Serdica
Mathematical Journal
Сердика
Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **36** (2010), 387–402

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

DENSE SUBDIFFERENTIABILITY AND TRUSTWORTHINESS FOR ARBITRARY SUBDIFFERENTIALS

Florence Jules, Marc Lassonde

Communicated by J. P. Revalski

Abstract. We show that the properties of dense subdifferentiability and of trustworthiness are equivalent for any subdifferential satisfying a small set of natural axioms. The proof relies on a remarkable property of the subdifferential of the inf-convolution of two (non necessarily convex) functions. We also show the equivalence of the dense subdifferentiability property with other basic properties of subdifferentials such as a weak* Lipschitz Separation property, a strong Compact Separation property and a Minimal property for the analytic approximate subdifferential of Ioffe.

1. Introduction. In the sixties, E. Asplund [1] introduced spaces characterized by *dense differentiability* properties of continuous convex functions. Two notions of differentiability were privileged, that of Fréchet, giving rise later to

2010 *Mathematics Subject Classification:* 49J52, 49J50, 58C20, 26B09.
Key words: lower semicontinuous function, inf-convolution, subdifferential, approximate sum rule, Asplund space, subdifferentiability space, trustworthy space, variational analysis.

the so-called Asplund spaces, and that of Gâteaux (equivalent to Dini-Hadamard as the functions are locally Lipschitz), leading to weak Asplund spaces. In an analogous manner, A. D. Ioffe [15] defined the so-called *subdifferentiability spaces*, as spaces characterized by properties of *dense subdifferentiability* for lower semicontinuous non necessarily convex functions. The chosen subdifferentials are the ε-subdifferentials of Fréchet and of Hadamard (Dini), leading respectively to the concepts of subdifferentiability space (S-space) and of weak subdifferentiability space (WS-space).

Clearly, the property for a subdifferential of a lower semicontinuous function to be not empty at sufficiently many points of the underlying space (*dense subdifferentiability*) is the minimal property we can expect from a subdifferential. On the other hand, such a subdifferential should have good calculus properties to be useful for analysis problems, at least it should satisfy an approximate sum rule like "the subdifferential of the sum of two lower semicontinuous functions is contained in the approximate sum of the subdifferentials of each function." Spaces possessing such a calculus rule are called *trustworthy spaces* by Ioffe [15]. Again, the subdifferentials considered at the origin were the ε-subdifferentials of Fréchet and of Hadamard (Dini), leading respectively to the notions of trustworthy space (T-space) and of weak trustworthy space (WT-space).

The equivalence between the properties of dense subdifferentiability and of trustworthiness, in some sense extreme, and other considerations of the same nature, have been the object of much investigation. Ioffe [15] showed that if X is a WT-space (respectively a T-space), then X is a WS-space (respectively an S-space), Fabian [12] proved the reverse implications. Ekeland-Lebourg [11] established that every S-space is an Asplund space, Fabian-Zhivkov [14] demonstrated the inverse.

Then, the question arose whether the equivalences in those results remain valid for the 'exact' subdifferentials of Fréchet and of Hadamard. Thanks to the recently proved smooth variational principle of Borwein-Preiss [3], Fabian [13] was able to give a positive answer to the question. Later, Ioffe [18] introduced the concepts of subdifferentiability space and of trustworthy space for arbitrary (exact) subdifferentials and announced the coincidence of both types of spaces in the case of smooth (viscosity) bornological subdifferentials, but without giving a detailed proof.

Our paper pursues the same line, its principal objective being to investigate the next natural question: does the equivalence between dense subdifferentiability and trustworthiness hold for any subdifferential satisfying a small set of natural axioms? The answer is positive (Theorem 3.2). The proof relies on a remarkable property (Theorem 2.1) of the subdifferential of the inf-convolution of two functions, valid for *analytic subdifferentials* like the Clarke, the proximal and the bornological ones. This property in fact enables us to prove the equivalence of the dense subdifferentiability property with five other properties including a weak* Lipschitz Separation property, a strong Compact Separation property and a Minimal property for the analytic approximate subdifferential of Ioffe.

Notation and definitions. In what follows, X stands for a real Banach space, B_X for its closed unit ball, X^* for its topological dual, and $\langle .,. \rangle$ for the duality pairing. All functions f on X are extended-real-valued and *proper*, i.e., they do not take the value $-\infty$ and their effective domain $\operatorname{dom} f := \{x \in X : f(x) < \infty\}$ is nonempty. For a function f on X, we define f^- by $f^-(x) := f(-x)$, and for $S \subset X$, we define f_S by

$$f_S(x) := \begin{cases} f(x), & \text{if} \quad x \in S \\ +\infty, & \text{if} \quad x \notin S. \end{cases}$$

We say that f is *lower semicontinuous near a point* $\bar{x} \in X$ if there is a closed neighborhood V of $\bar{x}$ such that f_V is lower semicontinuous, i.e., the sub-level sets $\{x \in V : f_V(x) \le \lambda\}$ $(\lambda \in \mathbb{R})$ are closed. We say that f is *inf-compact near a point* $\bar{x} \in X$ if there is a closed neighborhood V of $\bar{x}$ such that f_V is inf-compact, i.e., the sub-level sets $\{x \in V : f_V(x) \le \lambda\}$ $(\lambda \in \mathbb{R})$ are compact. A function $f : X \to (-\infty, +\infty]$ is said to attain a *strong local minimum at a point* $\bar{x} \in X$ if there exists a neighborhood V of $\bar{x}$ such that f_V attains a strong minimum at $\bar{x}$, i.e., for every sequence $\{x_n\} \subset V$, $f_V(x_n) \to \inf f_V$ implies $x_n \to \bar{x}$.

