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FREE BICOMMUTATIVE ALGEBRAS

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Abstract. Algebras with identities $a(bc) = b(ac)$, $(ab)c = (ac)b$ is called bicommutative. Bases and the cocharacter sequence for free bicommutative algebras are found. It is shown that the exponent of the variety of bicommutaive algebras is equal to 2.

Introduction. One of important problems of modern algebra is to study free algebras satisfying some identities. To construct bases as vector spaces, to find the cocharacter sequence, to construct multilinear components and to find their dimensions or asymptotics of the growth are parts of this problem. From this point of view varieties of associative algebras, varieties of associative and commutative algebras and varieties of Lie algebras are well understood. For example, free associative commutative algebras are polynomial algebras. Relations between homogeneous polynomials and symmetric polynomials are used in many

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branches of mathematics and physics. The free associative algebra is a tensor algebra, its multilinear components are isomorphic to the regular modules of the corresponding symmetric groups. Free Lie algebras as S_n- and GL_n-modules were studied by Klyachko [9]. Besides these results there are many other interesting classes of algebras for which problems on free algebras still remain a difficult task. For example, very little is known about free alternative algebras, free Malcev algebras, even about free commutative algebras.

Let $com = t_1t_2 + t_2t_1$ and $acom = t_1t_2 - t_2t_1$ be the commutative and anti-commutative polynomials, respectively, and let

$$ass = (t_1, t_2, t_3) = t_1(t_2t_3) - (t_1t_2)t_3$$

be the associator or the associative polynomial. An algebra with the identity $com = 0$ is called anti-commutative. Commutative algebras are defined by the identity $acom = 0$ and associative algebras by $ass = 0$. As we already mentioned almost everything is known for free algebras in the associative case. Free commutative and free anti-commutative algebras are less understood. For example, the structure of the free anti-commutative algebra as a module of the symmetric group is known only for degree ≤ 7, [1].

Recently the following generalizations of the identities of commutativity and associativity have become popular:

$$lcom = t_1(t_2t_3) - t_2(t_1t_3) \text{ (left-commutative)},$$

$$rsym = ass(t_1, t_2, t_3) - ass(t_1, t_3, t_2) \text{ (right-symmetric)}.$$

Similarly one defines the non-commutative non-associative polynomials

$$rcom = (t_1t_2)t_3 - (t_1t_3)t_2 \text{ (right-commutative)},$$

$$lsym = ass(t_1, t_2, t_3) - ass(t_2, t_1, t_3) \text{ (left-symmetric)}.$$

Algebras with identities $lsym = 0$ and $rsym = 0$ are called assosymmetric. Bases for free assosymmetric algebras are found in [8]. Bases of free right-symmetric algebras were described in [10], [2] and [4]. Bases of free right-commutative algebras and right-symmetric algebras can be described in terms of rooted trees. The cocharacter sequences of right-symmetric algebras and right-commutative algebras are equal.

An algebra with identities $lcom = 0$ and $rsym = 0$ is called Novikov. Free Novikov algebras were described in [4]. The structure of these algebras as S_n- or GL_n-modulesis not known. In [5] bases of free Novikov algebras in terms of Young diagrams are constructed (the frame is a Young diagram, but the filling rule is different). The lexicographic order on Young diagrams induces an order on such a base of the free Novikov algebra. This order induces a filtration and a grading in free Novikov algebras. It seems that graded Novikov algebra satisfy bicommutative identities. The connection between bicommutative algebras and the filtration and the grading of free Novikov algebras gives a motivation to study bicommutative algebras.

An algebra with identities $lcom = 0$, $rcom = 0$ is called *bicommutative.* The aim of our paper is to find bases and the cocharacter sequence for bicommutative algebras.

Let $\mathcal{B}icom$ be the variety of bicommutative algebras. The codimension sequence of the variety is defined as a sequence of dimensions of the multilinear components $c_q = \dim F_q^{\text{multi}}$, $q = 0, 1, 2, \ldots$. The exponent of the variety is defined as

$$\operatorname{Exp} \mathcal{B}icom = \lim_{n\to\infty} \sqrt[n]{c_n}.$$

In our paper we construct bases for free bicommutative algebras. We study F_q^{multi} as an S_q-module and construct its decomposition into irreducible components. We show that $\operatorname{Exp} \mathcal{B}icom = 2$. The main results of the paper were announced in [3]. Unfortunately, in [3] the formula for the dimensions (Theorem 2) was given in a non-correct form. Here we give the correction.

2. Statement of main results. Let $F(X)$ be a free bicommutative algebra over a field K of characteristic $p \geq 0$. For $\mathbf{m} = (m_1, \ldots, m_q)$ denote by $F^{\mathbf{m}}(X)$ the space spanned by the products of the elements $x_1, \ldots, x_q$, such that x_1 appears m_1 times, x_2 appears m_2 times, etc., x_q appears m_q times. Let F_n^{multi} be the multilinear component of the free bicommutative algebra generated by n elements, i.e., $F_n^{\text{multi}} = F_n^{11\cdots1}(X)$, with $|X| = n$.

Let $\mathbf{Z}_+$ be the set of non-negative integers and let $\mathbf{Z}_+^q = \mathbf{Z}_+ \oplus \cdots \oplus \mathbf{Z}_+$ be the direct sum of q copies of $\mathbf{Z}_+$. Let $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{Z}_+^q$, all components except the i-th being equal to 0.

