}{\partial\theta} = \frac{r}{\theta} - \sum_{i=1}^{r} x_i^{-\beta} + \sum_{i=1}^{r-1} \frac{m_i x_i^{-\beta}\exp(-\theta x_i^{-\beta})}{(1-\exp(-\theta x_i^{-\beta}))} + \frac{(\gamma_r - 1)x_i^{-\beta}\exp(-\theta x_i^{-\beta})}{(1-\exp(-\theta x_r^{-\beta}))} = 0, \tag{8}$$

$$\frac{\partial\ell(\theta,\beta \mid \underline{x})}{\partial\beta} = \frac{r}{\beta} - \sum_{i=1}^{r}\ln x_i - \theta\sum_{i=1}^{r-1} x_i^{-\beta}\ln x_i - \theta\sum_{i=1}^{r-1} \frac{m_i x_i^{-\beta}\exp(-\theta x_i^{-\beta})\ln x_i}{(1-\exp(-\theta x_i^{-\beta}))} + \frac{(\gamma_r - 1)x_r^{-\beta}\exp(-\theta x_r^{-\beta})\ln x_r}{(1-\exp(-\theta x_r^{-\beta}))} = 0. \tag{9}$$

2.2. The loss function. It is well known that, for Bayesian prediction, the result depends on the loss function assumed. So most authors use the simple quadratic loss function (squared error (SE)) and obtain the posterior mean as the Bayesian predictive estimate.

A number of asymmetric loss functions are proposed for use, among these, one of the most popular asymmetric loss function is (linear-exponential) loss function (LINEX). It is introduced by Varian [32].

Recently, many authors consider asymmetric loss functions in reliability and used it in different estimation problems, such as (Wahed, Abdus [33] and Jokiel-Rokita and Alicja [19]). This function rises approximately exponentially on one side of zero and approximately linearly on the other side. Under the assumption that the minimal loss occurs at $\widehat{\varphi} = \varphi$, the LINEX loss function for can be expressed as:

$$L(\Delta) \propto \exp(c\Delta) - c\Delta - 1; \quad c \neq 0, \tag{10}$$

where $\Delta = \widehat{\varphi} - \varphi$, $\widehat{\varphi}$ is an estimate of φ. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively (if $c > 0$, the overestimation is more serious than underestimation, and vice versa). For c closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric. The posterior expectation of the LINEX loss function (10) is

$$E_\varphi[L(\widehat{\varphi} - \varphi)] \propto \exp(c\widehat{\varphi})E_\varphi[\exp(-c\varphi)] - c(\widehat{\varphi} - E_\varphi(\varphi)) - 1, \tag{11}$$

where $E_\varphi(\cdot)$ denotes the posterior expectation with respect to the posterior density of φ. The Bayes estimator of φ, denoted by $\widehat{\varphi}_{BL}$ under the LINEX loss function is the value $\widehat{\varphi}$ which minimizes (11), it is

$$\widehat{\varphi}_{BL} = -\frac{1}{c}\ln\{E_\varphi[\exp(-c\varphi)]\}, \tag{12}$$

provided that the expectation $E_\varphi[\exp(-c\varphi)]$ exists and is finite.

2.3. Prior and posterior distribution. When both of the two parameters θ and β are assumed to be unknown, Soland [28] considered a family of joint prior distributions that places continuous distributions on the scale parameter and discrete distributions on the shape parameter. We assume that the shape parameter β is restricted to a finite number of values $\beta_1, \beta_2, \ldots, \beta_{\mathcal{L}}$ with respective prior probabilities $\xi_1, \xi_2, \ldots, \xi_{\mathcal{L}}$ such that $0 \le \xi_j \le 1$, $\sum_{j=1}^{\mathcal{L}} \xi_j = 1$ and $P(\beta = \beta_j) = \xi_j$. Further, suppose that conditional upon $\beta = \beta_j$, $j = 1, 2, \ldots, \mathcal{L}$, θ has a natural gamma (a_j, b_j) prior, with a density

$$\pi(\theta \mid \beta = \beta_j) = \frac{b_j^{a_j}}{\Gamma(a_j)}\theta^{a_j-1}\exp[-b_j\theta], \qquad a_j, b_j, \theta > 0. \tag{13}$$

Then the conditional posterior pdf of θ is given by

$$\begin{aligned} \pi^*(\theta \mid \beta = \beta_j, \underline{x}) = A_1\theta^{n+a_j-1}\exp\left[-\theta\left(\sum_{i=1}^{n} x_i^{-\beta_j} + b_j\right)\right] \\ \times \left[\prod_{i=1}^{n-1}\left(1 - \exp\left(-\theta x_i^{-\beta_j}\right)\right)^{m_i}\right]\left[1 - \exp\left(-\theta x_i^{-\beta_j}\right)\right]^{k-1}, \end{aligned} \tag{14}$$

where

$$A_1^{-1} = \sum_{q_1=0}^{m_1} \cdots \sum_{q_{n-1}=0}^{m_{n-1}} \sum_{d=0}^{k-1} \frac{D\Gamma(n + a_j)}{H\left(\beta_j\right)^{n+a_j}}. \tag{15}$$