2. Analytic subdifferentials and inf-convolution.

2.1. Analytic subdifferentials. We call *analytic subdifferential* any subdifferential directly defined through an analytic formula. Typical in this category are the Clarke, the proximal and the bornological subdifferentials.

The *Clarke subdifferential*, originally introduced by Clarke in his thesis [5] via a geometric construction, was given the following analytic formulation by Rockafellar [27]: for a proper function f on X and $\bar{x} \in \operatorname{dom} f$,

$$\partial_C f(\bar{x}) := \left\{ x^* \in X^* : \langle x^*, h \rangle \le \sup_{\delta > 0} \limsup_{\substack{t \searrow 0 \\ x \to_f \bar{x}}} \inf_{h' \in B_\delta(h)} \frac{f(x + th') - f(x)}{t}, \forall h \in X \right\}.$$

The *proximal subdifferential*, introduced explicitly by Rockafellar [28] in connection with the Clarke subdifferential, is defined by

$$\partial_P f(\bar{x}) := \left\{ x^* \in X^* : \liminf_{\|h\| \searrow 0} \frac{f(\bar{x}+h) - f(\bar{x}) - \langle x^*, h\rangle}{\|h\|^2} > -\infty \right\}.$$

We refer to the books by Clarke [6] and by Clarke *et al.* [7] for thorough study of these subdifferentials.

A *bornology* on X is a family β of closed, bounded and symmetric sets whose union is all of X and such that the union of any two members of β is contained in some member of β. Each bornology β gives rise to a concept of topology on X^* and to a concept of differentiability for functions on X. The *topology of β-convergence* on the dual space X^*, denoted by β^*, is the topology generated by the family of semi-norms $p_S(x^*) := \sup x^*(S)$, $S \in \beta$. We say that a function f on X is *β-differentiable* at a point $\bar{x} \in \operatorname{dom} f$, with *$\beta$-gradient* $x^* = f'_\beta(\bar{x}) \in X^*$, provided that for any $S \in \beta$

$$\lim_{t \searrow 0} \sup_{h \in S} \left| \frac{f(\bar{x}+th) - f(\bar{x})}{t} - \langle x^*, h\rangle \right| = 0.$$

Further, we say that f is *β-smooth* at $\bar{x} \in \operatorname{dom} f$ if f'_β exists in a neighborhood U of $\bar{x}$ and $f'_\beta : U \to (X^*, \beta^*)$ is continuous at $\bar{x}$.

Now, the *β-subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is defined by

$$\partial_\beta f(\bar{x}) := \left\{ x^* \in X^* : \liminf_{t \searrow 0} \inf_{h \in S} \left(\frac{f(\bar{x}+th) - f(\bar{x})}{t} - \langle x^*, h\rangle \right) \geq 0, \ \forall S \in \beta \right\},$$

while the *smooth* (or *viscosity) β-subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is given by

$$\partial^-_\beta f(\bar{x}) := \{ \varphi'_\beta(\bar{x}) : \varphi \text{ is locally Lipschitz, } \beta\text{-smooth at } \bar{x}, \\ \text{and } f - \varphi \text{ attains a local minimum at } \bar{x} \}.$$

See, e.g., Borwein-Preiss [3], Borwein-Zhu [4], Deville *et al.* [10], Jules [21] and Phelps [26] for more details.

Main examples of bornologies.

— The convex bounded or Fréchet bornology, denoted by F, consisting of all convex, closed, bounded and symmetric sets; F^* is the strong (norm) topology on X^*.

— The convex weak-compact or weak-Hadamard bornology, denoted by WH, consisting of all convex, weak-compact and symmetric sets; WH^* is the Mackey topology on X^*.

— The convex compact or Hadamard bornology, denoted by H, consisting of all convex, compact and symmetric sets; H^* is the bw^* topology on X^*.

— The polyhedral or convex-Gâteaux bornology, denoted by CG, consisting of all convex hulls of finite symmetric sets; CG^* is the w^* topology on X^*.

— The finite or Gâteaux bornology, denoted by G, consisting of all finite symmetric sets; G^* is the w^* topology on X^*.

2.2. Subdifferential of inf-convolution. We recall that the *inf-convolution* (also called the epi-sum) of two proper functions f_1 and f_2 on X is the function $f_1 \triangledown f_2 : X \to \overline{\mathbb{R}}$ given by

$$(f_1 \triangledown f_2)(z) := \inf\{f_1(x) + f_2(z - x) : x \in X\}.$$

The inf-convolution is said to be *exact at* $z \in X$ if the infimum in the definition of $(f_1 \triangledown f_2)(z)$ is attained, that is, if there exists $x \in X$ such that $(f_1 \triangledown f_2)(z) = f_1(x) + f_2(z - x)$.