Let $F(q)$ be the vector space with base

$$V = \{x_i \mid i = 1, \ldots, q\} \cup \left\{ v_{\alpha,\beta} \mid \alpha, \beta \in \mathbf{Z}_+^q,\ \sum_{i=1}^q \alpha_i \neq 0,\ \sum_{i=1}^q \beta_i \neq 0 \right\}.$$

We endow $F(q)$ with multiplication $\circ$ given by the following rules:

$$x_i \circ x_j = v_{\epsilon_i, \epsilon_j},$$

$$x_i \circ v_{\alpha,\beta} = v_{\alpha+\epsilon_i,\beta},$$

$$v_{\alpha,\beta} \circ x_j = v_{\alpha,\beta+\epsilon_j},$$

$$v_{\alpha,\beta} \circ v_{\gamma,\delta} = v_{\alpha+\gamma,\beta+\delta}.$$

Let $S^{[n]}$ be the trivial S_n-module, and let $S^{[n-i,i]}$ be the S_n-module corresponding to the Young diagram with two rows with $n-i$ and i boxes.

$$\begin{matrix} \bullet & \cdots & \cdots & \bullet \\ \bullet & \cdots & \bullet & \end{matrix}$$

In our paper we prove the following main result.

Theorem 2.1. *Let* $X = \{x_1, \ldots, x_q\}$ *be a set of generators and let* K *be a field of characteristic* $p \geq 0$. *Then the following statements are true.*

a. *The algebra* $F(q)$ *is isomorphic to the free bicommutative algebra* $F(X)$.

b. $(p = 0)$ *As a module of the symmetric group* S_n

$$F_n^{\text{multi}} \cong (n-1)S^{[n]} \oplus \bigoplus_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-2i+1)S^{[n-i,i]}, \quad n > 1,$$

where $S^{[n]} \cong K$ *is the trivial* S_n*-module,* $S^{[n-i,i]}$ *is the irreducible* S_n*-module corresponding to the partition* $\{n-i, i\} \vdash n$ *and* $\lfloor \alpha \rfloor$ *is the integer part of* α.

c. *The bicommutative operad is not Koszul.*

Let I_n be the set of sequences of non-decreasing positive integers $\mathbf{i} = i_1 i_2 \ldots i_n$, $i_1 \leq i_2 \leq \cdots \leq i_n$. For $\mathbf{i} \in I_n$ we say that its *content* is $\mathbf{m} = (m_1, \ldots, m_q)$ and write $\operatorname{cont}(\mathbf{i}) = \mathbf{m}$, if among the components of $\mathbf{i}$ there are m_i elements equal to i, $i = 1, 2, \ldots, q$.

Corollary 2.2. *Let* $X = \{x_1, \ldots, x_q\}$ *be a set of generators. Then the set of elements* x_s, *where* $s = 1, \ldots, q$, *and* $e_{\mathbf{i},\mathbf{j}}$, *where* $\mathbf{i} \in I_k$, $\mathbf{j} \in I_l$, $k+l = n$, *and* $\operatorname{cont}(\mathbf{i}) + \operatorname{cont}(\mathbf{j}) = \mathbf{m}$, *forms a base of* $F^{\mathbf{m}}(X)$.

Proof. The statement follows from Theorem 2.1, part **a**. □

Corollary 2.3. $(p = 0)$ *The cocharacter sequence for bicommutative algebras is given by*

$$\chi_n \cong \begin{cases} \rho_{[n]}, & \text{if } n = 1 \\ (n-1)\rho_{[n]} \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} (n-2i+1)\rho_{[n-i,i]}, & \text{if } n > 1 \text{ is odd} \\ (n-1)\rho_{[n]} \oplus \bigoplus_{i=1}^{\frac{n}{2}} (n-2i+1)\rho_{[n-i,i]}, & \text{if } n \text{ is even,} \end{cases}$$

where $\rho_{[i,j]}$ *is the character of the irreducible representation corresponding to the Young diagram with two rows with* i *and* j *boxes.*

P r o o f. It follows from Theorem 2.1, part **b**. □

Corollary 2.4. *For the Hilbert series*

$$H(bicom, t_1, \ldots, t_q) = \sum_{\mathbf{m}} \dim F^{\mathbf{m}}(X) t^{\mathbf{m}}$$

of $F(X)$ *the following formula holds*

$$H(bicom, t_1, \ldots, t_q) = 1 + \sum_{i=1}^{q} t_i + \prod_{i=1}^{q} \frac{1}{(1-t_i)^2} - 2 \prod_{i=1}^{q} \frac{1}{1-t_i}.$$

In particular,

$$\dim F^{\mathbf{m}}(X) = (m_1 + 1) \cdots (m_q + 1) - 2 + \delta_{|\mathbf{m}|,\mathbf{1}} + \delta_{|\mathbf{m}|,0}$$

and

$$\dim F_n(X) = \binom{n+2q-1}{n} - (2 - \delta_{n,1}) \binom{n+q-1}{n}.$$

The codimension sequence is given by the formula

$$c_q = \dim F^{\mathbf{1}}(X) = 2^q - 2 + \delta_{q,1}.$$

The exponent of the variety of bicommutative algebras is equal to 2 *.*

Here

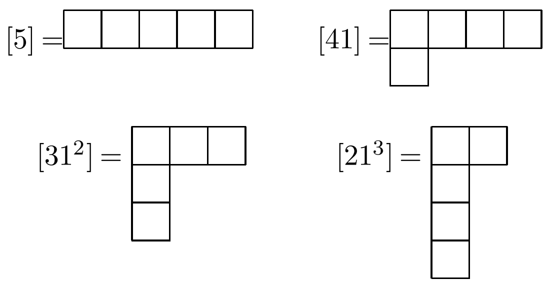
$$F_n(X) = \bigoplus_{|\mathbf{m}|=n} F^{\mathbf{m}}(X)$$

is the homogeneous component of degree n of $F(X)$ and $\delta_{x,y}$ denotes the Kronecker symbol. It is equal to 1 if $x = y$ and is equal to 0, if $x \neq y$.