On applying the discrete version of Bayes theorem, the marginal probability distribution of β is given by

$$p_j = P(\beta = \beta_j \mid \underline{x}) == A_2 \sum_{q_1=1}^{m_1} \cdots \sum_{q_{n-1}=1}^{m_{n-1}} \sum_{d=1}^{k-1} \frac{\nu_j b_j^{a_j} \beta_j^n \upsilon_j D\Gamma(n+a_j)}{\Gamma(a_j) H(\beta_j)^{n+a_j}}, \tag{16}$$

where

$$\left\{ \begin{aligned} A_2^{-1} &= \sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{m_1} \cdots \sum_{q_{n-1}=0}^{m_{n-1}} \sum_{d=0}^{k-1} \frac{D\nu_j b_j^{a_j} \beta_j^n \upsilon_j \Gamma(n+a_j)}{\Gamma(a_j)[H(\beta_j)]^{n+a_j}}, \\ D &= (-1)^{q_1+\cdots+q_{n-1}+d} \binom{m_1}{q_1} \cdots \binom{m_{n-1}}{q_{n-1}} \binom{k-1}{d}, \\ H(\beta_j) &= \sum_{i=1}^{n} x_i^{-\beta_j} + \sum_{i=1}^{n-1} q_i x_i^{-\beta_j} + d x_n^{-\beta_j} + b_j, \\ \upsilon_j &= \prod_{i=1}^{n} x_i^{-\beta_j - 1}. \end{aligned} \right. \tag{17}$$

Then from (14) and (16) the joint posterior of the parameters θ and β is given by

$$\pi^*(\theta, \beta \mid \underline{x}) = p_j \pi^*(\theta \mid \beta = \beta_j, \underline{x}). \tag{18}$$

2.4. Progressively type-II censored data. A progressively Type-II censored sample is observed as follows: n units are placed on a life-testing experiment and only $m \leq n$ are completely observed until failure. The censoring occurs progressively in m stages. The m stages are failure times of m completely observed units. At the time of the first failure (the first stage), R_1 of $(n-1)$ surviving units are randomly withdrawn from the experiment, R_2 of the $(n-R_1-2)$ surviving units are withdrawn at the time of the second failure (the second stage) and so on. Finally, at the time of the m^{th} failure (the m^{th} stage), all the remaining $(R_m = n - m - R_1 - \cdots - R_{m-1})$ surviving units are withdrawn. We will refer this to as progressively Type-II censoring scheme $(R_1, R_2, \ldots, R_m)$. Then, we shall denote the m completely observed failure times by $X_{i:m:n}^{(R_1,\ldots,R_m)}$, $i = 1, 2, \ldots, m$.

The progressively Type-II censored sample $X_{1:n:N}^{(R_1,\dots,R_r)},\dots,X_{n:n:N}^{(R_1,\dots,R_n)}$, with censoring scheme $\widetilde{R}=(R_1,\dots,R_n)$, and $R_i\in\mathbb{N}_0$, $1\le i\le n$, is a special case of the GOS with the parameters $m_i=R_i$, $i=1,2,\dots,n-1$ and $k=\gamma_n=R_n+1$, see Burkschat et al. [8].

3. Prediction. In this section we will reduce Bayesian prediction for inverse Weibull distribution based on GOS

3.1. Bayesian prediction based on GOS. Suppose that $X(1,n,\widetilde{m},k)$, $\dots,X(n,n,\widetilde{m},k)$, $k>0$, $\widetilde{m}=(m_1,m_2,\dots,m_{n-1})\in\Re^{n-1}$, are the n GOS, drawn from Inverse Weibull distribution, defined by (1). Let $Z(1,N,\widetilde{M},K),\dots,$ $Z(N,N,\widetilde{M},K)$, $K>0$, $\widetilde{M}=(M_1,M_2,\dots,M_{N-1})\in\Re^{N-1}$ be a second independent GOS of size N from the same distribution. Our aim is to develop a method to construct a prediction interval for a number of future. This is the two-sample prediction technique.

Let Z_s denotes the s^{th} GOS in the future sample of size N, $1\le s\le N$, the probability density function (*pdf*) of Z_s, $(M_1=\dots=M_{N-1}=M=-1)$, is given by

$$g_1(z_s\mid\theta,\beta)=\frac{k^s}{(s-1)!}[\overline{F}(z_s)]^{k-1}f(z_s)\,[g_M(F(z_s)]^{s-1}, \tag{19}$$

where

$$g_M(t)=h_M(t)-h_M(0), \tag{20}$$

and for $0<t<1$

$$h_M(t)=\begin{cases}-(1-t)^{M+1}/(M+1) & M\neq-1,\\ -\ln(1-t) & M=-1.\end{cases} \tag{21}$$

By substituting From (20) and (21) in (20), we obtain

$$g_1(z_s\mid\theta,\beta)=\frac{k^s}{(s-1)!}[\overline{F}(z_s)]^{k-1}f(z_s)\,[-\ln(1-F(z_s))]^{s-1}. \tag{22}$$

Applying (1) and (2) in (22), we obtain

$$\begin{aligned} g_1(z_s\mid\theta,\beta)=&\frac{k^s}{(s-1)!}\theta\beta z_s^{-\beta-1}\exp(-\theta z_s^{-\beta})\\ &\times[1-\exp(-\theta z_s^{-\beta})]^{k-1}[-\ln(1-\exp(-\theta z_s^{-\beta}))]^{s-1}.\end{aligned} \tag{23}$$

When both of the two parameters θ and β are unknown, then the Bayesian predictive density function of Z_s, $1 \le s \le N$, will be

$$g_2(z_s \mid \underline{x}) = \int_0^\infty \sum_{j=1}^{\mathcal{L}} g_1(z_s \mid \theta, \beta_j)\pi^*(\theta, \beta_j \mid \underline{x})d\theta. \tag{24}$$

It follows that the Bayesian prediction intervals for the future sample Z_s, $s = 1, 2, \ldots, N$ for some given value of λ_1, is given by

$$P[Z_s \ge \lambda_1 \mid \underline{x}] = \int_{\lambda_1}^\infty g_2(z_s \mid \underline{x})dz_s. \tag{25}$$