The result below is well known in the case of convex functions (see, e.g., Laurent [23, Proposition (6.6.4)] or Zălinescu [31, Corollary 2.4.7]) and in the case of the Fréchet subdifferential (Correa-Jofré-Thibault [9, Lemma 3.6]). We show that it is actually valid for any of the above analytic subdifferentials.

Theorem 2.1. *Let X be a Banach space, ∂ be any of the above analytic subdifferentials, f_1, f_2 be proper functions on X, $x \in \operatorname{dom} f_1$ and $y \in \operatorname{dom} f_2$. Suppose that $(f_1 \triangledown f_2)(x + y) = f_1(x) + f_2(y)$. Then*

$$\partial(f_1 \triangledown f_2)(x + y) \subset \partial f_1(x) \cap \partial f_2(y).$$

P r o o f. Let $g := f_1 \triangledown f_2$ and $z := x + y$, so that $g(z) = f_1(x) + f_2(z - x)$ by hypothesis. We show that if $z^* \in \partial g(z)$, then $z^* \in \partial f_1(x)$ and $z^* \in \partial f_2(z - x)$.

1/ *Case $\partial = \partial_\beta$.* Let $B \in \beta$. By definition of $\partial_\beta g(z)$, we have

$$\liminf_{t \searrow 0} \inf_{u \in B} \left(\frac{g(z + tu) - g(z)}{t} - \langle z^*, u \rangle \right) \geq 0.$$

Since $g(z+tu) = \inf\{f_1(v)+f_2(z+tu-v) : v \in X\}$ and $g(z) = f_1(x)+f_2(z-x)$, this gives

(1)
$$\forall v \in X, \quad \liminf_{t\searrow 0} \inf_{u\in B} \left(\frac{f_1(v) + f_2(z+tu-v) - f_1(x) - f_2(z-x)}{t} - \langle z^*, u\rangle \right) \geq 0.$$

Using (1) with $v = x + tu$ and $v = x$, we obtain respectively

(2)
$$\liminf_{t\searrow 0} \inf_{u\in B} \left(\frac{f_1(x+tu) - f_1(x)}{t} - \langle z^*, u\rangle \right) \geq 0,$$

(3)
$$\liminf_{t\searrow 0} \inf_{u\in B} \left(\frac{f_2(z+tu-x) - f_2(z-x)}{t} - \langle z^*, u\rangle \right) \geq 0.$$

Since (2) and (3) are true for any B in β, we conclude that $z^* \in \partial_\beta f_1(x)$ and $z^* \in \partial_\beta f_2(z-x)$.

2/ *Case* $\partial = \partial_\beta^-$. By definition of $\partial_\beta^- g(z)$, there exist $\varphi : X \to \mathbb{R}$ locally Lipschitz and β-smooth at z and $\lambda > 0$ such that

$$z^* = \varphi'_\beta(z), \quad (g-\varphi)(z) \leq (g-\varphi)(z+u), \ \forall u \in \lambda B_X.$$

Since $g(z) = f_1(x) + f_2(z-x)$ and $g(z+u) = \inf\{f_1(v)+f_2(z+u-v) : v \in X\}$, this gives

(4)
$$z^* = \varphi'_\beta(z),$$

(5)
$$f_1(x) + f_2(z-x) - \varphi(z) \leq f_1(v) + f_2(z+u-v) - \varphi(z+u), \ \forall u \in \lambda B_X, \ v \in X.$$

Putting $v = x + u$ in (5) yields

(6)
$$f_1(x) - \varphi(z) \leq f_1(x+u) - \varphi(z+u), \forall u \in \lambda B_X.$$

Consider $\chi : u \in X \mapsto \chi(u) = \varphi(z-x+u)$. Then, χ is locally Lipschitz and β-smooth at x with $\chi(x) = \varphi(z)$, $\chi(x+u) = \varphi(z+u)$ and $\chi'_\beta(x) = \varphi'_\beta(z)$. Rewriting (4) and (6) we get

$$z^* = \chi'_\beta(x), \quad f_1(x) - \chi(x) \leq f_1(x+u) - \chi(x+u), \ \forall u \in \lambda B_X,$$

proving that $z^* \in \partial_\beta^- f_1(x)$.

Similarly, letting $v = x$ in (5) gives

$$(7) \qquad f_2(z-x) - \varphi(z) \le f_2(z-x+u) - \varphi(z+u),\ \forall u \in \lambda B_X.$$

Consider $\psi : u \in X \mapsto \psi(u) = \varphi(u+x)$. Then, ψ is locally Lipschitz and β-smooth at $z-x$ with $\psi(z-x) = \varphi(z)$, $\psi(z-x+u) = \varphi(z+u)$ and $\psi'_\beta(z-x) = \varphi'_\beta(z)$. Rewriting (4) and (7) we get

$$z^* = \psi'_\beta(z-x), \quad f_2(z-x) - \psi(z-x) \le f_2(z-x+u) - \psi(z-x+u),\ \forall u \in \lambda B_X,$$

proving that $z^* \in \partial^-_\beta f_2(z-x)$.