Example. If $n = 1$, then

$$H(bicom, t) = \frac{t - t^2 + t^3}{(1-t)^2}.$$

Example. Let $n = 5$. Then $\dim F_5^{\text{multi}} = 30$. To construct the base elements we need to construct the bicommutative diagrams



Below is the list of multilinear base elements of degree 5:

$[5] = \{(((ab)c)d)e, (((ba)c)d)e, (((ca)b)d)e, (((da)b)c)e, (((ea)b)c)d\}$,

$$[41] = \{a(((bc)d)e), a(((cb)d)e), a(((db)c)e), a(((eb)c)d), b(((ca)d)e), b(((da)c)e), b(((ea)c)d), c(((da)b)e), c(((ea)b)d), d(((ea)b)c)\},$$

$$[31^2] = \{a(b((cd)e)), a(b((dc)e)), a(b((ec)d)), a(c((db)e)), a(c((eb)d)), a(d((ea)b)), b(c((da)e)), b(c((ea)d)), b(d((ea)c)), c(d((ea)b))\},$$

$$[21^3] = \{a(b(c(de))), a(b(c(ed))), a(b(d(ec))), a(c(d(eb))), b(c(d(ea)))\}.$$

Example. Let $n = 20$ or $n = 21$. Then

$$\chi_{20} = 19\rho_{[20]} \oplus \bigoplus_{i=1}^{10} (21 - 2i)\rho_{[20-i,i]}$$

$$= 19\rho_{[20]} \oplus 19\rho_{[19,1]} \oplus 17\rho_{[18,2]} \oplus 15\rho_{[17,3]} \oplus 13\rho_{[16,4]} \oplus 11\rho_{[15,5]}$$

$$\oplus 9\rho_{[14,6]} \oplus 7\rho_{[13,7]} \oplus 5\rho_{[12,8]} \oplus 3\rho_{[11,9]} \oplus \rho_{[10^2]},$$

$$\chi_{21} = 20\rho_{[21]} \oplus \bigoplus_{i=1}^{10} (22-2i)\rho_{[21-i,i]}$$

$$= 20\rho_{[21]} \oplus 20\rho_{[20,1]} \oplus 18\rho_{[19,2]} \oplus 16\rho_{[18,3]} \oplus 14\rho_{[17,4]} \oplus 12\rho_{[16,5]}$$

$$\oplus 10\rho_{[15,6]} \oplus 8\rho_{[14,7]} \oplus 6\rho_{[13,8]} \oplus 4\rho_{[12,9]} \oplus 2\rho_{[11,10]}.$$

Example. $\dim F^{234}(X) = 3 \cdot 4 \cdot 5 - 2 = 58\,.$

3. Multiplication in bicommutative algebras. In this section we establish some properties of bicommutative algebras which we need in the proof of Theorem 2.1, part **a**.

In Novikov algebras and in bicommutative algebras a base can be chosen of elements that are right-bracketed products of left-bracketed elements. To denote such products we shall use special notation. We write them as a sequence of rows arranged one above the other. The rows corresponds to left-bracketed elements and the columns to right-bracketed elements. The priority is given to the rows. For example,

$$a(b(cd)) = \begin{matrix} c & d \\ b & \\ a & \end{matrix}\,,$$

$$((ab)c)d = \begin{matrix} a & b & c & d \end{matrix}$$

$$((ab)c)(((xy)z)(((de)f)g)) = \begin{matrix} d & e & f & g \\ x & y & z & \\ a & b & c & \end{matrix}$$

So, we write the left-bracketed element $((a_1a_2)\cdots)a_n$ as the row

$$a_1 \quad a_2 \quad \cdots \quad a_n\,,$$

and the right-bracketed element $a_1(\cdots(a_{n-1}a_n)\cdots)$ as the column

$$\begin{matrix} a_{n-1} & a_n \\ \vdots & \\ a_1 & \end{matrix}\,,$$

and the following hook

$$L(a_1,\dots,a_k;b_1,\dots,b_l) = \begin{matrix} a_k & b_1 & \cdots & b_l \\ a_{k-1} & & & \\ \vdots & & & \\ a_1 & & & \end{matrix}$$

will denote the element $a_1(\cdots(a_{k-1}\Big((\cdots(a_kb_1)\cdots)b_l\Big))\cdots)$. For example,

$$L(a_1;b_1) = a_1b_1,$$

$$L(a_1;b_1,\dots,b_l) = (\cdots((a_1b_1)b_2)\cdots)b_l = L(L(a_1;b_1\dots,b_{l-1});b_l),$$
$$L(a_1,\dots,a_k;b_1) = a_1(\cdots(a_{k-1}(a_kb_1))\cdots) = L(a_1;L(a_2,\dots,a_k;b_1)).$$

Below notation of the form $a \stackrel{\text{rel}}{=} b$ will mean that $a=b$ because of the relation "rel".