The predictive bounds of two-sided interval with cover τ for the Z_s, may thus be obtained by solution the following two equations for the lower and upper Bayesian prediction bounds $L_s(\underline{x})$ and $U_s(\underline{x})$ for Z_s, $s = 1, 2, \ldots, N$:

$$P[Z_s \ge L_s(\underline{x}) \mid \underline{x}] = \frac{1+\tau}{2}, \qquad P[Z_s \ge U_s(\underline{x}) \mid \underline{x}] = \frac{1-\tau}{2}. \tag{26}$$

3.2. ML prediction for GOS. By replacing θ and β in the conditional density function (22) by $\widehat{\theta}$ and $\widehat{\beta}$ which we can find it from the numerical solution of the eq.s (8) and (9), then

$$\begin{aligned} g_3(z_s \mid \widehat{\theta}, \widehat{\beta}) = {} & \frac{k^s}{(s-1)!}\widehat{\theta}\widehat{\beta} z_s^{-\widehat{\beta}-1} \exp(-\widehat{\theta} z_s^{-\widehat{\beta}}) \\ & \times [1 - \exp(-\widehat{\theta} z_s^{-\widehat{\beta}})]^{k-1}[-\ln(1 - \exp(-\widehat{\theta} z_s^{-\widehat{\beta}}))]^{s-1}. \end{aligned} \tag{27}$$

The ML prediction intervals for Z_s, $s = 1, 2, \ldots, N$ are obtained by evaluating $P[Z_s \ge \lambda_2 \mid \underline{x}]$, for some given value of λ_2. It follows, from (27) that

$$P[Z_s \ge \lambda_2 \mid \underline{x}] = \int_{\lambda_2}^\infty g_3(z_s \mid \widehat{\theta}, \widehat{\beta})dz_s. \tag{28}$$

The predictive bounds of a two-sided interval with cover τ, for Z_s, $s = 1, 2, \ldots, N$ can be obtained by solving the following two lower $L_s(\underline{x})$ and upper $U_s(\underline{x})$ bounds:

$$P[Z_s \ge L_s(\underline{x}) \mid \underline{x}] = \frac{1+\tau}{2}, \text{ and } P[Z_s \ge U_s(\underline{x}) \mid \underline{x}] = \frac{1-\tau}{2}. \tag{29}$$

3.3. Bayesian prediction based on progressive type II censored. In this section we will predict of sample from lower record values based on progressive type II censored sample, then by putting $k = 1$ and replacing $\overline{F}(x)$ by

$F(x)$ in eq. (23), the *pdf* of future lower record values Z_s, is given by

$$g_4(z_s \mid \theta, \beta_j) = \frac{1}{\Gamma(s)} \theta^s \beta_j z_s^{-s\beta_j - 1} \exp(-\theta z_s^{-\beta_j}). \tag{30}$$

From eq. (29) and eq. (18) in progressive type II censored ($m_i = R_i$ and $k = R_n + 1$), then eq. (24), reduce to

$$g_5(z_s \mid \underline{x}) = \sum_{j=1}^{\mathcal{L}} \frac{A_3 p_{1j} \beta_j z_s^{-s\beta_j - 1}}{\Gamma(s)} \int_0^\infty \theta^{n+s+a_j-1} \exp\left[-\theta\left(\sum_{i=1}^n x_i^{-\beta_j} + z_s^{-\beta_j} + b_j\right)\right]$$

$$\times [\prod_{i=1}^n (1 - \exp(-\theta x_i^{-\beta_j}))^{R_i}] d\theta$$

$$= \sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{D_1 A_3 p_{1j} \beta_j z_s^{-s\beta_j - 1} \Gamma(n+s+a_j)}{\Gamma(s)[T(\beta_j) + z_s^{-\beta_j}]^{n+s+a_j}}, \tag{31}$$

where

$$\begin{cases} p_{1j} = A_4 \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{\nu_j b_j^{a_j} \beta_j^n \upsilon_j D_1 \Gamma(n+a_j)}{\Gamma(a_j)[T(\beta_j)]^{n+a_j}} \\ A_4^{-1} = \sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{\nu_j b_j^{a_j} \beta_j^n \upsilon_j D_1 \Gamma(n+a_j)}{\Gamma(a_j)[T(\beta_j)]^{n+a_j}}, \\ T(\beta_j) = \sum_{i=1}^n x_i^{-\beta_j} + \sum_{i=1}^n q_i x_i^{-\beta_j} + b_j, \\ A_3^{-1} = \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{D_1 \Gamma(n+a_j)}{[T(\beta_j)]^{n+a_j}}, \\ D_1 = (-1)^{q_1+\cdots+q_n} \binom{R_1}{q_1} \cdots \binom{R_n}{q_n}. \end{cases} \tag{32}$$

Then the Bayesian prediction intervals for the future lower record value

Z_s, $s = 1, 2, \ldots, N$, is given by

$$(33)\quad P[Z_s \geq \lambda_3 \mid \underline{x}]$$
$$= \sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{D_1 A_3 p_{1j} \Gamma(n+s+a_j)}{s\Gamma(s)\lambda_3^{s\beta_j}[T(\beta_j)]^{(n+s+a_j)}} \, {}_2F_1\left[s, n+s+a_j; s+1; -\frac{\lambda_3^{-\beta_j}}{T(\beta_j)}\right],$$

where

$${}_2F_1[a,b;c;z] = \sum_{\ell=0}^{\infty} \frac{(a)_\ell (b)_\ell}{(c)_\ell \ell!} z^\ell, \qquad (w)_\ell = w(w+1)\ldots(w+\ell-1),$$

is the hypergeometric function.