3/ The cases $\partial = \partial_C$ and $\partial = \partial_P$ are similar, so the proofs are omitted. □

Corollary. *Let X be a Banach space, ∂ be any of the above analytic subdifferentials, f_1, f_2 be proper functions on X, $x \in \operatorname{dom} f_1$ and $y \in \operatorname{dom} f_2$. If $(f_1 \triangledown f_2^-)(x-y) = f_1(x)+f_2(y)$ and $\partial(f_1 \triangledown f_2^-)(x-y) \neq \emptyset$, then $0 \in \partial f_1(x)+\partial f_2(y)$.*

Proof. Let $z^* \in \partial(f_1 \triangledown f_2^-)(x-y)$. It follows from the theorem that $z^* \in \partial f_1(x) \cap \partial f_2^-(-y)$. But, for any of the above analytic subdifferentials, we have $\partial f_2^-(-y) = -\partial f_2(y)$, whence the conclusion. □

Remark. For convex functions, the reverse implication in the above corollary is also true.

3. Characterizations of dense subdifferentiability. In this section, we call *subdifferential*, any operator ∂ which associates a subset $\partial f(x)$ of X^* with every function f defined on a Banach space X and every $x \in X$, such that the following axioms are satisfied:

(A1) If f is convex lower semicontinuous, then $\partial f(x) = \{x^* \in X^* : \langle x^*, y-x\rangle + f(x) \le f(y),\ \forall y \in X\}$.

(A2) If $V \subset X$ is a neighborhood of $x \in X$, then $\partial f_V(x) = \partial f(x)$.

(A3) If $(f_1 \triangledown f_2^-)(x-y) = f_1(x) + f_2(y)$ and $\partial(f_1 \triangledown f_2^-)(x-y) \neq \emptyset$, then $0 \in \partial f_1(x) + \partial f_2(y)$.

Observe that (A3) with $f_2 = 0$ reduces to the standard Fermat rule: *If* $x \in \operatorname{dom} f_1$ *is a minimum of* f_1*, then* $0 \in \partial f_1(x)$. Indeed, in this case $f_1 \nabla f_2^-$ is the constant function $z \mapsto \inf\{f_1(v) : v \in X\} = f_1(x)$, hence $\partial(f_1 \nabla f_2^-)(z) = \{0\}$ and $\partial f_2(z) = \{0\}$ for every $z \in X$ by (A1).

In view of the corollary of Theorem 2.1, all the analytic subdifferentials considered in Section 2 satisfy these axioms.

Before proceeding, we need to recall the definition of *closure* (of graphs) of subdifferentials. Given a function $f : X \to (-\infty, +\infty]$, we denote by d_f the associated graphical metric on $\operatorname{dom} f$ defined by $d_f(x,y) := \|x-y\|+|f(x)-f(y)|$ and write $x \to_f \bar{x}$ to mean that $x \to \bar{x}$ with respect to d_f. Given a topology τ^* on X^* and a point $\bar{x}$ in $\operatorname{dom} f$, we define the closure of ∂f at $\bar{x}$ with respect to the topology $d_f \times \tau^*$ on $\operatorname{dom} f \times X^*$, briefly the τ^**-closure of* ∂f *at* $\bar{x}$, as the set

$$\tau^*\text{–}\limsup_{x \to_f \bar{x}} \partial f(x) := \{\, x^* \in X^* : (\bar{x}, x^*) \in d_f \times \tau^*\text{–}\operatorname{cl} \partial f \,\},$$

where $d_f \times \tau^*\text{–}\operatorname{cl} \partial f$ denotes the closure of the graph of the mapping $x \mapsto \partial f(x)$ in $(\operatorname{dom} f, d_f) \times (X^*, \tau^*)$. In other words, x^* belongs to the above set if and only if there is a net $\{(x_\nu, x_\nu^*)\}$ in $\operatorname{dom} f \times X^*$ such that $x_\nu \to_f \bar{x}$, $x_\nu^* \xrightarrow{\tau^*} x^*$ and $x_\nu^* \in \partial f(x_\nu)$.
This definition naturally extends to the closure of sums of subdifferentials.

The sequential version of this closure, denoted by

$$\tau^*\text{–}\limsup_{x_n \to_f \bar{x}} \partial f(x_n),$$

is defined by requiring the net $\{(x_\nu, x_\nu^*)\}$ to be actually a sequence $\{(x_n, x_n^*)\}$ (in case τ^* is the norm topology on X^*, topological and sequential closures of course coincide). A widely used such 'limiting subdifferential' is the weak* sequential closure of the Fréchet subdifferential, see, e.g., the books by Mordukhovich [24] in infinite dimensional spaces and by Rockafellar-Wets [29] in finite dimensional spaces.