Lemma 3.1. *For any permutations $\sigma \in S_k$, $\tau \in S_l$, the following equality holds in bicommutative algebras*

$$L(a_{\sigma(1)},\dots,a_{\sigma(k)};b_{\tau(1)},\dots,b_{\tau(l)}) = L(a_1,\dots,a_k;b_1,\dots,b_l).$$

Proof. For $\sigma \in S_k$, $\tau \in S_l$, such that $\sigma(k) = k$ our statement is evident in virtue of the bicommutative identities. Suppose now that $\sigma(k) \neq k$. It is sufficient to consider the case when $\sigma = (k-1,k)$ is a transposition. To prove that

$$a_1(\cdots a_{k-2}(a_{k-1}((\cdots(a_kb_1)\cdots)b_l))\cdots))$$
$$= a_1(\cdots(a_{k-2}(a_k((\cdots(a_{k-1}b_1)\cdots)b_l))\cdots))$$

it is sufficient to establish the following relation

$$a_{k-1}((a_kb_1)\dots)b_l) = a_k((a_{k-1}b_1)\dots)b_l).$$

We have

$$a_{k-1}((a_kb_1)\dots)b_l) \stackrel{\text{lcom}}{=} ((a_kb_1)\dots)b_{l-1})(a_{k-1}b_l)$$
$$\stackrel{\text{rcom}}{=} ((a_k(a_{k-1}b_l))b_1)\dots)b_{l-1}) \stackrel{\text{lcom}}{=} ((a_{k-1}(a_kb_l))b_1)\dots)b_{l-1})$$
$$\stackrel{\text{rcom}}{=} ((a_{k-1}b_1)\dots)b_{l-1})(a_kb_l) \stackrel{\text{lcom}}{=} a_k((a_{k-1}b_1)\dots)b_l).$$ □

For the row

$$A = \ a_1 \quad a_2 \quad \cdots \quad a_n$$

we call

$$h(A) = a_1 = \textit{the head of } \ A$$

$$t(A) = \ a_2 \quad a_3 \quad \cdots \quad a_n \ = \textit{the tail of } \ A$$

$$b(A) = \ a_1 \quad a_2 \quad \cdots \quad a_{n-1} \ = \textit{the beginning part of } \ \ A$$

$$e(A) = a_n = \ \textit{the end part of } \ \ A$$

Lemma 3.2. *Let*

$$A_i = \ a_{i,1} \quad a_{i,2} \quad a_{i,3} \quad \cdots \quad a_{i,k_i}$$

be left-bracketed elements, $i = 1, \ldots, n$ *, and let*

$$B = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{matrix}$$

be their right-bracketed product. Then

$$B = L(h(A_1), \cdots, h(A_n); \{t(A_1), \ldots, t(A_n)\}).$$

In other words,

$$B = L(a_{n,1}, \ldots, a_{1,1}; a_{1,2}, \ldots, a_{1,k_1}, a_{2,2}, \ldots, a_{2,k_2}, \ldots, a_{n,2}, \ldots, a_{n,k_n}).$$

P r o o f. We use induction on n.
Let $n = 2$ and let

$$A_1 = \ a_{1,1} \quad a_{1,2} \quad a_{1,3} \quad \cdots \quad a_{1,k_1} \ ,$$

$$A_2 = \ a_{2,1} \quad a_{2,2} \quad a_{2,3} \quad \cdots \quad \cdots \quad a_{2,k_2} \ .$$

Let

$$b(A_2) = \ a_{2,1} \quad a_{2,2} \quad a_{2,3} \quad \cdots \quad \cdots \quad a_{2,k_2-1}$$

be the beginning part of A_2. Then

$$B = A_2A_1 = (b(A_2)a_{2,k_2})((a_{1,1}a_{1,2})\ldots)a_{1,k_1})$$

$$\overset{\text{rcom}}{=} (b(A_2)((a_{1,1}a_{1,2})\ldots)a_{1,k_1}))a_{2,k_2}$$

$$\overset{\text{lcom}}{=} (((a_{1,1}a_{1,2})\ldots)a_{1,k_1-1})(b(A_2)a_{1,k_1}))a_{2,k_2}$$

$$\overset{\text{rcom}}{=} (((a_{1,1}a_{1,2})\ldots)a_{1,k_1-1})a_{2,k_2})(b(A_2)a_{1,k_1})$$

$$\overset{\text{lcom}}{=} b(A_2)((a_{1,1}a_{1,2})\ldots)a_{1,k_1-1})a_{2,k_2})a_{1,k_1})$$

$$\overset{\text{rcom}}{=} b(A_2)((a_{1,1}a_{1,2})\ldots)a_{1,k_1-1})a_{1,k_1})a_{2,k_2})$$

$$\overset{\text{rcom}}{=} b(A_2)(A_1e(A_2)).$$

We can repeat this procedure $k_2 - 1$ times and obtain

$$B = A_2A_1 = b(A_2)(A_1e(A_2)) = b(b(A_2))((A_1e(A_2))e(b(A_2)))$$

$$= \cdots = a_{2,1}L(A_1; a_{2,k_2}, a_{2,k_2-1}, \ldots, a_{2,2}).$$

Therefore, by Lemma 3.1,

$$B = L(a_{2,1}, a_{1,1}; a_{1,2}, \ldots, a_{1,k_1}, a_{2,2}, \ldots, a_{2,k_2}).$$

Our lemma is proved for $n = 2$.

Suppose that our statement is true for $n - 1 > 1$. Then

$$B = A_nC,$$

where

$$C = A_{n-1}(\cdots(A_2A_1)\cdots).$$

By the inductive assumption

$$C = L(h(A_{n-1}), \ldots, h(A_1); \{t(A_1), \ldots, t(A_{n-1})\}).$$

Therefore,

$$B = A_nC = L(a_{n,1}, h(C); \{t(C), t(A_n)\}).$$

Note that

$$h(C) = a_{n-1}, t(C) = L(a_{n-2,1}, \ldots, a_{1,1}; \{t(A_1), \ldots, t(A_{n-1})\}).$$

Therefore,

$$L(a_{n,1}, h(C); \{t(C), t(A_n)\}) = L(a_{n,1}; L(h(C); \{t(C), t(A_n)\}))$$
$$= L(a_{n,1}, L(a_{n-1}; \{L(a_{n-2,1}, \ldots, a_{1,1}; t(A_1), \ldots), t(A_n)\}))$$
$$= \cdots = L(a_{n,1}, a_{n-1,1}, \ldots, a_{1,1}; \{t(A_1), \ldots, t(A_n)\}). \qquad \square$$