The τ 100% Bayesian prediction bounds the future lower record value Z_s, $s = 1, 2, \ldots, N$ are obtained by solution the following two nonlinear eq.s for lower bounds $L_s(\underline{x})$ and upper bounds $U_s(\underline{x})$:

$$(34)\quad \begin{cases} \displaystyle\sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{D_1 A_3 p_{1j} \Gamma(n+s+a_j)}{s\Gamma(s)[L_s(\underline{x})]^{s\beta_j}[T(\beta_j)]^{(n+s+a_j)}} \\ \qquad \times {}_2F_1\left[s, n+s+a_j; s+1; -\frac{[L_s(\underline{x})]^{-\beta_j}}{T(\beta_j)}\right] = \frac{1+\tau}{2}, \\ \displaystyle\sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum q_n = 0^{R_1} \frac{D_1 A_3 p_{1j} \Gamma(n+s+a_j)}{s\Gamma(s)[U_s(\underline{x})]^{s\beta_j}[T(\beta_j)]^{(n+s+a_j)}} \\ \qquad \times {}_2F_1\left[s, n+s+a_j; s+1; -\frac{[U_s(\underline{x})]^{-\beta_j}}{T(\beta_j)}\right] = \frac{1-\tau}{2}. \end{cases}$$

By using (31) the Bayes point predictor the future lower record value Z_s under SE and LINEX loss functions are given respectively, as

$$(35)\qquad \widetilde{Z}_{s(BS)} = \sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{D_1 A_3 p_{1j} \beta_j \Gamma(n+s+a_j)}{\Gamma(s)} I_1(z_s, \beta_j),$$

$$(36)\quad \widetilde{Z}_{s(BL)} = -\frac{1}{c} \text{Log}\left[\sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} \frac{D_1 A_3 p_{1j} \beta_j \Gamma(n+s+a_j)}{\Gamma(s)} I_2(z_s, \beta_j)\right],$$

where

$$
(37)\quad \begin{cases} I_1(z_s,\beta_j) = \displaystyle\int_0^\infty z_s^{-s\beta_j}[T(\beta_j)+z_s^{-\beta_j}]^{-(n+s+a_j)}dz_s, \\ I_2(z_s,\beta_j) = \displaystyle\int_0^\infty z_s^{-s\beta_j-1}e^{-cz_s}[T(\beta_j)+z_s^{-\beta_j}]^{-(n+s+a_j)}dz_s. \end{cases}
$$

Special case:

1. The τ 100% Bayesian prediction bounds for the first future lower record value Z_1 of the future sample of size N can be obtained by putting $s=1$, in (34), as

$$
(38)\quad \begin{cases} \displaystyle\sum_{j=1}^{\mathcal{L}}\sum_{q_1=0}^{R_1}\cdots\sum_{q_n=0}^{R_n}\frac{D_1A_3p_{1j}\Gamma(n+a_j+1)}{[L_1(x)]^{\beta_j}[T(\beta_j)]^{(n+a_j+1)}} \\ \qquad\qquad \times {}_2F_1\left[1,n+a_j+1;2;-\dfrac{[L_1(x)]^{-\beta_j}}{T(\beta_j)}\right] = \dfrac{1+\tau}{2}, \\ \displaystyle\sum_{j=1}^{\mathcal{L}}\sum_{q_1=0}^{R_1}\cdots\sum_{q_n=0}^{R_n}\frac{D_1A_3p_{1j}\Gamma(n+a_j+1)}{[U_1(x)]^{\beta_j}[T(\beta_j)]^{(n+a_j+1)}} \\ \qquad\qquad \times {}_2F_1\left[1,n+a_j+1;2;-\dfrac{[U_1(x)]^{-\beta_j}}{T(\beta_j)}\right] = \dfrac{1-\tau}{2}. \end{cases}
$$

2. The τ 100% Bayesian predictive bounds for the last future lower record value Z_N of the future sample of size N can be obtained by putting $s=N$, in (34), as

$$
(39)\quad \begin{cases} \displaystyle\sum_{j=1}^{\mathcal{L}}\sum_{q_1=1}^{R_1}\cdots\sum_{q_n=1}^{R_n}\frac{D_1A_3p_{1j}\Gamma(n+N+a_j)}{N\Gamma(N)[L_N(x)]^{N\beta_j}[T(\beta_j)]^{(n+N+a_j)}} \\ \qquad\qquad \times {}_2F_1\left[N,n+N+a_j;N+1;-\dfrac{[L_N(x)]^{-\beta_j}}{T(\beta_j)}\right] = \dfrac{1+\tau}{2}, \\ \displaystyle\sum_{j=1}^{\mathcal{L}}\sum_{q_1=0}^{R_1}\cdots\sum_{q_n=0}^{R_n}\frac{D_1A_3p_{1j}\Gamma(n+N+a_j)}{N\Gamma(N)[U_{n_1}(x)]^{N\beta_j}[T(\beta_j)]^{(n+N+a_j)}} \\ \qquad\qquad \times {}_2F_1\left[N,n+N+a_j;N+1;-\dfrac{[U_N(x)]^{-\beta_j}}{T(\beta_j)}\right] = \dfrac{1-\tau}{2}. \end{cases}
$$

3. The Bayesian point prediction for the first future lower record value Z_1 of the future sample of size N can be obtained by putting $s=1$, in (35) and

(36), as

$$(40)\qquad \widetilde{Z}_{1(BS)} = \sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} D_1 A_3 p_{1j} \beta_j \Gamma(n+a_j+1) I_1(z_1, \beta_j),$$

$$(41)\qquad \widetilde{Z}_{1(BL)} = -\frac{1}{c} \operatorname{Log}\left[\sum_{j=1}^{\mathcal{L}} \sum_{q_1=0}^{R_1} \cdots \sum_{q_n=0}^{R_n} D_1 A_3 p_{1j} \beta_j \Gamma(n+a_j+1) I_2(z_1, \beta_j)\right],$$

where $I_1(z_1, \beta_j)$ and $I_2(z_1, \beta_j)$ defined from (37) with $s = 1$.