In the list of properties stated below, the operator ∂_a^I plays a basic role. This operator is actually a subdifferential on the class of Lipschitz continuous functions, but not on the class of lower semicontinuous functions (it fails to satisfy (A1)). It was considered by Ioffe [16, 17] in the first step construction of his (geometric) *approximate subdifferential* (see also [2, 20, 19] for further developments and variants). This operator, called *Ioffe approximate analytic subdifferential*, is defined as follows: for $f : X \to (-\infty, +\infty]$ and $\bar{x} \in \operatorname{dom} f$,

$$\partial_a^I f(\bar{x}) := \bigcap_{L\in\mathcal{F}} w^*\!-\!\limsup_{x\to_f \bar{x}} \partial_H f_{x+L}(x), \tag{8}$$

where $\mathcal{F}$ denotes the collection of all finite dimensional linear subspaces of X. We may also express $\partial_a^I f(\bar{x})$ in terms of the more familiar Fréchet subdifferential (see, e.g., [21, Proposition A.5.]):

$$\partial_a^I f(\bar{x}) := \bigcap_{L\in\mathcal{F}} w^*\!-\!\limsup_{x\to_f \bar{x}} \partial_F f_{x+L}(x). \tag{9}$$

Given a subdifferential ∂ or any of its closures, we shall consider the following six properties for a Banach space X with respect to ∂.

(DS) *Dense Subdifferentiability.* For any lower semicontinuous $f : X \to (-\infty, +\infty]$ and $\bar{x} \in \operatorname{dom} f$, there exists a sequence $\{x_n\} \subset X$ such that $x_n \to_f \bar{x}$ and $\partial f(x_n) \neq \emptyset$.

(CS) *Compact Separation.* For any lower semicontinuous $f : X \to (-\infty, +\infty]$ and any $\varphi : X \to \mathbb{R}$ convex inf-compact near $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} \varphi$, if $f + \varphi$ admits a strong local minimum at $\bar{x}$, then there exist $x_n \to_f \bar{x}$ and $y_n \to_\varphi \bar{x}$ such that

$$0 \in \partial f(x_n) + \partial \varphi(y_n).$$

(w^*-CS) *w^*-Compact Separation.* For any lower semicontinuous $f : X \to (-\infty, +\infty]$ and any $\varphi : X \to \mathbb{R}$ convex inf-compact near $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} \varphi$, if $f + \varphi$ admits a strong local minimum at $\bar{x}$, then

$$0 \in w^*\!-\!\limsup_{\substack{x\to_f \bar{x}\\ y\to_\varphi \bar{x}}} (\partial f(x) + \partial \varphi(y)).$$

(w^*-LS) *w^*-Lipschitz Separation.* For any lower semicontinuous $f : X \to (-\infty, +\infty]$ and any $\varphi : X \to \mathbb{R}$ convex Lipschitz continuous near $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} \varphi$, if $f + \varphi$ admits a strong local minimum at $\bar{x}$, then

$$0 \in w^*\!-\!\limsup_{x\to_f \bar{x}} \partial f(x) + \partial \varphi(\bar{x}).$$

(w^*-MP) *w^*-Minimality Property of ∂_a^I.* For any lower semicontinuous $f : X \to (-\infty, +\infty]$ and $\bar{x} \in X$,

$$\partial_a^I f(\bar{x}) \subset w^*\!-\!\limsup_{x\to_f \bar{x}} \partial f(x).$$

(w^*-SR) w^*-*Sum Rule.* For any lower semicontinuous $f_i : X \to (-\infty, +\infty]$, $i = 1, \ldots, k$, and $\bar{x} \in X$,

$$\partial_a^I \left(\sum_{i=1}^{k} f_i \right) (\bar{x}) \subset w^*\text{–}\limsup_{x_i \to_{f_i} \bar{x}} \sum_{i=1}^{k} \partial f_i(x_i).$$

F. Jules proved in [21] that, for any subdifferential ∂ satisfying (A1), one has (w^*-CS)$\Leftrightarrow$ (w^*-MP) $\Leftrightarrow$ (w^*-SR) (Theorem 6.5, *loc. cit.*) and (w^*-LS) $\Rightarrow$ (DS) (Proposition 6.7, *loc. cit.*). It is easily seen that these results remain valid for any closure $\hat{\partial}$ of ∂. Since always (CS)$\Rightarrow$ (w^*-CS) and (w^*-SR) $\Rightarrow$ (w^*-LS), to get the equivalence of the above six properties it remains to prove that (DS) $\Rightarrow$ (CS). This is the object of the next theorem.

Theorem 3.1. *Let ∂ be any subdifferential or any of its closures, and let X be a Banach space. Then,* (DS) $\Rightarrow$ (CS).

Slightly modifying definitions from Ioffe [15, 18], we say that, given a subdifferential ∂ or any of its closures, a Banach space X is a *∂-subdifferentiability space* if the property (DS) is satisfied, and a *∂-trustworthy space* if the following property is satisfied:

(w^*-Tr) w^*-*Trustworthiness.* For any lower semicontinuous $f_i : X \to (-\infty, +\infty]$, $i = 1, \ldots, k$, and $\bar{x} \in X$,

$$\partial \left(\sum_{i=1}^{k} f_i \right) (\bar{x}) \subset w^*\text{–}\limsup_{x_i \to_{f_i} \bar{x}} \sum_{i=1}^{k} \partial f_i(x_i).$$

Theorem 3.2. *Let $\partial \subset \partial_a^I$ be any subdifferential or any of its closures, and let X be a Banach space. Then,* (DS) $\Leftrightarrow$ (w^*-Tr)*, that is, X is a ∂-subdifferentiability space if and only if X is a ∂-trustworthy space.*

The proofs of the theorems are postponed to the next section. Here are some comments on the properties considered and on the results obtained in this section.