Lemma 3.3. *Let* $k, l, m, n \geq 1$ *. Then*

$$L(a_1, \ldots, a_k; b_1, \ldots, b_l) L(c_1, \ldots, c_m; d_1, \ldots, d_n)$$
$$= L(a_1, \ldots, a_k, c_1, \ldots, c_m; b_1, \ldots, d_l, d_1, \ldots, d_n).$$

Proof. Set

$$R_{k,l} = L(a_1, \ldots, a_k; b_1, \ldots, b_l), \quad S_{n,m} = L(c_1, \ldots, c_m; d_1, \ldots, d_n).$$

Then by the left-commutativity

$$R_{k,l} S_{m,n} = c_1(\cdots(c_{m-1}(R_{k,l}((\cdots(c_m d_1)\cdots)d_n)))\cdots).$$

Further,

$$R_{k,l}((\cdots(c_m d_1)\cdots)d_n)$$
$$= [a_1(\cdots(a_{k-1}((\cdots(a_k b_1)\cdots)b_l))\cdots)]((\cdots(c_m d_1)\cdots)d_n)$$
$$\overset{\text{lcom}}{=} ((\cdots(c_m d_1)\cdots)d_{n-1})[(a_1(\cdots(a_{k-1}((\cdots(a_k b_1)\cdots)b_l))\cdots)d_n]$$
$$\overset{\text{rcom}}{=} ((\cdots(c_m d_1)\cdots)d_{n-1})[(a_1 d_n)(a_2(\cdots(a_{k-1}((\cdots(a_k b_1)\cdots)b_l))\cdots))]$$
$$\overset{\text{lcom}}{=} (a_1 d_n)[((\cdots(c_m d_1)\cdots)d_{n-1})(a_2(\cdots(a_{k-1}((\cdots(a_k b_1)\cdots)b_l))\cdots))]$$
$$\overset{\text{Lemma 3.2}}{=} a_1(\cdots(a_k(((\cdots((\cdots(c_m b_1)\cdots)b_l)d_1)\cdots)d_n)\cdots)\cdots).$$

So, by Lemma 3.1,

$$R_{k,l} S_{m,n} = L(a_1, \ldots, a_k, c_1, \ldots, c_m; b_1, \ldots, b_l, d_1, \ldots, d_n). \qquad \square$$

4. Proof of Theorem 2.1, part a. First of all we shall see that the algebra $(F(q), \circ)$ is bicommutative. Let us check the left-commutative identity. We have

$$(x_i \circ x_j) \circ x_s = v_{\epsilon_i, \epsilon_j} \circ x_s = v_{\epsilon_i, \epsilon_j + \epsilon_s} = (x_i \circ x_s) \circ x_j.$$

Further,

$$(x_i \circ x_j) \circ v_{\alpha,\beta} = v_{\epsilon_i,\epsilon_j} \circ v_{\alpha,\beta} = v_{\epsilon_i+\alpha,\epsilon_j+\beta},$$

$$(x_i \circ v_{\alpha,\beta}) \circ x_j = v_{\alpha+\epsilon_i,\beta} \circ x_j = v_{\alpha+\epsilon_i,\beta+\epsilon_j},$$

and

$$(x_i \circ x_j) \circ v_{\alpha,\beta} = (x_i \circ v_{\alpha,\beta}) \circ x_j.$$

Similarly,

$$(x_i \circ v_{\alpha,\beta}) \circ v_{\gamma,\delta} = v_{\alpha+\epsilon_i,\beta} \circ v_{\gamma,\delta} = v_{\alpha+\gamma+\epsilon_i,\beta+\delta} = (x_i \circ v_{\gamma,\delta}) \circ v_{\alpha,\beta}$$

and

$$(v_{\alpha,\beta}, v_{\gamma,\delta}) \circ v_{\mu,\nu} = v_{\alpha+\gamma+\mu,\beta+\delta+\nu} = (v_{\alpha,\beta}, v_{\mu,\nu}) \circ v_{\gamma,\delta}.$$

The right-commutative identity can be checked in a similar way.

Now we shall prove that $F(q)$ is isomorphic to the free bicommutative algebra $F(X)$ with generators $X = \{x_1, \ldots, x_q\}$. For $\mathbf{i} \in I_k, \mathbf{j} \in I_l$ set

$$e_{\mathbf{i},\mathbf{j}} = L(x_{i_1}, \ldots, x_{i_k}; x_{j_1}, \ldots, x_{j_l}) = \begin{matrix} x_{i_k} & x_{j_1} & \cdots & x_{j_l} \\ x_{i_{k-1}} & & & \\ \vdots & & & \\ x_{i_1} & & & \end{matrix} .$$

We use induction on $n = |\mathbf{m}| = \sum_{i=1}^{q} m_i$. For $n = 1$ the statement is trivial: any homogeneous element of degree 1 is an element of the form x_s.

Suppose that the statement is true for all $\mathbf{m}'$ with $|\mathbf{m}'| < n$. Let $|\mathbf{m}| = n > 1$. We have to prove that $F^{\mathbf{m}}(X)$ is a linear span of elements $e_{\mathbf{ij}}$ such that $\mathrm{cont}(\mathbf{i}) + \mathrm{cont}(\mathbf{j}) = \mathbf{m}$, where $i_1 \le \cdots \le i_k$, $k \ge 1$ $j_1 \le \cdots \le j_l$, $l \ge 1$, $i_1 + \cdots + i_k + j_1 + \cdots + j_l = n$. If $n = 2$ any element of content $\mathbf{m}$ with $|\mathbf{m}| = 2$ is a linear combination of elements of the form $x_{s'} x_{s''} = e_{s's''}$.