4. The Bayesian point prediction for the Last future lower record value Z_N of the future sample of size N can be obtained by putting $s = N$, in (35) and (36), as

$$(42)\qquad \widetilde{Z}_{N(BS)} = \sum_{j=1}^{\mathcal{L}} \sum_{q_1=1}^{R_1} \cdots \sum_{q_n=1}^{R_n} \frac{D_1 A_3 p_{1j} \beta_j \Gamma(n+N+a_j)}{\Gamma(N)} I_1(z_N, \beta_j),$$

$$(43)\qquad \widetilde{Z}_{N(BL)} = -\frac{1}{c} \operatorname{Log}\left[\sum_{j=1}^{\mathcal{L}} \sum_{q_1=1}^{R_1} \cdots \sum_{q_n=1}^{R_n} \frac{D_1 A_3 p_{1j} \beta_j \Gamma(n+N+a_j)}{\Gamma(N)} I_2(z_N, \beta_j)\right],$$

where $I_1(z_N, \beta_j)$ and $I_2(z_N, \beta_j)$ defined from (37) with $s = N$.

3.4. Bayesian prediction based on order statistics. In this subsection we will predict lower record values sample from the order statistics sample so let ($m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = 1$) in eq. (18) and ($M_1 = M_2 = \cdots = M_{N-1} = -1$) and $k = 1$ and replace $\overline{F}(z_s)$ by $F(z_s)$) in eq. (23), then eq. (24), reduce to

$$g_6(z_s \mid \underline{x}) = \sum_{j=1}^{\mathcal{L}} \frac{p_{2j} \beta_j [\phi(\beta_j)]^{n+a_j} z_s^{-s\beta_j - 1}}{\Gamma(s)\Gamma(n+a_j)} \int_0^{\infty} \theta^{n+s+a_j-1} \exp[\phi(\beta_j) + z_s^{-\beta_j}] d\theta,$$

$$(44)\qquad = \sum_{j=1}^{\mathcal{L}} \frac{p_{2j} \beta_j [\phi(\beta_j)]^{n+a_j} z_s^{-s\beta_j - 1}}{Bet(n+a_j, s)[\phi(\beta_j) + z_s^{-\beta_j}]^{n+s+a_j}},$$

where

$$(45)\quad \begin{cases} p_{2j} = A_4 \dfrac{\xi_j b_j^{a_j} \beta_j^n \upsilon_j \Gamma(n+a_j)}{\Gamma(a_j)[\phi(\beta_j)]^{n+a_j}}, \qquad \phi(\beta_j) = \sum_{i=1}^{n} x_i^{-\beta_j} + b_j, \\ A_4^{-1} = \sum_{j=1}^{\mathcal{L}} \dfrac{\xi_j b_j^{a_j} \beta_j^n \upsilon_j \Gamma(n+a_j)}{\Gamma(a_j)[\phi(\beta_j)]^{n+a_j}}. \end{cases}$$

Then the Bayesian prediction intervals for the future lower record value Z_s, $s = 1, 2, \ldots, N$, is given by

$$P[Z_s \geq \lambda_4 \mid \underline{x}] = \sum_{j=1}^{\mathcal{L}} \frac{p_{2j}\beta_j[\phi(\beta_j)]^{n+a_j}}{Bet(n+a_j, s)} \int_{\lambda_4}^{\infty} z_s^{-s\beta_j - 1}[\phi(\beta_j) + z_s^{-\beta_j}]^{-(n+s+a_j)} dz_s$$

$$(46)\quad = \sum_{j=1}^{\mathcal{L}} \frac{p_{2j}[\phi(\beta_j)]^{-s}[\lambda_4]^{-s\beta_j}}{sBet(n+a_j, s)} \; {}_2F_1\left[s, n+s+a_j; s+1; -\frac{[\lambda_4]^{-\beta_j}}{\phi(\beta_j)}\right].$$

The τ 100% Bayesian predictive bounds the future lower record value Z_s, $s = 1, 2, \ldots, N$, are obtained by solution the following two nonlinear equations for lower bounds $L_s(\underline{x})$ and upper bounds $U_s(\underline{x})$:

$$(47)\quad \begin{cases} \sum_{j=1}^{\mathcal{L}} \dfrac{p_{2j}[\phi(\beta_j)]^{-s}[L_s(\underline{x})]^{-s\beta_j}}{sBet(n+a_j, s)} \\ \qquad \times {}_2F_1\left[s, n+s+a_j; s+1; -\dfrac{[L_s(\underline{x})]^{-\beta_j}}{\phi(\beta_j)}\right] = \dfrac{1+\tau}{2}, \\ \sum_{j=1}^{\mathcal{L}} \dfrac{p_{2j}[\phi(\beta_j)]^{-s}[U_s(\underline{x})]^{-s\beta_j}}{sBet(n+a_j, s)} \\ \qquad \times {}_2F_1\left[s, n+s+a_j; s+1; -\dfrac{[U_s(\underline{x})]^{-\beta_j}}{\phi(\beta_j)}\right] = \dfrac{1-\tau}{2}. \end{cases}$$