• Subdifferentiability spaces (that is, Banach spaces verifying (DS)) were introduced by Ioffe, first for the ε-subdifferentials of Fréchet and Hadamard (Dini) [15], then for arbitrary subdifferentials [18].

• Properties of Compact Separation type (CS) and (w^*–CS) were considered for the first time by Ioffe [15]. Their connections with (DS) were investigated in Fabian [12, 13] and Ioffe [18].

• Strong forms of the Lipschitz Separation property (LS) (with the norm topology in X^* instead of the weak* topology) were first considered in [15, 13] for the ε-subdifferentials ($\varepsilon \geq 0$) of Fréchet and Dini (Hadamard), and then in [32, 18, 22, 21] for any subdifferential; in the latter papers, it is shown that this property (called 'basic fuzzy principle' in [18]) is equivalent to many other subdifferential rules involving the strong topology in X^*. The *sequential* weak* form of (LS) was considered in [30, 8]. The topological weak* property (w^*–LS) stated here was studied in [21]. We don't know whether this property is equivalent to its sequential weak* variant, neither a fortiori to its strong variant.

• The w^*-Minimality Property of ∂_a^I is discussed, e.g., in [17, 2, 19], for Lipschitz continuous functions and bornological subdifferentials. The general situation (lower semicontinuous functions, any subdifferential) is considered in [21].

• The (mixed) w^*-sum rule (w^*–SR) was introduced in [21].

• Theorem 3.1 generalizes Fabian [13, Lemma 3] which focuses on the Fréchet and Hadamard (Dini) subdifferentials (ε-subdifferentials were treated earlier in Fabian [12]). The idea of using convolution for the proof is taken from this paper. Ioffe [18, Theorem 4] states without proof the equivalence between (DS) and a strong Compact Separation type property for the smooth β-subdifferentials.

• Theorem 3.2 generalizes results of Ioffe [15, 18] and Fabian [13] that concerned only special subclasses of bornological subdifferentials.

To be complete, it is worth mentioning that the dense subdifferentiability property for a subdifferential on the class of lower semicontinuous functions forces the full subdifferentiability property for its w^*-closure on the class of locally Lipschitz continuous functions. More precisely, we have:

Proposition 3.3. *Let $\partial \subset \partial_C$ be any subdifferential. Let X be a ∂-subdifferentiability space and $\varphi : X \to \mathbb{R}$ be Lipschitz continuous near $\bar{x} \in X$. Then*

$$w^*\text{-}\limsup_{x \to \bar{x}} \partial\varphi(x) \neq \emptyset.$$

Moreover, if the unit ball of X^* is sequentially weak compact, then*

$$w^*\text{-}\limsup_{x_n \to \bar{x}} \partial\varphi(x_n) \neq \emptyset.$$

Proof. The arguments are standard. For φ locally Lipschitz continuous, the mapping $\partial_C\varphi$ is locally bounded, hence also is $\partial\varphi$. By (SD), there exists a sequence $\{(x_n, x_n^*)\} \subset \partial\varphi$ such that $x_n \to \bar{x}$. The sequence $\{x_n^*\}$ is therefore bounded, hence it admits a subnet $\{x_\nu^*\}$ w^*-convergent to some $\bar{x}^*$. By definition, $\bar{x}^* \in w^*\text{–}\limsup_{x\to\bar{x}} \partial\varphi(x)$, showing the nonemptyness of this set.

In the case where the unit ball of X^* is sequentially weak* compact, then the above sequence $\{x_n^*\}$ admits a w^*-convergent subsequence whose limit of course belongs to the set $w^*\text{–}\limsup_{x_n\to\bar{x}} \partial\varphi(x)$. □

4. Proofs of the theorems of Section 3.

Proof of Theorem 3.1. To prove (CS), let $f : X \to (-\infty, +\infty]$ be lower semicontinuous, $\varphi : X \to \mathbb{R}$ be convex inf-compact near $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom}\varphi$, and assume that $f + \varphi$ admits a strong local minimum at $\bar{x}$. Take $\lambda > 0$ so that, on the closed ball $\bar{x} + 2\lambda B_X$, $\bar{x}$ realizes the strong minimum of $f + \varphi$, f is lower bounded by μ and φ is inf-compact (hence lower semicontinuous). Set $V := \bar{x} + \lambda B_X$ and denote by f_V the function equal to f in V and to $+\infty$ outside V. Then f_V is lower semicontinuous and lower bounded, so the inf-convolution $g = f_V \nabla \varphi^-$ is lower semicontinuous and exact at every point of X (see, e.g., Moreau [25, Proposition 4.e] or Laurent [23, Proposition (6.5.5)]). Notice that $g(0) = \inf(f_V + \varphi) = (f + \varphi)(\bar{x})$.