Suppose that $n > 2$. If the monomial $u \in F(X)$ of content $\mathbf{m}$ has the form $u = x_s e_{\mathbf{ij}}$, then $u = e_{\mathrm{sort}(\{s,\mathbf{i}\})\mathbf{j}}$, where $\mathrm{sort}(\{s, \mathbf{i}\})$ means that the elements of $\{s, \mathbf{i}\}$ are ordered in non-decreasing way. If the monomial $u \in F^{\mathbf{m}}(X)$ has the form $u = e_{\mathbf{ij}} x_s$, then $u = e_{\mathbf{i}\,\mathrm{sort}(\{s,\mathbf{j}\})}$. If the monomial $u \in F^{\mathbf{m}}(X)$ has the form $e_{\mathbf{i}'\mathbf{j}'} e_{\mathbf{i}''\mathbf{j}''}$ then by Lemma 3.3 $u = e_{\mathrm{sort}(\{\mathbf{i}',\mathbf{i}''\})\mathrm{sort}(\{\mathbf{j}',\mathbf{j}''\})}$.

So, we have established that the map $F(X) \to F(q)$, $|X| = q$, given by the rules

$$x_i \mapsto x_i, \quad e_{\mathbf{ij}} \mapsto v_{\mathrm{cont}(\mathbf{i})\mathrm{cont}(\mathbf{j})}$$

defines an isomorphism of the algebras $F(X)$ and $F(q)$. $\square$

5. Cocharacter sequence. Let $M_{k,n-k}$ be the linear span of the multilinear base elements of shape $[n-k+1,1^{k-1}]$ The action of the symmetric group S_n on $M_{k,n-k}$ is natural:

$$\sigma L(x_{i_1},\dots,x_{i_k},x_{j_1},\dots,x_{j_{n-k}}) = L(x_{\sigma(i_1)},\dots,x_{\sigma(i_k)},x_{\sigma(j_1)},\dots,x_{\sigma(j_{n-k})}).$$

Recall that the irreducible modules of S_n are described by partitions $\lambda \in P(n)$. Any irreducible S_n-module is isomorphic to the Specht module S^λ constructed by Young tabloids of shape λ. Recall that the Young tabloid $\{t\}$ is a class of Young tableaux t under the equivalence relation given by $t \sim t'$ if the corresponding rows of t and t' contain the same integers, maybe in different order. For example,

$$\begin{array}{ccccc} \hline 1 & 3 & 6 & 7 & 4 \\ \hline 2 & 4 & 5 & & \\ \hline \end{array} \sim \begin{array}{ccccc} \hline 7 & 3 & 4 & 1 & 6 \\ \hline 2 & 4 & 5 & & \\ \hline \end{array} \not\sim \begin{array}{ccccc} \hline 1 & 3 & 2 & 7 & 4 \\ \hline 6 & 4 & 5 & & \\ \hline \end{array}$$

Let $C(t)$ be the column-stabilizer of the Young tableau t. Then S^λ is the vector space with base

$$v_t = \sum_{\sigma \in C(t)} \operatorname{sign}\sigma\,\{\sigma(t)\},$$

where t runs on the standard Young tables of shape λ. It has the structure of S_n-module. Moreover it is irreducible and any irreducible S_n-module is isomorphic to such a module. The S_n-module S^λ is called a Specht module. For details see [6].

Lemma 5.1.

$$M_{k,n-k} \cong S^{[n]} \oplus \bigoplus_{i=1}^{k} S^{[n-i,i]}, \qquad k \le n/2.$$

P r o o f. We use induction on $k = 1,\dots,\lfloor n/2 \rfloor$.

Let $k=1$. The n-dimensional standard S_n-module is a direct sum of a 1-dimensional and an $(n-1)$-dimensional irreducible submodule. Therefore, since $M_{1,n-1}$ is isomorphic to the standard n-dimensional S_n-module,

$$M_{1,n-1} \cong S^{[n]} \oplus S^{[n-1,1]}.$$

Now, suppose that our statement is true for $k-1>1$. By the hook formula,

$$\dim S^{[n-i,i]} = \frac{n!(n-2i+1)}{(n-i+1)!i!}.$$

We see that

$$\binom{n}{k} = 1 + \sum_{i=1}^{k} \frac{n!(n-2i+1)}{(n-i+1)!i!}.$$

Therefore

$$\dim M_{k,n-k} = \dim S^{[n]} + \sum_{i=1}^{k} \dim S^{[n-i,i]}.$$

So, to prove our statement it is sufficient to find in $M_{k,n-k}$ S_n-submodules isomorphic to $S^{[n]}, S^{[n-1,1]}, \dots, S^{[n-k,k]}$.