By using (44) the Bayes point predictor the future lower record value Z_s under SE and LINEX loss functions are given, respectively, as

$$(48)\quad \widetilde{Z}_{s(BS)} = \sum_{j=1}^{\mathcal{L}} \frac{p_j^* \beta_j [\phi(\beta_j)]^{n+a_j} I_3(z_s, \beta_j)}{Bet(n+a_j, s)},$$

$$\widetilde{Z}_{s(BL)} = -\frac{1}{c} Log\left[\sum_{j=1}^{\mathcal{L}} \frac{p_j^* \beta_j [\phi(\beta_j)]^{n+a_j} I_4(z_s,\beta_j)}{Bet(n+a_j,s)}\right], \tag{49}$$

where

$$\begin{cases} I_3(z_s,\beta_j) = \displaystyle\int_0^\infty z_s^{-s\beta_j}[\phi(\beta_j)+z_s^{-\beta_j}]^{-(n+s+a_j)}dz_s, \\ I_4(z_s,\beta_j) = \displaystyle\int_0^\infty z_s^{-s\beta_j-1}e^{-cz_s}[\phi(\beta_j)+z_s^{-\beta_j}]^{-(n+s+a_j)}dz_s. \end{cases} \tag{50}$$

Special case:

1. The τ 100% Bayesian prediction bounds for the first future lower record value Z_1 of the future sample of size N can be obtained by putting $s = 1$, in (47), as

$$\begin{cases} \displaystyle\sum_{j=1}^{\mathcal{L}} \frac{p_{2j}(n+a_j)[L_1(\underline{x})]^{-\beta_j}}{[\phi(\beta_j)]}\, {}_2F_1\left[1, n+a_j+1; 2; -\frac{[L_1(\underline{x})]^{-\beta_j}}{\phi(\beta_j)}\right] = \frac{1+\tau}{2}, \\ \displaystyle\sum_{j=1}^{\mathcal{L}} \frac{p_{2j}(n+a_j)[U_1(\underline{x})]^{-\beta_j}}{[\phi(\beta_j)]}\, {}_2F_1\left[1, n+a_j+1; 2; -\frac{[U_1(\underline{x})]^{-\beta_j}}{\phi(\beta_j)}\right] = \frac{1-\tau}{2}. \end{cases} \tag{51}$$

2. The τ 100% Bayesian prediction bounds for the last future lower record value Z_N of the future sample of size N can be obtained by putting $s = N$, in (47), as

$$\begin{cases} \displaystyle\sum_{j=1}^{\mathcal{L}} \frac{p_{2j}[\phi(\beta_j)]^{-N}[L_N(\underline{x})]^{-N\beta_j}}{NBet(n+a_j,N)} \\ \qquad\qquad \times {}_2F_1\left[N, n+N+a_j; N+1; -\frac{[L_N(\underline{x})]^{-\beta_j}}{\phi(\beta_j)}\right] = \frac{1+\tau}{2}, \\ \displaystyle\sum_{j=1}^{\mathcal{L}} \frac{p_{2j}[\phi(\beta_j)]^{-N}[U_N(\underline{x})]^{-N\beta_j}}{NBet(n+a_j,N)} \\ \qquad\qquad \times {}_2F_1\left[N, n+N+a_j; N+1; -\frac{[U_N(\underline{x})]^{-\beta_j}}{\phi(\beta_j)}\right] = \frac{1-\tau}{2}. \end{cases} \tag{52}$$

3. The Bayesian point prediction for the first future lower record value Z_1 of the future sample of size N can be obtained by putting $s = 1$, in (48) and (49), as:

$$\widetilde{Z}_{1(BS)} = \sum_{j=1}^{\mathcal{L}} \frac{p_j^* \beta_j [\phi(\beta_j)]^{n+a_j}}{Bet(n+a_j, 1)} I_3(z_1, \beta_j), \tag{53}$$

$$\widetilde{Z}_{1(BL)} = -\frac{1}{c} \operatorname{Log} \left[\sum_{j=1}^{\mathcal{L}} \frac{p_j^* \beta_j [\phi(\beta_j)]^{n+a_j}}{Bet(n+a_j, 1)} I_4(z_1, \beta_j) \right], \tag{54}$$

where $I_3(z_1, \beta_j)$ and $I_4(z_1, \beta_j)$ defined from (50) by putting $s = 1$.

4. The Bayesian point prediction for the Last Z_N observation of the future sample of size N can be obtained by putting $s = N$, in (48) and (49), as

$$\widetilde{Z}_{N(BS)} = \sum_{j=1}^{\mathcal{L}} \frac{p_j^* \beta_j [\phi(\beta_j)]^{n+a_j}}{Bet(n+a_j, N)} I_3(z_N, \beta_j), \tag{55}$$

$$\widetilde{Z}_{N(BL)} = -\frac{1}{c} \operatorname{Log} \left[\sum_{j=1}^{\mathcal{L}} \frac{p_j^* \beta_j [\phi(\beta_j)]^{n+a_j}}{Bet(n+a_j, N)} I_3(z_N, \beta_j) \right], \tag{56}$$

where $I_3(z_N, \beta_j)$ and $I_4(z_N, \beta_j)$ defined from (50) by putting $s = N$.