First consider the case of a subdifferential ∂ satisfying Axioms (A1)–(A3). The property (DS) applied to the function g at point $0 \in \operatorname{dom} g$ produces a sequence $\{z_n\} \subset X$ such that $z_n \to 0$, $g(z_n) \to g(0)$ and $\partial g(z_n) \neq \emptyset$. The inf-convolution $f_V \nabla \varphi^-$ being exact at every point, for every $n \in \mathbb{N}$ there exists $x_n \in V$ such that

$$g(z_n) = f(x_n) + \varphi(x_n - z_n). \tag{10}$$

We have $f(x_n) \geq \mu$ and $f(x_n) + \varphi(x_n - z_n) \to g(0)$, hence, for n large enough, the sequence $\{x_n - z_n\}$ lies in the compact set

$$\{z \in \bar{x} + 2\lambda B_X : \varphi(z) \leq g(0) + 1 - \mu\}.$$

We may therefore suppose, without loss of generality, that the two sequences $\{x_n - z_n\}$ and $\{x_n\}$ converge to some point $x_0 \in \bar{x} + 2\lambda B_X$, in fact $x_0 \in V$ because $\{x_n\} \subset V$. From the lower semicontinuity of f and φ on $\bar{x} + 2\lambda B_X$ we derive that

$$\begin{aligned} f(x_0) + \varphi(x_0) &\leq \liminf_{n\to\infty} f(x_n) + \liminf_{n\to\infty} \varphi(x_n - z_n) \\ &\leq \lim_{n\to\infty} (f(x_n) + \varphi(x_n - z_n)) = g(0). \end{aligned}$$

Thus, $(f+\varphi)(x_0) = g(0) = \inf(f_V+\varphi)$, whence $x_0 = \bar{x}$ since $\bar{x}$ is a strong minimum of $f_V+\varphi$. Finally, the sequences $\{x_n\}$ and $\{x_n - z_n\}$ satisfy

$$x_n \to \bar{x}, \quad f(x_n) \to f(\bar{x}), \tag{11}$$

$$x_n - z_n \to \bar{x}, \quad \varphi(x_n - z_n) \to \varphi(\bar{x}). \tag{12}$$

On the other hand, $g(z_n) = f(x_n) + \varphi(x_n - z_n)$ by (10) and $\partial g(z_n)$ is non empty, so we may apply (A3) with $x := x_n$, $y := x_n - z_n$, $f_1 := f_V$, $f_2 := \varphi$, to obtain

$$0 \in \partial f_V(x_n) + \partial\varphi(x_n - z_n).$$

Now, since $x_n \to \bar{x}$, V is a neighborhood of x_n for n large enough, hence $\partial f_V(x_n) = \partial f(x_n)$ for n large enough by (A2). This completes the proof in the case of a subdifferential ∂.

Consider next the case of any closure $\hat{\partial}$ of a subdifferential ∂ satisfying Axioms (A1)–(A3). The property (DS) applied to g at $0 \in \operatorname{dom} g$ produces a sequence $\{\bar{z}_n\} \subset X$ such that $\bar{z}_n \to 0$, $g(\bar{z}_n) \to g(0)$ and $\hat{\partial} g(\bar{z}_n) \neq \emptyset$. By definition of $\hat{\partial} g$, we can find a sequence $\{(z_n, z_n^*)\} \subset \partial g$ such that, for every $n \in \mathbb{N}$, $d_g(\bar{z}_n, z_n) \leq 1/n$, hence $z_n \to 0$, $g(z_n) \to g(0)$ and $\partial g(z_n) \neq \emptyset$. Thus we have arrived at the same situation as above, so there are sequences $\{x_n\}$ and $\{x_n - z_n\}$ satisfying (11), (12) and

$$0 \in \partial f(x_n) + \partial\varphi(x_n - z_n) \subset \hat{\partial} f(x_n) + \hat{\partial}\varphi(x_n - z_n).$$

The proof is complete. □

P r o o f o f T h e o r e m 3.2. By the observation preceding Theorem 3.1, the first six properties are equivalent for any subdifferential ∂ or any of its closures $\hat{\partial}$. In particular, (DS) ⇒ (w^*-SR). Obviously, (w^*-SR) ⇒ (w^*-Tr) since $\partial \subset \partial_a^I$, hence also $\hat{\partial} \subset \partial_a^I$. On the other hand, always (w^*-Tr) ⇒ (w^*-LS), and, thanks again to the above equivalence, (w^*-LS) ⇒ (DS). This shows that (w^*-Tr) is equivalent to the first six properties. □

REFERENCES

[1] E. Asplund. Fréchet differentiability of convex functions. *Acta Math.* **121** (1968), 31–47.

[2] J. M. Borwein, A. D. Ioffe. Proximal analysis in smooth spaces. *Set-Valued Anal.* **4** (1996), 1–24.

[3] J. M. Borwein, D. Preiss. A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions. *Trans. Amer. Math. Soc.* **303** (1987), 517–527.