To any Young tabloid $\{t\}$ we associate a bicommutative element $f(t)$ by the rule

$$t = \begin{array}{lllll} \hline j_1 & j_2 & \cdots & \cdots & j_l \\ \hline i_1 & i_2 & \cdots & i_k & \\ \hline \end{array} \mapsto f(t) = \begin{array}{llllll} x_{i_1} & x_{j_1} & x_{j_2} & \cdots & \cdots & x_{j_l} \\ x_{i_2} & & & & & \\ \vdots & & & & & \\ x_{i_k} & & & & & \end{array}.$$

Since the order of the entries in the rows of the Young tableau t is not essential for the definition of the Young tabloid $\{t\}$, the element $f(t)$ is an element of bicommutative algebra which is equal to the base element $e_{\mathbf{i},\mathbf{j}}$, where $\mathbf{i} = i_1 \dots i_k, \mathbf{j} = j_1 \dots j_l$. So, the map f is well-defined. Now we are able to extend the map f to an imbedding of S_n-modules

$$f : S^{[n-k,k]} \to M_{k,n-k}$$

by

$$v_t = \sum_{\sigma \in C_t} \operatorname{sign} \sigma \{\sigma(t)\} \mapsto f(v_t) = \sum_{\sigma \in C_t} \operatorname{sign} \sigma \, f(\sigma(t)).$$

For example, if

$$t = \begin{array}{lll} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array},$$

then $f(v_t)$ is equal to

$$\begin{array}{llll} x_1 & x_2 & x_4 & x_5 \\ x_3 & & & \end{array} - \begin{array}{llll} x_1 & x_2 & x_3 & x_4 \\ x_5 & & & \end{array} - \begin{array}{llll} x_2 & x_1 & x_4 & x_5 \\ x_3 & & & \end{array} + \begin{array}{llll} x_2 & x_1 & x_3 & x_4 \\ x_5 & & & \end{array};$$

in other words,

$$f(v_t) = e_{13\,245} - e_{15\,234} - e_{23\,145} + e_{25\,134}.$$

So, we have established that $M_{k,n-k}$ contains an S_n-submodule isomorphic to $S^{[n-k,k]}$.

The following map is an S_n-module imbedding

$$M_{k-1,n-k+1} \to M_{k,n-k}, \quad k \le n/2,$$

given by

$$e_{\mathbf{i},\mathbf{j}} \mapsto \sum_{s=1}^{n-k+1} e_{sort(\mathbf{i}\cup\{s\})\,\mathbf{j}(\hat{s})},$$

where $\mathbf{i} = i_1 \ldots i_{k-1}$, $\mathbf{j} = j_1 \ldots j_{n-k+1}$, $\mathbf{j}(\hat{s}) = j_1 \ldots j_{s-1} j_{s+1} \ldots j_{n-k+1}$ and $\mathbf{i} \cup \{s\} = i_1 \ldots i_{k-1} s$. So, the S_n-module $M_{k,n-k}$ contains an S_n-submodule isomorphic to $M_{k-1,n-k+1}$.

By the inductive assumption

$$M_{k-1,n-k+1} \cong S^{[n]} \oplus \oplus_{i=1}^{k-1} S^{[n-i,i]}.$$

Therefore, the following isomorphism is S_n-module isomorphism

$$M_{k,n-k} \cong S^{[n]} \oplus \bigoplus_{i=1}^{k} S^{[n-i,i]}, \qquad k \le n/2.$$

The proof of the lemma is completed. □

6. Proof of Theorem 2.1, part b. Note that the following isomorphism is an S_n-module isomorphism

$$M_{k,n-k} \cong M_{n-k,k}.$$

By Theorem 2.1, part **a**, the following isomorphisms are also S_n-module isomorphisms

$$F_1^{\text{multi}} \cong K, \qquad F_n^{\text{multi}} \cong \bigoplus_{k=1}^{n-1} M_{k,n-k}, \qquad n > 1.$$

Let $\alpha_k = [n-k+1, 1^{k-1}]$ be a hook.

Below we use Lemma 5.1.

If $n > 1$ is odd, then

$$F_n^{\text{multi}} \cong 2 \bigoplus_{k=1}^{\frac{n-1}{2}} M_{k,n-k} = 2 \bigoplus_{k=1}^{\frac{n-1}{2}} \left(S^{[n]} \bigoplus_{i=1}^{k} S^{[n-i,i]} \right)$$

$$= (n-1)S^{[n]} \oplus 2 \bigoplus_{k=1}^{\frac{n-1}{2}} \bigoplus_{i=1}^{k} S^{[n-i,i]}$$

$$= (n-1)S^{[n]} \bigoplus_{i=1}^{\frac{n-1}{2}} (n-2i+1)S^{[n-i,i]}.$$

If n is even, then

$$F_n^{\text{multi}} \cong M_{\frac{n}{2},\frac{n}{2}} \oplus 2 \bigoplus_{k=1}^{\frac{n-2}{2}} S^{\alpha_k}$$

$$= M_{\frac{n}{2},\frac{n}{2}} \oplus 2 \bigoplus_{k=1}^{\frac{n-2}{2}} (S^{[n]} \bigoplus_{i=1}^{k} S^{[n-i,i]})$$

$$= M_{\frac{n}{2},\frac{n}{2}} \oplus (n-2)S^{[n]} \oplus 2 \bigoplus_{k=1}^{\frac{n-2}{2}} \bigoplus_{i=1}^{k} S^{[n-i,i]}$$

$$= M_{\frac{n}{2},\frac{n}{2}} \oplus (n-2)S^{[n]} \oplus \bigoplus_{k=1}^{\frac{n-2}{2}} (n-2i)S^{[n-i,i]}$$

$$= (n-1)S^{[n]} \bigoplus_{i=1}^{\frac{n}{2}} (n-2i+1)S^{[n-i,i]}.$$

Theorem 2.1, part **b**, is proved. □

7. Proof of Theorem 2.1, part c. It is easy to see that the bicommutative operad is self-dual. In other words, the dual of the bicommutative operad is defined by the identities $lcom = 0$ and $rcom = 0$ which are the same as for the bicommutative operad. Let F be a free bicommutative algebra and let G be a free algebra of the dual of the bicommutative operad. Then $F \otimes G$ should be Lie-admissible (for details see [7]). The multilinear component of degree 3 of the free bicommutative algebra F has dimension 6 and is generated by the elements $a(bc), a(cb), b(ca), (ab)c, (ba)c$ and $(ca)b$. The Jacobi identity for $a \otimes u, b \otimes v$ and $c \otimes w$ gives us several conditions for $u, v, w \in G$ which are equivalent to the bicommutative identities.