3.5. ML prediction for record values. From eq. (27) the density function for future lower record values Z_s, $s = 1, 2, \ldots, N$, is given by

$$q(z_s \mid \widehat{\theta}, \widehat{\beta}) = \frac{1}{(s-1)!} \widehat{\theta}^s \widehat{\beta} z_s^{-s\widehat{\beta}-1} \exp(-\widehat{\theta} z_s^{-\widehat{\beta}}). \tag{57}$$

Then from eq.s (28) and (29) the τ 100% ML prediction intervals for the future lower record values Z_s , $s = 1, 2, \ldots, N$, is given by numerical solution of the following eq.s

$$\begin{cases} \dfrac{\widehat{\theta}^s}{\Gamma(s)} InGamma\ (s, \widehat{\theta}^s, [L_s(\underline{x})]^{-\widehat{\beta}}) = \dfrac{1+\tau}{2}, \\[2ex] \dfrac{\widehat{\theta}}{\Gamma(s)} InGamma\ (s, \widehat{\theta}^s, [U_s(\underline{x})]^{-\widehat{\beta}}) = \dfrac{1-\tau}{2}. \end{cases} \tag{58}$$

where $InGamma\ (t_1, t_2, \varphi)$ is the incomplete Gamma function defined by

$$InGamma(t_1, t_2, \psi) = \int_0^{\psi} y^{t_1-1} \exp[-t_2 y] dy. \tag{59}$$

By using (57) the ML point predictor for the future lower record values Z_s, $s = 1, 2, \ldots, N$, is given by

$$E(Z_s) = \frac{\widehat{\theta}^s \widehat{\beta}}{(s-1)!} \int_0^{\infty} z_s^{-s\widehat{\beta}} \exp(-\widehat{\theta} z_s^{-\widehat{\beta}}) dz_s = \frac{\Gamma[s - (1/\widehat{\beta})]\ \widehat{\theta}^{\frac{1}{\widehat{\beta}}}}{\Gamma(s)}. \tag{60}$$

Special case:

1. The τ 100% ML prediction intervals for the first future lower record value Z_1 of the future sample of size N can be obtained by putting $s = 1$, in (58), as

$$\begin{cases} \widehat{\theta} InGamma(1, \widehat{\theta}, [L_1(\underline{x})]^{-\widehat{\beta}}) = \dfrac{1+\tau}{2}, \\ \widehat{\theta} InGamma(1, \widehat{\theta}, [U_1(\underline{x})]^{-\widehat{\beta}}) = \dfrac{1-\tau}{2}. \end{cases} \tag{61}$$

2. The τ 100% ML prediction intervals for the last future lower record value Z_N of the future sample of size N can be obtained by putting $s = N$, in (58), as

$$\begin{cases} \dfrac{\widehat{\theta}^N}{\Gamma(N)} InGamma(N, \widehat{\theta}, [L_N(\underline{x})]^{-\widehat{\beta}}) = \dfrac{1+\tau}{2}, \\ \dfrac{\widehat{\theta}^N}{\Gamma(N)} InGamma(N, \widehat{\theta}, [U_N(\underline{x})]^{-\widehat{\beta}}) = \dfrac{1-\tau}{2}. \end{cases} \tag{62}$$

3. The ML point prediction for the first and last future lower record value Z_1 and Z_N of the future sample of size N can be obtained by putting $s = 1$and $s = N$ respectively, in (60), as

$$E(Z_1) = \widehat{\theta}^{\frac{1}{\widehat{\beta}}} \Gamma[1 - (1/\widehat{\beta})], \tag{63}$$

$$E(Z_N) = \widehat{\theta}^{\frac{1}{\widehat{\beta}}} \Gamma[N - (1/\widehat{\beta})]/\Gamma(N). \tag{64}$$

4. Application examples.

Example (Real data).

I) Prediction based on progressive type II censored. Consider the data given by Dumonceaux and Antle [13], represents the maximum flood levels (in millions of cubic feet per second) of the Susquehenna River at Harrisburg, Pennsylvenia over 20 four-year periods (1890−1969) as: 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265. Therefore, we observe the following order statistics : 0.265, 0.269, 0.297, 0.315, 0.324, 0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423, 0.449, 0.484, 0.494, 0.613, 0.654, 0.74. We can obtain the values of (a_j, b_j) by using the expected values of the reliability $S(t)$;

$$E[S(t) \mid \beta = \beta_j] = \int_\theta (1 - \exp(-\theta t^{-\beta_j})) \frac{b_j^{a_j} \theta^{a_j - 1} \exp[-b_j \theta]}{\Gamma(a_j)} d\theta$$

$$= 1 - \left(1 + \frac{t^{-\beta_j}}{b_j}\right)^{-a_j}, \qquad t > 0. \tag{65}$$

Now suppose that the prior beliefs about the distribution enable one to specify two values $(S(t_1), t_1)$ and $(S(t_2), t_2)$. Then the values of a_j , b_j can by obtained numerically from (65). If there are no prior beliefs, a nonparametric approach can be used to estimate the two values of $S(t)$ by using

$$S(t_i = X_i) = \frac{n - i + 0.625}{n + 0.25}. \tag{66}$$

See Martez and Waller [18].

By using the nonparametric approach of the reliability function , we set $t_1 = 0.412$ and $t_2 = 0.338$ in (66), we get $S(t_1) = 0.47$ and $S(t_2) = 0.72$.

For $\mathcal{L} = 10$ concerning the value of the MLE of the parameter β, ($\widehat{\beta} = 2.5743$), we assume that β_j takes the values: 2.3 (0.1) 3.2 with equal probabilities each of 0.1. Then the values of the hyper-parameters a_j, b_j at each value of β_j are obtained by solving the following equations using Newton-Raphson method.

$$1 - (1 + \frac{0.412^{-\beta_j}}{b_j})^{-a_j} = 0.47, \tag{67}$$

$$1 - (1 + \frac{0.338^{-\beta_j}}{b_j})^{-a_j} = 0.72. \tag{68}$$

Table 1. Progressive type II censored sample ($m = 8$, $n = 20$)

i	1	2	3	4	5	6	7	8
$x_{i,m,n}$	0.265	0.269	0.297	0.392	0.402	0.484	0.494	0.613
R_i	0	0	5	0	5	0	0	2

Table 1 shows the values of the progressive type II censored data

In Table 2 and Table 3 we reduce the 90% , 95% Bayesian prediction intervals (BPI) for the future lower record Z_s, $s = 1, 2, 3, 4, 5$ and the Bayes point prediction, under SE and LINEX loss function based on progressive type II censored data.