[4] J. M. Borwein, Q. J. Zhu. Viscosity solutions and viscosity subderivatives in smooth Banach spaces with applications to metric regularity. *SIAM J. Control Optim.* **34** (1996), 1568–1591.

[5] F. H. Clarke. Necessary conditions for nonsmooth problems in optimal control and the calculus of variations. Ph.D. Thesis, Univ. of Washington, 1973.

[6] F. H. Clarke. Optimization and Nonsmooth Analysis. Wiley-Interscience, New-York, 1983.

[7] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski. Nonsmooth Analysis and Control Theory. Graduate Texts in Mathematics Vol. **178**, Springer-Verlag, New York, 1998.

[8] R. Correa, P. Gajardo, L. Thibault. Links between directional derivatives through multidirectional mean value inequalities. *Math. Program., Ser. B* **116** (2009), 57–77.

[9] R. Correa, A. Jofré, L. Thibault. Characterization of lower semicontinuous convex functions. *Proc. Amer. Math. Soc.* **116** (1992), 67–72.

[10] R. Deville, G. Godefroy, V. Zizler. A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions. *J. Funct. Anal.* **111** (1993), 197–212.

[11] I. Ekeland, G. Lebourg. Generic Fréchet differentiability and pertubed optimization problems. *Trans. Amer. Math. Soc.* **47** (1974), 324–353.

[12] M. Fabian. On classes of subdifferentiability spaces of Ioffe. *Nonlinear Anal.* **12** (1988), 63–74.

[13] M. Fabian. Subdifferentiability and trustworthiness in the light of the new variational principle of Borwein and Preiss. *Acta Univ. Carolin. Math. Phys.* **30**, *2* (1989), 51–56.

[14] M. Fabian, N. V. Zhivkov. A characterization of Asplund spaces with help of local ε-supports of Ekeland and Lebourg. *C. R. Acad Bulg. Sci.* **38**, 671–674 (1985).

[15] A. D. Ioffe. *On subdifferentiability spaces*, Ann. New York Acad. Sci., **410**, (1983), 107–119.

[16] A. D. Ioffe. Approximate subdifferentials and applications, Part II. *Mathematika* **33** (1986), 111–128.

[17] A. D. Ioffe. Approximate subdifferentials and applications, Part III. *Mathematika* **36** (1989), 1–38.

[18] A. D. Ioffe. Fuzzy principles and characterization of trustworthiness. *Set-Valued Anal.* **6** (1998), 265–276.

[19] A. D. Ioffe. Codirectional compactness, metric regularity and subdifferential calculus. In: Constructive, experimental, and nonlinear analysis, Limoges, 1999, CMS Conf. Proc. vol. **27**, Amer. Math. Soc., Providence, RI, 2000, 123–163.

[20] A. D. Ioffe, J.-P. Penot. Subdifferentials of performance functions and calculus of coderivatives of set-valued mappings. *Serdica Math. J.* **22**, *3* (1996), 359–384.

[21] F. Jules. Sur la somme de sous-différentiels de fonctions semi-continues inférieurement. Dissertationes Mathematicae **423** (2003), 62 pp.

[22] M. Lassonde. First order rules in nonsmooth contrained optimization. *Nonlinear Anal.* **44**, *8* (2001), Ser. A: Theory Methods, 1031–1056.

[23] P.-J. Laurent. Approximation et optimisation. Collection Enseignement des Sciences Vol. **13**, Hermann, Paris, 1972.

[24] B. S. Mordukhovich. Variational analysis and generalized differentiation I & II, Fundamental Principles of Mathematical Sciences Vol **330** & **331**, Springer-Verlag, Berlin, 2006.

[25] J.-J. Moreau. Fonctionnelles convexes, séminaire équations aux dérivées partielles. Collège de France, Paris, 1966.

[26] R. R. Phelps. Convex Functions, Monotone Operators and Differentiability. Lect. Notes in Math. Vol **1364**, Springer-Verlag, Berlin, 1993.

[27] R. T. Rockafellar. Generalized directional derivatives and subgradients of nonconvex functions. *Canad. J. Math.* **32** (1980), 257–280.

[28] R. T. Rockafellar. Proximal subgradients, marginal values and augmented Lagrangians in nonconvex optimisation. *Math. Oper. Res.* **6** (1981), 424–436.

[29] R. T. Rockafellar, R. Wets. Variational Analysis. Springer, New York, 1997.

[30] L. Thibault, D. Zagrodny. Integration of subdifferentials of lower semicontinuous functions. *J. Math. Anal. Appl.* **189** (1995), 33–58.

[31] C. Zălinescu. Convex analysis in general vector spaces. World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[32] Q. J. Zhu. The equivalence of several basic theorems on subdifferentials. *Set-Valued Anal.* **6** (1998), 171–185.

Florence Jules, Marc Lassonde
LAMIA
Université des Antilles et de la Guyane
97159 Pointe à Pitre, France
e-mail: `florence.jules@univ-ag.fr`
e-mail: `marc.lassonde@univ-ag.fr`

Received November 11, 2010
Revised January 13, 2011