First five dimensions of the multilinear components of the free bicommutative algebra are $1, 2, 6, 14, 30$. Therefore the corresponding exponential generating function of the codimensions of bicommutative algebras has the form

$$f(x) = \sum_{q\geq 1}(-1)^q c_q \frac{x^q}{q!}$$

$$= -x + 2x^2/2 - 6x^3/3! + 14x^4/4! - 30x^5/5! + O(x^6),$$

and

$$f(f(x)) = x + x^5 + O(x^6) \neq x.$$

So, by the results of [7] the bicommutative operad is not Koszul.

8. Proof of Corollary 2.4.

Lemma 8.1. *Let M be the set of compositions of n , i.e., the set of non-negative integral solutions of the equation $m_1 + \cdots + m_q = n$. Then*

$$\sum_{(m_1,\ldots,m_q)\in M} (m_1 \cdots m_q)^l = \binom{n+q-1}{(l+1)q-1}$$

if $l = 0, 1$.

P r o o f. It is well known that number of solutions in non-negative integers of the equation $m_1+\cdots+m_q = n$ is $\binom{n+q-1}{n}$. Therefore, for $l = 0$ our statement is true.

Let $l = 1$. We use induction on q . If $q = 1$, then statement is trivial. Suppose that for $q - 1 > 0$ out statement is true. Then

$$\sum_{(m_1,\ldots,m_q)\in M} m_1 \cdots m_q = \sum_{m_q=0}^{n} m_q \sum_{m_1+\ldots+m_{q-1}=n-m_q} m_1 \cdots m_{q-1}$$

$$= \sum_{m_q=0}^{n} m_q \binom{n - m_q + q - 2}{2q - 3}.$$

So, by the well-known formula

$$\sum_i \binom{a+i}{k}\binom{b-i}{l} = \binom{a+b+1}{k+l+1}$$

we have

$$\sum_{(m_1,\ldots,m_q)\in M} m_1\cdots m_q = \sum_{m_q=0}^{n} \binom{m_q}{1}\binom{n-m_q+q-2}{2q-3}$$
$$= \binom{n+q-1}{2q-1}.$$

Our statement is proved also for $l=1$. $\square$

Corollary 8.2.

$$\sum_{m_1+\cdots+m_q=n} (m_1+1)\cdots(m_q+1) - 2 = \binom{n+2q-1}{n} - (2-\delta_{n,1})\binom{n+q-1}{n}.$$

Proof. Let $n>1$. By Lemma 8.1

$$\sum_{m_1+\cdots+m_q=n} 2 = 2\binom{n+q-1}{n}.$$

and

$$\sum_{m_1+\cdots+m_q=n} (m_1+1)\cdots(m_q+1) = \sum_{\substack{m_1+\cdots+m_q=n+q\\ m_1>0,\ldots,m_q>0}} m_1\cdots m_q$$
$$= \sum_{\substack{m_1+\cdots+m_q=n+q\\ m_1\geq 0,\ldots,m_q\geq 0}} m_1\cdots m_q = \binom{n+q+q-1}{2q-1} = \binom{n+2q-1}{n}.$$

Therefore,

$$\sum_{m_1+\cdots+m_q=n} (m_1+1)\cdots(m_q+1) - 2 = \binom{n+2q-1}{n} - 2\binom{n+q-1}{n}.$$

For $n=1$ statement is evident. $\square$

Proof of Corollary 2.4. In how many ways one can choose pairs (α,β) such that $\alpha,\beta\in \mathbf{Z}_+^q$, $\alpha+\beta=\mathbf{m}$, $\alpha,\beta\neq(0,\ldots,0)$? For α_i there are m_i+1 possibilities: α_i may be equal to $0,1,\ldots,m_i$. We have to exclude the two extremal cases: $\alpha=(0,\ldots,0)$ and $\alpha=\mathbf{m}$. Therefore by the product rule the base vectors $v_{\alpha,\beta}$ such that $\alpha_i+\beta_i=m_i$, $i=1,\ldots,q$, can be chosen in $\prod_{i=1}^q(m_i+1)-2$ ways. So,

$$\dim F^{\mathbf{m}}(X) = \prod_{i=1}^{q}(m_i+1) - 2 + \delta_{|\mathbf{m}|,1} + \delta_{|\mathbf{m}|,0},$$

and

$$H(bicom, t_1, \dots, t_q) = \sum_{m_1 \geq 0} \cdots \sum_{m_q \geq 0} \prod_{i=1}^{q} (m_i + 1) t_i^{m_i} - 2 \prod_{i=1}^{q} t_i^{m_i} + \sum_{i=1}^{q} t_i + 1$$

$$= \left(\prod_{i=1}^{q} t_i \right) \sum_{m_1 \geq 0} \cdots \sum_{m_q \geq 0} \left(\prod_{i=1}^{q} t_i^{m_i+1} \right)' - 2 \prod_{i=1}^{q} t_i^{m_i} + \sum_{i=1}^{q} t_i + 1$$

$$= \prod_{i=1}^{q} (\frac{t_i}{1-t_i})' - 2 \prod_{i=1}^{q} \frac{1}{1-t_i} + \sum_{i=1}^{q} t_i$$

$$= \prod_{i=1}^{q} \frac{1}{(1-t_i)^2} - 2 \prod_{i=1}^{q} \frac{1}{1-t_i} + \sum_{i=1}^{q} t_i + 1.$$

The other statements of Corollary 2.4 follow from Corollary 8.2. □

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