Table 2. 90% and 95% BPI for lower records Z_s, $s = 1, 2, 3, 4, 5$ based on progressive type II censored data

s	90% BPI	Length	95% BPI	Length
1	$[0.2791, 1.3424]$	1.0633	$[0.2564, 1.7676]$	1.5110
2	$[0.2332, 0.6480]$	0.4147	$[0.2174, 0.7533]$	0.5358
3	$[0.2082, 0.4743]$	0.2660	$[0.1955, 0.5295]$	0.3340
4	$[0.1913, 0.3920]$	0.2006	$[0.1803, 0.4286]$	0.2483
5	$[0.1787, 0.3427]$	0.1639	$[0.1688, 0.3699]$	0.2011

Table 3. Bayesian point prediction for lower records Z_s, $s = 1, 2, 3, 4, 5$ based on progressive type II censored data under SE and LINEX loss function

s	SE	LINEX		
		$c = 0.1$	$c = 0.5$	$c = 1$
1	0.6235	0.6103	0.5785	0.5530
2	0.3872	0.3861	0.3822	0.3777
3	0.3142	0.3139	0.3124	0.3106
4	0.2749	0.2747	0.2739	0.2729
5	0.2491	0.2490	0.2485	0.2478

In Table 4 we reduce the 90% , 95% ML prediction intervals (MLPI) for the future lower record Z_s, $s = 1, 2, 3, 4, 5$ and the ML point prediction based on Progressive type II censored sample.

Table 4. 90% and 95% MLPI and ML point prediction for lower records Z_s, $s = 1, 2, 3, 4, 5$ based on progressive type II censored data

s	90% MLPI	Length	95% MLPI	Length	Point P.
1	$[0.2781, 1.3500]$	1.0719	$[0.2565, 1.7760]$	1.5195	0.6231
2	$[0.2326, 0.6365]$	0.4039	$[0.2185, 0.7387]$	0.5202	0.3811
3	$[0.2084, 0.4605]$	0.2521	$[0.1975, 0.5132]$	0.3156	0.3071
4	$[0.1922, 0.3772]$	0.1850	$[0.1832, 0.4118]$	0.2286	0.2673
5	$[0.1802, 0.3272]$	0.1470	$[0.1725, 0.3528]$	0.1803	0.2413

II) Prediction based on order statistics. By using the real data in (I) and for $\mathcal{L} = 10$ concerning the value of the MLE of the parameter β, ($\widehat{\beta} = 4.314$), we assume that β_j takes the values: 4 (0.1) 4.9 with equal probabilities each of 0.1.

Then the values of the hyper-parameters a_j, b_j at each value of β_j are obtained by using the same way in (I).

In Table 5 and Table 6 we reduce the 90% , 95% Bayesian prediction intervals for the future lower record Z_s, $s = 1, 2, 3, 4, 5$ and the Bayes point prediction, under SE and LINEX loss function based on order statistics.

Table 5. 90% and 95% BPI for lower records Z_s, $s = 1, 2, 3, 4, 5$ based on order statistics

s	90% BPI	Length	95% BPI	Length
1	$[0.2780, 0.7189]$	0.4409	$[0.2641, 0.8472]$	0.5830
2	$[0.2492, 0.4618]$	0.2126	$[0.2391, 0.5051]$	0.2659
3	$[0.2328, 0.3821]$	0.1493	$[0.2244, 0.4080]$	0.1836
4	$[0.2213, 0.3403]$	0.1190	$[0.2139, 0.3589]$	0.1451
5	$[0.2125, 0.3135]$	0.1009	$[0.2057, 0.3282]$	0.1225

Table 6. Bayesian point prediction for lower records Z_s, $s = 1, 2, 3, 4, 5$ based on order statistics

s	SE	LINEX		
		$c = 0.1$	$c = 0.5$	$c = 1$
1	0.4337	0.4323	0.4274	0.4221
2	0.3340	0.3338	0.3328	0.3317
3	0.2957	0.2956	0.2951	0.2946
4	0.2731	0.2730	0.2727	0.2724
5	0.2574	0.2574	0.2572	0.2569

In Table 7 we reduce the 90%, 95% ML prediction intervals for the future lower record Z_s, $s = 1, 2, 3, 4, 5$ and the ML point prediction based on order statistics.

Table 7. 90% and 95% MLPI and ML point prediction for lower records Z_s, $s = 1, 2, 3, 4, 5$ based on order statistics

s	90% MLPI	Length	95% MLPI	Length	Point P.
1	$[0.2779, 0.7133]$	0.4355	$[0.2648, 0.8402]$	0.5754	0.4307
2	$[0.2498, 0.4555]$	0.2057	$[0.2407, 0.4978]$	0.2571	0.3309
3	$[0.2339, 0.3755]$	0.1415	$[0.2266, 0.4005]$	0.1739	0.2925
4	$[0.2229, 0.3333]$	0.1104	$[0.2166, 0.3512]$	0.1346	0.2699
5	$[0.2145, 0.3062]$	0.0917	$[0.2090, 0.3202]$	0.1113	0.2543

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